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Analysis and numerical solution of control problems in descriptor form

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Analysis and numerical solution of control problems in descriptor form[1]

Peter Kunkel Volker Mehrmann Werner Rath

Abstract

We study linear variable coefficient control problems in descriptor form. Based on a behaviour approach and the general theory for linear differential algebraic systems we give the theoretical analysis and describe numerically stable methods to determine the structural properties of the system like solvability, regularity, model consistency and redundancy. We also discuss regularization via feedback.

Keywords: descriptor systems, differential-algebraic equations, s-index, regularization, feedback design

AMS(MOS) subject classification: 93C50, 65L05, 34H05, 93B10, 93B11, 93B40

1 Introduction

In this paper we study control problems of the form

$$\begin{aligned} E(t)\dot{x} &= A(t)x + B(t)u + f(t), && (1)\\ y &= C(t)x + g(t) && (2) \end{aligned}$$

in a given interval $[t_0, t_f]$, with initial condition

$$x(t_0) = x^0. \tag{3}$$

Here x is the state, u is the input, y is the output of the system. If we denote by $C^r([t_0, t_f], \mathbb{C}^{n,\ell})$ the set of r-times continuously differentiable functions from the interval $[t_0, t_f]$ to the vector space $\mathbb{C}^{n,\ell}$ of complex $n \times \ell$ matrices, then we assume that $E \in C([t_0, t_f], \mathbb{C}^{n,\ell})$, $A \in C([t_0, t_f], \mathbb{C}^{n,\ell})$, $B \in C([t_0, t_f], \mathbb{C}^{n,m})$, $C \in C([t_0, t_f], \mathbb{C}^{p,\ell})$.

Control problems of this form arise in mechanical multibody systems [20, 19, 35], electrical circuits [18] or mixed systems, where different models are coupled together. In this general form they allow to model very complex dynamical systems with constraints, models that are automatically generated with redundant equations or combinations of models of different types, see, e.g., [19]. In particular systems of the form (1)–(2) also arise as linearizations of general nonlinear control problems of the form

$$\begin{aligned} \mathcal{F}(t, x, u, \dot{x}) &= 0, && (4)\\ y &= \mathcal{G}(t, x), && (5) \end{aligned}$$

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where the linearization yields

$$\begin{aligned} E(t) &= \frac{\partial \mathcal{F}}{\partial \dot{x}}|_{(\hat{x}(t),\hat{u}(t))}, \quad A(t) = \frac{\partial \mathcal{F}}{\partial x}|_{(\hat{x}(t),\hat{u}(t))}, \\ B(t) &= \frac{\partial \mathcal{F}}{\partial u}|_{(\hat{x}(t),\hat{u}(t))}, \quad C(t) = \frac{\partial \mathcal{G}}{\partial x}|_{(\hat{x}(t),\hat{u}(t))}. \end{aligned} \tag{6}$$

To analyze the system behaviour and to control systems of the form (1)–(2), we need to develop the mathematical theory as well as numerical methods that can be used for the analysis, design and simulation.

Since for a given input function u, the system (1)–(2) represents a differential-algebraic equation (DAE), it is clear that the theory for the control problem is related to the theory of DAEs. This theory has undergone major changes in the last 10 years, see [17, 9, 10, 3, 23, 30, 31, 24, 25, 28]. Several attempts have been made to transfer this theory to the study of control problems [11, 12, 8, 32] but a major drawback of these attempts was that they did not lead to a practical numerical method.

Recently in [23, 26, 28] a new theoretical analysis and an associated numerical method have been introduced that allow to study over- and underdetermined systems and have the potential to study systems of the form (1)–(2) via a behaviour approach (see [38]). Such an approach combines state, input and output variables into one system vector and then studies the combined system. The main difficulty with this approach for descriptor systems is that it needs derivatives of the system and thus also, via the chain rule, derivatives of the inputs u. But in practice the input function is usually only piecewise continuous (like in bang-bang control). This makes a general analysis very difficult. A way out of this dilemma may be the use of generalized functions as suggested in [31] for linear time varying DAEs, but this approach has neither been extended to nonlinear systems nor to linear systems that are over- or underdetermined.

We follow a different direction in this paper and extend the concepts introduced in [23, 26, 27] to control problems. We will analyze several aspects of the control system (1)–(2). First we give a theoretical analysis using local and global condensed forms under equivalence transformations and second we construct numerical methods for the computation of the invariants of these condensed forms. In both cases we discuss solvability, regularity, consistency and also regularization by feedback. Also we discuss how the new approach can be used for model verification and model reduction. Furthermore, if the system properties are not as desired, then they can sometimes be changed by feedback.

Since the concepts for DAEs have changed in recent years, we need to recall some of the terminology that we will be using.

Definition 1 Given an input function u, a function $x : [t_0, t_f] \to \mathbb{C}^n$ is called *solution* of (1) if $x \in C^1([t_0, t_f], \mathbb{C}^n)$ and x satisfies (1) pointwise. It is called *solution of the initial value problem* (1), (3) if x is solution of (1) and x satisfies (3). An initial condition (3) is called *consistent* if the corresponding initial value problem has at least one solution.

While solvability is associated with a particular input function we also need a terminology that allows to check whether the system is solvable for some input function or for every piecewise continuous input function and every (with this input function) consistent initial value.

Definition 2 A control problem of the form (1) is called *consistent* if there exists an input function u for which there exists a solution.

It is called *regular* if it has a unique solution for every sufficiently smooth input function u and inhomgeneity f and every initial value that is consistent for the system with this input function.

Note that in the case of constant coefficients regularity is usually associated with the matrix pair (E, A), but redundancy is usually excluded by assuming that the block matrix $[E, A, B]$ has full row rank, and that the system is homogeneous. In the general nonlinear case or the time-varying case, in particular when the problem arises from automatic model generation, such an assumption would not be appropriate, since it would be one of the tasks of the analysis to determine redundancies and inconsistencies in the model.

2 Theoretical analysis.

In this section we analyse the control problem via equivalence transformations and condensed forms. This is a straightforward generalization of the work in [23] and has been carried out in detail in [33, 32]. The forms that we discuss here are ordered slightly different than those in [32] but can be obtained in a similar way. After having obtained the condensed forms, we analyze the properties of the system and discuss how these properties can be modified by feedback.

2.1 Condensed forms

The standard variable coefficient equivalence transformations that we can apply to a linear descriptor system with variable coefficients are given in the following definition.

Definition 3 Let (E_i, A_i, B_i, C_i), $i = 1, 2$, be two quadruples of matrix functions with $E_i, A_i \in C([t_0, t_f], \mathbb{C}^{n,l})$, $B_i \in C([t_0, t_f], \mathbb{C}^{n,m})$, $C_i \in C([t_0, t_f], \mathbb{C}^{p,l})$. These quadruples are called *(globally) equivalent* if there exist $N \in C([t_0, t_f], \mathbb{C}^{p,p})$, $P \in C([t_0, t_f], \mathbb{C}^{n,n})$, $Q \in C^1([t_0, t_f], \mathbb{C}^{l,l})$ and $R \in C([t_0, t_f], \mathbb{C}^{m,m})$ with N, P, Q, R pointwise nonsingular, such that

$$\begin{array}{rcl} E_2 & = & PE_1Q, \\ A_2 & = & PA_1Q - PE_1\dot{Q}, \\ B_2 & = & PB_1R, \\ C_2 & = & NC_1Q. \end{array} \tag{7}$$

As in the case of linear DAEs [23], we get the corresponding local equivalence by choosing $\dot{Q}(t)$ independently of $Q(t)$ at a fixed point $t \in [t_0, t_f]$.

Definition 4 Two quadruples of matrices (E_i, A_i, B_i, C_i), $i = 1, 2$, with $E_i, A_i \in \mathbb{C}^{n,l}$, $B_i \in \mathbb{C}^{n,m}$, $C_i \in \mathbb{C}^{p,l}$ are called *(locally) equivalent* if there exist matrices $N \in \mathbb{C}^{p,p}$, $P \in \mathbb{C}^{n,n}$, $Q, S \in \mathbb{C}^{l,l}$, $R \in \mathbb{C}^{m,m}$ with N, P, Q, R nonsingular such that

$$\begin{array}{rcl} E_2 &=& PE_1Q, \\ A_2 &=& PA_1Q - PE_1S, \\ B_2 &=& PB_1R, \\ C_2 &=& NC_1Q. \end{array} \tag{8}$$

Using the local equivalence transformation in (8), we obtain the following local invariants and condensed form for a quadruple of matrices (E, A, B, C).

Theorem 5 *Let $E, A \in \mathbb{C}^{n,l}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{p,l}$ and*

$$\begin{array}{ll} (a) & T \textit{ basis of } \operatorname{kernel} E, \\ (b) & Z \textit{ basis of } \operatorname{corange} E = \operatorname{kernel} E^*, \\ (d) & V \textit{ basis of } \operatorname{corange}(Z^*AT), \\ (e) & W \textit{ basis of } \operatorname{kernel}(Z^*AT), \\ (f) & K \textit{ basis of } \operatorname{corange}(V^*Z^*B). \end{array} \tag{9}$$

Then, the quantities

$$\begin{array}{lll} (a) & r = \operatorname{rank} E & (\textit{rank}), \\ (b) & a = \operatorname{rank}(Z^*AT) & (\textit{algebraic part}), \\ (c) & \phi = \operatorname{rank}(V^*Z^*B) & (\textit{state feedback part}), \\ (d) & \omega = \operatorname{rank}(CTW) & (\textit{output feedback part}), \\ (e) & s = \operatorname{rank}(K^*V^*Z^*AT') & (\textit{strangeness}), \\ (f) & d = r - s & (\textit{differential part}), \\ (g) & u^l = n - r - a - s - \phi & (\textit{left undetermined part}), \\ (h) & u^r = l - r - a - \omega & (\textit{right undetermined part}), \end{array} \tag{10}$$

are invariant under (8) and (E, A, B, C) is equivalent to the condensed form

$$\begin{array}{c} s \\ d \\ a \\ s \\ \phi \\ u^l \end{array} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \\ 0 & * \\ 0 & 0 \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \right. \tag{11}$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right).$$

Here the last column in the first, second and fourth matrix has width u^r.

Proof.
The proof is a simple extension of the proof for the DAE case or the approach discussed in [32]. The complete proof from [33] with all details is given in Appendix A. □

Note that (11) is not a canonical form but rather a condensed form, since the blocks denoted by $*$ can be partitioned further, when proceeding to a canonical form, see [33].

If we apply the results for the local condensed form (11) to equation (1)–(2) then we obtain functions $r, a, \phi, \omega, s : [t_1, t_2] \to \mathbb{N}_0$ (the other values depend on these invariants). For the following analysis, we assume that these quantities are constant, i.e.,

$$r(t) \equiv r,\ a(t) \equiv a,\ \phi(t) \equiv \phi,\ \omega(t) \equiv \omega,\ s(t) \equiv s \tag{12}$$

on $[t_0, t_f]$, i.e., we require the local characteristic values at a fixed point to bear global information of the solution. Then we get the following global condensed form:

Theorem 6 *Let E, A, B, C in (1)–(2) be sufficiently smooth and let (12) hold. Then the quadruple (E, A, B, C) is equivalent to a quadruple of matrix functions of the form*

$$\begin{array}{c} s \\ d \\ a \\ s \\ \phi \\ u^l \end{array} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \\ 0 & 0 \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 \end{bmatrix} \right). \tag{13}$$

Here the columns in the first, second and fourth matrix have dimensions s, d, a, ω and u^r, respectively.

Proof. The proof is analogous to the proof of Theorem 4.3 in [23]. For details see Appendix B or [33]. □

Writing down the descriptor system equations that belong to the matrix quadruple (13), (by transforming the inhomogeneities accordingly), we get

$$\begin{array}{ll} (a) & \dot{x}_1 = A_{12}(t)x_2 + A_{14}(t)x_4 + A_{15}(t)x_5 + B_{12}(t)u_2 + f_1(t), \\ (b) & \dot{x}_2 = A_{24}(t)x_4 + A_{25}(t)x_5 + B_{22}(t)u_2 + f_2(t), \\ (c) & 0 = x_3 + B_{32}(t)u_2 + f_3(t), \\ (d) & 0 = x_1 + f_4(t), \\ (e) & 0 = A_{52}(t)x_2 + u_1 + f_5(t), \\ (f) & 0 = f_6(t) \end{array} \tag{14}$$

and the corresponding output equations

$$\begin{array}{ll} (a) & y_1 = x_4 + g_1(t), \\ (b) & y_2 = C_{21}(t)x_1 + C_{22}(t)x_2 + C_{23}(t)x_3 + g_2(t). \end{array} \tag{15}$$

From equation (14d) we see that $x_1 \equiv f_4$, i.e., x_1 is fixed. If the inhomogeneity is smooth enough, then by inserting $\dot{x}_1 \equiv \dot{f}_4$ in (14a) we get an algebraic equation (note that the inhomogeneity changes). Doing the same in the output equations leads to zeroing out the block C_{21} and changing the inhomogeneity g_2.

Remark 1 Note that the equations in (14d) are uncontrollable, i.e., independent of the input function u. It is very important that only these equations are differentiated, since otherwise derivatives of the input functions would be needed, which would restrict the set of admissable input functions. Note furthermore, that equations (14)–(15) indicate some of the problems which occur in the analysis of descriptor systems that exhibit a coupling of algebraic and differential equations as via x_1 in (14) (see also [8]). Theoretically the state components in x_1 are fixed via (14d). But in the descriptor system (1)–(2) there exist linearization errors and there are usually already modelling errors in (4)–(5). Hence we cannot expect that (14d) is fulfilled exactly in the practical problem for which we use our system as a model. Small perturbations may occur on both sides of the equation and may lead to structural changes.

For the resulting system, obtained from (14) by differentiating the equations in (14d) and inserting in (14a) and (15), we can again compute characteristic values $r, a, \phi, o, s, d, u^l, u^r$ and the condensed form (14)–(15) and we can then proceed inductively. This procedure leads to a sequence of quadruples of matrix functions $(E_i, A_i, B_i, C_i),\ i \in \mathbb{N}_0$. Starting with the quadruple $(E_0, A_0, B_0, C_0) = (E, A, B, C)$ the matrix quadruple $(E_{i+1}, A_{i+1}, B_{i+1}, C_{i+1})$ is derived from (E_i, A_i, B_i, C_i) by bringing it into the form (14)–(15) and inserting equation (14d) into (14a) and (15). Note that although (E, A, B, C) does not determine a unique sequence (E_i, A_i, B_i, C_i), the corresponding characteristic quantities $r_i, a_i, \phi_i, \omega_i, s_i$ are uniquely determined, see [32]. Clearly for this we must assume that in every step i we have

$$r_i(t) \equiv r_i,\ a_i(t) \equiv a_i,\ \phi_i(t) \equiv \phi_i,\ \omega_i(t) \equiv \omega_i,\ s_i(t) \equiv s_i. \tag{16}$$

If these conditions hold then, since the sequence (r_i) strictly decreases unless s_i becomes 0, there exists $\mu \in \mathbb{N}_0$, such that this process becomes stationary with $s_\mu = 0$.

This index s_μ is called the *strangeness index* or *s-index* of the system in [23, 33, 32]. It is defined if (16) holds for $i = 0, \ldots, \mu$. In [25] it was shown that for systems with unique solutions condition (16) can be significantly relaxed, only the final numbers a_μ, d_μ have to be constant. The proof in [25], however, does not extend to underdetermined systems.

If the s-index μ is well defined, then we have the following condensed form.

Theorem 7 *If the s-index μ is well defined for the quadruple of matrix functions (E, A, B, C) in (1)–(2), then system (1)–(2) is equivalent (in the sense that the solution* *sets are in one-to-one correspondence via a scaling by nonsingular matrix functions) to a descriptor system of the form*

$$\begin{array}{lrcl}(a) & \dot{x}_1 & = & A_{13}(t)x_3 + A_{14}(t)x_4 + B_{12}(t)u_2 + f_1(t), \\ (b) & 0 & = & x_2 + B_{22}(t)u_2 + f_2(t), \\ (c) & 0 & = & A_{31}(t)x_1 + u_1 + f_3(t), \\ (d) & 0 & = & f_4(t), \end{array} \tag{17}$$

with associated output equations

$$\begin{array}{lrcl}(a) & y_1 & = & x_3 + g_1(t), \\ (b) & y_2 & = & C_{21}(t)x_1 + C_{22}(t)x_2 + g_2(t) \end{array} \tag{18}$$

and initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ x_4(t_0) \end{bmatrix} = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \end{bmatrix}. \tag{19}$$

Here d_μ, a_μ, ω_μ and u_μ^r are the number of the differential, algebraic, output feedback and undetermined components of the state x, whereas ϕ_μ and u_μ^l are the number of equations in (17c) and (17d).

Proof. The proof is analogous to the proof for DAEs in [23]. Since it is very technical we refer the reader for details to [33]. □

Remark 2 It should be noted that in general the s-index and the characteristic quantities a, d of the control system (1-2) are not equal to corresponding quantities of the DAE obtained by fixing an input function u. Consider the following example. Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then the system has $\mu = 0, \phi_0 = 1, a_0 = 0, d_0 = 1, u_0^l = 0$ while the DAE obtained for a fixed input function has $\mu = 1, a_1 = 0, d_1 = 1, u_1^l = 1$.

A direct consequence of Theorem 7 is the following integral equation obtained by integrating and inserting the first equation of (17):

$$\begin{array}{lrcl}(a) & 0 & = & x_1(t) - \int_{t_0}^t [A_{13}(\tau)x_3(\tau) + A_{14}(\tau)x_4(\tau) + B_{12}(\tau)u_2(\tau) + f_1(\tau)]\, d\tau + x_1^0, \\ (b) & 0 & = & x_2(t) + B_{22}(t)u_2(t) + f_2(t), \\ (c) & 0 & = & A_{31}(t)\int_{t_0}^t [A_{13}(\tau)x_3(\tau) + A_{14}(\tau)x_4(\tau)]\, d\tau + u_1(t) + f_3(t) \\ & & & + A_{31}(t)\int_{t_0}^t [B_{12}(\tau)u_2(\tau) + f_1(\tau)]\, d\tau + A_{31}(t)x_1^0, \\ (d) & 0 & = & f_4(t). \end{array} \tag{20}$$

2.2 Solvability, regularity and consistency

From the condensed form (17)–(18) we can read off several of the system properties.

Corollary 8 *If a descriptor system has the condensed form (17)–(18), then we have the following properties :*

1. *The system is consistent if and only if either* $u^l_\mu = 0$ *or* $f_4 \equiv 0$*. If* $u^l_\mu \neq 0$ *and* $f_4(t) \equiv 0$ *then the equations in (17d) describe redundancies in the system that can be omitted.*

2. *If the system is consistent and if* $\phi_\mu = 0$*, then for a given input function* u*, an initial condition is consistent if and only if it satisfies (17b). Solutions of the corresponding initial value problem will in general not be unique, since the state components in* x_3, x_4 *are not determined.*

3. *The system is regular and of s-index* 0 *(as free system, i.e., for* $u = 0$*), if and only if* $u^l_\mu = 0$*,* $\phi_\mu = 0$ *and* $d_\mu + a_\mu = l$*.*

Proof.

1. If $f_4 \not\equiv 0$ and $u^l_\mu \neq 0$, then clearly the system has no solution, regardless how we choose the input function. Conversely if either $u^l_\mu = 0$ or $f_4 \equiv 0$ then we can determine an imput u for which the system is solvable. Setting $u_2 = 0$, $x_3 = 0$ and $x_4 = 0$, system (17 a) is an ordinary differential equation for x_1. Having fixed x_1 by solving this equation, we obtain x_2 from (17b) and u_1 from (17c)

2. A consistent system with $\phi_\mu = 0$ reduces to (17a,b) which is a differential-algebraic system with s-index 0 for every input function u and (17b) represents the algebraic part which gives the consistency condition for the initial value, see, e.g., [24]. Note that the solution will in general not be unique.

3. Let us first assume that $u^l_\mu = \phi_\mu = 0$ and $d_\mu + a_\mu = l$. In this case (17) reduces to the system
$$\begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{12}(t)u_2 + f_1(t) \\ B_{22}(t)u_2 + f_2(t) \end{bmatrix}$$
which is uniquely solvable for every input function and every inhomogeneity. Moreover, it has s-index 0 for $u_2 = 0$. Conversely let the system be regular and have s-index 0 for $u = 0$. If $\phi_\mu \neq 0$ then (17c) gives either consistency conditions for the inhomogeneity or an s-index larger than 0, in contradiction to the assumptions. Hence $\phi_\mu = 0$. If $u^l_\mu \neq 0$, then the system is not sovable for every inhomogeneity. Thus u^l_μ. Finally if $d_\mu + a_\mu \neq l$, then fixing an input u yields a nonsquare differential-algebraic system for x resulting either in conistency conditions for the inhomogeniety or free solution components. Hence we also have $d_\mu + a_\mu \neq l$. □

Ideally we would like our system to be regular and have s-index 0 (as a free system with $u = 0$.) While regularity guarantees unique solvability for a large class of input functions, it does not guarantee that the s-index is 0 (as a free system). If the s-index is not 0 for the free system there may be hidden manifolds leading to differentiability constraints for the input function or extra constraints for the initial values or redundancies. For regular systems of s-index 0 (as a free system), differential and algebraic components (fast and slow modes) can be decoupled, see [8]. If the system does not have these nice properties already, and Corollary 8 tells us when it has or has not, then we can modify the system to obtain these properties. This can be done in several different ways.

The first possibility is a reinterpretation of variables in the condensed form (17)–(18), see also [7, 8]. The components in x_3, x_4 can be viewed as inputs, since they can be chosen freely. So the 'real' input of the system is given by x_3, x_4 and u_2. Furthermore for given x_3, x_4 and u_2 the components in x_1, x_2 and u_1 are fixed and hence they represent the 'real' states of the system. Also, the system has a *feedthrough term*, since the 'new' input x_3 is directly obtained as $y_1 = x_3 + g_1$. If it is not intended that we measure again the input variables in x_3, then this part can be omitted from the output equation.

If the system is consistent, i.e., either $f_4 \equiv 0$ or $u^l_\mu = 0$, then by omitting the redundancies and reinterpreting the variables, we obtain an *underlying square system* (see also [7, 8]) which is regular and of s-index 0 (as a free system). This system has the form:

$$
\begin{aligned}
\begin{bmatrix} I_{d_\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u}_1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{a_\mu} & 0 \\ A_{31} & 0 & I_{\phi_\mu} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix} \\
&\quad + \begin{bmatrix} A_{13} & A_{14} & B_{12} \\ 0 & 0 & B_{22} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ u_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \qquad (21) \\
y_2 &= \begin{bmatrix} C_{21} & C_{22} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix} + g_2.
\end{aligned}
$$

Remark 3 The square system (21) can be viewed as a regularization of the original system in the sense that it is regular and has s-index 0 as a free system. In principle further regularization by elimination or differentiation of the algebraic equations is possible. But elimination of the algebraic equations may increase the sensitivity of the system to perturbations (since the elimination process may be ill-conditioned), while differentiation would change the structure of the solution space, in particular one would need further initial conditions.

2.3 Regularization by proportional state or output feedback

If the original system was consistent and is square after removing the redundancies, i.e., $\omega_\mu + u^r_\mu = \phi_\mu$, then we can also obtain a regular system that is of s-index 0 (as a free system) by using state or output feedback. This problem has been studied for constant coefficient systems in detail in [5, 6, 7, 13] and, based on different condensed forms, also in [8, 32].

To study this question we need to analyze how the characteristic quantities in Theorem 7 behave under feedback. It is obvious already from the local condensed form that the quantities a_i, ϕ_i and ω_i cannot be invariants under proportional state or output feedback but we will show in the next section that the quantities μ, d_μ, u^l_μ, u^r_μ are invariant. The regularization via feedback can be obtained directly from Theorem 7.

Corollary 9 *Given a descriptor system with coefficients $E(t), A(t), B(t)$ in the form (17)–(18). There exists a proportional state feedback $u = F(t)x + w$ such that the closed loop system*

$$E(t)\dot{x} = (A(t) + B(t)F(t))x + B(t)w + f(t) \tag{22}$$

is regular and has s-index 0 (as a free system) if and only if $u^l_\mu = 0$ and $d_\mu + a_\mu = l$.

Proof. If $d_\mu + a_\mu \neq l$ then with the feedback gain matrix

$$F = \begin{bmatrix} 0 & 0 & I_{\omega_\mu} & 0 \\ 0 & 0 & 0 & I_{u^r_\mu} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the state feedback

$$u = F \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{23}$$

we obtain the closed loop system

$$\begin{bmatrix} I_{d_\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13} & A_{14} \\ 0 & I_{a_\mu} & 0 & 0 \\ 0 & 0 & I_{\omega_\mu} & 0 \\ 0 & 0 & 0 & I_{u^r_\mu} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & B_{12} \\ 0 & 0 & B_{22} \\ I_{\omega_\mu} & 0 & 0 \\ 0 & I_{u^r_\mu} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_{31} \\ f_{32} \end{bmatrix},$$

which is regular and has s-index 0 as a free system.

For the converse observe first of all that $u^l_\mu = 0$ is necessary to obtain a system that is regular. For a given

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \end{bmatrix}$$

partitioned conformally with (17), we obtain that

$$A+BF=\left[\begin{array}{cccc} B_{12}F_{21} & B_{12}F_{22} & A_{13}+B_{12}F_{23} & A_{14}+B_{12}F_{24} \\ B_{22}F_{21} & I_{a_\mu}+B_{22}F_{22} & B_{22}F_{23} & B_{22}F_{24} \\ A_{31}+F_{11} & F_{12} & F_{13} & F_{14} \end{array}\right].$$

For the free system to be regular and of s-index 0 we need that

$$\left[\begin{array}{ccc} I_{a_\mu}+B_{22}F_{22} & B_{22}F_{23} & B_{22}F_{24} \\ F_{12} & F_{13} & F_{14} \end{array}\right]$$

has full column rank, i.e., $d_\mu + a_\mu = l$. □

We also have a characterization when the system can be regularized by output feedback.

Corollary 10 *Given a descriptor system with coefficients $E(t), A(t), B(t), C(t)$ in the form (17)–(19). There exists an output feedback $u = F(t)y + w$ such that the closed loop system*

$$E(t)\dot{x} = (A(t) + B(t)F(t)C(t))x + B(t)w + f(t) + B(t)F(t)g(t) \tag{24}$$

is regular and has s-index 0 (as a free system) if and only if $u^l_\mu = 0$, $u^r_\mu = 0$ and $\omega_\mu = \phi_\mu$.

Proof. If $u^l_\mu = 0$, $u^r_\mu = 0$ and $\omega_\mu = \phi_\mu$, then with

$$F=\left[\begin{array}{cc} I_{\phi_\mu} & 0 \\ 0 & 0 \end{array}\right]$$

and the output feedback

$$u = F\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] + \left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] = \left[\begin{array}{c} x_3 + g_1 + w_1 \\ w_2 \end{array}\right]$$

we obtain the closed loop system

$$\begin{aligned}\left[\begin{array}{ccc} I_{d_\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array}\right] &= \left[\begin{array}{ccc} 0 & 0 & A_{13} \\ 0 & I_{a_\mu} & 0 \\ 0 & 0 & I_{\phi_\mu} \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] \\ &+ \left[\begin{array}{cc} 0 & B_{12} \\ 0 & B_{22} \\ I_{\phi_\mu} & 0 \end{array}\right]\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] + \left[\begin{array}{c} f_1 \\ f_2 \\ f_3 + g_1 \end{array}\right].\end{aligned}$$

This system is clearly regular and has a free system of s-index 0.

For the converse again it is clear that $u^l_\mu = 0$ is necessary. For given

$$F=\left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array}\right]$$

partitioned conformally with (17) and (18), we obtain that

$$A + BFC = \begin{bmatrix} B_{12}F_{22}C_{21} & B_{12}F_{22}C_{22} & A_{13} + B_{12}F_{23} & A_{14} + B_{12}F_{21} \\ B_{22}F_{22}C_{21} & I_{a_\mu} + B_{22}F_{22}C_{22} & B_{22}F_{21} & 0 \\ A_{31} + F_{12}C_{21} & F_{12}C_{22} & F_{11} & 0 \end{bmatrix}.$$

The same argument as in the proof of Corollary 8, shows that we must have that

$$\begin{bmatrix} I_{a_\mu} + B_{22}F_{22}C_{22} & B_{22}F_{21} & 0 \\ F_{12}C_{22} & F_{11} & 0 \end{bmatrix}$$

is square nonsingular, which implies $u_\mu^r = 0$ and $\omega_\mu = \phi_\mu$. □

Remark 4 Note that Corollaries 9 and 10 provide the existence of regularizing feedbacks only for the condensed forms (17)–(19). Note that, using backtransformation, corresponding results can also be obtained for the original system. Since the proofs become simpler with the techniques of the next section we defer the presentation of these results.

In this section we have discussed condensed forms for linear descriptor systems with variable coefficients and we have demonstrated that the approach introduced in [23] for DAEs can be generalized to descriptor systems. We have also shown how a reinterpretation of variables can be used to obtain an underlying square system that has s-index 0 (as a free system) and how the system can be made regular and of s-index 0 via state or output feedback.

However, as is obvious from the results in this section, this approach is not directly feasible for numerical computation, since we would need a series of variable coefficient transformations and their derivatives as well as rank decisions based on these transformations. We also have used transformations which do not have orthonormal columns, so numerical backward stability cannot be guaranteed.

For the numerical treatment of DAEs, therefore, in [26] a different approach was developed and we will modify this approach for descriptor systems in the next section.

3 Numerical methods

In this section we describe an approach to transform the control problem (1)–(2) in a way that can be implemented also as a backwards stable numerical procedure. For linear DAEs with variable coefficients it was shown in [28] that the invariants can be determined without changes of basis from a derivative array, similar to that introduced in [10]. This derivative array uses the DAE and its derivatives up to a certain order to determine the invariants and a condensed from that has the same solution set as the original DAE, but displays the major invariants. This condensed form can also be used for numerical simulation, since it displays the manifold, including all the hidden manifolds that exist for higher DAEs with nonvanishing s-index, see [26, 29]. Furthermore with some extra work it also displays the free variables, which can be interpreted as controls. But on first sight this approach has one major disadvantage when applied to control problems. Since a derivative array has to be formed, also derivatives of the controls u have to be used and this usually restricts the set of admissable controls. This is the difficulty that is faced also in the approach discussed in [11, 12]. In the following we show how we can modify the approach of [26] and avoid the differentiation of controls.

3.1 Analysis of the behaviour approach

If we wanted to apply directly the approach of [26, 29], this would mean that we perform a behaviour approach (see [38]) by introducing a behaviour vector

$$z = \begin{bmatrix} x \\ u \\ y \end{bmatrix}$$

and rewriting system (1)–(2) as

$$\mathcal{E}(t)\dot{z} = \mathcal{A}(t)z + \gamma(t), \tag{25}$$

with

$$\mathcal{E}(t) = \begin{bmatrix} E(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{A}(t) = \begin{bmatrix} A(t) & B(t) & 0 \\ C(t) & 0 & -I_p \end{bmatrix}, \ \gamma(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \tag{26}$$

and initial condition

$$\begin{bmatrix} I_n & 0 & 0 \end{bmatrix} z(t_0) = x^0. \tag{27}$$

This is a general nonsquare linear DAE with variable coefficients. Note that the derivative of the original input u and the output y occur only formally.

If we ingore the fact that the input variables may not be differentiable, then we can successively differentiate (25) and obtain the following inflated system

$$M_k(t)\dot{z}_k = N_k(t)z_k + \xi_k(t), \ k = 0, 1, \ldots, \tag{28}$$

where

$$\begin{aligned} (M_k)_{i,j} &= \tbinom{i}{j}\mathcal{E}^{(i-j)} - \tbinom{i}{j+1}\mathcal{A}^{(i-j-1)}, \ i,j = 0,\ldots,k, \\ (N_k)_{i,j} &= \begin{cases} \mathcal{A}^{(i)} & \text{for } i = 0,\ldots,k, \ j = 0, \\ 0 & \text{otherwise}, \end{cases} \\ (z_k)_i &= z^{(i)}, \ i = 0,\ldots,k, \\ (\xi_k)_i &= \gamma^{(i)}, \ i = 0,\ldots,k, \end{aligned} \tag{29}$$

see also [26]. Here we use the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$. For convenience of notation, in the following we leave out the explicit dependence on t.

In [26] it was shown that global equivalence transformations to $(\mathcal{E}, \mathcal{A})$ establish local equivalence transformations in (M_k, N_k) and how the global characteristic quantities of $(\mathcal{E}, \mathcal{A})$ and the local characteristic quantities of (M_k, N_k) are related via recursion formulas. Using these formulas, the global structural information of the system (25) can be determined from the inflated system (28). Furthermore a system of s-index 0 with the same solution set as the original system can be extracted from the inflated system built up to the level μ, where μ is the s-index of the system. This extraction procedure determines smooth matrix valued functions Z_1, Z_2, Z_3 with orthonormal columns and maximal rank such that

$$\begin{bmatrix} Z_2 & Z_3 \end{bmatrix}^* M_\mu = 0, \; Z_3^* N_\mu \begin{bmatrix} I_l \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0, \tag{30}$$

where the projected matrices

$$\hat{A}_2 = Z_2^* N_\mu \begin{bmatrix} I_\ell \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \hat{E}_1 = Z_1^* \mathcal{E} \tag{31}$$

have full row rank and the system associated with

$$\left(\begin{bmatrix} \hat{E}_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ 0 \end{bmatrix} \right) \tag{32}$$

has s-index 0. Here $\hat{A}_1 = Z_1^* \mathcal{A}$.

The first observation that one can make from this system is that if we form the inflated pair from $(\mathcal{E}, \mathcal{A})$, then the output equation contributes to the extracted s-index 0 system in a very particular way. The parts in the inflated pair that arise from the output equation always have full row rank, so they will not contribute to the left nullspace of M_k at any level other than $k = 0$. This means that the output equation will occur unchanged in the extracted s-index 0 system. To demonstrate this observation consider the array up to level 2 which is

$$M_2 = \left[\begin{array}{ccc|ccc|ccc} E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A - \dot{E} & B & 0 & E & 0 & 0 & 0 & 0 & 0 \\ C & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \dot{A} - 2\ddot{E} & \dot{B} & 0 & A - \dot{E} & B & 0 & E & 0 & 0 \\ \dot{C} & 0 & 0 & C & 0 & I & 0 & 0 & 0 \end{array}\right],$$

$$N_2 = \left[\begin{array}{ccc|ccc|ccc} A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \dot{A} & \dot{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{C} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \ddot{A} & \ddot{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \ddot{C} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Due to the structure of M_k the output equation contributes to the left nullspace of M_k only in the first block, where the whole equation is used. Hence it makes no sense to include the output equation in the computations, and we may rather consider the analysis of the system

$$\mathcal{E}(t)\dot{z} = \mathcal{A}(t)z + \gamma(t), \tag{33}$$

where

$$z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} E & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & B \end{bmatrix}, \quad \gamma = f.$$

For this system the theory of [24, 26] applies and one obtains the following condensed form.

Theorem 11 *Let the s-index μ be well defined for the system given by $(\mathcal{E}, \mathcal{A})$ in (33). Setting*

$$\hat{a} = a_\mu, \;\; \hat{d} = d_\mu, \;\; \hat{u}^l = u_0^l + \ldots + u_\mu^l, \tag{34}$$

the inflated pair (M_μ, N_μ) associated with $(\mathcal{E}, \mathcal{A})$ has the following properties:

1. *For all $t \in [t_0, t_f]$ we have* $\operatorname{rank} M_\mu(t) = (\mu+1)n - \hat{a} - \hat{u}^l$. *This implies the existence of a smooth matrix function Z with orthonormal columns and size $((\mu+1)n, \hat{a} - \hat{u}^l)$ satisfying $Z^* M_{\hat{\mu}} = 0$.*

2. *For all $t \in [t_0, t_f]$ we have* $\operatorname{rank} Z^* N_\mu [I_{\ell+m}\, 0 \,\cdots\, 0]^* = \hat{a}$ *and without loss of generality Z can be partitioned as $[Z_2, Z_3]$, with Z_2 of size $((\mu+1)n, \hat{a})$ and Z_3 of size $((\mu+1)n, \hat{u}^l)$, such that $A_2 = Z_2^* N_\mu [I_{\ell+m}\, 0 \,\cdots\, 0]^*$ has full row rank $\hat{a}$ and that $Z_3^* N_\mu [I_{\ell+m}\, 0 \,\cdots\, 0]^* = 0$. Furthermore there exists a smooth matrix function T_2 with orthonormal columns and size $(\ell+m, \hat{d})$, $\hat{d} = l + m - \hat{a}$ satisfying $\hat{A}_2 T_2 = 0$.*

3. *For all $t \in [t_0, t_f]$ we have that* $\operatorname{rank} \mathcal{E}(t) T_2(t) = \hat{d}$. *This implies the existence of a smooth matrix function Z_1 with orthonormal columns and size $(n, \hat{d})$ so that $\hat{E}_1 = Z_1^* \mathcal{E}$ has constant rank $\hat{d}$.*

Furthermore, system (33) has the same solution set as the s-index 0 system

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} z + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix}, \tag{35}$$

where $\hat{A}_1 = Z_1^* \mathcal{A}$, $\hat{f}_1 = Z_1^* f$, $\hat{f}_i = Z_i^* \xi_\mu$ for $i = 2, 3$.

Proof. The proof is given in [26] for the square case but the proof there is also valid in the rectangular case. □

Note that the third block row in (35) has $\hat{u}^l$ equations, which in general is larger than u^l in (17). The function $\hat{f}_3$ contains the part f_4 from (17) and parts of its derivatives.

An immediate observation that can be made from Theorem 11 is that the constructed submatrices $\hat{A}_1$ and $\hat{A}_2$ have been obtained from the block matrix

$$\begin{bmatrix} A & B \\ \dot{A} & \dot{B} \\ \vdots & \vdots \\ A^{(\mu)} & B^{(\mu)} \end{bmatrix}$$

by transformations from the left. This has three immediate consequences.

First of all this means that derivatives of u are nowhere needed, just derivatives of the coefficient matrices, i.e., although formally the derivatives of u occur in the inflated equations (33), they are not used for the form (35).

Second, it follows from the construction of $\hat{A}_1$ and $\hat{A}_2$, that the partitioning into the part stemming from the original states x and the original controls u is not mixed up and hence the system that we have extracted from the inflated pair has the form

$$\begin{array}{lrcl} (a) & E_1(t)\dot{x} & = & A_1(t)x + B_1(t)u + \hat{f}_1(t), \\ (b) & 0 & = & A_2(t)x + B_2(t)u + \hat{f}_2(t), \\ (c) & 0 & = & \hat{f}_3(t), \\ (d) & x(t_0) & = & x_0, \end{array} \tag{36}$$

where $E_1 = \hat{E}_1 \begin{bmatrix} I_\ell \\ 0 \end{bmatrix}$, $A_i = \hat{A}_i \begin{bmatrix} I_\ell \\ 0 \end{bmatrix}$, $B_i = \hat{A}_i \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, for $i = 1, 2$.

The third observation that we obtain form comparing (36b) and (17b,c) is that

$$\hat{d} = d_\mu, \;\; \hat{a} = a_\mu + \phi_\mu, \;\; u^r = \ell - \hat{d} - \hat{a} = u^r_\mu. \tag{37}$$

It was observed in [25] for the case of uniquely solvable systems, that the quantity μ may even vary as long as the quantities $\hat{a}, \hat{d}$ stay constant. The same is true for over- or underdetermined systems, if $\hat{a}, \hat{d}, \hat{u}^l$ are constant. $A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$.

We have seen so far that part of the structural information in Theorem 7 can be determined from the inflated system (29).

To determine the remaining information we have to perform changes of basis. We can proceed via the following algorithm.

Algorithm 1 *Given a system in the form (36) for which the s-index μ is defined. At a fixed point $\hat{t} \in [t_0, t_f]$ we proceed as follows.*

1. *Determine a unitary matrix $Q = [Q_1, Q_2]$ of size (ℓ, ℓ) such that*

$$E_1(\hat{t}) \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \end{bmatrix},$$

where E_{11} has size $(\hat{d}, \hat{d})$ and is nonsingular.

2. Determine unitary matrices $U = [U_1, U_2]$, of size $(n - \hat{d}, n - \hat{d})$ and $V = [V_1, V_2]$ of size $(\ell - \hat{d}, \ell - \hat{d})$ such that

$$U^* A_2(\hat{t}) Q_2 V = \begin{bmatrix} A_{22} & 0 \\ 0 & 0 \end{bmatrix},$$

where A_{22} is of size (a, a) and nonsingular. Set $\phi = \hat{a} - a$.

3: Determine the column rank ω of $C(\hat{t}) Q_2 V_2$, i.e., determine a unitary $W = [W_1, W_2]$, of size $(\ell - a - \hat{d}, \ell - a - \hat{d})$ such that

$$C(\hat{t}) Q_2 V_2 W = \begin{bmatrix} C_3 & 0 \end{bmatrix},$$

where C_3 has full column rank ω.

The computation of the unitary transformation matrices Q, U, V, W in Algorithm 1 can be done via singular value decompositions or rank revealing QR decompositions, see [16].

Remark 5 If we observe a change in the characteristic quantities for consecutive values $t_1, t_2 \in [t_0, t_f]$, which clearly can only be determined within the range of uncertainty of numerical rank computation, then this indicates that the s-index is not well-defined.

Since Theorem 11 yields the characteristic quantities $\hat{d} = d_\mu$ and $\hat{a} = a_\mu + \phi_\mu$ of (17) it follows immediately that the further quantities that we have determined in Algorithm 1, as long as they are constant, also determine the quantities $\phi = \phi_\mu$ and $\omega = \omega_\mu$. Hence via local computation we can determine all the global characteristic values of our problem.

In this subsection we have used a slight modification of the methods introduced in [26] to analyse the control problem in the form (33) and to determine the structural invariants. We have seen that the combination of state and control variables is not really necessary, i.e., the analysis can be carried out without mixing these quantities. This means that for this numerical method, which we can view as model verification, model reduction and index reduction, a behaviour approach can be used formally without having to worry about the differentiability properties of the input functions.

3.2 Solvability, consistency and reintrepretation of variables

Since we have established that the global characteristic quantities can be determined via local computation from the inflated system, we also immediately have the conclusions of Corollary 8 in terms of the condensed form (36), i.e.,

1. The system is consistent if and only if either $\hat{u}^l = 0$ or $\hat{f}_3 \equiv 0$, where $\hat{f}_3$ is defined in (36). If $\hat{u}^l \neq 0$ and $\hat{f}_3 \equiv 0$ then the equations in (36c) describe redundancies in the system that can be omitted.

2. If the system is consistent and if $\phi = 0$, then for a given input function u, an initial condition is consistent if and only if it satisfies (36b). Solutions of the corresponding initial value problem will in general not be unique.

3. The system is regular and of s-index 0 (as a free system) if and only if $\hat{u}^l = 0$, $\phi = 0$ and $\hat{d} + \hat{a} = l$.

Remark 6 If $\hat{u}^l \neq 0$ and $\hat{f}_3(t) \equiv 0$, then the system is consistent and the redundancies can in principle be removed. But note that in the presence of modelling, linearization and roundoff errors, redundancies and inconsistencies have to be viewed very critically. The presented procedure will identify these redundancies and inconsistencies within the range of uncertainty that is present in any numerical rank determination, i.e., the procedure can be used for a model verification or model reduction within these limitations.

The structure obtained from Algorithm 1 also allows a reinterpretation of variables. At every point $\hat{t} \in [t_0, t_f]$ we obtain which variables can be considered as free variables, i.e., controls, and which variables are fixed by choosing some of the others.

If at a fixed point $\hat{t}$ Algorithm 1 is carried out followed by a column compression $B_3(\hat{t})P_1 = [B_{31}, 0]$ and a row compression $P_2 C_3(\hat{t}) = \begin{bmatrix} C_{13} \\ 0 \end{bmatrix}$ with nonsingular B_{31}, C_{13} then we end up with the following transformed quantities: Let

$$
\begin{aligned}
T &= \begin{bmatrix} Q_1 & Q_2 V_1 & Q_2 V_2 W_1 & Q_2 V_2 W_2 \end{bmatrix}, \\
x(\hat{t}) &= T\tilde{x} = T \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}, \quad u(\hat{t}) = P_1 \tilde{u} = [P_{11}, P_{12}] \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \\
y(\hat{t}) &= P_2 \tilde{y} = [P_{21}, P_{22}] \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}.
\end{aligned}
$$

Then one obtains that $\tilde{x}_1, \tilde{x}_2, \tilde{u}_1$ correspond to the 'real' state variables, while $\tilde{u}_2, \tilde{x}_3, \tilde{x}_4$ correspond to the 'real' input variables and $\tilde{y}_1$ represents a feedthrough term. This approach can therefore be used for model verification.

3.3 Regularization by state or output feedback

As we have already seen in Section 2, we can also modify the system properties by feedback. To construct such feedbacks in a numerically stable manner has been the topic of many recent papers, see e.g., [5, 6, 13] for the case of constant coefficients and [1, 2] for the variable coefficient case. Here we present a new and more generally applicable approach based on the condensed form (36), but before this we give a proof that the s-index is invariant under feedback.

Theorem 12 *Consider a linear variable coefficient control system of the form (1) and suppose that the s-index μ of the system is well defined. Then the characteristic quantities $\hat{d}$, $\hat{a}$ and $\hat{u}^l$ are invariant under proportional state feedback $u = F(t)x + w$ and proportional output feedback $u = F(t)y + w$.*

Proof. Proportional state feeback is just a change of basis in the behaviour approach applied to (33), i.e., in (33) we set

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \\ F(t) & I_m \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{u} \end{bmatrix}.$$

Proportional output feedback is an equivalence transformation in the behaviour approach applied to (25), i.e., in (25) we set

$$\begin{bmatrix} x \\ u \\ y \end{bmatrix} = \begin{bmatrix} I_\ell & 0 & 0 \\ 0 & I_m & F(t) \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{u} \\ \tilde{y} \end{bmatrix}$$

and premultiply by the nonsingular matrix $\begin{bmatrix} I_n & B(t)F(t) \\ 0 & I_p \end{bmatrix}$.

It follows that the characteristic quantities μ, d, α are invariant under both types of feedbacks. □

If we want to make the system (36) regular and of s-index 0 (as a free system) then clearly the system has to satisfy $\hat{u}^l = 0$. If this is the case, then the extracted system (36) is given by

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix} x + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} u + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \end{bmatrix} \tag{38}$$

and we can determine proportional state and output feedbacks that make the corresponding closed loop systems s-index 0 (as a free system). This is more complicated than in the theoretical case, since the condensed form (36) is not refined enough. Fortunately, we can determine the feedbacks locally at fixed points $\hat{t}$ via the refined structure obtained from Algorithm 1.

Corollary 13 *Consider a descriptor system in the form (1) for which the s-index is well defined and let $\hat{u}^l, \hat{d}, \hat{a}$ be the characteristic quantities obtained locally at fixed times $\hat{t}$ from Algorithm 1 be globally constant in the interval $[t_0, t_f]$. There exists a state feedback $u = F(t)x + w$ such that the closed loop system*

$$E(t)\dot{x} = (A(t) + B(t)F(t))x + B(t)w + f(t) \tag{39}$$

is regular (as a free system) if and only if $\hat{u}^l = 0$ and $\hat{d} + \hat{a} = l$.

Proof. Observe that Theorem 12 implies that applying a state feedback to the original system and then computing the corresponding reduced system (33) gives the same as first bringing the original system to the reduced form (33) and then applying the feedback. Thus (39) is regular as a free system if and only if the form (36) with inserted feedback is regular and of s-index 0 as a free system. Hence it suffices to construct the feedback for the reduced system (33).

We determine piecewise constant feedbacks by constructing them locally in the neighborhood of a fixed time $\hat{t}$ and using continuity and a finite covering of the interval we get a global piecewise constant feedback.

If $\hat{u}^l = 0$ and $\hat{d} + \hat{a} = l$, then at a fixed point $\hat{t}$ we can construct the required $F(\hat{t})$ as follows. Let Q, V, W be the unitary matrices determined in Algorithm 1 and set $\tilde{B}_3 = U_2^* B_2$. By assumption $\tilde{B}_3$ has size (ϕ, m) and full row rank. Thus, there exists a matrix F_3 of size (m, ϕ) such that $\tilde{B}_3 F_3$ is nonsingular. Set

$$F(\hat{t}) = \begin{bmatrix} 0 & 0 & F_3 \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & V^* \end{bmatrix} Q^*. \tag{40}$$

Then in the closed loop system (39) we have that $E_1(\hat{t})Q \begin{bmatrix} I_d & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} E_{11} & 0 & 0 \end{bmatrix}$ and

$$U^*(A_2(\hat{t}) + B_2(\hat{t})F(\hat{t}))Q \begin{bmatrix} I_d & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} 0 & A_{22} & * \\ 0 & 0 & \tilde{B}_3 F_3 \end{bmatrix}$$

with nonsingular A_{22} and $\tilde{B}_3 F_3$ and hence the closed loop system is regular and has s-index 0 at $\hat{t}$. By assumption the characteristic quantities are constant and by continuity, in a neighborhood $\mathcal{U}$ of $\hat{t}$ the constant transformation matrices Q, U, V will not change the quantities and hence if we apply the same constant feedback in the whole neighborhood $\mathcal{U}$, we will obtain that the system is regular and of s-index 0 in the whole neighborhood. By taking a finite covering of the interval $[t_0, t_f]$ with such small neighborhoods, we then have constructed a piecewise constant global feedback $F(t)$ such that the system is regular and of s-index 0 as a free system.

For the converse observe, $\hat{u}^l = 0$ is clearly necessary. Furthermore, the condition $\hat{d} + \hat{a} = l$ is also necessary, since otherwise for any given feedback $F(t)$ the closed loop system is nonsquare and hence not uniquely solvable. □

We also have the characterization when the system can be regularized by output feedback.

Corollary 14 *Given a descriptor system in the form (1) for which the s-index is well defined and let There exists an output feedback $u = F(t)y + w$ such that the closed loop system*

$$E(t)\dot{x} = (A(t) + B(t)F(t)C(t))x + B(t)w + f(t) + B(t)F(t)g(t) \tag{41}$$

is regular (as a free system) if and only if $\hat{u}^l = 0$, $u^r = 0$ and $\phi = \omega$.

Proof. Again, it suffices to study the system in the reduced form (33), and we also construct piecewise constant feedbacks by constructing them locally in the neighborhood of a fixed time $\hat{t}$ and using continuity and a finite covering of the interval we get a global piecewise constant feedback.

If $\hat{u}^l = 0$, $u^r = 0$ and $\phi = \omega$ then we can construct the feedback as follows. For a fixed $\hat{t} \in [t_0, t_f]$ let Q, V, W be the unitary matrices determined in Algorithm 1 and set $\tilde{B}_3 = U_2^* B_2$. Since $\tilde{B}_3$ has full row rank and C_3 has full column rank, we can determine a matrix F_{33} of size (ω, ϕ) such that $\tilde{B}_3 F_{33} C_3$ is nonsingular. With

$$F(\hat{t}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & W^* \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & V^* \end{bmatrix} Q^*, \tag{42}$$

we obtain

$$E_1(\hat{t}) Q \begin{bmatrix} I_d & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & W \end{bmatrix} = \begin{bmatrix} E_{11} & 0 & 0 \end{bmatrix}$$

and

$$U^*(A_2(\hat{t}) + B_2(\hat{t}) F(\hat{t}) C(\hat{t})) Q \begin{bmatrix} I_d & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & W \end{bmatrix} = \begin{bmatrix} 0 & A_{22} & * \\ 0 & 0 & \tilde{B}_3 F_{33} C_3 \end{bmatrix},$$

where A_{22} and $\tilde{B}_3 F_{33} C_3$ are nonsingular. The same arguments as in the proof of Corollary 13 yields the existence of a feedback $F(t)$ such that the closed loop system is regular and has s-index 0 as a free system. For the converse the argument is as in the proof of Corollary 13. □

Remark 7 If we use Algorithm 1 and the construction given in the proofs of Corollaries 13 and 14 there is some freedom in the choice of $F(\hat{t})$ which can be used to make the property that the system is regular and has s-index 0 (as a free system) robust to perturbations. For constant coefficient systems this has been done in [5, 6]. How to obtain a maximally robust closed loop system in the case of variable coefficients or in the nonlinear case is currently under investigation.

In this section we have discussed the construction of feedbacks that make the system regular and of s-index 0 (as a free system). The regularizing feedbacks can be constructed via numerical procedures that can be implemented locally as numerically backwards stable procedures.

3.4 Regularization via derivative feedback

In the constant coefficient case also the regularization via derivative feedback is important and has been analyzed in [5, 6, 13, 14]. If derivative feedback is used then one can also modify the number of algebraic and dynamic variables. A general proportional and derivative feedback takes the form

$$u = F(t)x + G(t)\dot{x} + w \tag{43}$$

or in the output case

$$u = F(t)y + G(t)\dot{y} + w. \tag{44}$$

The difficulty with derivative feedback, however, is that the characteristic quantities μ, $\hat{d}$, $\hat{a}$, and ω are not invariant under derivative feedback, as is demonstrated in the following example.

Example 1 Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then $\mu = 1, \hat{a} = 0, \hat{d} = 1, \hat{u}^l = 1, u^r = 1$. With the derivative feedback $u = F\dot{x} = [1, -1]\dot{x}$ we obtain the closed loop system matrices

$$E + BF = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for which $\mu = 0, \hat{a} = 1, \hat{d} = 1, \hat{u}^l = u^r = 0$.

Due to this problem it is very difficult to decide how to use derivative feedback for regularization. It will depend strongly on the application whether an increase or decrease of the number of dynamic variables is desirable or not and hence in this general setting we cannot discuss this topic.

3.5 Impulse controllability and observability

In the constant coefficient case a square system can be made regular and of s-index 0 (as a free system) via state feedback if it is controllable at infinity or impulse controllable, [5]. There exists an algebraic condition for this property, i.e.,

$$\operatorname{rank}[E \;\, AS_\infty \;\, B] = n, \tag{45}$$

where S_∞ is a matrix whose columns span kernel(E). As we have seen in (13), in the variable coefficient case this algebraic condition is replaced by the requirement that the system has a well defined s-index, and satisfies $\hat{u}^l = 0$, $\hat{d} + \hat{a} = l$. Note that consistency (i.e., the condition $\hat{u}^l = 0$) is only needed if an inhomogeneity is present, hence we can define a linear variable coefficient system to be *impulse controllable* if it satisfies $\hat{d} + \hat{a} = l$.

For output feedback in the constant coefficient case (see [6]) also the dual condition to (45) is needed, called observability at infinity or impulse observability. The associated algebraic condition is

$$\operatorname{rank} \begin{bmatrix} E \\ T_\infty^* A \\ C \end{bmatrix} = n, \tag{46}$$

where T_∞ is a matrix whose columns span kernel E^*. In the variable coefficient case we then define analogously the system to be *impulse observable* if it has a well defined s-index and $\omega = \phi$.

It follows from Corollary 14 that there exists an output feedback such that the closed loop system is regular and has s-index (as a free system) if and only it is impulse controllable and impulse observable and satisfies $\hat{u}^l = 0$.

For the controllability and observability of the dynamic part of the system we can then use the standard terminology of ODE control systems, see [22, 37, 34].

4 Numerical examples

In this section we present some numerical examples that were obtained with the code DGELDS which is a modification of the code DGELDA of [29] for the structural analysis and simulation of descriptor systems of the form (1)–(2).

We have tested the described procedures for several well known benchmark problems in control and obtained the following results.

For the model of a three-link manipulator model of [21] with dimensions $n = 8$, $l = 8$, $m = 3$, $p = 3$, and characteristic quantities $\mu = 2$, $d_\mu = 2$, $a_\mu = 9$, $u^l_\mu = 0$, $u^r_\mu = 3$, the code DGELDS combined with Algorithm 1 returned correctly $\hat{d} = 2$, $\hat{u}^l = 0$, $\hat{a} = a = 6$, $\phi = u^r = \omega = 0$.

For the model of a simple RLC circuit [15, 4] with dimensions $n = 4$, $l = 4$, $m = 1$, $p = 1$, and characteristic quantities $\mu = 0$, $d_\mu = 2$, $a_\mu = 3$, $u^l_\mu = 0$, $u^r_\mu = 1$, the code DGELDS combined with Algorithm 1 returned correctly $\hat{d} = 2$, $\hat{u}^l = 0$, $\hat{a} = a = 2$, $\phi = u^r = \omega = 0$.

Finally we discuss the model of a bearring transmission [36], which has the form

$$\begin{bmatrix} I_3 & & \\ & M & \\ & & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & I_3 & 0 \\ -Q & -P & G^T \\ H & G & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ S \\ \alpha \end{bmatrix},$$

with M being a diagonal mass matrix,

$$\begin{aligned} P &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_1 & -d_1 \\ 0 & -d_1 & d_1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_1 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ G &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -v_u & 0 & 0 \end{bmatrix} \end{aligned}$$

For the exact parameters of the original model, which included $\alpha = 0$, see [36]. The characteristic quantities of the system are $\mu = 1$, $d_\mu = 5$, $a_\mu = 2$, $u^l_\mu = 0$, $u^r_\mu = 1$ and DGELDS combined with Algorithm 1 returned correctly $\hat{d} = 5$, $\hat{u}^l = 0$, $\hat{a} = a = 2$, $\phi = 1$, $u^r = \omega = 0$. In this case the system is not controllable at infinity, so there is no state feedback that makes the original system regular and of s-index 0 directly. But clearly, as Corollary 13 states, there exists a state feedback so that after reduction to the form (33) the system is regular and of s-index 0.

If we slightly modify the system by choosing $\alpha = 1$, then the characteristic quantities are $\mu = 0$, $d_\mu = 6$, $a_\mu = 1$, $u^l_\mu = 0$, $u^r_\mu = 1$ and DGELDS combined with Algorithm 1 returned correctly $\hat{d} = 6$, $\hat{u}^l = 0$, $\hat{a} = 1$, $a = 0$, $\phi = 1$, $u^r = \omega = 0$. In this case the system is controllable at infinity, so there is a state feedback that makes directly the original system regular and of s-index 0.

5 Conclusions and outlook

In this paper we have presented the theoretical analysis as well as numerical methods that allow to perform analysis and regularization for linear control problems with variable coefficients. We have shown that a behaviour approach can be used to apply the DAE-based methods also for the analysis of linear descriptor control systems with variable coefficients. We have given necessary and sufficient conditions that are numerically verifiable for the existence of regularizing feedbacks and we have described the construction of such feedbacks. Future research includes the generalization of this apporach to the general nonlinear case. This generalization faces several difficulties. First of all the general theory and the numerical methods developed in [28] work only for uniquely solvable systems. Thus we cannot directly apply a behaviour approach, since the solution is in general not unique. Similar difficulties also occur if the system is inconsistent or has redundant equations. Another future problem is the generalization of the presented theory to include the case that the output also depends explicitely on the input u or that we have a disturbance term in the system that has to be compensated.

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A Proof of Theorem 5

Let (E_i, A_i, B_i, C_i), $i = 1, 2$, be equivalent. Since $\operatorname{rank}(E_2) = \operatorname{rank}(PE_1Q) = \operatorname{rank}(E_1)$, r is invariant.

Note that a as defined in (10) is identical to the a defined for DAEs in [23]. Hence, we only need to show that ϕ, ω and s are invariant. The invariance of d, u^l and u^r follows then immediately from (10).

For ϕ, ω and s we must first show that they are well–defined with respect to the choice of the bases. Each change of bases can be represented by

$$\begin{array}{l}\tilde{Z} = ZM_Z,\ \tilde{T} = TM_T,\ \tilde{T}' = T'M_{T'},\ \tilde{V} = M_Z^{-1}VM_V,\\ \tilde{W} = M_T^{-1}WM_W,\ \tilde{K} = M_V^{-1}KM_K,\end{array}$$

with nonsingular matrices $M_Z,\ M_T,\ M_{T'},\ M_V,\ M_W$ and M_K. Then from

$$\operatorname{rank}(\tilde{V}^*\tilde{Z}^*B) = \operatorname{rank}(M_V^*V^*M_Z^{-*}M_Z^*Z^*B) = \operatorname{rank}(V^*Z^*B),$$

$$\operatorname{rank}(C\tilde{T}\tilde{W}) = \operatorname{rank}(CTM_TM_T^{-1}WM_W) = \operatorname{rank}(CTW),$$

and

$$\begin{array}{rl}\multicolumn{2}{l}{\operatorname{rank}(\tilde{K}^*\tilde{V}^*\tilde{Z}^*A\tilde{T}')}\\ = & \operatorname{rank}(M_K^*K^*M_V^{-*}M_V^*V^*M_Z^{-*}M_Z^*Z^*AT'M_{T'})\\ = & \operatorname{rank}(K^*V^*Z^*AT')\end{array}$$

it follows that the quantities ϕ, ω and s are well-defined. Let now bases Z_2, T_2, T_2', V_2, W_2 and K_2 be given for (E_2, A_2, B_2, C_2), e.g.

$$\operatorname{rank}(E_2 T_2) = 0,\ T_2^* T_2 \text{ nonsingular},\ \operatorname{rank}(T_2^* T_2) = n - r.$$

Using (8) and setting

$$Z_1^* = Z_2^* P,\ T_1 = QT_2,\ T_1' = QT_2', V_1 = V_2,\ W_1 = W_2,\ K_1 = K_2$$

the above $Z_1, T_1, T_1', V_1, W_1, K_1$ form bases according to (9). Since

$$\begin{aligned} \phi_2 &= \operatorname{rank}(V_2^* Z_2^* B_2) \\ &= \operatorname{rank}(V_2^* Z_2^* P B_1 R) \\ &= \operatorname{rank}(V_1^* Z_1^* B_1) = \phi_1, \end{aligned}$$

we get the invariance of ϕ. For o we get

$$\begin{aligned} o_2 &= \operatorname{rank}(C_2 T_2 W_2) \\ &= \operatorname{rank}(N C_1 Q T_2 W_2) \\ &= \operatorname{rank}(C_1 T_1 W_1) = o_1. \end{aligned}$$

Then s is invariant, since

$$\begin{aligned} s_2 &= \operatorname{rank}(K_2^* V_2^* Z_2^* A_2 T_2') \\ &= \operatorname{rank}(K_2^* V_2^* Z_2^* P A_1 Q T_2' - K_2^* V_2^* Z_2^* P E_1 S T_2') \\ &= \operatorname{rank}(K_1^* V_1^* Z_1^* A_1 T_1') = s_1, \end{aligned}$$

where we used $Z_1^* E_1 = 0$.

For the derivation of the condensed form (11) we use nonsingular transformation matrices, i.e., in the first step we take a basis Z' of $\operatorname{range} E$ and set $P = [Z'\ Z]^*$, etc. As result we obtain the following sequence of equivalent ($\sim$) matrix quadruples:

$$\begin{aligned} &(E, A, B, C) \\ &\sim \left(\begin{bmatrix} Z'^* E T' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} Z'^* A T' & Z'^* A T \\ Z^* A T' & Z^* A T \end{bmatrix}, \begin{bmatrix} Z'^* B \\ Z^* B \end{bmatrix}, \begin{bmatrix} CT' & CT \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ Z^* A T' & Z^* A T \end{bmatrix}, \begin{bmatrix} (Z'^* E T')^{-1} Z'^* B \\ Z^* B \end{bmatrix}, \begin{bmatrix} CT' & CT \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ V'^* Z^* A T' & V'^* Z^* A T W' & 0 \\ V^* Z^* A T' & 0 & 0 \end{bmatrix}, \right. \\ &\qquad \left. \begin{bmatrix} (Z'^* E T')^{-1} Z'^* B \\ V'^* Z^* B \\ V^* Z^* B \end{bmatrix}, \begin{bmatrix} CT' & CTW' & CTW \end{bmatrix} \right) \end{aligned}$$

with W' basis of cokernel(Z^*AT)

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ V'^*Z^*AT' & I_a & 0 \\ V^*Z^*AT' & 0 & 0 \end{bmatrix}, \begin{bmatrix} (Z'^*ET')^{-1}Z'^*B \\ V'^*Z^*B \\ V^*Z^*B \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} CT' & (V'^*Z^*ATW')^{-1}CTW' & CTW \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & I_a & 0 \\ K'^*V^*Z^*AT' & 0 & 0 \\ K^*V^*Z^*AT' & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} * & (Z'^*ET')^{-1}Z'^*BL \\ * & V'^*Z^*BL \\ K'^*V^*Z^*BL' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & CTW \end{bmatrix} \right)$$

with K' basis of range(V^*Z^*B), L basis of kernel(V^*Z^*B) and L' basis of cokernel(V^*Z^*B)

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & I_a & 0 \\ K'^*V^*Z^*AT' & 0 & 0 \\ K^*V^*Z^*AT' & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ 0 & V'^*Z^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & CTW \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & I_a & 0 & 0 \\ K'^*V^*Z^*AT' & 0 & 0 & 0 \\ K^*V^*Z^*AT' & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ 0 & V'^*Z^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & I_\omega & 0 \\ * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ 0 & I_a & 0 & 0 \\ K'^*V^*Z^*AT' & 0 & 0 & 0 \\ K^*V^*Z^*AT' & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ 0 & V'^*Z^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & I_\omega & 0 \\ * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ K'^*V^*Z^*AT' & 0 & 0 & 0 \\ K^*V^*Z^*AT' & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ 0 & V'^*Z^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & I_\omega & 0 \\ * & * & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ * & * & 0 & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & [I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL \\ 0 & [0\ I_d][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL \\ 0 & KV'^*Z^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ * & * & * & 0 & 0 \end{bmatrix} \right)$$

with Y basis of kernel$(K^*V^*Z^*AT')$ and Y' basis of cokernel$(K^*V^*Z^*AT')$, which at last leads to (11) by a final transformation step.

B Appendix B: Proof of Theorem 6

In the following, we will omit the argument t in the occurring matrices functions. Furthermore, since we are only interested in the block structure of the matrices we change the notation of the blocks in each step. We need the following well-known Lemma, see [30].

Lemma 15 . *Let $E \in C^\ell([t_0, t_f], \mathbb{C}^{n,n})$, $\ell \in \mathbb{N}_0$ and $\operatorname{rank} E(t) = r$ for all $t \in [t_0, t_f]$. Then there exist $U, V \in C^\ell([t_0, t_f], \mathbb{C}^{n,n})$ with $U(t), V(t)$ nonsingular (unitary) for every $t \in [t_0, t_f]$ such that*

$$U(t)^*E(t)V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad t \in [t_0, t_f],$$

where $\Sigma(t) \in C^\ell([t_0, t_f], \mathbb{C}^{r,r})$ is nonsingular.

Proof of Theorem 6. Applying Lemma 15 to the matrix E and setting $N = I, P = \Sigma^\dagger U^*, Q = V$ and $R = I$, where $\Sigma^\dagger := \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}$, we find that (E, A, B) is equivalent to

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \right).$$

Now, we apply Lemma 15 to A_{22} and set $N = I, P = \operatorname{diag}(I, \Sigma^\dagger U^*)$, $Q = \operatorname{diag}(I, V)$ and $R = I$, which yields

$$\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix} \right).$$

Proceeding with B_{31} we get

$$\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & I_a & 0 \\ A_{31} & 0 & 0 \\ A_{41} & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix} \right).$$

Applying Lemma 15 to C_{13} and setting $N = \Sigma^\dagger U^*$, $P = I$, $Q = \operatorname{diag}(I_{r+a}, V)$, $R = I$ we obtain

$$\left(\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & I_a & 0 & 0 \\ A_{31} & 0 & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} C_{11} & C_{12} & I_\omega & 0 \\ C_{21} & C_{22} & 0 & 0 \end{bmatrix} \right).$$

Choosing

$$N = I,\ P = \begin{bmatrix} I & & -B_{11} & \\ & I & -B_{21} & \\ & & I & \\ & & & I \end{bmatrix},\ Q = \begin{bmatrix} I & & & \\ & I & & \\ -C_{11} & -C_{12} & I & \\ & & & I \end{bmatrix},\ R = I$$

we eliminate B_{11} and B_{21} in the matrix B, C_{11} and C_{12} in the matrix C and the coarse block structure of E and A remains unchanged. Now, we can proceed with the last block row and get

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & I_a & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \\ I_\phi & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 \end{bmatrix} \right)$$

by applying Lemma 15 to A_{41}.

Furthermore, we use the identities in A to eliminate the other entries in the first and third block column of A. Choosing

$$N = I,\ P = I,\ Q = \begin{bmatrix} I & & & & \\ & I & & & \\ & -A_{32} & I & & \\ & & & I & \\ & & & & I \end{bmatrix},\ R = I$$

we eliminate A_{32} and we have $PE\dot{Q} = 0$, i.e. we get no fill in. Using a block permutation, which moves the fourth block row of E, A and B to the fifth block row we obtain

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & A_{22} & 0 & A_{24} & A_{25} \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \\ 0 & 0 \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 \end{bmatrix} \right).$$

Setting $N = I$, $P = I$, $Q = \operatorname{diag}(I_s, Q_2, I_{\ell-r})$, $R = I$, where Q_2 is chosen to be the solution of the initial value problem

$$\dot{Q}_2(t) = A_{22}(t) Q_2(t), \quad Q_2(t_0) = I,$$

which is nonsingular at every point $t \in [t_1, t_2]$, we get

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \\ 0 & 0 \\ I_\phi & 0 \\ 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & I_\omega & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 \end{bmatrix} \right).$$

Finally, by setting $N = I$, $P = \operatorname{diag}(I_s, Q_2^{-1}, I_{n-r})$, $Q = I$, $R = I$ we get the desired result.

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