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Numerische Simulation auf massiv parallelen Rechnern

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Some remarks to large deformation elasto–plasticity (continuum formulation)

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Abstract

The continuum theory of large deformation elasto–plasticity is summarized as far as it is necessary for the numerical treatment with the *Finite–Element–Method*. Using the calculus of modern differential geometry and functional analysis, the fundamental equations are derived and the proof of most of them is shortly outlined. It was not our aim to give a contribution to the development of the theory, rather to show the theoretical background and the assumptions to be made in *state of the art* elasto–plasticity.

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1 Introduction

From a *theoretical* point of view, a *physical body* can be treated as a set $\mathcal{B}$ of *material points* representing the *atoms* or *molecules* of the *body*'s material. Considering this *body* as a *continuum*, the theory shortly outlined in the following applies. For more details see [Tri81].
First we introduce a system $\mathcal{U}$ of open subsets of $\mathcal{B}$ with the following properties:

a) $\emptyset \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{U}$.

b) Finite intersections of subsets from $\mathcal{U}$ are again subsets of $\mathcal{U}$.

c) Any unions of subsets from $\mathcal{U}$ are again subsets of $\mathcal{U}$.

$\mathcal{U}$ is called a *topology* of $\mathcal{B}$ and $\mathcal{A}_{\mathbf{x}} \in \mathcal{U}$ is called *neighbourhood* of a point $\mathbf{x} \in \mathcal{B}$ if $\mathbf{x} \in \mathcal{A}_{\mathbf{x}}$.
$\mathcal{B}$ is called a *Hausdorff space*, if for each two points $\mathbf{x} \in \mathcal{B}$ and $\mathbf{y} \in \mathcal{B}$ there exist *neighbourhoods* $\mathcal{A}_{\mathbf{x}}$ and $\mathcal{A}_{\mathbf{y}}$ with $\mathcal{A}_{\mathbf{x}} \bigcap \mathcal{A}_{\mathbf{y}} = \emptyset$, i.e. the *topology* of $\mathcal{B}$ is rich enough to separate the points of $\mathcal{B}$.
The *Hausdorff space* $\mathcal{B}$ is said to have the *dimension* n, if for each $\mathbf{x} \in \mathcal{B}$ there exists a *neighbourhood* $\mathcal{A}_{\mathbf{x}}$ that can be mapped onto an open subset of R^n by a bijective mapping, called *chart* or *local coordinate system*, with components $\{x^k\}_{k=1\ldots n}$.
If this mapping can be choosen to be C^∞ and orientation preserving, the *n–dimensional Hausdorff space* $\mathcal{B}$ equipped with the *chart* $\{x^k\}$ is called a *n–dimensional* C^∞*–manifold* [1].
The *mathematical* assumptions made above correspond to the common *physical* understanding of *bodies* and *physical spaces*. So, from a general point of view, including also shells, rods and "exotic" materials like liquid crystals, a *physical body* $\mathcal{B}$ and the *physical space* $\mathcal{S}$ containing $\mathcal{B}$ can be considered to be special cases of *manifolds*.
To have a unified approach, as well as for conceptual clarity, it is usefull to think geometrically and to represent bodies in terms of manifolds [MH83].

2 Some Differential Geometry

The tangent space

Let $\mathcal{M}$ be a *n–dimensional* C^∞*–manifold* and let $\mathbf{x} \in \mathcal{M}$. Then the *tangent space* $T_x\mathcal{M}$ to $\mathcal{M}$ at $\mathbf{x}$ is the *vector space* R^n of *vectors* regarded as emanating from $\mathbf{x}$. So *vectors* like the *velocity* and the *acceleration* (see ch. 3) can be understood as elements of $T_x\mathcal{S}$ to $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$, where $\mathcal{S}$ denotes the *physical space* described as C^∞*–manifold*.
The *dual space* $T_x^*\mathcal{M}$ to $T_x\mathcal{M}$ is called the *cotangent space*. It's elements $\boldsymbol{\alpha} \in T_x^*\mathcal{M}$ will be called *covectors*. A *covector* $\boldsymbol{\alpha}$ is a functional $\boldsymbol{\alpha}(\mathbf{x}) : T_x\mathcal{M} \mapsto \mathrm{R}$. In this sence $T_x^*\mathcal{M}$ is *dual* to $T_x\mathcal{M}$ [2]. A *covector* is said to be *"covariant"*, and a *vector* to be *"contravariant"*.

[1]This is not the most general definition of a C^∞*–manifold*, but it should be suitable for the understandig of the present subject (for more details see [Tri81]).

[2]After the introduction of *cotangent spaces* $T_x^*\mathcal{M}$, a *vector* $\mathbf{v}$ also can be defined as a functional $\mathbf{v}(\mathbf{x}) : T_x^*\mathcal{M} \mapsto \mathrm{R}$.

Tensors

A *tensor* $\mathbf{t}$ of the type $\binom{p}{q}$ at $\mathbf{x}\in\mathcal{M}$ is a multilinear mapping

$$\mathbf{t}:\overbrace{T_x^*\mathcal{M}\times\ldots\times T_x^*\mathcal{M}}^{p\ copies}\times\underbrace{T_x\mathcal{M}\times\ldots\times T_x\mathcal{M}}_{q\ copies}\mapsto \mathrm{R}\ .$$

It is said that $\mathbf{t}$ is *contravariant* of rank p and *covariant* of rank q.
Let $\{\mathbf{e}_j\}\subset T_x\mathcal{M}$ and $\{\mathbf{e}^i\}\subset T_x^*\mathcal{M}$ be the *base vectors* in $T_x\mathcal{M}$ and the *dual base vectors* in $T_x^*\mathcal{M}$ respectively. Then the components of $\mathbf{t}$ are defined by

$$t^{i_1\ldots i_p}_{j_1\ldots j_q}:=\mathbf{t}(\mathbf{e}^{i_1},...,\mathbf{e}^{i_p},\mathbf{e}_{j_1},...,\mathbf{e}_{j_q})\ ,$$

and

$$\mathbf{t}(\mathbf{w}^1,\ldots,\mathbf{w}^p,\mathbf{v}_1,\ldots,\mathbf{v}_q)=t^{i_1\ldots i_p}_{j_1\ldots j_q}w^1_{i_1}\cdots w^p_{i_p}v^{j_1}_1\cdots v^{j_q}_q$$

holds for all $\mathbf{w}^i=w^i_a\mathbf{e}^a\in T_x^*\mathcal{M}$ and $\mathbf{v}_j=v^b_j\mathbf{e}_b\in T_x\mathcal{M}$.

The Riemannian metric

For $\mathbf{x}\in\mathcal{M}$ let $\mathbf{g}$ be a *covariant tensor* of rank 2 (i.e., a *tensor* of type $\binom{0}{2}$) with the properties

$$\begin{array}{rcll}\mathbf{g}(\mathbf{u},\mathbf{v}) & = & \mathbf{g}(\mathbf{v},\mathbf{u}); & \mathbf{u},\mathbf{v}\in T_x\mathcal{M}\\ \mathbf{g}(\mathbf{u},\mathbf{u}) & > & 0; & \mathbf{0}\neq\mathbf{u}\in T_x\mathcal{M}\ .\end{array}\tag{2.1}$$

With this symmetric and positive definite (and therefore invertible) *metric tensor* $\mathbf{g}$ the functional $\langle\mathbf{u},\mathbf{v}\rangle:=\mathbf{g}(\mathbf{u},\mathbf{v})$ is an *inner product* on $T_x\mathcal{M}$. Written in components it reads

$$\langle\mathbf{u},\mathbf{v}\rangle=u^av^bg_{ab}\quad .\tag{2.2}$$

Introducing such a *tensor* $\mathbf{g}$ for each $\mathbf{x}\in\mathcal{M}$ we get a *Riemannian metric* on $\mathcal{M}$.

Tensor and vector fields

The associated tensor and vector fields

For $\boldsymbol{\alpha}:=\alpha_b\mathbf{e}^b\in T_x^*\mathcal{M}$ the *associated vector field* $\boldsymbol{\alpha}^\sharp:=\boldsymbol{\alpha}^a\mathbf{e}_a\in T_x\mathcal{M}$ is defined by it's components $\alpha^a:=g^{ab}\alpha_b$ with g^{ab} denoting the components of $\mathbf{g}^{-1}$. The *associated covector field* $\mathbf{u}^\flat:=u_a\mathbf{e}^a\in T_x^*\mathcal{M}$ to $\mathbf{u}:=u^b\mathbf{e}_b\in T_x\mathcal{M}$ is defined by $u_a:=g_{ab}u^b$. Similarly $\mathbf{t}^\flat$ means the *tensor associated* to $\mathbf{t}$ with all indices *lowered* and $\mathbf{t}^\sharp$ with all indices *raised*, especially $\mathbf{g}^{-1}\equiv\mathbf{g}^\sharp$ and $\mathbf{g}\equiv\mathbf{g}^\flat$.

Scalar products

Using this notation and equation (2.2), we define the *dual paring* "$(\cdot\ ,\ \cdot)$" between elements of $T_x^*\mathcal{M}$ and $T_x\mathcal{M}$ and the *scalar product* "$\rangle\cdot\ ,\ \cdot\langle$" in $T_x^*\mathcal{M}$ by $(\mathbf{u},\mathbf{v}):=\langle\mathbf{u}^\sharp,\mathbf{v}\rangle$ for $\mathbf{u}\in T_x^*\mathcal{M}$, $\mathbf{v}\in T_x\mathcal{M}$ and $\rangle\mathbf{u},\mathbf{v}\langle:=\langle\mathbf{u}^\sharp,\mathbf{v}^\sharp\rangle$ for $\mathbf{u},\mathbf{v}\in T_x^*\mathcal{M}$. These definitions yield to the identity $\langle\mathbf{u}^\sharp,\mathbf{v}^\sharp\rangle=(\mathbf{u},\mathbf{v}^\sharp)=\rangle\mathbf{u},\mathbf{v}\langle$ for any $\mathbf{u},\mathbf{v}\in T_x^*\mathcal{M}$ as well as $\langle\mathbf{u},\mathbf{v}\rangle=(\mathbf{u}^\flat,\mathbf{v})=\rangle\mathbf{u}^\flat,\mathbf{v}^\flat\langle$ for any $\mathbf{u},\mathbf{v}\in T_x\mathcal{M}$.

The dual base

In two or three dimensions, the *dual base* is defined by the vector product $\mathbf{e}^a := \frac{\mathbf{e}_b \times \mathbf{e}_c}{[\mathbf{e}_a \mathbf{e}_b \mathbf{e}_c]}$. Due to this definition the equation $(\mathbf{e}^a, \mathbf{e}_b) = \delta^a_b$ holds, and once the *base* $\{\mathbf{e}_b\}$ is given, the *base vectors* $\mathbf{e}^a$ are uniquely defined. Since we deal here with *spaces* of arbitrary dimenson, we have to go another way:
Let the *base* $\{\mathbf{e}_b\}$ in $T_x\mathcal{M}$ be given. On page *2* we just introduced and used a *base* $\{\mathbf{e}^a\}$ in $T^*_x\mathcal{M}$. From now on we postulate, that the *base vectors* $\{\mathbf{e}^a\}$ fulfill $(\mathbf{e}^a, \mathbf{e}_b) = \delta^a_b$. This uniquely determins the $\mathbf{e}^a$, since in case of n dimensions n^2 conditions have to be fulfilled by choosing n components of n *(co-)vectors*. Note that this doesn't mean to have any orthonormalized system[3] in $T_x\mathcal{M}$ or in $T^*_x\mathcal{M}$.

The covariant derivative

Let $\mathbf{x} \in \mathcal{M}$. The *covariant derivative* of a *vector* $\mathbf{v} \in T_x\mathcal{M}$ along a *vector* $\mathbf{w} \in T_x\mathcal{M}$ is a bilinear mapping defined by $grad_{\mathbf{w}}\mathbf{v} : T_x\mathcal{M} \times T_x\mathcal{M} \mapsto T_x\mathcal{M}$, fulfilling

$$\begin{aligned} grad_{f\mathbf{w}}\mathbf{v} &= f\, grad_{\mathbf{w}}\mathbf{v} \\ grad_{\mathbf{w}} f\mathbf{v} &= f\, grad_{\mathbf{w}}\mathbf{v} + (\nabla f, \mathbf{w})\, \mathbf{v} \end{aligned} \tag{2.3}$$

for all scalar functions f, where $\nabla f = \dfrac{\partial f}{\partial x^a}\mathbf{e}^a \in T^*_x\mathcal{M}$ and $(\nabla f, \mathbf{w}) = \dfrac{\partial f}{\partial x^a} w^a$ [4]. Defining this for all $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{w}$ have to be regarded as *vector fields* and we get a *connection* on the *manifold* $\mathcal{M}$. The *Christoffel symbols* γ^c_{ab} of this *connection* on $\mathcal{M}$ with the *coordinate system* $\{x^c\}$ are defined by $grad_{\mathbf{e}_b}\mathbf{e}_a = \gamma^c_{ab}\mathbf{e}_c$. Using the properties (2.3) for $\mathbf{v} = v^a\mathbf{e}_a$ and $\mathbf{w} = w^b\mathbf{e}_b$, the identity $grad_{\mathbf{w}}\mathbf{v} = w^b\left[v^a grad_{\mathbf{e}_b}\mathbf{e}_a + (\nabla v^a, \mathbf{e}_b)\mathbf{e}_a\right]$ follows. Introducing the *Christoffel symbols* we get

$$grad_{\mathbf{w}}\mathbf{v} = \left(\frac{\partial v^c}{\partial x^b} + \gamma^c_{ab}v^a\right) w^b\mathbf{e}_c = v^c_{|b} w^b \mathbf{e}_c \tag{2.4}$$

with $v^c_{|b} := \dfrac{\partial v^c}{\partial x^b} + \gamma^c_{ab}v^a$.
Defining for each $\mathbf{v} \in T_x\mathcal{M}$ another *tensor grad* $\mathbf{v}$ of type $\binom{1}{1}$ with components

$$(grad\ \mathbf{v})^a_b := \frac{\partial v^a}{\partial x^b} + \gamma^a_{cb}v^c = v^a_{|b}\ , \tag{2.5}$$

the *covariant derivative* can be expressed by $grad_{\mathbf{w}}\mathbf{v} \equiv (grad\mathbf{v}, \mathbf{w})$.[5]
We define the *covariant derivative* of a *covector* $\mathbf{u} \in T^*_x\mathcal{M}$ along a *vector* $\mathbf{w} \in T_x\mathcal{M}$ via the

[3] As an example, in two dimensions, $\mathbf{e}_1 = (1\ 0)^T$, $\mathbf{e}_2 = (1\ 1)^T$, $\mathbf{e}^1 = (1\ -1)^T$ and $\mathbf{e}^2 = (0\ 1)^T$ fulfills this conditions.

[4] $(\nabla f, \mathbf{w}) = \langle(\nabla f)^\sharp, \mathbf{w}\rangle = ((\nabla f)^\sharp)^a w^b g_{ab} = g^{ac}\dfrac{\partial f}{\partial x^c} w^b g_{ab} = \dfrac{\partial f}{\partial x^c} w^b g^{ca} g_{ab} = \dfrac{\partial f}{\partial x^c} w^b \delta^c_b$

[5] Here and in the following we use the symbol $(\cdot,\cdot)$, originally defined for the *dual paring*, also in the sense of $(grad\mathbf{v}, \mathbf{w})^a := (grad\mathbf{v})^a_b w^b$, since the components of the resulting *vector* can be regarded as *dual parings* of the "columns" of $grad\mathbf{v}$ with $\mathbf{w}$.

associated vector fields:

$$grad_{\mathbf{w}}\mathbf{u} := \left(grad_{\mathbf{w}}\mathbf{u}^{\sharp}\right)^{\flat} \quad , \quad \mathbf{u} = u_a\mathbf{e}^a \quad , \quad \mathbf{w} = w^b\mathbf{e}_b\ . \tag{2.6}$$

Using the rules from page 2, the identity $g_{ab}g^{ac} = \delta^c_b$, the properties (2.3) and the representation (2.4) some calculus shows that the components r_f of $\mathbf{r} := grad_{\mathbf{w}}\mathbf{u} = r_f\mathbf{e}^f$ will be $r_f = w^b\left[\frac{\partial u_f}{\partial x^b} + u_d\left(g_{cf}\gamma^c_{ab}g^{ad} + g_{cf}\frac{\partial g^{cd}}{\partial x^b}\right)\right]$. Applying (2.10) to $g_{cf}\gamma^c_{ab}$ and taking for the rightmost term into account that $0 = \frac{\partial \delta^d_f}{\partial x^b} = \frac{\partial\left(g_{cf}g^{cd}\right)}{\partial x^b} = g_{cf}\frac{\partial g^{cd}}{\partial x^b} + g^{cd}\frac{\partial g_{cf}}{\partial x^b}$, we find $r_f = w^b\left[\frac{\partial u_f}{\partial x^b} - u_d g^{ad}\frac{1}{2}\left(\frac{\partial g_{ab}}{\partial x^f} + \frac{\partial g_{af}}{\partial x^b} - \frac{\partial g_{bf}}{\partial x^a}\right)\right] = w^b\left(\frac{\partial u_f}{\partial x^b} - u_d g^{ad}g_{ca}\gamma^c_{fb}\right)$ (again (2.10) and the symmetry of $\mathbf{g}$ was used). So find the analogon to (2.4) for *covectors*:

$$grad_{\mathbf{w}}\mathbf{u} = \left(\frac{\partial u_c}{\partial x^b} - \gamma^a_{cb}u_a\right)w^b\mathbf{e}^c = u_{c|b}w^b\mathbf{e}^c \tag{2.7}$$

with $u_{c|b} := \frac{\partial u_c}{\partial x^b} - \gamma^a_{cb}u_a$.

Regarding a *tensor* $\mathbf{t}$ of type $\binom{p}{q}$ as a *multivector* of p *vectors* and q *covectors*[6], it is natural to define the components of the *covariant derivative* of a *tensor* by a generalization of (2.4) and (2.7):

$$\begin{gathered}(grad_{\mathbf{v}}\mathbf{t})^{a\ldots b}_{c\ldots d} := t^{a\ldots b}_{c\ldots d|e}v^e := \\ \left[\frac{\partial t^{a\ldots b}_{c\ldots d}}{\partial x^e} + \left(t^{f\ldots b}_{c\ldots d}\gamma^a_{fe} + \ldots + t^{a\ldots g}_{c\ldots d}\gamma^b_{ge}\right) - \left(t^{a\ldots b}_{f\ldots d}\gamma^f_{ce} + \ldots + t^{a\ldots b}_{c\ldots g}\gamma^g_{de}\right)\right]v^e.\end{gathered} \tag{2.8}$$

The Riemannian space

If a *connection* is defined on a *manifold* $\mathcal{M}$, $\mathcal{M}$ is called an *affine space*. Introducing a *Riemannian metric* $\mathbf{g}$ on an *affine space* $\mathcal{M}$, $\mathcal{M}$ becomes *torsion free*, i.e. the *Christoffel symbols* are symmetric ($\gamma^c_{ab} = \gamma^c_{ba}$). A *torsion free affine space* is called a *Riemannian space*.

As can be shown [Tri81], in a *Riemannian space* $\mathcal{M}$ for each $\mathbf{x} \in \mathcal{M}$ a local *Cartesian coordinate system* with the *base vectors* $\mathbf{i}_j \in T_x\mathcal{M}$ can be found with $\gamma^i_{jk} = 0$ and $\mathbf{g}(\mathbf{i}_i, \mathbf{i}_j) = \delta_{ij}$. Therefore the *metric tensor* $\mathbf{g}$ is uniquely determined. Its components may be computed by $g_{ab} := \mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = e^i_a e^j_b\,\mathbf{g}(\mathbf{i}_i, \mathbf{i}_j) = e^i_a e^i_b$. With components e^i_a of $\mathbf{e}_a$ (with respect to the *base* $\{\mathbf{i}_i\}$ $\mathbf{e}_a = e^i_a\mathbf{i}_i$) and the components x^a and z^i for any $\mathbf{x} = x^a\mathbf{e}_a = z^i\mathbf{i}_i$, we get $z^i = x^a e^i_a$. This gives $e^i_a = \frac{\partial z^i}{\partial x^a}$, and the first part of (2.9) is proved. The *Christoffel symbols* γ^a_{bc} are defined by $grad_{\mathbf{e}_a}\mathbf{e}_b = \gamma^c_{ab}\mathbf{e}_c$. To compute $grad_{\mathbf{e}_a}\mathbf{e}_b$, we formulate (2.4) in the *Cartesian coordinate system* and use it for $\mathbf{w} := \mathbf{e}_a = e^k_a\mathbf{i}_k$ and $\mathbf{v} := \mathbf{e}_b = e^j_b\mathbf{i}_j$ so

[6] roughly spoken, write the *vectors* and *covectors* one besides the other

$grad_{\mathbf{e}_a}\mathbf{e}_b = \left(\frac{\partial e^i_b}{\partial z^j} + \gamma^i_{kj} e^k_b\right) e^j_a \mathbf{i}_i = \frac{\partial e^i_b}{\partial z^j} e^j_a \mathbf{i}_i$. The resulting equation $\frac{\partial e^i_b}{\partial z^j} e^j_a \mathbf{i}_i = \gamma^c_{ab} e^i_c \mathbf{i}_i$ is equivalent to $\gamma^c_{ab} \frac{\partial z^i}{\partial x^c} = \frac{\partial}{\partial z^j}\left(\frac{\partial z^i}{\partial x^b}\right) \frac{\partial z^j}{\partial x^a} = \frac{\partial x^d}{\partial z^j} \frac{\partial^2 z^i}{\partial x^d \partial x^b} \frac{\partial z^j}{\partial x^a} = \frac{\partial^2 z^i}{\partial x^d \partial x^b} \delta^d_a = \frac{\partial^2 z^i}{\partial x^a \partial x^b}$. So the second part of (2.9) is verified:

$$
\begin{aligned}
g_{ab} &= \frac{\partial z^i}{\partial x^a} \frac{\partial z^j}{\partial x^b} \delta_{ij} \\
\gamma^c_{ab} &= \frac{\partial^2 z^i}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial z^i} \,.
\end{aligned}
\tag{2.9}
$$

The combination of both formulas in (2.9) gives

$$
2 g_{ab} \gamma^a_{dc} = \frac{\partial g_{cb}}{\partial x^d} + \frac{\partial g_{db}}{\partial x^c} - \frac{\partial g_{dc}}{\partial x^b}. \tag{2.10}
$$

Consequently, taking into account the local *Euclidean structure* of *torsion free affine spaces*, the components of the *metric tensor* $\mathbf{g}$ and the *Christoffel symbols* γ^a_{bc} of the *coordinate system* $\{x^a\}$ are uniquely determined and so is the *connection* on $\mathcal{M}$.

The divergence of a vector in noncartesian coordinates

For any *vector* $\mathbf{v} \in T_x\mathcal{M}$ we have $\mathbf{v} = v^i \mathbf{i}_i = v^b \mathbf{e}_b = v^b e^i_b \mathbf{i}_i = v^b \frac{\partial z^i}{\partial x^b} \mathbf{i}_i$[7] and therefore as immediately can be seen, the components of a *vector* transform as $v^i = \frac{\partial z^i}{\partial x^b} v^b$, and consequently we have $\frac{\partial}{\partial z^j} v^i = \frac{\partial x^a}{\partial z^j} \frac{\partial}{\partial x^a}\left(\frac{\partial z^i}{\partial x^b} v^b\right) = \frac{\partial x^a}{\partial z^j}\left(\frac{\partial^2 z^i}{\partial x^a \partial x^b} v^b + \frac{\partial z^i}{\partial x^b} \frac{\partial v^b}{\partial x^a}\right)$. Using the notation from (2.9), (2.4) and (2.5) we get the *divergence* $div\ \mathbf{v} := \frac{\partial v^i}{\partial z^i}$ in *curvilinear coordinates* as

$$
div\ \mathbf{v} = \gamma^b_{ab} v^a + \frac{\partial v^a}{\partial x^a} = v^a_{|a} = (grad\ \mathbf{v})^a_a = trace(grad\ \mathbf{v}) \,. \tag{2.11}
$$

Push forward and pull back

Let $\mathcal{B}$ and $\mathcal{S}$ be two (not necessary different) *manifolds*, $\mathbf{X} \in \mathcal{B}$, $\mathbf{x} \in \mathcal{S}$ and $\boldsymbol{\varphi} : \mathcal{B} \mapsto \mathcal{S}$ $(\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}))$ a regular mapping in the sense, that $\boldsymbol{\varphi}$ has a C^1 inverse. For $\mathbf{U} \in T_X\mathcal{B}$ the *vector field* $\boldsymbol{\varphi}_*\mathbf{U} \in T_{\varphi(X)}\mathcal{S}$ with components (2.12) is called the *push forward* of $\mathbf{U}$ by $\boldsymbol{\varphi}$:

$$
(\boldsymbol{\varphi}_*\mathbf{U})^a_{|\mathbf{x}} := \left(\frac{\partial(\boldsymbol{\varphi})^a}{\partial X^A} U^A\right)_{|\boldsymbol{\varphi}^{-1}(\mathbf{x})} \,. \tag{2.12}
$$

The components of the *pull back* $\boldsymbol{\varphi}^*\mathbf{u} \in T_X\mathcal{B}$ of some *vector field* $\mathbf{u} \in T_{\varphi(X)}\mathcal{S}$ by the mapping $\boldsymbol{\varphi}$ are defined in (2.13):

$$
(\boldsymbol{\varphi}^*\mathbf{u})^A_{|\mathbf{X}} := \left(\frac{\partial(\boldsymbol{\varphi}^{-1})^A}{\partial x^a} u^a\right)_{|\boldsymbol{\varphi}(\mathbf{X})} \,. \tag{2.13}
$$

[7] cf. the previous section

In the same way for *covectors* $\mathbf{V}\in T^*_X\mathcal{B}$ and $\mathbf{v}\in T^*_{\varphi(X)}\mathcal{S}$ the *push forward* $\boldsymbol{\varphi}_*\mathbf{V}\in T^*_{\varphi(X)}\mathcal{S}$ and the *pull back* $\boldsymbol{\varphi}^*\mathbf{v}\in T^*_X\mathcal{B}$ are defined by their components

$$(\boldsymbol{\varphi}_*\mathbf{V})_{a\,|\mathbf{x}} := \left(\frac{\partial(\boldsymbol{\varphi}^{-1})^A}{\partial x^a}\right)_{|\mathbf{x}} (V_A)_{|\boldsymbol{\varphi}^{-1}(\mathbf{x})} \quad , \quad (\boldsymbol{\varphi}^*\mathbf{v})_{A\,|\mathbf{X}} := \left(\frac{\partial(\boldsymbol{\varphi})^a}{\partial X^A}\right)_{|\mathbf{X}} (v_a)_{|\boldsymbol{\varphi}(\mathbf{X})} \,. \tag{2.14}$$

If $\mathbf{T}$ is a *tensor* of type $\binom{p}{q}$ acting on $\mathcal{B}$, its *push forward* $\boldsymbol{\varphi}_*\mathbf{T}$ is a *tensor* of the same type on $\boldsymbol{\varphi}(\mathcal{B})$ defined by:

$$(\boldsymbol{\varphi}_*\mathbf{T})_{|\mathbf{x}}(\mathbf{v}^1,\ldots,\mathbf{v}^p,\mathbf{u}_1,\ldots,\mathbf{u}_q)_{|\mathbf{x}} := \mathbf{T}_{|\boldsymbol{\varphi}^{-1}(\mathbf{x})}(\boldsymbol{\varphi}^*\mathbf{v}^1,\ldots,\boldsymbol{\varphi}^*\mathbf{v}^p,\boldsymbol{\varphi}^*\mathbf{u}_1,\ldots,\boldsymbol{\varphi}^*\mathbf{u}_q)_{|\boldsymbol{\varphi}^{-1}(\mathbf{x})} \tag{2.15}$$

with $\mathbf{v}^i\in T^*_x\mathcal{S}$ and $\mathbf{u}_i\in T_x\mathcal{S}$, and the *pull back* of a *tensor* $\mathbf{t}$ defined on $\boldsymbol{\varphi}(\mathcal{B})$ is:

$$(\boldsymbol{\varphi}^*\mathbf{t})_{|\mathbf{X}}(\mathbf{V}^1,\ldots,\mathbf{V}^p,\mathbf{U}_1,\ldots,\mathbf{U}_q)_{|\mathbf{X}} := \mathbf{t}_{|\boldsymbol{\varphi}(\mathbf{X})}(\boldsymbol{\varphi}_*\mathbf{V}^1,\ldots,\boldsymbol{\varphi}_*\mathbf{V}^p,\boldsymbol{\varphi}_*\mathbf{U}_1,\ldots,\boldsymbol{\varphi}_*\mathbf{U}_q)_{|\boldsymbol{\varphi}(\mathbf{X})} \,. \tag{2.16}$$

The *pull back* and the *push forward* of scalar functions $f(\mathbf{x})$ and $F(\mathbf{X})$ are defined by

$$\boldsymbol{\varphi}^* f(\mathbf{x}) := f(\boldsymbol{\varphi}(\mathbf{X})) \qquad , \qquad \boldsymbol{\varphi}_* F(\mathbf{X}) := F(\boldsymbol{\varphi}^{-1}(\mathbf{x}))\,. \tag{2.17}$$

The material time derivative

Let $c(t)$ be a *integral curve* in $\mathcal{M}$, i.e. the tangent to $c(t)$ can be found as $\mathbf{v}=\frac{dc}{dt}\in T_{c(t)}\mathcal{M}$. Then the *material time derivative* of $\mathbf{a}=a^c\mathbf{e}_c\in T_{c(t)}\mathcal{M}$ and $\mathbf{b}=b_c\mathbf{e}^c\in T^*_{c(t)}\mathcal{M}$ with a^c , b_c , $\mathbf{e}^c$ and $\mathbf{e}_c$ depending on $\mathbf{x}(t):=c(t)$ is given by

$$\begin{aligned} \frac{d}{dt}\mathbf{a} &= v^b\frac{\partial a^c}{\partial x^b}\mathbf{e}_c + v^b a^a\frac{\partial}{\partial x^b}\mathbf{e}_a = v^b\left(\frac{\partial a^c}{\partial x^b}+a^a\gamma^c_{ab}\right)\mathbf{e}_c = a^c{}_{|b}v^b\mathbf{e}_c = grad_{\mathbf{v}}\mathbf{a} \\ \frac{d}{dt}\mathbf{b} &= v^b\frac{\partial b_c}{\partial x^b}\mathbf{e}^c + v^b b_a\frac{\partial}{\partial x^b}\mathbf{e}^a = v^b\left(\frac{\partial b_c}{\partial x^b}-b_a\gamma^a_{bc}\right)\mathbf{e}^c = b_{c|b}v^b\mathbf{e}^c = grad_{\mathbf{v}}\mathbf{b} \end{aligned} \tag{2.18}$$

with the *covariant derivative* "*grad*" from (2.4) and (2.7). To verify (2.18) we only have to remember[8], that $\frac{\partial}{\partial x^b}\mathbf{e}_a = \frac{\partial}{\partial x^b}e^i_a\mathbf{i}_i = \frac{\partial^2 z^i}{\partial x^a\partial x^b}e^c_i\mathbf{e}_c = \frac{\partial^2 z^i}{\partial x^a\partial x^b}\frac{\partial x^c}{\partial z^i}\mathbf{e}_c = \gamma^c_{ab}\mathbf{e}_c$ and $\frac{\partial}{\partial x^b}\mathbf{e}^a = \frac{\partial}{\partial x^b}e^a_i\mathbf{i}^i = \frac{\partial}{\partial x^b}\frac{\partial x^a}{\partial z^i}\mathbf{i}^i = \frac{\partial}{\partial x^b}\frac{\partial x^a}{\partial z^i}e^i_c\mathbf{e}^c \overset{9}{=} -\frac{\partial x^a}{\partial z^i}\frac{\partial^2 z^i}{\partial x^b\partial x^c}\mathbf{e}_c = -\gamma^a_{bc}\mathbf{e}^c$ is valid. In a similar way, the *material time derivative* of a *tensor* $\mathbf{t}$ can be proved to be

$$\frac{d}{dt}\mathbf{t} = grad_{\mathbf{v}}\mathbf{t}\,. \tag{2.19}$$

If $\mathbf{t}$, $\mathbf{a}$, $\mathbf{b}$ explicitely depend on t, their *material derivatives* are given by

$$\frac{d}{dt}\mathbf{t} = \frac{\partial}{\partial t}\mathbf{t} + grad_{\mathbf{v}}\mathbf{t} \quad , \quad \frac{d}{dt}\mathbf{a} = \frac{\partial}{\partial t}\mathbf{a} + grad_{\mathbf{v}}\mathbf{a} \quad , \quad \frac{d}{dt}\mathbf{b} = \frac{\partial}{\partial t}\mathbf{b} + grad_{\mathbf{v}}\mathbf{b}\,. \tag{2.20}$$

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[8] cf. pages 4, 5

[9] taking into accout $\frac{\partial x^a}{\partial z^i}\frac{\partial z^i}{\partial x^c}=\delta^a_c$ and consequently
$0=\frac{\partial}{\partial x^b}\delta^a_c=\frac{\partial}{\partial x^b}\left[\frac{\partial x^a}{\partial z^i}\frac{\partial z^i}{\partial x^c}\right]=\frac{\partial x^a}{\partial z^i}\frac{\partial}{\partial x^b}\frac{\partial z^i}{\partial x^c}+\frac{\partial z^i}{\partial x^c}\frac{\partial}{\partial x^b}\frac{\partial x^a}{\partial z^i}$

The transport of vectors and tensors along curves

Let $\boldsymbol{\psi}_{t,s} : \mathcal{M} \to \mathcal{M}$ for real s and t be a collection of *maps*, such that for an *integral curve* $\mathbf{x} = c(s)$ of $\mathbf{v}$ (cf. page 6) $c(t) := \boldsymbol{\psi}_{t,s}(c(s))$ is an *integral curve* of $\mathbf{v}$ again. Assume in addition, that $\boldsymbol{\psi}_{s,s}(\mathbf{x}) = \mathbf{x}$ and $\boldsymbol{\psi}_{t,s} \circ \boldsymbol{\psi}_{s,r} = \boldsymbol{\psi}_{t,r}$ holds[10].

Based on this construction, we introduce a linear mapping $\mathbf{\Psi}_{t,s} : T_{c(s)}\mathcal{M} \mapsto T_{c(t)}\mathcal{M}$ with $\mathbf{\Psi}_{t,s} \circ \mathbf{\Psi}_{s,r} = \mathbf{\Psi}_{t,r}$ and $\mathbf{\Psi}_{s,s}$ beeing the *identical mapping*. Then $\mathbf{\Psi}_{t,s}$ transports a *vector* $\mathbf{a}_s \in T_x\mathcal{M}$ emanating from $\mathbf{x} := c(s)$ to $\mathbf{x}' := c(t)$, i.e. $\mathbf{a}_t := \mathbf{\Psi}_{t,s}\mathbf{a}_s \in T_{x'}\mathcal{M}$ with $a_t^a := (\mathbf{\Psi}_{t,s})^a_b a^b_s$.

Assuming the *transport* to be done in a *parallel* manner, i.e. $\frac{d}{ds}(\mathbf{\Psi}_{s,r}\mathbf{a}_r) = \frac{d}{ds}\mathbf{a}_s = \mathbf{0}$, $\mathbf{\Psi}_{t,s}$ is called *shifter* and denoted by $\mathbf{S}_{t,s}$. For the case of *parallel transport*, we get from (2.18) $0 = \left(\frac{d}{ds}\mathbf{a}_s\right)^a = \left[\frac{d}{ds}(\mathbf{S}_{s,r})^a_b + \gamma^a_{cd}v^c(\mathbf{S}_{s,r})^d_b\right] a^b_r$, or $\lim\limits_{s\to t}\frac{d}{ds}(\mathbf{S}_{s,t})^a_b = -\gamma^a_{cb}v^c$. Since $\mathbf{S}_{t,s} = \mathbf{S}^{-1}_{s,t}$ and therefore $(\mathbf{S}_{t,s})^a_c(\mathbf{S}_{s,t})^c_b = \delta^a_b$, the relation $\left[\frac{d}{ds}(\mathbf{S}_{t,s})^a_c\right](\mathbf{S}_{s,t})^c_b = -(\mathbf{S}_{t,s})^a_c\left[\frac{d}{ds}(\mathbf{S}_{s,t})^c_b\right]$ can be found, tending to $\lim\limits_{s\to t}\frac{d}{ds}(\mathbf{S}_{t,s})^a_c\,\delta^c_b = \delta^a_c\gamma^c_{db}v^d$ if s tends to t. Applying this to the equation $\lim\limits_{s\to t}\frac{d}{ds}\left[(\mathbf{S}_{t,s})^a_b\, a^b_s\right] = \lim\limits_{s\to t}\left[a^b_s\frac{d}{ds}(\mathbf{S}_{t,s})^a_b + (\mathbf{S}_{t,s})^a_b\,\frac{d}{ds}a^b_s\right] = a^b_t\lim\limits_{s\to t}\frac{d}{ds}(\mathbf{S}_{t,s})^a_b + \delta^b_a\frac{d}{dt}a^b_t$ and regarding (2.18), (2.20), we get

$$\lim_{s\to t}\frac{d}{ds}(\mathbf{S}_{t,s}\mathbf{a}_s)^a = \frac{d}{dt}a^a_t + \gamma^a_{cb}v^c a^b_t = \left(\frac{d}{dt}\mathbf{a}_t\right)^a . \;{}^{11} \tag{2.21}$$

Applying the same calculus to a *tensor* $\mathbf{t}$ yields to

$$\lim_{s\to t}\frac{d}{ds}(\mathbf{S}_{t,s}\mathbf{t}_s) = \frac{d}{dt}\mathbf{t}. \tag{2.22}$$

The Lie derivative

Using in (2.22) the *push forward* induced by $\boldsymbol{\psi}_{t,s}$ instead of a *shifter*, we get the *Lie derivative*:

Let the mapping $\boldsymbol{\varphi}$ used in (2.14) be $\boldsymbol{\psi}_{t,s}$. Then, the *transport* $\mathbf{\Psi}_{t,s} := \boldsymbol{\psi}_{*_{t,s}}$ is well defined and

$$L_v\mathbf{t} := \lim_{s\to t}\frac{d}{ds}\left(\boldsymbol{\psi}_{*_{t,s}}\mathbf{t}_s\right) \tag{2.23}$$

is called the *Lie derivative* $L_v\mathbf{t}$ of the *tensor* $\mathbf{t}$. Owing to $\boldsymbol{\psi}_{*_{t,s}}\mathbf{t}_s = \boldsymbol{\psi}^*_{s,t}\mathbf{t}_s$, the *Lie derivative* defined in (2.23) is equaivalent to $L_v\mathbf{t} = \lim\limits_{s\to t}\frac{d}{ds}\left(\boldsymbol{\psi}^*_{s,t}\mathbf{t}_s\right)$.

Holding s fixed in $\mathbf{t}_s$ at $s = t$, i.e. $\tilde{\mathbf{t}}_t := \mathbf{t}(t, c(s))$, we get the *autonomous Lie derivative*

$$\mathcal{L}_v\mathbf{t} := \lim_{s\to t}\frac{d}{ds}\left(\boldsymbol{\psi}_{*_{t,s}}\tilde{\mathbf{t}}_t\right) . \tag{2.24}$$

[10] then $\boldsymbol{\psi}_{t,s}$ is called the *flow* or *evolution operator* of $\mathbf{v}$

[11] Note, that for $\mathbf{a}_t := \mathbf{S}_{t,s}\mathbf{a}_s$ we have $\left(\frac{d\mathbf{a}_t}{ds}\right)^a = \left(\frac{d[(\mathbf{S}_{t,s})^c_b a^b_s \mathbf{e}_c(t)]}{ds}\right)^a = \left(\frac{d[(\mathbf{S}_{t,s})^c_b a^b_s)]}{ds}\mathbf{e}_c(t)\right)^a = \frac{d[(\mathbf{S}_{t,s})^a_b a^b_s)]}{ds} = \frac{da^a_t}{ds}$

In the general case, the *autonomous Lie derivative* is related to $L_v\mathbf{t}$ by

$$L_v\mathbf{t} = \frac{\partial}{\partial t}\mathbf{t} + \mathcal{L}_v\mathbf{t}\ . \tag{2.25}$$

<u>The *autonomous Lie derivative* of a *tensor*</u>

The components of the *autonomous Lie derivative* of a *tensor* $\mathbf{t}$ are

$$(\mathcal{L}_v\mathbf{t})^{a\ldots b}_{c\ldots d} = t^{a\ldots b}_{c\ldots d|e}v^e - \left(t^{f\ldots b}_{c\ldots d}v^a_{|f} + \ldots + t^{a\ldots f}_{c\ldots d}v^b_{|f}\right) + \left(t^{a\ldots b}_{f\ldots d}v^f_{|c} + \ldots + t^{a\ldots b}_{c\ldots f}v^f_{|d}\right) \tag{2.26}$$

(for notation see (2.18, 2.8)). In the special case of the *metric tensor* $\mathbf{g}$ the *autonomous Lie derivative* can be simplified to

$$(\mathcal{L}_v\mathbf{g})_{ab} = \frac{\partial g_{ab}}{\partial x^c}v^c + g_{cb}\frac{\partial v^c}{\partial x^a} + g_{ac}\frac{\partial v^c}{\partial x^b}, \tag{2.27}$$

taking into account (2.26, 2.8, 2.10).

<u>The *Lie derivative* of a function</u>

Let $f_s := f(s, c(s))$ be a function on $\mathcal{S}$. Then, the *push forward* of f_s induced by $\boldsymbol{\psi}_{t,s}$ reads $\boldsymbol{\psi}_{*_{t,s}}f_s = f\left(s, \boldsymbol{\psi}^{-1}_{t,s}(c(t))\right) = f(s, c(s))$. Due to $\mathbf{v}_s := \mathbf{v}(s, c(s)) = \frac{dc_s}{ds}$ we have

$$L_v f_t := \lim_{s\to t}\left(\frac{d}{ds}\boldsymbol{\psi}_{*_{t,s}}f_s\right) = \lim_{s\to t}\left(\frac{\partial f_s}{\partial s} + \frac{\partial f_s}{\partial x^a}\frac{d(\boldsymbol{\psi}^{-1}_{t,s})^a}{ds}\right) = \left(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial x^a}v^a_t\right) = \frac{df_t}{dt}\ . \tag{2.28}$$

<u>The *Lie derivative* of a *vector field*</u>

For a *vector field* $\mathbf{w}$ on $\mathcal{S}$ we get $\left(\boldsymbol{\psi}_{*_{t,s}}\mathbf{w}_s\right)^a = \left(\frac{\partial(\boldsymbol{\psi}_{t,s})^a}{\partial x^b}w^b_s\right)|_{\psi^{-1}_{t,s}}$ for the *push forward*'s components. Some simple calculus gives $\frac{d}{ds}\left(\boldsymbol{\psi}_{*_{t,s}}\mathbf{w}_s\right)^a = \frac{\partial(\boldsymbol{\psi}_{t,s})^a}{\partial x^b}\frac{d}{ds}w^b_s + w^b_s\frac{d}{ds}\left(\frac{\partial(\boldsymbol{\psi}_{t,s})^a}{\partial x^b}\right)$ with $\frac{\partial(\boldsymbol{\psi}_{t,s})^a}{\partial x^b}\frac{d}{ds}w^b_s = \frac{\partial(\boldsymbol{\psi}_{t,s})^a}{\partial x^b}\left(\frac{\partial w^b_s}{\partial s} + \frac{\partial w^b_s}{\partial x^c}\frac{d(\boldsymbol{\psi}^{-1}_{t,s})^c}{ds}\right) \underset{(s\to t)}{\longrightarrow} \delta^a_b\left(\frac{\partial w^b_t}{\partial t} + \frac{\partial w^b_t}{\partial x^c}v^c_t\right)$, and using[12] $\frac{\partial(\boldsymbol{\psi}^{-1}_{t,s})^a}{\partial x^c}\frac{d}{ds}\left(\frac{\partial(\boldsymbol{\psi}_{t,s})^c}{\partial x^b}\right) = -\frac{\partial(\boldsymbol{\psi}_{t,s})^c}{\partial x^b}\frac{\partial}{\partial x^c}\frac{d(\boldsymbol{\psi}^{-1}_{t,s})^a}{ds} = -\frac{\partial(\boldsymbol{\psi}_{t,s})^c}{\partial x^b}\frac{\partial v^a_s}{\partial x^c} \underset{(s\to t)}{\longrightarrow} -\delta^c_b\frac{\partial v^a_t}{\partial x^c}$, we get

$$(L_v\mathbf{w})^a = \frac{\partial w^a_t}{\partial t} + \frac{\partial w^a_t}{\partial x^b}v^b_t - w^b_t\frac{\partial v^a_t}{\partial x^b}. \tag{2.29}$$

<u>The *Lie derivative* of a *covector field*</u>

Let $\mathbf{u}$ be a *covector field* on $\mathcal{S}$. Then the components of its *push forward* can be found as $\left(\boldsymbol{\psi}_{*_{t,s}}\mathbf{u}_s\right)_a = \left(\frac{\partial(\boldsymbol{\psi}^{-1}_{t,s})^b}{\partial x^a}\right)|_{c(t)}u_{sb}|_{\psi^{-1}_{t,s}}$, an analogous computation as above leads to

[12] c.f. page 6

$$\frac{d}{ds}\left(\boldsymbol{\psi}_{*_{t,s}}\mathbf{u}_s\right)_a = \frac{d}{ds}\left(\frac{\partial(\boldsymbol{\psi}^{-1}_{t,s})^b}{\partial x^a}\right)u_{sb} + \frac{\partial(\boldsymbol{\psi}^{-1}_{t,s})^b}{\partial x^a}\frac{du_{sb}}{ds} = \frac{\partial v^b_s}{\partial x^a}u_{sb} + \frac{\partial(\boldsymbol{\psi}^{-1}_{t,s})^b}{\partial x^a}\left(v^c_s\frac{\partial u_{sb}}{\partial x^c} + \frac{\partial u_{sb}}{\partial s}\right)$$

and consequently

$$(L_v\mathbf{u})_a = \frac{\partial u_{ta}}{\partial t} + \frac{\partial u_{ta}}{\partial x^b}v^b_t + u_{tb}\frac{\partial v^b_t}{\partial x^a}. \tag{2.30}$$

Linearization of tensor fields

The *Taylor's Theorem*

$$\mathbf{t}_t = \mathbf{t}_s + \left(\frac{d}{ds}\mathbf{t}_s\right)(t-s) + \ldots + \frac{1}{n!}\left(\frac{d^n}{ds^n}\mathbf{t}_s\right)(t-s)^n + \ldots \tag{2.31}$$

(used for $(t-s)$ sufficiently small) remains valid also for sufficiently smooth *tensor fields* $\mathbf{t}$ over *curves* $c(r)$ on *manifolds*, i.e. for $\mathbf{t}:=\mathbf{t}_r=\mathbf{t}(c(r))$ ([MH83]). So, according to (2.19) and (2.3), the *linearization* $\tilde{\mathbf{t}}$ of such a *tensor field* $\mathbf{t}$ can be written as

$$\tilde{\mathbf{t}} = \mathbf{t} + grad_{\mathbf{u}}\mathbf{t}, \tag{2.32}$$

with the *tangent vector* $\mathbf{u}:=\mathbf{u}_s:=(t-s)\mathbf{v}_s$ to the *curve* $c(s)$ on whitch $\mathbf{t}_s$ is defined. With (2.22) an alternative formulation

$$\tilde{\mathbf{t}}_t = \mathbf{t}_s + \lim_{r\to s}\left(\frac{d}{dr}(\mathbf{S}_{s,r}\mathbf{t}_r)\right)(t-s), \tag{2.33}$$

usefull for farther computations, can be gained from (2.31).

3 Kinematics of Finite Deformations

In the following the positions of the material points of a *body* shall be described by its *reference configuration* $\mathcal{B}\subset\mathcal{S}$. $\mathcal{B}$ has to be an open set in the *Riemannian space* $\mathcal{S}$ with a piecewise smooth boundary. *Material points* in $\mathcal{B}$ are denoted by $\mathbf{X}=(X^1,\ldots,X^N)$, while *spatial points* in $\mathcal{S}$ are denoted by $\mathbf{x}=(x^1,\ldots,x^n)$. The *dimensions* of $\mathcal{B}$ and $\mathcal{S}$ are assumed to be the same ($n=N$). Any *motion* of a *body* $\mathcal{B}$ may be regarded as a time–dependent family of *configurations*, defined as sufficiently smooth, orientation preserving and invertible mappings $\boldsymbol{\Phi}_t:\mathcal{B}\longrightarrow\mathcal{S}$ (i.e. $\mathbf{x}:=\mathbf{x}_t=\boldsymbol{\Phi}(\mathbf{X},t):=\boldsymbol{\Phi}_t(\mathbf{X})$) [13]. According to this definition, the identification of the *body* $\mathcal{B}$ with the *reference configuration* $\boldsymbol{\Phi}_0(\mathcal{B})$ makes sense ($\mathbf{X}\equiv\boldsymbol{\Phi}_0(\mathbf{X})$). Let additionally $\{X^A\}$ and $\{x^a\}$ denote *coordinate systems* on $\mathcal{B}$and $\mathcal{S}$, respectively. Component wise representations will be assumed always with respect to these *coordinate systems* in the following chapters.

[13] We denote the function $\boldsymbol{\Phi}(\mathbf{X},t)$ with t fixed by $\boldsymbol{\Phi}_t(\mathbf{X})$ and with $\mathbf{X}$ fixed by $\boldsymbol{\Phi}_{\mathbf{X}}(t)$

Velocity and acceleration

The *material velocity* $\mathbf{V_X}(t)$ and the *material acceleration* $\mathbf{A_X}(t)$ at some point $\mathbf{X}$ are defined via its motion $\mathbf{x} = \mathbf{\Phi_X}(t)$ in $\mathcal{S}$:

$$\mathbf{V_X}(t) := \frac{d}{dt}\mathbf{\Phi_X}(t) \qquad , \qquad \mathbf{A_X}(t) := \frac{d}{dt}\mathbf{V_X}(t)\ . \tag{3.1}$$

They will be regarded as vectors based at the point $\mathbf{x} = \mathbf{\Phi_X}(t)$ with components

$$V^a = \frac{d}{dt}\Phi^a_{\mathbf{X}}(t) \quad , \quad A^a = \frac{d}{dt}V^a + \gamma^a_{bc}V^bV^c\ .^{14} \tag{3.2}$$

The *spatial velocity* and *spatial acceleration* are defined as

$$\mathbf{v_X}(t) := \mathbf{V}_{\mathbf{\Phi}^{-1}_{\mathbf{X}}(t)}(t) \qquad , \qquad \mathbf{a_X}(t) := \mathbf{A}_{\mathbf{\Phi}^{-1}_{\mathbf{X}}(t)}(t)\ . \tag{3.3}$$

Some calculus shows, that $\mathbf{a_X}(t)$ is the *material time derivative* $\frac{d}{dt}\mathbf{v}$ of $\mathbf{v}$:

$$\mathbf{a_X}(t) = \frac{d}{dt}\mathbf{v_X}(t) = \frac{\partial \mathbf{v_X}}{\partial t} + grad_{\mathbf{v}}\mathbf{v} \tag{3.4}$$

with the *covariant derivative* $grad_{\mathbf{v}}\mathbf{v}$ from (2.4) in the *current configuration*. The components of $\mathbf{v}$ and $\mathbf{a}$ are

$$v^a = V^a \quad \text{and} \quad a^a = \frac{\partial v^a}{\partial t} + v^a_{|b}v^b \quad \text{with} \quad v^a_{|b} := \frac{\partial v^a}{\partial x^b} + \gamma^a_{bc}v^c\ . \tag{3.5}$$

As in [Wri86] and using $\boldsymbol{\psi}_{*_{t,s}}\mathbf{t}_s = \mathbf{\Phi}_{*t}\mathbf{\Phi}^*_s\,\mathbf{t}_s$, the *Lie derivative* $L_v\mathbf{t}$ from (2.23) of a *tensor* $\mathbf{t}$ on $\mathcal{S}$ with respect to the *velocity* $\mathbf{v}$ can be found as

$$L_v\mathbf{t} = \mathbf{\Phi}_{*\,t}\left(\frac{d}{dt}\left(\mathbf{\Phi}^*_t\mathbf{t}_t\right)\right). \tag{3.6}$$

The displacements

Let $\mathbf{S} := \mathbf{S}(\mathbf{X}) : T_X\mathcal{B} \mapsto T_{\Phi_t(X)}\mathcal{S}$ be a *shifter*, [15] transporting a *vector* emanating from $\mathbf{X}$ to a *vector* emanating from $\mathbf{x} = \mathbf{\Phi}_t(\mathbf{X})$. Using the existence of *local Cartesian coordinate systems* $\{z^i\}$ and $\{Z^I\}$ corresponding to $\{x^a\}$ and $\{X^A\}$ the components of $\mathbf{S}(\mathbf{X})$ are

$$S^a_A = \frac{\partial x^a}{\partial z^i}\frac{\partial Z^I}{\partial X^A}\delta^i_I \tag{3.7}$$

[14] Note, that A^a from (3.2) coincides with the *material time derivative* from (2.18) and (2.20), with the only difference, that the term $\frac{\partial V^a}{\partial X^A}\frac{d}{dt}X^A$, araising from (2.18) for $\mathbf{a} := \mathbf{V}$, disappears in (3.2) since the *reference configuration* does not change in time

[15] In the notation of page 7 it reads $\mathbf{S}_{t,0}$ with $\boldsymbol{\psi}_{t,0} := \mathbf{\Phi}(\mathbf{X}, t)$.

with δ^i_I denoting *Kronecker*'s symbol. Note, that $\mathbf{S}$ is *orthogonal* ($\mathbf{S}^T = \mathbf{S}^{-1}$). Now, on the *reference configuration*, the *displacements* $\mathbf{U}$ can be defined as

$$\mathbf{U} := \mathbf{S}^T \mathbf{x}_t - \mathbf{X} \quad \text{with components} \quad U^A = S^A_a x^a_t - X^A \,. \tag{3.8}$$

In the *current configuration* the *displacements* $\mathbf{u}$ is

$$\mathbf{u} := \mathbf{x}_t - \mathbf{S}\mathbf{X} \quad \text{with components} \quad u^a = x^a_t - S^a_A X^A. \tag{3.9}$$

In (3.8) and (3.9) no difference is made in descriptors for $\mathbf{X} \in \mathcal{B}$ and $\mathbf{X} \in T_X\mathcal{B}$ and also not for $\mathbf{x}_t \in \mathcal{S}$ and $\mathbf{x}_t \in T_{\Phi_t(X)}\mathcal{S}$. This is possible because of the supposed underlying *local Eucledian structure* of $\mathcal{B}$, implicating an isomorphism between $\mathcal{B}$ and $T_X\mathcal{B}$.
The computation of the *velocity* and the *acceleration* using the *displacements* is possible, but seems to make not so much sense, as can be seen in the following. On page 7 was shown, that for a *shifter* $\mathbf{S}^T_{0,t}$ used here $\frac{d}{dt}S^A_a = S^A_c \gamma^c_{ba} V^b$ is valid. [16] The time derivative of $\mathbf{S}_{t,0}$ can be found, taking into account $\frac{d}{dt}S^a_A = -S^b_A S^a_B \frac{d}{dt}S^B_b$, as $\frac{d}{dt}S^a_A = -S^b_A \gamma^a_{cb} v^c$. So the time derivative of *displacements* reads in components

$$\left(\frac{d}{dt}\mathbf{U}\right)^A = \frac{d}{dt}U^A \qquad = S^A_a V^a + S^A_c \gamma^c_{ba} V^b x^a \qquad = S^A_a \left(V^a + \gamma^a_{bc} V^b x^c\right)$$

$$\left(\frac{d}{dt}\mathbf{u}\right)^a = \frac{d}{dt}u^a + \gamma^a_{bc} v^b u^c = v^a + S^c_A \gamma^a_{cb} v^b X^A + \gamma^a_{bc} v^b \left(x^c - S^c_A X^A\right) = \qquad v^a + \gamma^a_{bc} v^b x^c.$$

The deformation gradient

Another kind of *mapping* between $T_X\mathcal{B}$ and $T_{\Phi_t(X)}\mathcal{S}$ is the *deformation gradient* $\mathbf{F}$, $\mathbf{F} := \mathbf{F}(\mathbf{X}, t) : T_X\mathcal{B} \mapsto T_{\Phi_t(X)}\mathcal{S}$ with components

$$F^a_A = \frac{\partial \Phi^a}{\partial X^A} \tag{3.10}$$

In terms of *displacements* the *deformation gradient* $\mathbf{F}$, its inverse $\mathbf{F}^{-1}$ and their components can be expressed by

$$\begin{array}{lcl} \mathbf{F} = \mathbf{S}(\mathbf{I} + GRAD\mathbf{U}) & , & \mathbf{F}^{-1} = \mathbf{S}^T(\mathbf{i} - grad\mathbf{u}) \\ F^a_A = S^a_B(\delta^A_B + U^B_{|A}) & , & (F^{-1})^A_a = S^A_b(\delta^b_a - u^b_{|a}) \end{array} \tag{3.11}$$

with $GRAD\mathbf{U}$ and $grad\mathbf{u}$ according to (2.5), $U^B_{|A}$ and $u^b_{|a}$ from (3.5) and $\mathbf{I}$, $\mathbf{i}$ denoting the identity operator.
The *transpose*, or *adjoint* of $\mathbf{F}$ [17] is the linear transformation $\mathbf{F}^T : T_{\Phi_t(X)}\mathcal{S} \mapsto T_X\mathcal{B}$ such that $\langle \mathbf{F}\mathbf{W}, \mathbf{v} \rangle = \langle \mathbf{W}, \mathbf{F}^T\mathbf{v} \rangle$ for all $\mathbf{W} \in T_X\mathcal{B}$ and $\mathbf{v} \in T_{\Phi_t(X)}\mathcal{S}$. Consequently $\mathbf{F}^T(\mathbf{x}, t)$ is given in components by

$$(F^T)^A_a = g_{ab} F^b_B G^{AB}. \tag{3.12}$$

The *deformation gradient* and its *adjoint* play a fundamental role in the subsequent theory.

[16] This can also be seen by the following calculation, using (3.7, 3.2, 2.9):
$\frac{d}{dt}S^A_a = \frac{d}{dt}\left(\frac{\partial z^i}{\partial x^a}\frac{\partial X^A}{\partial Z^I}\delta^I_i\right) = \frac{\partial^2 z^i}{\partial x^a \partial x^b} V^b \frac{\partial X^A}{\partial Z^I}\delta^I_i = \gamma^c_{ab} V^b \frac{\partial z^i}{\partial x^c}\frac{\partial X^A}{\partial Z^I}\delta^I_i = S^A_c \gamma^c_{ba} V^b.$

[17] and any other linear transformation $\mathbf{A} : T_X\mathcal{B} \mapsto T_{\Phi_t(X)}\mathcal{S}$

The deformation tensors

On the *reference configuration* we define the *right Cauchy–Green tensor* $\mathbf{C}(\mathbf{X},t)$, also called *Green deformation tensor*, to be

$$\mathbf{C} := \mathbf{F}^T\mathbf{F} \qquad , \qquad C^A_B = g_{ab}G^{AC}F^b_C F^a_B. \tag{3.13}$$

If $\mathbf{C}$ is invertible, $\mathbf{B} := \mathbf{C}^{-1}$ is called the *Piola deformation tensor*. On the *current configuration* the *left Cauchy–Green tensor*, also called *Finger deformation tensor*, $\mathbf{b}(\mathbf{x},t)$ is defined as

$$\mathbf{b} := \mathbf{F}\mathbf{F}^T \qquad , \qquad b^a_b = g_{bc}G^{AB}F^c_A F^a_B \tag{3.14}$$

with the inverse $\mathbf{c} := \mathbf{b}^{-1}$. The *material* or *Lagrangian strain tensor* $\mathbf{E}$ is defined by

$$\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{I}) \qquad , \qquad E^A_B = \frac{1}{2}\left(C^A_B - \delta^A_B\right) \tag{3.15}$$

and the *spatial* or *Eulerian strain tensor* $\mathbf{e}$ by

$$\mathbf{e} := \frac{1}{2}(\mathbf{i} - \mathbf{c}) \qquad , \qquad e^a_b = \frac{1}{2}\left(\delta^a_b - c^a_b\right)\ . \tag{3.16}$$

In terms of *pull backs* and *push forwards* the various *deformation tensors* (3.13)–(3.16) can be redefined by:

$$\begin{array}{lll} \mathbf{C}^\flat := \mathbf{\Phi}^*\mathbf{g} & \mathbf{B}^\sharp := \mathbf{\Phi}^*\mathbf{g}^\sharp & \mathbf{E}^\flat := \mathbf{\Phi}^*\mathbf{e}^\flat \\ \mathbf{c}^\flat := \mathbf{\Phi}_*\mathbf{G} & \mathbf{b}^\sharp := \mathbf{\Phi}_*\mathbf{G}^\sharp & \mathbf{e}^\flat := \mathbf{\Phi}_*\mathbf{E}^\flat\ . \end{array} \tag{3.17}$$

The *material* (or *Lagrangian*) *rate of deformation tensor* $\mathbf{D}$ is defined by

$$\mathbf{D} := \frac{1}{2}\frac{d}{dt}\mathbf{C}. \tag{3.18}$$

Using the formulation of the *Lie derivative* from (3.6), the *associated material rate of deformation tensor* $\mathbf{D}^\flat$ can be found as

$$\mathbf{D}^\flat = \frac{1}{2}\mathbf{\Phi}^*\mathcal{L}_{\mathbf{V}}\mathbf{g}\ . \tag{3.19}$$

At last, the *spatial* (or *Eulerian*) *rate of deformation tensor* $\mathbf{d}$ can be defined using

$$\mathbf{d}^\flat := \frac{1}{2}\mathcal{L}_{\mathbf{V}}\mathbf{g}. \tag{3.20}$$

Remark:
From an empirical point of view, the changes in the lenght of a *line element* during a *motion* of a *body* $\mathcal{B}$ are a measure of *deformation*. Some computation gives

$$dS^2 - ds^2 = G_{AB}\ dX^A dX^B - g_{ab}\ dx^a dx^b.$$

From this we get

$$dS^2 - ds^2 = [\, G_{AB} - g_{ab}\, F^a_A\, F^b_B \,]\, dX^A dX^B = G_{AC} E^C_B\, dX^A dX^B$$

as well as

$$dS^2 - ds^2 = [\, G_{AB}\, (F^{-1})^A_a\, (F^{-1})^B_b - g_{ab} \,]\, dx^a dx^b = g_{ac} e^c_b\, dx^a dx^b,$$

i.e., the deformation can completely be described in terms only related to the *reference configuration* or to the *current configuration* by using $\mathbf{E}$ or $\mathbf{e}$ from above and the corresponding metric tensors.

4 The stress tensor and balance of momentum

In the following we will assume that for a given sufficiently smooth *motion* $\mathbf{\Phi}(\mathbf{X}, t)$ of a *body* $\mathcal{B} \subset \mathcal{S} = \mathrm{R}^n$ there exist

- a *mass density* function $\rho(\mathbf{x}, t)$,
- a continuous vector field $\mathbf{r}(\mathbf{x}, t, \mathbf{n})$, called the *Cauchy traction vector* (representing the *force per unit area* exerted on a *surface element* of $\boldsymbol{\partial\Phi}_t(\mathcal{A})$, oriented with *unit outward normal* $\mathbf{n}$) and
- an *external force* field $\mathbf{l}(\mathbf{x}, t)$.

Then, the *balance of momentum* is satified, if for every sufficiently smooth open set $\mathcal{A} \subset \mathcal{B}$ the equation (4.1) is true:

$$\frac{d}{dt} \int_{\Phi_t(\mathcal{A})} \rho\, \mathbf{v}\, dv = \int_{\Phi_t(\mathcal{A})} \rho\, \mathbf{l}\, dv \;+ \int_{\partial\Phi_t(\mathcal{A})} \mathbf{r}\, da\ . \qquad (4.1)$$

If (4.1) and *conservation of mass* $\left(\frac{d}{dt}\rho + \rho div \mathbf{v} = 0\right)$ holds, there exists a unique[18] symmetric *Cauchy stress tensor* $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ satisfying

$$\mathbf{r} = \langle \boldsymbol{\sigma}, \mathbf{n} \rangle\ ^{19} \qquad \text{and} \qquad \rho \frac{d}{dt}\mathbf{v} = \rho\, \mathbf{l} + div\ \boldsymbol{\sigma}\ , \qquad (4.2)$$

or, written in components[20],

$$r^a = \sigma^{ac} g_{bc} n^b \quad , \quad \rho \left(\frac{\partial v^a}{\partial t} + v^a_{|b} v^b \right) = \rho\, l^a + (div\ \boldsymbol{\sigma})^a \quad , \quad (div\ \boldsymbol{\sigma})^a = \sigma^{ba}_{|b}\ ,$$
$$\sigma^{ba}_{|b} = \frac{\partial \sigma^{ba}}{\partial x^b} + \sigma^{ca} \gamma^b_{cb} + \sigma^{bc} \gamma^a_{cb}\ . \qquad (4.3)$$

[18]For a proof see page 31 in the appendix.

[19]Here and in the following we use the symbol $\langle \cdot, \cdot \rangle$, originally defined for the inner product, also in the sense of the leftmost part of (4.3), since the components of the resulting vector may be regarded as inner products of the "columns" of $\boldsymbol{\sigma}$ with $\mathbf{n}$.

[20]See page 5 for a hint on how to derive the components of $div\ \boldsymbol{\sigma}$.

With the *Jacobian* (9.8) we define the *Piola transform* $\mathbf{P}(\mathbf{X}, t)$ of $\boldsymbol{\sigma}$ with components

$$P^{aA} := J\ (F^{-1})^A_b\ \sigma^{ab} \tag{4.4}$$

which is called the *first Piola–Kirchhoff stress tensor*. This tensor is related to the *Cauchy stress tensor* $\boldsymbol{\sigma}$ by means of the *Piola Identity*

$$DIV\mathbf{P} = J\ div\boldsymbol{\sigma}\ , \tag{4.5}$$

what can be proved by some calculus. Using the theorem of *Gauss* and *Ostrogradski* (9.7), taking into account the underlying *Euclidean structure* of $\boldsymbol{\Phi}_t(\mathcal{A})$, the transformation behaviour of *domain integrals* (9.5), (9.6) and making use of (4.5) it can be shown, that the *balance of momentum* (4.1) is equivalent to

$$\frac{d}{dt}\int_{\mathcal{A}} \rho_{Ref}\mathbf{V}\ dV = \int_{\mathcal{A}} \rho_{Ref}\mathbf{L}\ dV + \int_{\partial\mathcal{A}} \mathbf{R}\ dA \tag{4.6}$$

with the *density* $\rho_{Ref} := \rho\ J$ in the *reference configuration*, $\mathbf{N}$ denoting the *unit outward normal* to $\boldsymbol{\partial}\mathcal{A}$, $\mathbf{L}(\mathbf{X}, t) = \mathbf{l}(\boldsymbol{\Phi}_t(\mathbf{X}), t)$ and $\mathbf{R} = \langle\mathbf{P}, \mathbf{N}\rangle = P^{aA}N_A$. The same analysis used to deduce (4.2) from (4.1) gives (4.7) from (4.6):

$$\rho_{Ref}\frac{d}{dt}\mathbf{V} = \rho_{Ref}\mathbf{L} + DIV\mathbf{P}$$

in coordinates:

$$\rho_{Ref}\left(\frac{dV^a}{dt} + \gamma^a_{bc}V^bV^c\right) = \rho_{Ref}L^a + P^{aA}_{|A} \tag{4.7}$$

with

$$P^{aA}_{|A} := \frac{\partial P^{aA}}{\partial X^A} + F^c_A\gamma^a_{bc}P^{bA} + \Gamma^A_{AC}P^{aC}.$$

The *second Piola–Kirchhoff stress tensor* $\mathbf{T}(\mathbf{X}, t)$ is defined by

$$T^{AB} := (F^{-1})^A_a P^{aB}. \tag{4.8}$$

The symmetry of $\mathbf{T}$ follows from the symmetry of $\boldsymbol{\sigma}$. The *first Piola–Kirchhoff stress tensor* is symmetric in the sense of

$$P^{aA}F^b_A = P^{bA}F^a_A\ . \tag{4.9}$$

On the *current configuration* it is also usefull to introduce a fourth stress tensor $\boldsymbol{\tau}$ called *Kirchhoff stress tensor*, defined by

$$\boldsymbol{\tau} := J\boldsymbol{\sigma}. \tag{4.10}$$

5 Balance of energy and principle of virtual work

Balance of momentum (4.1) explicitly uses the linear structure of R^n, because vector functions are integrated. It is correct to interpret this equation component-by-component in *Cartesian coordinates* $\{z^i\}$ but not in a general coordinate system, because the assumption of total forces like $\mathbf{l}$ and $\mathbf{r}$ in (4.1) acting on a body doesn't directly make sense, when the containing space $\mathcal{S}$ is curved. However, *energy balance* is sensefull on *manifolds* and can be used as a *covariant* basis for elasticity. *Covariance* may be explained in general terms in the following:

Suppose we have a theory described by a number of *tensor fields* $\mathbf{a}$, $\mathbf{b}$, ... on some space $\mathcal{S}$, and the equations of our theory (partial differential equations, integral equations, ...) take the form $\Lambda(\mathbf{a}, \mathbf{b}, ...) = 0$. The equations are called *covariant* or *form invariant*, if for any *diffeomorphism* [21] $\boldsymbol{\varphi} : \mathcal{S} \mapsto \mathcal{S}$ the equation $\boldsymbol{\varphi}^*\Lambda(\mathbf{a}, \mathbf{b}, ...) := \Lambda(\boldsymbol{\varphi}^*\mathbf{a}, \boldsymbol{\varphi}^*\mathbf{b}, ...) = 0$ holds with the *pull back* $\boldsymbol{\varphi}^*\mathbf{a}$ of some tensor $\mathbf{a}$ by the mapping $\boldsymbol{\varphi}$ as defined on page 5.

The balance of energy principle

We take into account only *mechanical effects* with functions $\rho(\mathbf{x}, t)$, $\mathbf{l}(\mathbf{x}, t)$ and $\mathbf{r}(\mathbf{x}, t, \mathbf{n})$, given for $\mathbf{x} \in \mathbf{\Phi}_t(\mathcal{B})$ and $\mathbf{n} \in T_x\mathcal{S}$, as they were described at the beginning of chapter 4. Let $e := e(\mathbf{x}, t)$ be the density of *internal energy*. Then, the *balance of energy principle* is satisfied if, for each sufficiently smoot $\mathcal{A} \subset \mathcal{B}$, the equation (5.1) holds:

$$\frac{d}{dt} \int\limits_{\Phi_t(\mathcal{A})} \rho \left[e + \frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle \right] \, dv = \int\limits_{\Phi_t(\mathcal{A})} \rho \langle \mathbf{l}, \mathbf{v} \rangle \, dv + \int\limits_{\partial\Phi_t(\mathcal{A})} \langle \mathbf{r}, \mathbf{v} \rangle \, da. \tag{5.1}$$

Superposed motions

Let the motion $\mathbf{\Phi}_t$, $\mathbf{x} := \mathbf{\Phi}(\mathbf{X}, t)$ of our body $\mathcal{S}$ be superposed by another motion or a *change of observer* $\boldsymbol{\varphi}_t : \mathcal{S} \mapsto \mathcal{S}$, $\tilde{\mathbf{x}} := \tilde{\mathbf{x}}_t := \boldsymbol{\varphi}(\mathbf{x}_t, t) = \boldsymbol{\varphi}(\mathbf{\Phi}_{\mathbf{X}}(t), t) =: \tilde{\mathbf{\Phi}}_{\mathbf{X}}(t)$ with $\boldsymbol{\varphi}(\mathbf{x}, t_0) = \tilde{\mathbf{x}}_{t_0} = \tilde{\mathbf{\Phi}}_{\mathbf{X}}(t_0) = \mathbf{\Phi}_{\mathbf{X}}(t_0) = \mathbf{x}_{t_0}$. Under this superposed motion the *metric tensor* $\mathbf{g}$ changes to

$$\tilde{\mathbf{g}} = \boldsymbol{\varphi}_*\mathbf{g} \qquad \text{with} \qquad \tilde{g}_{ab} = \frac{\partial(\boldsymbol{\varphi}^{-1})^c}{\partial \tilde{x}^a} \frac{\partial(\boldsymbol{\varphi}^{-1})^d}{\partial \tilde{x}^b} g_{cd} \, . \tag{5.2}$$

To proof this, we start with (2.9) and get $\tilde{g}_{ab} := \frac{\partial z^i}{\partial \tilde{x}^a}\frac{\partial z^i}{\partial \tilde{x}^b} = \frac{\partial z^i}{\partial x^c}\frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial z^i}{\partial x^d}\frac{\partial x^d}{\partial \tilde{x}^b} = \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}$ with $\mathbf{x} = \boldsymbol{\varphi}^{-1}(\tilde{\mathbf{x}})$. According to (3.1), the velocity $\tilde{\mathbf{V}}$ of $\tilde{\mathbf{\Phi}}$ has the components

$$\tilde{V}^a_{\mathbf{X}}(t) = \left(\frac{d}{dt} \tilde{\mathbf{\Phi}}_{\mathbf{X}}(t) \right)^a = \frac{\partial \varphi^a}{\partial t}|_{\mathbf{\Phi}_{\mathbf{X}}(t)} + \frac{\partial \varphi^a}{\partial x^b_t}|_{\mathbf{\Phi}_{\mathbf{X}}(t)} V^b_{\mathbf{X}}(t). \tag{5.3}$$

[21] a sufficiently smooth bijective mapping

Using (3.3) and (2.12) we get the *spatial velocity* $\tilde{\mathbf{v}}$ as

$$\tilde{\mathbf{v}}_{\tilde{\mathbf{X}}}(t) = \boldsymbol{\varphi}_* \mathbf{v}_{\mathbf{X}}(t) + \boldsymbol{\xi}(t) \qquad \text{with} \qquad \tilde{v}^a = \frac{\partial \varphi^a}{\partial x^b} v^b + \xi^a \tag{5.4}$$

where $\boldsymbol{\xi} := \frac{d\boldsymbol{\varphi}}{dt}$ is the *velocity* of $\tilde{\mathbf{x}}$ relative to $\mathbf{x}$.
As proved on page 32, the *spatial acceleration* $\tilde{\mathbf{a}}$ reads

$$\tilde{\mathbf{a}} = \boldsymbol{\varphi}_* \mathbf{a} + \frac{\partial \boldsymbol{\xi}}{\partial t} + \widetilde{grad}_{\boldsymbol{\xi}} \boldsymbol{\xi} + 2\, \widetilde{grad}_{(\boldsymbol{\varphi}_* \mathbf{v})} \boldsymbol{\xi} \,, \tag{5.5}$$

with $\frac{\partial \boldsymbol{\xi}}{\partial t} + \widetilde{grad}_{\boldsymbol{\xi}} \boldsymbol{\xi}$ denoting the accelaration of $\tilde{\mathbf{x}}$ relative to $\mathbf{x}$. Due to (3.2)–(3.5) and (2.18), the components of $\tilde{\mathbf{a}}$ are

$$\tilde{a}^a = \frac{d}{dt} \tilde{v}^a + \tilde{\gamma}^a_{cd} \tilde{v}^c \tilde{v}^d \,. \tag{5.6}$$

We assume, that the forces and the *Cauchy stress vector* transform as in (5.7):

$$\tilde{\mathbf{l}} - \tilde{\mathbf{a}} = \boldsymbol{\varphi}_* (\mathbf{l} - \mathbf{a}) \qquad , \qquad \tilde{\mathbf{r}} = \boldsymbol{\varphi}_* \mathbf{r} \,. \tag{5.7}$$

At time $t = t_0$ equations (5.4-5.7) read

$$\begin{aligned} \tilde{\mathbf{v}} &= \mathbf{v} + \boldsymbol{\xi} \qquad , \qquad & \tilde{\mathbf{a}} &= \mathbf{a} + \frac{\partial \boldsymbol{\xi}}{\partial t} + grad_{\boldsymbol{\xi}} \boldsymbol{\xi} + 2\, grad_{\mathbf{v}} \boldsymbol{\xi} \\ \tilde{\mathbf{l}} - \tilde{\mathbf{a}} &= \mathbf{l} - \mathbf{a} \qquad , \qquad & \tilde{\mathbf{r}} &= \mathbf{r} \end{aligned} \tag{5.8}$$

If the transformation $\boldsymbol{\varphi}$ is not a *rigid* body motion, $\boldsymbol{\varphi}$ changes the metric (cf. (5.2)) and influences the *acceleration* (cf. (5.5)). Therefore, the *internal energy* e must depend parametrically on the metric $\mathbf{g}$, and it is natural to suppose the transformation

$$\tilde{e} := e(\boldsymbol{\varphi}_t^{-1}(\tilde{\mathbf{x}}), t, \boldsymbol{\varphi}^* \tilde{\mathbf{g}}) \,. \tag{5.9}$$

Then, as proved in the appendix page 33, the time derivative of $\tilde{e}$ at$=t_0$ where $\boldsymbol{\varphi} = identity$ can be found as

$$\left(\frac{d}{dt} \tilde{e} \right)_{|t_0} = \frac{d}{dt} e + \frac{\partial e}{\partial g_{ab}} \left(\mathcal{L}_\xi \mathbf{g} \right)_{ab} = \frac{d}{dt} e + \frac{\partial e}{\partial \mathbf{g}} : \mathcal{L}_\xi \mathbf{g} \tag{5.10}$$

with the *autonomous Lie derivative* $\mathcal{L}_\xi \mathbf{g}$ from (2.27). Comparing the *balance of energy principle* in the original and in the transformed state, on page 33 the identity

$$\int_{\Phi_t(\mathcal{A})} \left[\left(\frac{d}{dt} \rho + \rho\, div \mathbf{v} \right) \left(\frac{1}{2} \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle + \langle \mathbf{v}, \boldsymbol{\xi} \rangle \right) + \rho \left(\frac{\partial e}{\partial \mathbf{g}} : \mathcal{L}_\xi \mathbf{g} + \langle \mathbf{a} - \mathbf{l}, \boldsymbol{\xi} \rangle \right) \right] dv = \int_{\partial \Phi_t(\mathcal{A})} \langle \mathbf{r}, \boldsymbol{\xi} \rangle da \tag{5.11}$$

is proved. Introducing the *Cauchy stress tensor* $\mathbf{r} = \langle \boldsymbol{\sigma}, \mathbf{n} \rangle$ from (4.2) and applying the divergency theorem[22]

$$\begin{gathered} div \langle \boldsymbol{\sigma}, \boldsymbol{\xi} \rangle = \langle div\, \boldsymbol{\sigma}, \boldsymbol{\xi} \rangle + \boldsymbol{\sigma} : \boldsymbol{\omega}^\flat_\xi + \tfrac{1}{2} \boldsymbol{\sigma} : \mathcal{L}_\xi \mathbf{g} \\ \text{with the spin } \boldsymbol{\omega}^\flat_\xi,\ \omega_{\xi ab} = \tfrac{1}{2} \left((g_{ac} \xi^c)_{|b} - (g_{bc} \xi^c)_{|a} \right) = \tfrac{1}{2} \left(g_{ac} \xi^c_{|b} - g_{cb} \xi^c_{|a} \right) , \end{gathered} \tag{5.12}$$

[22]For a proof see page 31 in the appendix.

and $div\,\boldsymbol{\sigma}$ from (4.3) to the right hand side of (5.11) it reads

$$\int\limits_{\Phi_t(\mathcal{A})} \left[\left(\frac{d}{dt}\rho + \rho\, div \mathbf{v} \right) \langle \frac{1}{2}\boldsymbol{\xi} + \mathbf{v}, \boldsymbol{\xi} \rangle + \left(\rho\, \frac{\partial e}{\partial \mathbf{g}} - \frac{1}{2}\boldsymbol{\sigma} \right) : \mathcal{L}_\xi \mathbf{g} - \boldsymbol{\sigma} : \boldsymbol{\omega}_\xi^\flat + \langle \rho \mathbf{a} - \rho \mathbf{l} - div \boldsymbol{\sigma}, \boldsymbol{\xi} \rangle \right] dv = 0. \tag{5.13}$$

Since $\mathcal{A}$ is arbitrary, (5.13) results in a differential equation in $\boldsymbol{\xi}$ at any point. This violates the assumption of the arbitraryness of $\boldsymbol{\xi}$, unless the whole term to be integrated vanishes in each point. So (5.13) is valid only if we have

$$\begin{aligned}
&\frac{d}{dt}\rho + \rho div \mathbf{v} = 0 && \Longrightarrow \quad \textit{conservation of mass} \\
&\rho \mathbf{a} - \rho \mathbf{l} - div\, \boldsymbol{\sigma} = 0 && \Longrightarrow \quad \textit{conservation of momentum} \\
&\boldsymbol{\sigma} \text{ is symmetric} && \Longrightarrow \quad \textit{conservation of moment of momentum} \\
&\boldsymbol{\sigma} = 2\rho \frac{\partial e}{\partial \mathbf{g}} && \Longrightarrow \quad \textit{Doyle–Ericksen–Formula.}
\end{aligned} \tag{5.14}$$

So we see, that the *conservation of mass*, the *conservation of momentum* and the *conservation of moment of momentum*, as assumed in the previous chapter, can be shown to follow from *balance of energy* and the principle of *covariance*.

The principal of virtual work

Inserting the *Doyle–Ericksen–Formula* and the *conservation of momentum* from (5.14) into (5.11) with $\mathcal{A} = \mathcal{B}$ we get the *principal of virtual work*

$$\int\limits_{\Phi_t(\mathcal{B})} \left[\boldsymbol{\sigma} : \mathbf{d}_\xi^\flat + \rho \langle \mathbf{a} - \mathbf{l}, \boldsymbol{\xi} \rangle \right] dv - \int\limits_{\partial\Phi_t(\mathcal{B})} \langle \mathbf{r}, \boldsymbol{\xi} \rangle da = 0 \tag{5.15}$$

with $\mathbf{d}_\xi^\flat := \frac{1}{2}\mathcal{L}_\xi \mathbf{g}$ according to (3.20). *Pulling back* (5.1) to the *reference configuration*, it yields to

$$\frac{d}{dt}\int\limits_{\mathcal{A}} \rho_{Ref} \left[E + \frac{1}{2}\langle \mathbf{V}, \mathbf{V} \rangle \right] dV = \int\limits_{\mathcal{A}} \rho_{Ref} \langle \mathbf{L}, \mathbf{V} \rangle \, dV + \int\limits_{\partial\mathcal{A}} \langle \langle \mathbf{P}, \mathbf{N} \rangle, \mathbf{V} \rangle \, dA, \tag{5.16}$$

the analogon to (5.1), with $E := \boldsymbol{\Phi}^* e = e(\boldsymbol{\Phi}_t(\mathbf{X}), t, \boldsymbol{\Phi}_* \mathbf{C}^\flat) = E(\mathbf{X}, t, \mathbf{C}^\flat)$, as sketched on page 34. From this we get the equation (5.17):

$$\int\limits_{\mathcal{A}} \left[\left(2\rho_{Ref} \frac{\partial E}{\partial \mathbf{C}^\flat} - \mathbf{T} \right) : \mathbf{D}_\Xi^\flat - \mathbf{T} : \boldsymbol{\Omega}_\Xi^\flat + \langle \rho_{Ref} \mathbf{A} - \rho_{Ref} \mathbf{L} - DIV \mathbf{P}, \boldsymbol{\Xi} \rangle \right] dV = 0. \tag{5.17}$$

Following the argumentation used to derive (5.14) from (5.13) we see, that

$$\begin{aligned} &\rho_{Ref}\mathbf{A} - \rho_{Ref}\mathbf{L} - DIV\mathbf{P} = 0\ , \\ &\mathbf{T}\text{ is symmetric} \qquad\qquad \text{and} \\ &\mathbf{T} = 2\rho_{Ref}\frac{\partial E}{\partial \mathbf{C}^\flat}\ . \end{aligned} \tag{5.18}$$

Inserting the last line of (5.18) into (9.25) we finally get the *principle of virtual work* on the *reference configuration* :

$$\int\limits_{\mathcal{B}} \left(\mathbf{T}:\mathbf{D}^\flat_\Xi + \rho_{Ref}\langle \mathbf{A}-\mathbf{L}, \Xi\rangle\right)\ dV - \int\limits_{\partial\mathcal{B}} \langle \mathbf{R}, \Xi\rangle\ dA = 0 \tag{5.19}$$

with $\mathbf{R} := \langle \mathbf{P}, \mathbf{N}\rangle$.

6 The second law of thermodynamics

In *thermodynamics* of irreversible processes, one of the important objectives is to relate the change of *specific entropy* η to the various irreversible phenomena which may occur inside the system. The *second law of thermodynamics* is introduced by the ad–hoc dissipation inequality

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} \rho\eta\ dv \geq \int\limits_{\Phi_t(\mathcal{A})} \frac{\rho s}{\vartheta}\ dv + \int\limits_{\partial\Phi_t(\mathcal{A})} \frac{h}{\vartheta}\ da, \tag{6.1}$$

for each sufficiently smoot $\mathcal{A} \subset \mathcal{B}$, with the heat supply per unit mass $s(\mathbf{x}, t)$, the heat flux (across a surface with normal $\mathbf{n}$) $h(\mathbf{x}, t, \mathbf{n})$ and the *absolute temperature* $\vartheta(\mathbf{x}, t)$. The *first law of thermodynamics*, as given in (5.1), doesn't reflect the influence of *thermal effects* as introduced now. So it has to be rewritten as

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} \rho\left[e + \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle\right]\ dv = \int\limits_{\Phi_t(\mathcal{A})} \rho\left[\langle \mathbf{l}, \mathbf{v}\rangle + s\right]\ dv + \int\limits_{\partial\Phi_t(\mathcal{A})} \left[\langle \mathbf{r}, \mathbf{v}\rangle + h\right]\ da. \tag{6.2}$$

Assume, that there exists a *heat flux vector* $\mathbf{q}(\mathbf{x}, t)$ with $h(\mathbf{x}, t, \mathbf{n}) = -\langle \mathbf{q}(\mathbf{x}, t), \mathbf{n}\rangle$ and that *conservation of mass* holds. Then

$$\rho\frac{d}{dt}\eta \geq \frac{\rho s}{\vartheta} - div\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{\rho s}{\vartheta} - \frac{1}{\vartheta}\left[div\ \mathbf{q} - \frac{1}{\vartheta}\langle \mathbf{q}, \nabla\vartheta\rangle\right] \tag{6.3}$$

and

$$\rho\frac{d}{dt}e - \boldsymbol{\sigma}:\mathbf{d}^\flat - \rho s + div\ \mathbf{q} = 0 \tag{6.4}$$

can be shown [23] to follow from (6.1) and (6.2). Combining (6.3) and (6.4) we get

$$\rho\left[\frac{d}{dt}e - \vartheta\frac{d}{dt}\eta\right] - \boldsymbol{\sigma}:\mathbf{d}^\flat + \frac{1}{\vartheta}\langle \mathbf{q}, \nabla\vartheta\rangle \leq 0,$$

[23] cf. page 37

and with the *specific free energy* $\zeta := e - \vartheta\eta$ the *reduced dissipation inequality*

$$\rho\left[\frac{d}{dt}\zeta + \eta\frac{d}{dt}\vartheta\right] - \boldsymbol{\sigma}:\mathbf{d}^\flat + \frac{1}{\vartheta}\langle\mathbf{q}, \nabla\vartheta\rangle \leq 0 \tag{6.5}$$

follows.
Pulling back [24] (6.1) and (6.2), the *second* and the *first law of thermodynamics* on the *reference configuration* are obtained as

$$\frac{d}{dt}\int_{\mathcal{A}}\rho_{Ref}\,\mathcal{E}\,dV \geq \int_{\mathcal{A}}\frac{\rho_{Ref}S}{\mathcal{T}}\,dV + \int_{\partial\mathcal{A}}\frac{H}{\mathcal{T}}\,dA, \tag{6.6}$$

$$\frac{d}{dt}\int_{\mathcal{A}}\rho_{Ref}\left[E + \frac{1}{2}\langle\mathbf{V},\mathbf{V}\rangle\right]\,dV = \int_{\mathcal{A}}\rho_{Ref}\,[\langle\mathbf{L},\mathbf{V}\rangle + S]\,dV + \int_{\partial\mathcal{A}}[\langle\mathbf{R},\mathbf{V}\rangle + H]\,dA, \tag{6.7}$$

with $\mathcal{E}(\mathbf{X}, t, \mathbf{C}^\flat, \mathcal{T}) := \eta\left(\boldsymbol{\Phi}_t(\mathbf{X}), t, \boldsymbol{\Phi}_*\mathbf{C}^\flat, \vartheta\right)$,[25] $\mathcal{T}(\mathbf{X}, t) := \vartheta(\boldsymbol{\Phi}_t(\mathbf{X}), t)$, $S(\mathbf{X}, t) := s(\boldsymbol{\Phi}_t(\mathbf{X}), t)$, $H(\mathbf{X}, t, \mathbf{N}) := -\langle\mathbf{Q}(\mathbf{X}, t), \mathbf{N}\rangle$, $\mathbf{Q} := J\mathbf{F}^{-1}\cdot\mathbf{q}$, $\mathbf{R}(\mathbf{X}, t, \mathbf{N}) := \langle\mathbf{P}(\mathbf{X}, t), \mathbf{N}\rangle$, $\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$. Following the ideas sketched on page 37, the *localized forms* of (6.6) and (6.7) can be found as

$$\rho_{Ref}\frac{d}{dt}\mathcal{E} \geq \frac{\rho_{Ref}S}{\mathcal{T}} - DIV\left(\frac{\mathbf{Q}}{\mathcal{T}}\right), \tag{6.8}$$

$$\rho_{Ref}\frac{d}{dt}E - \mathbf{T}:\mathbf{D}^\flat - \rho_{Ref}S + DIV\mathbf{Q} = 0\,, \tag{6.9}$$

and the *reduced dissipation inequality* with $\mathcal{Z} := E - \mathcal{T}\mathcal{E}$ reads

$$\rho_{Ref}\left[\frac{d}{dt}\mathcal{Z} + \mathcal{E}\frac{d}{dt}\mathcal{T}\right] - \mathbf{T}:\mathbf{D}^\flat + \frac{1}{\mathcal{T}}\langle\mathbf{Q}, \nabla\mathcal{T}\rangle \leq 0\,. \tag{6.10}$$

The inequalities (6.5) and (6.10) are also called *spatial* and *material Clausius–Duhem inequality*, respectively.

7 Linearization of nonlinear elasticity

Applying (2.33) and (2.21) to the *deformation gradient* $\mathbf{F}$ defined in (3.10) we get

$$\widetilde{F}^a_A = F^a_A + \lim_{r\to s}\left[\frac{d}{dr}\left((S_{s,r})^a_b\,\frac{\partial\Phi^b_r}{\partial X^A}\right)\right](t-s) = F^a_A + \frac{\partial\Psi^a_s}{\partial X^A} + \gamma^a_{bc}\Psi^b_s F^c_A = F^a_A + \Psi^a_{s|A}. \tag{7.1}$$

with $\Psi^a_s := \frac{d}{ds}\Phi^a_s(t-s)$ and assuming $s \approx t$ (cf. page 7), or, in a more compact notation,

$$\widetilde{\mathbf{F}} = \mathbf{F} + GRAD\boldsymbol{\Psi}. \tag{7.2}$$

[24] specially using (9.5) and (9.6), for details see page 34 and the definitions made there.
[25] demanding the *principle of covariance* to apply to the *second law of thermodynamics*, the *specific entropy* η is not permitted to depend on the *metric* [MH83], so we must write $\mathcal{E}(\mathbf{X}, t, \mathcal{T}) := \eta(\boldsymbol{\Phi}_t(\mathbf{X}), t, \vartheta)$. But this doesn't infer the following theory.

We saw that, assuming an *infinitesimal deformation* $\mathbf{\Psi}$ *imposed on the finite deformation* $\mathbf{\Phi}_t$, the *deformation gradient* changes to (7.2).
The combination of (5.18) and (4.8) gives $\mathbf{P} = 2\rho_{Ref}\mathbf{F}\dfrac{\partial E}{\partial \mathbf{C}^\flat}$ with $E = E(\mathbf{X}, t, \mathbf{C}^\flat)$, showing us that, in the general case, the stresses $\mathbf{P} = \mathbf{P}(\mathbf{F}(t), E)$ depend on the deformation $\mathbf{F}$ as well as on space and time. Assuming the investigated material to be homogeneous in space and time means, that $\mathbf{P}$ is assumed to be a tensorial function of $\mathbf{F}$ only, and using the linear terms of (2.31) the *first Piola–Kirchhoff stress tensor* $\mathbf{P}$ can be found as

$$\widetilde{\mathbf{P}} = \mathbf{P} + \frac{\partial \mathbf{P}}{\partial \mathbf{F}} : GRAD\mathbf{\Psi} \ . \tag{7.3}$$

Taking into account the *linearization* of $\mathbf{V}$

$$\widetilde{V}^a = V^a + \lim_{r \to s}\left[\frac{d}{dr}\left((\mathbf{S}_{s,r})^a_b \frac{d}{dr}\Phi^a_r\right)\right](t-s) = V^a + \frac{d}{dt}\Psi^a + \gamma^a_{bc}\Psi^b V^c = V^a + \left(\frac{d}{dt}\mathbf{\Psi}\right)^a \tag{7.4}$$

the *linearization* of the *equation of motion* (4.7) will be

$$\rho_{Ref}\left[\frac{d}{dt}\left(\mathbf{V} + \frac{d}{dt}\mathbf{\Psi}\right) - \mathbf{L}\right] = DIV\left(\mathbf{P} + \frac{\partial \mathbf{P}}{\partial \mathbf{F}} : GRAD\mathbf{\Psi}\right) \ . \tag{7.5}$$

As the next step we have to compute $\dfrac{\partial \mathbf{P}}{\partial \mathbf{F}}$:
With $C_{AB} = g_{ab}F^a_A F^b_B$ from (3.13) and (3.17), we get

$$\frac{\partial C_{AB}}{\partial F^c_C} = g_{ab}\left(F^a_A \delta^b_c \delta^B_C + F^b_B \delta^a_c \delta^A_C\right) = 2g_{ab}F^b_A \delta^a_c \delta^B_C = 2g_{bc}F^b_A \delta^B_C \tag{7.6}$$

due to the symmetry of $\mathbf{g}$ and $\mathbf{C}$. Since E depends on $\mathbf{F}$ only by $\mathbf{C}$ we have

$$\frac{\partial E}{\partial F^a_A} = \frac{\partial E}{\partial C_{BC}}\frac{\partial C_{BC}}{\partial F^a_A} = 2\frac{\partial E}{\partial C_{BC}} g_{ab}F^b_B \delta^C_A = 2\frac{\partial E}{\partial C_{AB}} g_{ab}F^b_B \ . \tag{7.7}$$

Combining (5.18) and (4.8) we get

$$P^{aA} = 2\rho_{Ref}F^a_C \frac{\partial E}{\partial C_{AC}} \ . \tag{7.8}$$

From (7.7) and (7.8) we get $\dfrac{g_{ce}}{\rho_{Ref}}P^{cE} = \dfrac{\partial E}{\partial F^e_E}$. Multiplying this by g^{de} supplies for the components of $\mathbf{P}$, $\rho_{Ref}g^{de}\dfrac{\partial E}{\partial F^e_E} = \delta^d_c P^{cE} = P^{dE}$. So we find the first expression for $\dfrac{\partial \mathbf{P}}{\partial \mathbf{F}}$:

$$\frac{\partial P^{eE}}{\partial F^b_B} = \rho_{Ref}\ g^{ec}\frac{\partial^2 E}{\partial F^c_E \partial F^b_B} \ . \tag{7.9}$$

Using (7.7) and (7.6) helps us to compute $\dfrac{\partial^2 E}{\partial F^b_B \partial F^c_A} = \dfrac{\partial}{\partial F^c_A}\dfrac{\partial E}{\partial F^b_B} = 2g_{bd}\dfrac{\partial}{\partial F^c_A}\left(\dfrac{\partial E}{\partial C_{BC}}F^d_C\right) = 2g_{bd}\left(\delta^d_c\delta^C_A\dfrac{\partial E}{\partial C_{BC}} + F^d_C\dfrac{\partial^2 E}{\partial C_{BC}\partial C_{DE}}\dfrac{\partial C_{DE}}{\partial F^c_A}\right) = 2g_{bc}\dfrac{\partial E}{\partial C_{AB}} + 4g_{bd}g_{ce}\dfrac{\partial^2 E}{\partial C_{BC}\partial C_{AD}}F^d_C F^e_D$. Inserting this into (7.9) we get the second expression for $\dfrac{\partial \mathbf{P}}{\partial \mathbf{F}}$:

$$\frac{\partial P^{aA}}{\partial F^b_B} = \rho_{Ref}\, g^{ac}\frac{\partial^2 E}{\partial F^b_B \partial F^c_A} = \rho_{Ref}\, g^{ac}\left(2\frac{\partial E}{\partial C_{AB}}g_{cb} + 4\frac{\partial^2 E}{\partial C_{AC}\partial C_{DB}}g_{cd}g_{be}F^d_C F^e_D\right) . \qquad (7.10)$$

A third representation for $\dfrac{\partial \mathbf{P}}{\partial \mathbf{F}}$, introducing the components of the *elasticity tensor* $\boldsymbol{\mathcal{C}}$, is gained from (4.8) and (7.6):
Due to $\dfrac{\partial P^{aA}}{\partial F^b_B} = \dfrac{\partial (F^a_C T^{AC})}{\partial F^b_B} = T^{AB}\delta^a_b + F^a_C\dfrac{\partial T^{AC}}{\partial C_{DE}}\dfrac{\partial C_{DE}}{\partial F^b_B} = T^{AB}\delta^a_b + 2\dfrac{\partial T^{AC}}{\partial C_{BD}}g_{bc}F^a_C F^c_D$ we have

$$\mathcal{C}^{aBA}_b := \frac{\partial P^{aA}}{\partial F^b_B} = 2\frac{\partial T^{AC}}{\partial C_{DB}}F^a_C F^e_D g_{be} + T^{AB}\delta^a_b \, . \qquad (7.11)$$

Using the widely in common use notation $\dfrac{\partial T^{AC}}{\partial C_{DB}}F^a_C F^e_D g_{be} =: \dfrac{\partial \mathbf{T}}{\partial \mathbf{C}^\flat}\!\cdot\!\mathbf{F}\!\cdot\!\mathbf{F}\!\cdot\!\mathbf{g}$ and $T^{AB}\delta^a_b =: \mathbf{T}\otimes\mathbf{1}$ with $(\mathbf{1})^a_b := \delta^a_b$, equation (7.5) becomes

$$\rho_{Ref}\left[\frac{d}{dt}\left(\mathbf{V}+\frac{d}{dt}\boldsymbol{\Psi}\right)-\mathbf{L}\right] = DIV\left[\mathbf{P}+\left(2\frac{\partial \mathbf{T}}{\partial \mathbf{C}^\flat}\cdot\mathbf{F}\cdot\mathbf{F}\cdot\mathbf{g} + \mathbf{T}\otimes\mathbf{1}\right):GRAD\boldsymbol{\Psi}\right] . \qquad (7.12)$$

Next we linearize (4.2): From (3.3), (7.4) and $\boldsymbol{\psi} = \boldsymbol{\Psi}(\boldsymbol{\Phi}^{-1}(\mathbf{x}))$ we get

$$\tilde{\mathbf{v}} = \mathbf{v} + \frac{d}{dt}\boldsymbol{\psi} \, . \qquad (7.13)$$

Due to (4.4) we have $\boldsymbol{\sigma} = \dfrac{1}{J}\mathbf{F}\mathbf{P}$ and therefore

$$\tilde{\boldsymbol{\sigma}} = \frac{1}{J}\mathbf{F}\tilde{\mathbf{P}} \qquad (7.14)$$

with $\tilde{\mathbf{P}}$ from (7.3). From (4.2) we see, that $\rho\left(\dfrac{d\widetilde{\mathbf{V}}}{dt} - \mathbf{l}\right) = div\, \tilde{\boldsymbol{\sigma}}$, and the insertion (7.13), (7.14) and (7.3) supplies

$$\rho\left[\frac{d}{dt}\left(\mathbf{v} + \frac{d}{dt}\boldsymbol{\psi}\right) - \mathbf{l}\right] = div\left[\boldsymbol{\sigma} + \frac{1}{J}\mathbf{F}\frac{\partial \mathbf{P}}{\partial \mathbf{F}} : GRAD\ \boldsymbol{\Psi}\right] . \qquad (7.15)$$

For the *infinitesimal deformation* $\boldsymbol{\Psi}(\mathbf{X}) = \boldsymbol{\psi}(\boldsymbol{\Phi}(\mathbf{X}))$ we have

$$\Psi^a_{|B} = \frac{\partial \Psi^a}{\partial X^B} + \gamma^a_{bc}\Psi^c F^b_B = \frac{\partial \psi^a}{\partial x^b}\frac{\partial \Phi^b}{\partial X^B} + \gamma^a_{bc}\psi^c F^b_B = \left(\frac{\partial \psi^a}{\partial x^b} + \gamma^a_{bc}\psi^c\right)F^b_B = F^b_B\psi^a_{|b} \, .$$

Using the above equation and (7.10) we get the components of the inverse *Piola transform* of $\dfrac{\partial \mathbf{P}}{\partial \mathbf{F}} : GRAD\boldsymbol{\Psi}$ as

$$\frac{1}{J} F^c_A \frac{\partial P^{aA}}{\partial F^b_B} \Psi^b_{|B} = \frac{1}{J} F^c_A \frac{\partial P^{aA}}{\partial F^b_B} F^d_B \psi^b_{|d} = \frac{\rho_{Ref}}{J} F^c_A \left(2 \frac{\partial E}{\partial C_{AB}} \delta^a_b + 4 \frac{\partial^2 E}{\partial C_{AC} \partial C_{DB}} g_{be} F^a_C F^e_D \right) F^d_B \psi^b_{|d} . \tag{7.16}$$

Supposing the *internal energy* to satisfy the *principle of covariance*, as done in the previous chapters, we have $\boldsymbol{\Phi}^* e(\mathbf{x}, t, \mathbf{g}) = E(\mathbf{X}, t, \mathbf{C}^\flat)$ and consequently

$$\frac{\partial E}{\partial \mathbf{C}^\flat} = \boldsymbol{\Phi}^* \frac{\partial e}{\partial \mathbf{g}} \quad \text{and} \quad \frac{\partial^2 E}{\partial \mathbf{C}^{\flat 2}} = \boldsymbol{\Phi}^* \frac{\partial^2 e}{\partial \mathbf{g}^2} . \tag{7.17}$$

Inserting (7.17) with (9.2) into (7.16) we can write

$$\frac{1}{J} F^c_A \frac{\partial P^{aA}}{\partial F^b_B} \Psi^b_{|B} = \rho \left(2 \frac{\partial e}{\partial g_{cd}} \delta^a_b + 4 \frac{\partial^2 e}{\partial g_{ca} \partial g_{ed}} g_{be} \right) \psi^b_{|d} .$$

Finally, from (5.14) and (4.10) we get $2\rho \dfrac{\partial e}{\partial g_{cd}} = \sigma^{cd}$ as well as $2 \dfrac{\partial^2 e}{\partial g_{ca} \partial g_{ed}} = \frac{1}{\rho_{Ref}} \dfrac{\partial \tau^{ca}}{\partial g_{ed}}$, ending up with

$$\frac{1}{J} F^c_A \frac{\partial P^{aA}}{\partial F^b_B} \Psi^b_{|B} = \left(\sigma^{cd} \delta^a_b + \frac{2}{J} \frac{\partial \tau^{ca}}{\partial g_{ed}} g_{be} \right) \psi^b_{|d} . \tag{7.18}$$

Inserting (7.18) into (7.15), we get the desired linearization of (4.2) with notation explained in (7.12) and (7.18):

$$\rho \left[\frac{d}{dt} \left(\mathbf{v} + \frac{d}{dt} \boldsymbol{\psi} \right) - \mathbf{l} \right] = div \left[\boldsymbol{\sigma} + \left(\frac{2}{J} \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{g}} \cdot \mathbf{g} + \boldsymbol{\sigma} \otimes \mathbf{1} \right) : grad \boldsymbol{\psi} \right] . \tag{7.19}$$

8 Multiplicative Elastoplasticity at Finite Strains

The following theory is founded on the basic assumption of the *multiplicative split* (cf. [Sim93]) of the *deformation gradient* $\mathbf{F}$ in an *elastic part* $\overset{\mathrm{e}}{\mathbf{F}}$ and a *plastic part* $\overset{\mathrm{p}}{\mathbf{F}}$

$$\mathbf{F} = \overset{\mathrm{e}}{\mathbf{F}} \cdot \overset{\mathrm{p}}{\mathbf{F}} \quad , \quad F^a_A = \overset{\mathrm{e}}{F}{}^a_\alpha \overset{\mathrm{p}}{F}{}^\alpha_A \tag{8.1}$$

where the *plastic part* $\overset{\mathrm{p}}{\mathbf{F}}$ is obtained by elastic unloading all infinitesimal neighbourhoods of the body. This has the effect of introducing a new *configuration* with *coordinates* $\{\tilde{x}^\alpha\}$ and the *metric* $\tilde{\mathbf{g}}$ into the formulation, commonly termed the *intermediate configuration.* Obviously, the inverse tensor to (8.1) has the components

$$\left(F^{-1} \right)^A_{\ a} = \left(\overset{\mathrm{e}}{F}{}^{-1} \right)^\alpha_{\ a} \left(\overset{\mathrm{p}}{F}{}^{-1} \right)^A_{\ \alpha} . \tag{8.2}$$

Under the above assumption and heeding (9.1), the *right Cauchy –Green tensor* from (3.17) can be found as

$$C_{AB} = (\mathbf{\Phi}^*\mathbf{g})_{AB} = F^a_A F^b_B\, g_{ab} = \overset{\mathrm{p}}{F}{}^\alpha_A \overset{\mathrm{p}}{F}{}^\beta_B g_{ab} \overset{\mathrm{e}}{F}{}^a_\alpha \overset{\mathrm{e}}{F}{}^b_\beta = \overset{\mathrm{p}}{F}{}^\alpha_A \overset{\mathrm{p}}{F}{}^\beta_B \overset{\mathrm{e}}{C}_{\alpha\beta} \tag{8.3}$$

or

$$\mathbf{C}^\flat = \overset{\mathrm{p}}{\mathbf{\Phi}}{}^* \overset{\mathrm{e}}{\mathbf{C}}{}^\flat \qquad , \qquad \overset{\mathrm{e}}{C}_{\alpha\beta} := g_{ab} \overset{\mathrm{e}}{F}{}^a_\alpha \overset{\mathrm{e}}{F}{}^b_\beta \ , \tag{8.4}$$

and the *left Cauchy–Green tensor* from (3.17) reads

$$b^{ab} = \left(\mathbf{\Phi}_* \mathbf{G}^\sharp\right)^{ab} = F^a_A F^b_B G^{AB} = \overset{\mathrm{e}}{F}{}^a_\alpha \overset{\mathrm{e}}{F}{}^b_\beta\, G^{AB} \overset{\mathrm{p}}{F}{}^\alpha_A \overset{\mathrm{p}}{F}{}^\beta_B = \overset{\mathrm{e}}{F}{}^a_\alpha \overset{\mathrm{e}}{F}{}^b_\beta \overset{\mathrm{p}}{b}{}^{\alpha\beta} \tag{8.5}$$

or

$$\mathbf{b}^\sharp = \overset{\mathrm{e}}{\mathbf{\Phi}}{}_* \overset{\mathrm{p}}{\mathbf{b}}{}^\sharp \qquad , \qquad \overset{\mathrm{p}}{b}{}^{\alpha\beta} := G^{AB} \overset{\mathrm{p}}{F}{}^\alpha_A \overset{\mathrm{p}}{F}{}^\beta_B \ . \tag{8.6}$$

According to (3.18), the *associated material rate of deformation tensor* is given by

$$D_{AB} = \frac{1}{2}\left[\overset{\mathrm{p}}{F}{}^\alpha_A \overset{\mathrm{p}}{F}{}^\beta_B \frac{d}{dt}\overset{\mathrm{e}}{C}_{\alpha\beta} + \overset{\mathrm{e}}{C}_{\alpha\beta}\left(\overset{\mathrm{p}}{F}{}^\alpha_A \frac{d}{dt}\overset{\mathrm{p}}{F}{}^\beta_B + \overset{\mathrm{p}}{F}{}^\beta_B \frac{d}{dt}\overset{\mathrm{p}}{F}{}^\alpha_A\right)\right], \tag{8.7}$$

and consequently, with (see e.g. [Hac92])

$$\overset{\mathrm{e}}{\mathbf{D}} := \frac{1}{2}\frac{d}{dt}\overset{\mathrm{e}}{\mathbf{C}} \tag{8.8}$$

we may write

$$\mathbf{D}^\flat = \overset{\mathrm{p}}{\mathbf{\Phi}}{}^* \overset{\mathrm{e}}{\mathbf{D}}{}^\flat + \overset{\mathrm{p}}{\mathbf{D}}{}^\flat, \text{ with } \overset{\mathrm{p}}{D}_{AB} := \frac{1}{2}\overset{\mathrm{e}}{C}_{\alpha\beta}\left(\overset{\mathrm{p}}{F}{}^\alpha_A \frac{d}{dt}\overset{\mathrm{p}}{F}{}^\beta_B + \overset{\mathrm{p}}{F}{}^\beta_B \frac{d}{dt}\overset{\mathrm{p}}{F}{}^\alpha_A\right). \tag{8.9}$$

Combining (3.19) and (3.20) we see that $\mathbf{d}^\flat = \mathbf{\Phi}_*\mathbf{D}^\flat$, and together with (8.9) we get the *spatial rate of deformation tensor*

$$\mathbf{d}^\flat = \overset{\mathrm{e}}{\mathbf{d}}{}^\flat + \overset{\mathrm{p}}{\mathbf{d}}{}^\flat \ \text{ with } \ \overset{\mathrm{e}}{\mathbf{d}}{}^\flat := \mathbf{\Phi}_* \overset{\mathrm{p}}{\mathbf{\Phi}}{}^* \overset{\mathrm{e}}{\mathbf{D}}{}^\flat = \overset{\mathrm{e}}{\mathbf{\Phi}}{}_* \overset{\mathrm{e}}{\mathbf{D}}{}^\flat \ \text{ and } \ \overset{\mathrm{p}}{\mathbf{d}}{}^\flat := \Phi_* \overset{\mathrm{p}}{\mathbf{D}}{}^\flat \ . \tag{8.10}$$

Using the equations from above, their components can be easyly found as

$$\overset{\mathrm{p}}{d}_{ab} = \frac{1}{2}\Big(g_{ac}(F^{-1})^A_b + g_{cb}(F^{-1})^A_a\Big)\overset{\mathrm{e}}{F}{}^c_\alpha \frac{d}{dt}\overset{\mathrm{p}}{F}{}^\alpha_A \ \text{ and } \ \overset{\mathrm{e}}{d}_{ab} = \frac{1}{2}(F^{-1})^\alpha_a (F^{-1})^\beta_b \frac{d}{dt}\Big(g_{cd}\overset{\mathrm{e}}{F}{}^c_\alpha \overset{\mathrm{e}}{F}{}^d_\beta\Big). \tag{8.11}$$

Analogically to (3.6) we define the *"elastic" Lie derivative*

$$\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{v}}\mathbf{g} := \overset{\mathrm{e}}{\mathbf{\Phi}}{}_* \frac{d}{dt}\left(\overset{\mathrm{e}}{\mathbf{\Phi}}{}^*\mathbf{g}\right) \tag{8.12}$$

and see from (9.1) that

$$\overset{\mathrm{e}}{\mathbf{d}}{}^\flat = \frac{1}{2}\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{v}}\mathbf{g} \tag{8.13}$$

holds.

The yield criterion

Let $\overset{i}{\mathbf{\Pi}}{}^\sharp$ be a set of k *contravariant tensors* of any rank, describing the *hardening*, and let the *Stress space* of $\left\{\mathbf{T}, \overset{i}{\mathbf{\Pi}}{}^\sharp\right\}$ be defined to be the space R^m, where m complies to the sum of the number of components in $\mathbf{T}$ and in all the $\overset{i}{\mathbf{\Pi}}{}^\sharp$'s, counting symmetrical components only once. For the sake of shortness and without lack of generality, we will restrict to $k = 1$ and drop the index i in the sequel. Let's assume, that the stress level of the *second Piola–Kirchhoff stress tensor* $\mathbf{T}$, at which *plastic deformation* begins, is determined by a *convex hyperplane*

$$\Upsilon(\mathbf{T}, \mathbf{\Pi}^\sharp) = 0 \tag{8.14}$$

in the *stress space*. For stress levels with $\Upsilon(\mathbf{T}, \mathbf{\Pi}^\sharp) < 0$ the material is regarded to behave *hyperelastic*, that is the last line of (5.18) is assumed to hold, and $\Upsilon(\mathbf{T}, \mathbf{\Pi}^\sharp) > 0$ will be forbidden. From (4.4) and (4.8) we get $\mathbf{T} = J\mathbf{\Phi}^*\boldsymbol{\sigma}$ and therefore $\boldsymbol{\sigma} = \frac{1}{J}\mathbf{\Phi}_*\mathbf{T}$ holds. According to this we define the *internal variables* describing *hardening* in the *spatial formulation* by

$$\boldsymbol{\pi}^\sharp := \frac{1}{J}\mathbf{\Phi}_*\mathbf{\Pi}^\sharp \ . \tag{8.15}$$

With $\upsilon(\boldsymbol{\sigma}, \boldsymbol{\pi}^\sharp) := \Upsilon(J\mathbf{\Phi}^*\boldsymbol{\sigma}, J\mathbf{\Phi}^*\boldsymbol{\pi}^\sharp)$ the correlating *yield criterion* in the *spatial formulation* reads

$$\upsilon(\boldsymbol{\sigma}, \boldsymbol{\pi}^\sharp) = 0 \ . \tag{8.16}$$

The principle of maximal dissipation

Since *plastic deformation* is an irreversible process, the *internal energy* as discussed on page 15 does not fully describe the appearing phenomena. Energy will be dissipated, the *entropy* of the system increases, although thermal effects further on are assumed to be neglectable. This can be taken into consideration by the *free energy* as follows from *thermodynamics* (cf. ch. 6).

As already stated on pages 17–19, the *internal energy* E and the *entropy* $\mathcal{E}$ in the general case depend on $\mathbf{X}, t$ and $\mathbf{C}^\flat$. Restricting to homogeneous and stationary isothermal problems, we have for the *free energy* $\mathcal{Z} = E - \mathcal{T}\mathcal{E}$, $\mathcal{Z} = \mathcal{Z}(\mathbf{C}^\flat)$. Here we introduce some additional internal variables $\mathbf{\Theta}^\flat$ explained below[26], the *specific free energy* in the *material formulation* may depend on:

$$\mathcal{Z} = \mathcal{Z}(\mathbf{C}^\flat, \mathbf{\Theta}^\flat) \ . \tag{8.17}$$

Since the *intermediate configuration* is an appropriate configuration [Hac92] for describing the material behaviour, it is suggestive to formulate the *free energy function*

$$\mathcal{Z} := \overset{\mathrm{p}}{\mathbf{\Phi}}{}^* \,\widetilde{\mathcal{Z}}(\overset{\mathrm{e}}{\mathbf{C}}{}^\flat, \widetilde{\mathbf{\Theta}}{}^\flat)^{27} \tag{8.18}$$

[26] cf. (8.20).

[27] $\overset{\mathrm{p}}{\mathbf{\Phi}}{}^* \,\widetilde{\mathcal{Z}}(\overset{\mathrm{e}}{\mathbf{C}}{}^\flat, \widetilde{\mathbf{\Theta}}{}^\flat) = \widetilde{\mathcal{Z}}(\overset{\mathrm{p}}{\mathbf{\Phi}}_*\mathbf{C}^\flat, \overset{\mathrm{p}}{\mathbf{\Phi}}_*\mathbf{\Theta}^\flat) =: \mathcal{Z}(\mathbf{C}^\flat, \mathbf{\Theta}^\flat)$

with $\overset{e}{\mathbf{C}}{}^\flat = \overset{p}{\mathbf{\Phi}}_*\mathbf{C}^\flat$ from (8.4) and $(\widetilde{\mathbf{\Theta}}^\flat)_{\alpha\ldots\beta} := (\overset{p}{\mathbf{\Phi}}_*\mathbf{\Theta}^\flat)_{\alpha\ldots\beta} = (\overset{p}{F}{}^{-1})^A_\alpha \cdots (\overset{p}{F}{}^{-1})^B_\beta \Theta_{A\ldots B}$.
In the *spatial* configuration we have

$$\zeta := \mathbf{\Phi}_* \mathcal{Z} = \mathcal{Z}(\mathbf{\Phi}^*\mathbf{g}, \mathbf{\Phi}^*\boldsymbol{\theta}^\flat) = \zeta(\mathbf{g}, \boldsymbol{\theta}^\flat) \tag{8.19}$$

with $(\boldsymbol{\theta}^\flat)_{a\ldots b} := (\mathbf{\Phi}_*\mathbf{\Theta}^\flat)_{a\ldots b} = (F^{-1})^A_a \cdots (F^{-1})^B_b \Theta_{A\ldots B}$.
Now we require the *covariant* tensors $\mathbf{\Pi}^\sharp$ and $\boldsymbol{\pi}^\sharp$ from the previous section to be conjugate to $\mathbf{\Theta}^\flat$ and $\boldsymbol{\theta}^\flat$, respectively:

$$\mathbf{\Pi}^\sharp := -\rho_{Ref} \frac{\partial \mathcal{Z}}{\partial \mathbf{\Theta}^\flat} \quad , \quad \boldsymbol{\pi}^\sharp := -\rho \frac{\partial \zeta}{\partial \boldsymbol{\theta}^\flat} \ . \tag{8.20}$$

In components, this reads $\Pi^{A\ldots B} := -\rho_{Ref} \frac{\partial \mathcal{Z}}{\partial \Theta_{A\ldots B}}$ and $\pi^{a\ldots b} := -\rho \frac{\partial \zeta}{\partial \theta_{a\ldots b}}$.
Combining (8.15) with (8.19), the transformations

$$\mathbf{\Pi}^\sharp = -\rho\ J\ \mathbf{\Phi}^* \frac{\partial \zeta}{\partial \boldsymbol{\theta}^\flat} = -\rho_{Ref} \frac{\partial \mathcal{Z}}{\partial \mathbf{\Theta}^\flat}$$

can be found. So the assumptions (8.20) are in correspondence to each other.
The *Drucker postulate* or the *principle of maximal dissipation* implies, that the *local dissipation function*

$$\mathcal{D}_M := \mathbf{T} : \mathbf{D}^\flat - \rho_{Ref} \frac{d}{dt} \mathcal{Z} \geq 0 \quad , \quad \mathcal{D}_S := \boldsymbol{\sigma} : \mathbf{d}^\flat - \rho \frac{d}{dt} \zeta \geq 0 \tag{8.21}$$

will become maximal during plastic deformation. Note, that (8.21) is the restriction of the *reduced dissipation inequalities* (6.10), (6.5) to *isothermal* processes.
Since $\frac{\partial \mathcal{Z}}{\partial C_{AB}} = \frac{\partial \widetilde{\mathcal{Z}}}{\partial \overset{e}{C}_{\alpha\beta}} \frac{\partial \overset{e}{C}_{\alpha\beta}}{\partial C_{AB}} = \frac{\partial \widetilde{\mathcal{Z}}}{\partial \overset{e}{C}_{\alpha\beta}} (\overset{p}{F}{}^{-1}\,)^A_\alpha (\overset{p}{F}{}^{-1}\,)^B_\beta$ holds, we have $\frac{\partial \widetilde{\mathcal{Z}}}{\partial \overset{e}{C}_{\alpha\beta}} = \overset{p}{F}{}^\alpha_A \overset{p}{F}{}^\beta_B \frac{\partial \mathcal{Z}}{\partial C_{AB}}$.
Using this, we get $\frac{d\mathcal{Z}}{dt} = \frac{d\widetilde{\mathcal{Z}}}{dt} = \frac{\partial \widetilde{\mathcal{Z}}}{\partial \overset{e}{C}_{\alpha\beta}} \frac{d\overset{e}{C}_{\alpha\beta}}{dt} + \frac{\partial \widetilde{\mathcal{Z}}}{\partial \widetilde{\Theta}_{\alpha\ldots\beta}} \frac{d\widetilde{\Theta}_{\alpha\ldots\beta}}{dt} =$

$\overset{p}{F}{}^\alpha_A \overset{p}{F}{}^\beta_B \frac{\partial \mathcal{Z}}{\partial C_{AB}} \frac{d\overset{e}{C}_{\alpha\beta}}{dt} + \overset{p}{F}{}^\alpha_A \cdots \overset{p}{F}{}^\beta_B \frac{\partial \mathcal{Z}}{\partial \Theta_{A\ldots B}} \frac{d}{dt} (\overset{p}{\mathbf{\Phi}}_*\mathbf{\Theta}^\flat)_{\alpha\ldots\beta} = \left(\overset{p}{\mathbf{\Phi}}_* \frac{\partial \mathcal{Z}}{\partial \mathbf{C}^\flat} \right) : \frac{d\overset{e}{\mathbf{C}}{}^\flat}{dt} + \frac{\partial \mathcal{Z}}{\partial \mathbf{\Theta}^\flat} \vdots \overset{p}{\mathbf{\Phi}}{}^* \frac{d}{dt} (\overset{p}{\mathbf{\Phi}}_* \mathbf{\Theta}^\flat)$.
This and an analogous calculation gives

$$\frac{d\mathcal{Z}}{dt} = 2 \left(\overset{p}{\Phi}_* \frac{\partial \mathcal{Z}}{\partial \mathbf{C}^\flat} \right) : \overset{e}{\mathbf{D}}{}^\flat + \frac{\partial \mathcal{Z}}{\partial \mathbf{\Theta}^\flat} \vdots \left[\overset{p}{\mathbf{\Phi}}{}^* \frac{d}{dt} \left(\overset{p}{\mathbf{\Phi}}_* \mathbf{\Theta}^\flat \right) \right] \tag{8.22}$$

and

$$\frac{d\zeta}{dt} = 2 \frac{\partial \zeta}{\partial \mathbf{g}} : \overset{e}{\mathbf{d}}{}^\flat + \frac{\partial \zeta}{\partial \boldsymbol{\theta}^\flat} \vdots \left[\overset{e}{\mathbf{\Phi}}_* \frac{d}{dt} \left(\overset{e}{\mathbf{\Phi}}{}^* \boldsymbol{\theta}^\flat \right) \right] . \tag{8.23}$$

with $\overset{e}{\mathbf{D}}{}^\flat$ from (8.8) and $\overset{e}{\mathbf{d}}{}^\flat$ from (8.13).
Although it doesn't coincide with the *Lie derivative* from (3.6) or (8.12), we define

$$\overset{p}{\mathcal{L}}_{\mathbf{V}} \mathbf{\Theta}^\flat := \overset{p}{\mathbf{\Phi}}{}^* \frac{d}{dt} \left(\overset{p}{\mathbf{\Phi}}_* \mathbf{\Theta}^\flat \right) . \tag{8.24}$$

Using this, (8.20), (8.9) and the evident identity $\mathbf{T}:(\overset{\mathrm{p}}{\boldsymbol{\Phi}}{}^{*}\overset{\mathrm{e}}{\mathbf{D}}{}^{\flat}) = (\overset{\mathrm{p}}{\boldsymbol{\Phi}}_{*}\mathbf{T}):\overset{\mathrm{e}}{\mathbf{D}}{}^{\flat}$, the left part of (8.21) reads

$$\mathcal{D}_M = \left[\overset{\mathrm{p}}{\boldsymbol{\Phi}}_{*}\left(\mathbf{T} - 2\rho_{_{Ref}}\frac{\partial \mathcal{Z}}{\partial \mathbf{C}^{\flat}}\right)\right] : \overset{\mathrm{e}}{\mathbf{D}}{}^{\flat} + \mathbf{T} : \overset{\mathrm{p}}{\mathbf{D}}{}^{\flat} + \boldsymbol{\Pi}^{\sharp}\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^{\flat} \ . \tag{8.25}$$

The same way, including (8.10), (8.20) and $\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^{\flat} := \overset{\mathrm{e}}{\boldsymbol{\Phi}}_{*}\frac{d}{dt}\left(\overset{\mathrm{e}}{\boldsymbol{\Phi}}{}^{*}\boldsymbol{\theta}^{\flat}\right)$ [28] the right part of (8.21) reads

$$\mathcal{D}_S = \left(\boldsymbol{\sigma} - 2\rho\frac{\partial \zeta}{\partial \mathbf{g}}\right) : \overset{\mathrm{e}}{\mathbf{d}}{}^{\flat} + \boldsymbol{\sigma} : \overset{\mathrm{p}}{\mathbf{d}}{}^{\flat} + \boldsymbol{\pi}^{\sharp}\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^{\flat} \ . \tag{8.26}$$

Taking into account that, if no *plastic deformation* occours, also no *dissipation* should take place, e.g. $\mathcal{D}_S = \mathcal{D}_M = 0$. Then

$$\mathbf{T} = 2\rho_{_{Ref}}\frac{\partial \mathcal{Z}}{\partial \mathbf{C}^{\flat}} \ \text{ and } \ \boldsymbol{\sigma} = 2\rho\frac{\partial \zeta}{\partial \mathbf{g}} \tag{8.27}$$

hold and (8.27) replaces the *Doyle–Ericksen–Formula* in (5.14) and (5.18) for *plasticity* problems. Now suppose, that (8.27) holds. Then, (8.25) and (8.26) reads

$$\mathcal{D}_M = \mathbf{T} : \overset{\mathrm{p}}{\mathbf{D}}{}^{\flat} + \boldsymbol{\Pi}^{\sharp}\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^{\flat} \quad \text{and} \quad \mathcal{D}_S = \boldsymbol{\sigma} : \overset{\mathrm{p}}{\mathbf{d}}{}^{\flat} + \boldsymbol{\pi}^{\sharp}\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^{\flat} \ . \tag{8.28}$$

Now suppose, that the *yield criterions* (8.14) and (8.16) are fullfilled, and that the *stresses* and *internal variables* $\mathbf{T}_{max}$, $\boldsymbol{\sigma}_{max}$, $\boldsymbol{\Pi}^{\sharp}_{max}$ and $\boldsymbol{\pi}^{\sharp}_{max}$ adopt values, maximizing the *dissipation* (8.28). Then, as necessary conditions,

$$\frac{\partial}{\partial \mathbf{T}}(\Lambda\Upsilon - \mathcal{D}_M) = \mathbf{0} \quad , \quad \frac{\partial}{\partial \boldsymbol{\Pi}^{\sharp}}(\Lambda\Upsilon - \mathcal{D}_M) = \mathbf{0} \ , \tag{8.29}$$

$$\frac{\partial}{\partial \boldsymbol{\sigma}}(\lambda\upsilon - \mathcal{D}_S) = \mathbf{0} \quad \text{and} \quad \frac{\partial}{\partial \boldsymbol{\pi}^{\sharp}}(\lambda\upsilon - \mathcal{D}_S) = \mathbf{0} \tag{8.30}$$

must be fullfilled. Holding $\overset{\mathrm{p}}{\mathbf{D}}{}^{\flat}$, $\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^{\flat}$, $\overset{\mathrm{p}}{\mathbf{d}}{}^{\flat}$, $\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^{\flat}$ tight, the extrem of (8.27) is described by the equations

$$\overset{\mathrm{p}}{\mathbf{D}}{}^{\flat} = \Lambda\frac{\partial \Upsilon}{\partial \mathbf{T}} \quad , \quad \overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^{\flat} = \Lambda\frac{\partial \Upsilon}{\partial \boldsymbol{\Pi}^{\sharp}} \ , \tag{8.31}$$

$$\overset{\mathrm{p}}{\mathbf{d}}{}^{\flat} = \lambda\frac{\partial \upsilon}{\partial \boldsymbol{\sigma}} \quad \text{and} \quad \overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^{\flat} = \lambda\frac{\partial \upsilon}{\partial \boldsymbol{\pi}^{\sharp}} \ , \tag{8.32}$$

and the *dissipation* will be maximal if and only if the *yield surface* is *convex*, as assumed on page 24.

[28] Note, that this complies to (3.6) and (8.12).

The evolution of stresses

Applying (8.22) for $\dfrac{\partial \mathcal{Z}}{\partial \mathbf{C}^\flat}$ instead of $\mathcal{Z}$, the *material time derivative* of the *second Piola–Kirchhoff* stress tensor (8.27) can be found as

$$\frac{d\mathbf{T}}{dt} = 2\rho_{Ref}\left[\frac{\partial^2 \mathcal{Z}}{\partial {\mathbf{C}^\flat}^2}\, \overset{\mathrm{p}}{\boldsymbol{\Phi}}{}^{*}\frac{d}{dt}\left(\overset{\mathrm{p}}{\boldsymbol{\Phi}}{}_{*}\mathbf{C}^\flat\right) + \frac{\partial^2 \mathcal{Z}}{\partial \mathbf{C}^\flat \partial \boldsymbol{\Theta}^\flat}\, \overset{\mathrm{p}}{\boldsymbol{\Phi}}{}^{*}\frac{d}{dt}\left(\overset{\mathrm{p}}{\boldsymbol{\Phi}}{}_{*}\boldsymbol{\Theta}^\flat\right)\right], \tag{8.33}$$

and therefore the *Lie derivative* of the *Kirchhoff* stress tensor reads

$$\mathcal{L}_{\mathbf{V}}\boldsymbol{\tau} = \boldsymbol{\Phi}_{*}\frac{d}{dt}(\boldsymbol{\Phi}^{*}\boldsymbol{\tau}) = \boldsymbol{\Phi}_{*}\frac{d\mathbf{T}}{dt} = 2\rho_{Ref}\left[\frac{\partial^2 \zeta}{\partial \mathbf{g}^2}\, \overset{\mathrm{e}}{\boldsymbol{\Phi}}{}_{*}\frac{d}{dt}\left(\overset{\mathrm{e}}{\boldsymbol{\Phi}}{}^{*}\mathbf{g}\right) + \frac{\partial^2 \zeta}{\partial \mathbf{g}\partial \boldsymbol{\theta}^\flat}\, \overset{\mathrm{e}}{\boldsymbol{\Phi}}{}_{*}\frac{d}{dt}\left(\overset{\mathrm{e}}{\boldsymbol{\Phi}}{}^{*}\boldsymbol{\theta}^\flat\right)\right]. \tag{8.34}$$

Consequently using (8.9), (8.11), (8.20),

$$\frac{d\mathbf{T}}{dt} = 2\frac{\partial \mathbf{T}}{\partial \mathbf{C}^\flat}:\left(\mathbf{D} - \overset{\mathrm{p}}{\mathbf{D}}{}^\flat\right) + \frac{\partial \mathbf{T}}{\partial \boldsymbol{\Theta}^\flat}\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^\flat, \text{ or } \frac{d\mathbf{T}}{dt} = 2\frac{\partial \mathbf{T}}{\partial \mathbf{C}^\flat}:\left(\mathbf{D} - \overset{\mathrm{p}}{\mathbf{D}}{}^\flat\right) - 2\frac{\partial \boldsymbol{\Pi}^\sharp}{\partial \mathbf{C}^\flat}\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^\flat, \tag{8.35}$$

$$\mathcal{L}_{\mathbf{V}}\boldsymbol{\tau} = 2\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{g}}:\left(\mathbf{d}^\flat - \overset{\mathrm{p}}{\mathbf{d}}{}^\flat\right) + \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}^\flat}\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^\flat, \text{ or } \mathcal{L}_{\mathbf{V}}\boldsymbol{\tau} = 2\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{g}}:\left(\mathbf{d}^\flat - \overset{\mathrm{p}}{\mathbf{d}}{}^\flat\right) - 2\frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{g}}\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^\flat \tag{8.36}$$

can be shown, with $\boldsymbol{\kappa} := J\boldsymbol{\pi}^\sharp = -\rho_{Ref}\dfrac{\partial \zeta}{\partial \boldsymbol{\theta}^\flat}$ [29].

The plastic spin

As easy can be seen, the set of equations describing the *plastic material behaviour* in the *material* configuration (8.31) (8.35) and in the *spatial* configuration (8.32) (8.36) is not entirelly complete, additional assumptions are necessary with respect to the *plastic spin* $\overset{\mathrm{p}}{\boldsymbol{\Omega}}{}^\flat$, $\overset{\mathrm{p}}{\boldsymbol{\omega}}{}^\flat$ to construct the *plastic* and *elastic Lie derivative* $\overset{\mathrm{p}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\Theta}^\flat$ and $\overset{\mathrm{e}}{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\theta}^\flat$. Following the strategie splitting the rate of deformation tensor (cf. page 23) in an *elastic* and a *plastic* part, the skew symmetric *spin* $\boldsymbol{\Omega}^\flat := skew(\mathbf{D}^\flat)$ can be splitted by:

$$\begin{aligned}
\Omega_{AB} &:= \frac{1}{2}g_{ab}\left(F^a_A\frac{d}{dt}F^b_B - F^b_B\frac{d}{dt}F^a_A\right) \\
&= \frac{1}{2}g_{ab}\left[\overset{\mathrm{p}}{F}{}^\alpha_A\overset{\mathrm{p}}{F}{}^\beta_B\left(\overset{\mathrm{e}}{F}{}^a_\alpha\frac{d}{dt}\overset{\mathrm{e}}{F}{}^b_\beta - \overset{\mathrm{e}}{F}{}^b_\beta\frac{d}{dt}\overset{\mathrm{e}}{F}{}^a_\alpha\right) + \overset{\mathrm{e}}{F}{}^a_\alpha\overset{\mathrm{e}}{F}{}^b_\beta\left(\overset{\mathrm{p}}{F}{}^\alpha_A\frac{d}{dt}\overset{\mathrm{p}}{F}{}^\beta_B - \overset{\mathrm{p}}{F}{}^\beta_B\frac{d}{dt}\overset{\mathrm{p}}{F}{}^\alpha_A\right)\right] \\
&= \overset{\mathrm{p}}{F}{}^\alpha_A\overset{\mathrm{p}}{F}{}^\beta_B\overset{\mathrm{e}}{\Omega}_{\alpha\beta} + \overset{\mathrm{p}}{\Omega}_{AB}.
\end{aligned}$$

Or in a more compact notation

$$\boldsymbol{\Omega}^\flat = \overset{\mathrm{p}}{\boldsymbol{\Phi}}{}^{*}\overset{\mathrm{e}}{\boldsymbol{\Omega}}{}^\flat + \overset{\mathrm{p}}{\boldsymbol{\Omega}}{}^\flat, \tag{8.37}$$

[29] like $\boldsymbol{\tau} := J\boldsymbol{\sigma} = 2\rho_{Ref}\dfrac{\partial \zeta}{\partial \mathbf{g}}$

with $\overset{\mathrm{p}}{\Omega}_{AB} := \overset{\mathrm{e}}{C}_{\alpha\beta}\left(\overset{\mathrm{p}}{F}{}^{\alpha}_{A}\frac{d}{dt}\overset{\mathrm{p}}{F}{}^{\beta}_{B} - \overset{\mathrm{p}}{F}{}^{\beta}_{B}\frac{d}{dt}\overset{\mathrm{p}}{F}{}^{\alpha}_{A}\right), \ \overset{\mathrm{e}}{\Omega}_{\alpha\beta} := \frac{1}{2}g_{ab}\left(\overset{\mathrm{e}}{F}{}^{a}_{\alpha}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{b}_{\beta} - \overset{\mathrm{e}}{F}{}^{b}_{\beta}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{a}_{\alpha}\right)$.
And using $\boldsymbol{\omega}^\flat := \boldsymbol{\Phi}_*\boldsymbol{\Omega}^\flat$ one finds

$$\omega_{ab} := \frac{1}{2}\left[\left(g_{ac}\overset{\mathrm{e}}{F}{}^{-1\beta}_{\ \ b}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{c}_{\beta} - g_{cb}\overset{\mathrm{e}}{F}{}^{-1\beta}_{\ \ a}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{c}_{\beta}\right) + \left(g_{ac}F^{-1A}_{\ \ b} - g_{cb}F^{-1A}_{\ \ a}\right)\overset{\mathrm{e}}{F}{}^{c}_{\alpha}\frac{d}{dt}\overset{\mathrm{p}}{F}{}^{\alpha}_{A}\right], \text{ or}$$

$$\boldsymbol{\omega}^\flat = \overset{\mathrm{e}}{\boldsymbol{\omega}}{}^\flat + \overset{\mathrm{p}}{\boldsymbol{\omega}}{}^\flat \tag{8.38}$$

with

$$\overset{\mathrm{p}}{\omega}_{ab} := \frac{1}{2}\left(g_{ac}F^{-1A}_{\ \ b} - g_{cb}F^{-1A}_{\ \ a}\right)\overset{\mathrm{e}}{F}{}^{c}_{\alpha}\frac{d}{dt}\overset{\mathrm{p}}{F}{}^{\alpha}_{A}, \quad \overset{\mathrm{e}}{\omega}_{ab} := \frac{1}{2}\left(g_{ac}\overset{\mathrm{e}}{F}{}^{-1\beta}_{\ \ b}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{c}_{\beta} - g_{cb}\overset{\mathrm{e}}{F}{}^{-1\beta}_{\ \ a}\frac{d}{dt}\overset{\mathrm{e}}{F}{}^{c}_{\beta}\right).$$

The simpliest assumptions is to let the *plastic spin* $\overset{\mathrm{p}}{\boldsymbol{\Omega}}{}^\flat, \overset{\mathrm{p}}{\boldsymbol{\omega}}{}^\flat$ to be zero (see [Hac92]) until more information is available.
A more detailled discussion of this subject, beside other modells where the *plastic spin* is not explicitely included (e.g. [Sim93]), can be found in [MG98].

9 Appendix

Pull back, push forward and the deformation gradient

On several places of this article we use a representation of the *pull back* and the *push forward* of some *tensors* in terms of the *deformation gradient* (3.10).
Let $\mathbf{t}$ and $\mathbf{T}$ two tensors of type $\binom{2}{0}$ and $\mathbf{r}$ and $\mathbf{R}$ of type $\binom{0}{2}$ without any special physical meaning in this chapter but connected by $\mathbf{T} = \boldsymbol{\Phi}^*\mathbf{t}, \ \mathbf{t} = \boldsymbol{\Phi}_*\mathbf{T}, \ \mathbf{R} = \boldsymbol{\Phi}^*\mathbf{r}$ and $\mathbf{r} = \boldsymbol{\Phi}_*\mathbf{R}$. Then, from the formulas given on page 5 we derive

$$\begin{array}{lll} T^{AB} = (F^{-1})^A_a\,(F^{-1})^B_b\,t^{ab} & , & t^{ab} = F^a_A F^b_B T^{AB} \\ R_{AB} = F^a_A F^b_B r_{ab} & , & r_{ab} = (F^{-1})^A_a\,(F^{-1})^B_b\,R_{AB}\ . \end{array} \tag{9.1}$$

In the same way for $\mathbf{t}$ and $\mathbf{T}$ of type $\binom{4}{0}$ with $\mathbf{T} = \boldsymbol{\Phi}^*\mathbf{t}$ and $\mathbf{t} = \boldsymbol{\Phi}_*\mathbf{T}$ we have

$$T^{ABCD} = \left(F^{-1}\right)^A_a\left(F^{-1}\right)^B_b\left(F^{-1}\right)^C_c\left(F^{-1}\right)^D_d t^{abcd} \text{ and } t^{abcd} = F^a_A F^b_B F^c_C F^d_D T^{ABCD}\ . \tag{9.2}$$

For "*two–point–tensors*" $\mathbf{W}$ of type $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ with components $W^a{}_B$, acting from $T_X\mathcal{B} \times T_x\mathcal{M}$ to R^1, the *push forward's* components are

$$(\boldsymbol{\Phi}_*\mathbf{W})^a_b = (F^{-1})^B_b W^a{}_B \tag{9.3}$$

and it's *pull back* is

$$(\boldsymbol{\Phi}^*\mathbf{W})^A_B = (F^{-1})^A_a W^a{}_B. \tag{9.4}$$

Domain integrals

As well known [MK70], volume integrals transform under change of *coordinate systems* $\{z^i\}\rightarrow\{x^a\}$ as

$$\int\limits_{\mathcal{A}_{\mathbf{z}}} f(\mathbf{z})\ dz^1\cdots dz^n = \int\limits_{\mathcal{A}_{\mathbf{x}}} f(\mathbf{z}(\mathbf{x}))\ det\left(\frac{\partial z^i}{\partial x^a}\right) dx^1\cdots dx^n = \int\limits_{\mathcal{A}_{\mathbf{x}}} f(\mathbf{z}(\mathbf{x}))\ \sqrt{det(g_{ab})}\ dx^1\cdots dx^n,$$

because, assuming the $\{z^i\}$ to be *Carthesian coordinates* we have

$$\sqrt{det(g_{ab})} = \sqrt{det\left(\frac{\partial z^i}{\partial x^a}\frac{\partial z^i}{\partial x^b}\right)} = det\left(\frac{\partial z^i}{\partial x^a}\right).$$

Therefore, in *general coordinates*, the *volume element* can be written as

$$dv := \sqrt{det(g_{ab})}\ dx^1\cdots dx^n.$$

The same way, considering the transformation behaviour of a volume integral under a *motion* $\mathbf{\Phi}_t$, we may write

$$\int\limits_{\Phi_t(\mathcal{A})} h(\mathbf{x})\ dx^1\cdots dx^n = \int\limits_{\mathcal{A}} h(\mathbf{\Phi}_t(\mathbf{X}))\ det\left(\frac{\partial \Phi^a}{\partial X^A}\right) dX^1\cdots dX^n,$$

or, for $h(\mathbf{x}) := f(\mathbf{x})\sqrt{det(g_{ab})}$,

$$\int\limits_{\Phi_t(\mathcal{A})} f(\mathbf{x})\ dv = \int\limits_{\mathcal{A}} f(\mathbf{\Phi}_t(\mathbf{X}))\ \frac{\sqrt{det(g_{ab})}}{\sqrt{det(G_{AB})}}\ det\left(\frac{\partial \Phi^a}{\partial X^A}\right) dV. \tag{9.5}$$

The *surface integral of 2. kind*

$$\int\limits_{\partial\mathcal{A}_{\mathbf{z}}} \langle \mathbf{b}(\mathbf{z}), \mathbf{n}_{\mathbf{z}}\rangle da_{\mathbf{z}} := \int\limits_{\partial\mathcal{A}_{\mathbf{z}}} det\begin{pmatrix} b^1 & \cdots & b^n \\ \frac{\partial z^1}{\partial p^1} & \cdots & \frac{\partial z^n}{\partial p^1} \\ \vdots & \cdots & \vdots \\ \frac{\partial z^1}{\partial p^{n-1}} & \cdots & \frac{\partial p^n}{\partial p^{n-1}} \end{pmatrix} dp^1\cdots dp^{n-1}$$

over some vector $\mathbf{b}$ with components b^i related to $\{z^i\}$ transforms to

$$\int\limits_{\partial\mathcal{A}_{\mathbf{x}}} \langle \mathbf{b}(\mathbf{z}(\mathbf{x})), \mathbf{n}_{\mathbf{x}}\rangle da_{\mathbf{x}} := \int\limits_{\partial\mathcal{A}_{\mathbf{x}}} det\begin{pmatrix} \beta^1 & \cdots & \beta^n \\ \frac{\partial x^1}{\partial q^1} & \cdots & \frac{\partial x^n}{\partial q^1} \\ \vdots & \cdots & \vdots \\ \frac{\partial x^1}{\partial q^{n-1}} & \cdots & \frac{\partial x^n}{\partial q^{n-1}} \end{pmatrix} det\left(\frac{\partial z^i}{\partial x^a}\right) dq^1\cdots dq^{n-1},$$

with $\beta^a := \dfrac{\partial x^a}{\partial z^i} b^i$ related to $\{x^a\}$. This follows from the transformation of volume integrals as stated above, from the matrix product

$$\begin{pmatrix} \beta^1 & \cdots & \beta^n \\ \dfrac{\partial x^1}{\partial p^1} & \cdots & \dfrac{\partial x^n}{\partial p^1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial x^1}{\partial p^{n-1}} & \cdots & \dfrac{\partial x^n}{\partial p^{n-1}} \end{pmatrix} \left(\frac{\partial z^i}{\partial x^a} \right) = \begin{pmatrix} b^1 & \cdots & b^n \\ \dfrac{\partial z^1}{\partial p^1} & \cdots & \dfrac{\partial z^n}{\partial p^1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial z^1}{\partial p^{n-1}} & \cdots & \dfrac{\partial z^n}{\partial p^{n-1}} \end{pmatrix}$$

and the relation

$$det \begin{pmatrix} \beta^1 & \cdots & \beta^n \\ \dfrac{\partial x^1}{\partial p^1} & \cdots & \dfrac{\partial x^n}{\partial p^1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial x^1}{\partial p^{n-1}} & \cdots & \dfrac{\partial x^n}{\partial p^{n-1}} \end{pmatrix} = det \begin{pmatrix} \beta^1 & \cdots & \beta^n \\ \dfrac{\partial x^1}{\partial q^1} & \cdots & \dfrac{\partial x^n}{\partial q^1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial x^1}{\partial q^{n-1}} & \cdots & \dfrac{\partial x^n}{\partial q^{n-1}} \end{pmatrix} det \left(\frac{\partial q^a}{\partial p^i} \right) .$$

So, in *general coordinates*, the *surface element* reads

$$da := \sqrt{det(g_{ab})}\; dq^1 \cdots dq^{n-1},$$

and with $\mathbf{B} := \mathbf{\Phi}^* \mathbf{b}$, the transformation behaviour of a *surface integral* under a *motion* $\mathbf{\Phi}_t$ can be written as

$$\int\limits_{\partial\Phi_t(\mathcal{A})} \langle \mathbf{b}, \mathbf{n} \rangle \; da = \int\limits_{\partial\mathcal{A}} \langle \mathbf{B}, \mathbf{N} \rangle \frac{\sqrt{det(g_{ab})}}{\sqrt{det(G_{AB})}} \, det \left(\frac{\partial \Phi^a}{\partial X^A} \right) dA \; . \tag{9.6}$$

Using the same calculus as above, the theorem of *Gauss* and *Ostrogradski* can be written in the form

$$\int\limits_{\Phi_t(\mathcal{A})} div \; \mathbf{b} \; dv = \int\limits_{\partial\Phi_t(\mathcal{A})} \langle \mathbf{b}, \mathbf{n} \rangle \; da. \tag{9.7}$$

The time derivative of the *Jacobian* J

$$J := \frac{\sqrt{det(g_{ab})}}{\sqrt{det(G_{AB})}} \, det \left(\frac{\partial \Phi^a}{\partial X^A} \right) = \frac{\sqrt{det(g_{ab})}}{\sqrt{det(G_{AB})}} \, det(F^a_A) \tag{9.8}$$

can be found using the identities

$$\frac{\partial}{\partial F^a_A} det\,(F^a_A) = det\,(F^a_A) \left(F^{-1} \right)^A_{\ a} \quad \text{and} \quad \frac{\partial}{\partial g_{ab}} det\,(g_{ab}) = det\,(g_{ab})\, g^{ab} \tag{9.9}$$

Since G_{AB} doesn't depend on t, we have

$$\frac{dJ}{dt} = \frac{1}{\sqrt{det(G_{AB})}} \left[\frac{1}{2} \frac{det \left(\dfrac{\partial \Phi^a}{\partial X^A} \right)}{\sqrt{det(g_{ab})}} \frac{d}{dt} det(g_{ab}) + \sqrt{det(g_{ab})} \frac{d}{dt} det \left(\frac{\partial \Phi^a}{\partial X^A} \right) \right]$$

$$= J \left[\frac{1}{2} g^{ab} \frac{d}{dt} g_{ab} + \frac{\partial X^A}{\partial x^a} \frac{d}{dt} \frac{\partial \Phi_a}{\partial X^A} \right] = J \left[\frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial x^c} v^c + \frac{\partial v^a}{\partial x^a} \right]$$

due to $\frac{d}{dt}g_{ab} = \frac{\partial g_{ab}}{\partial x^c}v^c$ and $\frac{\partial X^A}{\partial x^a}\frac{d}{dt}\frac{\partial \Phi^a}{\partial X^A} = \frac{\partial X^A}{\partial x^a}\frac{\partial}{\partial X^A}\frac{d\Phi^a}{dt} = \frac{\partial v^a}{\partial x^a}$. Inserting (2.10) and (2.11), we get

$$\frac{\partial}{\partial t}J = J div\, \mathbf{v}\ . \tag{9.10}$$

Proof of equation (4.2)

Due to the theorem of *Gauss* and *Ostrogradski* (9.7), taking into account the underlying *Euclidean structure* of $\mathbf{\Phi}_t(\mathcal{A})$ the equation $\int\limits_{\Phi_t(\mathcal{A})} div\ \boldsymbol{\sigma}\ dv = \int\limits_{\partial\Phi_t(\mathcal{A})} \langle\boldsymbol{\sigma}, \mathbf{n}\rangle da$ holds[30] component-by-component for every sufficiently smooth tensor $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}$ can be choosen to be symmetric and to fulfill $\langle\boldsymbol{\sigma}, \mathbf{n}\rangle = \mathbf{r}$. Applying the transport theorem (9.11) to the left hand side of (4.1) and using the *conservation of mass* $\left(\frac{d}{dt}\rho + \rho\ div\ \mathbf{v} = 0\right)$ we get

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} \rho\ \mathbf{v}\ dv = \int\limits_{\Phi_t(\mathcal{A})} \left[\rho\frac{d}{dt}\mathbf{v} + \mathbf{v}\left(\frac{d}{dt}\rho + \rho div\mathbf{v}\right)\right] dv = \int\limits_{\Phi_t(\mathcal{A})} \rho\frac{d}{dt}\mathbf{v}\ dv.$$

So (4.1) can be stated as $\int\limits_{\Phi_t(\mathcal{A})} \rho\frac{d}{dt}\mathbf{v}dv = \int\limits_{\Phi_t(\mathcal{A})} [\rho\mathbf{l} + div\ \boldsymbol{\sigma}]\, dv$. Since $\boldsymbol{\sigma}$ has to satisfy $div\ \boldsymbol{\sigma} = \rho(\frac{d}{dt}\mathbf{v} - \mathbf{l})$ as well as $\langle\boldsymbol{\sigma}, \mathbf{n}\rangle = \mathbf{r}$ we have 6 conditions to determine the components of $\boldsymbol{\sigma}$. Due to the required symmetry of $\boldsymbol{\sigma}$, the number of those components is reduced to 6. So $\boldsymbol{\sigma}$ is unique.

Proof of equation (5.12)

Due to (2.11) and (4.3) we have $div\langle\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle = (grad\langle\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle)^a_a = \frac{\partial\langle\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle^a}{\partial x^a} + \gamma^a_{ab}\langle\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle^b = \frac{\partial(\sigma^{ac}g_{bc}\xi^b)}{\partial x^a} + \gamma^a_{ab}\sigma^{bc}g_{dc}\xi^d = \nu_c\left(\frac{\partial\sigma^{ac}}{\partial x^a} + \gamma^a_{ab}\sigma^{bc}\right) + \sigma^{ac}\frac{\partial\nu_c}{\partial x^a} = \nu_c\left(\frac{\partial\sigma^{ac}}{\partial x^a} + \sigma^{bc}\gamma^a_{ab} + \sigma^{ab}\gamma^c_{ab}\right) + \sigma^{ac}\frac{\partial\nu_c}{\partial x^a} - \nu_c\sigma^{ab}\gamma^c_{ab}$ with the substitution $\nu_c := g_{dc}\xi^d$. Introducing (4.3) and hereafter (2.2) and (2.7) we get $div\langle\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle = \nu_c(div\boldsymbol{\sigma})^c + \sigma^{ac}\left(\frac{\partial\nu_c}{\partial x^a} - \gamma^d_{ac}\nu_d\right) = \langle div\boldsymbol{\sigma}, \boldsymbol{\xi}\rangle + \sigma^{ac}\nu_{c|a}$. The last addend is equal to $\sigma^{ab}\nu_{b|a} = \sigma^{ab}\nu_{a|b} = \frac{1}{2}\sigma^{ab}\left(\nu_{a|b} - \nu_{b|a} + \nu_{b|a} + \nu_{a|b}\right) = \boldsymbol{\sigma} : \boldsymbol{\omega}^\flat_\xi + \frac{1}{2}\sigma^{ab}\left(\nu_{b|a} + \nu_{a|b}\right) = \boldsymbol{\sigma} : \boldsymbol{\omega}^\flat_\xi + \frac{1}{2}\sigma^{ab}\left(g_{bc}\frac{\partial\xi^c}{\partial x^a} + g_{ac}\frac{\partial\xi^c}{\partial x^b} + \xi^c(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - 2\gamma^d_{ab}g_{cd})\right)$. Including (2.27), for the first line of (5.12), it remains to show, that $\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - 2\gamma^d_{ab}g_{cd} = \frac{\partial g_{ab}}{\partial x^c}$, and this is just (2.10). For the second line of (5.12), some simple computation gives $g_{ac}\xi^c_{|b} - g_{cb}\xi^c_{|a} = g_{ac}\frac{\partial\xi^c}{\partial x^b} - g_{bc}\frac{\partial\xi^c}{\partial x^a} + \xi^d(g_{ac}\gamma^c_{bd} - g_{cb}\gamma^c_{da})$ and $(g_{ac}\xi^c)_{|b} - (g_{bc}\xi^c)_{|a} =$

[30] see the footnote on page 13

$g_{ac}\frac{\partial \xi^c}{\partial x^b} - g_{bc}\frac{\partial \xi^c}{\partial x^a} + \xi^d(\frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{bd}}{\partial x^a})$. Extracting the terms common to both expressions we see, that the proof is complete when $\frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{bd}}{\partial x^a} = g_{ac}\gamma^c_{bd} - g_{cb}\gamma^c_{da}$ has been shown, and this again is a consequence of (2.10).

The transport theorem

For any function $f_t(\mathbf{x})$ we have $\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} f dv = \frac{d}{dt}\int\limits_{\mathcal{A}} f_t(\mathbf{\Phi}_t(\mathbf{X}))J_t(\mathbf{X})dV = \int\limits_{\mathcal{A}} \left(\frac{df_t}{dt}J + f_t\frac{dJ}{dt}\right) dV =$ $\int\limits_{\Phi_t(\mathcal{A})} \left(\frac{df_t}{dt}J + f_t\frac{dJ}{dt}\right) J^{-1}dv$ with $\frac{dJ}{dt} = \frac{\partial J}{\partial t} = J div\, \mathbf{v}$ from (9.10). This gives the transport theorem

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} f_t\ dv = \int\limits_{\Phi_t(\mathcal{A})} \left(\frac{df_t}{dt} + f_t\ div\ \mathbf{v}\right) dv \tag{9.11}$$

with $\mathbf{v}$ denoting the velocity of the motion $\mathbf{\Phi}_t$.
For superposed motions $\boldsymbol{\varphi}_t$ we have

$$\int\limits_{\varphi_t(\Phi_t(\mathcal{A}))} \tilde{f}(\tilde{\mathbf{x}})\ d\tilde{v} = \int\limits_{\Phi_t(\mathcal{A})} \tilde{f}(\boldsymbol{\varphi}_t(\mathbf{x}))\tilde{J}_t(\mathbf{x})\ dv \quad \text{with} \quad \tilde{J}_t(\mathbf{x}) \equiv 1 \quad \forall \mathbf{x} \quad \text{and} \quad \forall t \tag{9.12}$$

because we get $det(\tilde{g}_{ab}) = det(g_{ab})det\left(\frac{\partial(\boldsymbol{\varphi}_t^{-1})^a}{\partial x^b}\right)det\left(\frac{\partial(\boldsymbol{\varphi}_t^{-1})^a}{\partial x^b}\right)$ from (5.2)[31], leading to $\tilde{J}_t = \sqrt{\frac{det(\tilde{g}_{ab})}{det(g_{ab})}}\, det\left(\frac{\partial \varphi_t^a}{\partial x^b}\right) = det\left(\frac{\partial(\boldsymbol{\varphi}_t^{-1})^a}{\partial x^b}\right)det\left(\frac{\partial \varphi_t^a}{\partial x^b}\right) = det\left(\frac{\partial(\boldsymbol{\varphi}_t^{-1})^a}{\partial x^c}\frac{\partial \varphi_t^c}{\partial x^b}\right) = det\,(\delta^a_b) \equiv 1.$
Using (9.12) and then (9.11) we find the transport theorem for superposed motions

$$\frac{d}{dt}\int\limits_{\varphi_t(\Phi_t(\mathcal{A}))} \tilde{f}_t(\tilde{\mathbf{x}})\ d\tilde{v} = \int\limits_{\Phi_t(\mathcal{A})} \left(\frac{d\tilde{f}_t(\boldsymbol{\varphi}_t(\mathbf{x}))}{dt} + \tilde{f}_t(\boldsymbol{\varphi}_t(\mathbf{x}))div\mathbf{v}\right) dv\ . \tag{9.13}$$

Proof of equation (5.5)

Using (5.4), (3.1) and (3.5), we get for the *material acceleration* $\widetilde{\mathbf{A}}$ the equation $\widetilde{\mathbf{A}}_{\mathbf{X}}(t) = \frac{d}{dt}\widetilde{\mathbf{V}}_{\mathbf{X}}(t) = \frac{d}{dt}\left[\left(\frac{\partial \varphi^a}{\partial t} + \frac{\partial \varphi^a}{\partial x^b}V^b\right)\tilde{\mathbf{e}}_a\right]$ with $\frac{d}{dt}\tilde{\mathbf{e}}_a = \tilde{\gamma}^c_{ab}\left(\frac{\partial \varphi^b}{\partial t} + \frac{\partial \varphi^b}{\partial x^d}V^d\right)\tilde{\mathbf{e}}_c$ (cf. (2.18)) and therefore we get $\left(\frac{\partial \varphi^a}{\partial t} + \frac{\partial \varphi^a}{\partial x^b}V^b\right)\frac{d}{dt}\tilde{\mathbf{e}}_a = \tilde{\gamma}^a_{bc}\left(\xi^c + \frac{\partial \varphi^c}{\partial x^d}V^d\right)\left(\xi^b + \frac{\partial \varphi^b}{\partial x^e}V^e\right)\tilde{\mathbf{e}}_a$. Since $\frac{d}{dt}\left(\frac{\partial \varphi^a}{\partial t} + \frac{\partial \varphi^a}{\partial x^b}V^b\right) = \frac{d\xi^a}{dt} + \frac{\partial \varphi^a}{\partial x^b}\frac{dV^b}{dt} + \frac{\partial \xi^a}{\partial x^b}V^b + \frac{\partial^2 \varphi^a}{\partial x^b \partial x^c}V^bV^c$, $\frac{d\xi^a}{dt} = \frac{\partial \xi^a}{\partial t} + \frac{\partial \xi^a}{\partial \tilde{x}^b}\left(\xi^b + \frac{\partial \tilde{x}^b}{\partial x^c}V^c\right)$

[31]Note, that this doesn't apply to motions $\mathbf{\Phi}$ because these are mappings between different spaces.

and $\frac{\partial\xi^a}{\partial x^b}V^b = \frac{\partial\xi^a}{\partial\tilde{x}^c}\frac{\partial\varphi^c}{\partial x^b}V^b$ the *material acceleration* $\widetilde{\mathbf{A}}$ in the transformed state will be

$$\widetilde{\mathbf{A}} = \left[\frac{\partial\varphi^a}{\partial x^b}\frac{dV^b}{dt} + \left(\tilde{\gamma}^a_{bc}\frac{\partial\varphi^c}{\partial x^f}\frac{\partial\varphi^b}{\partial x^e} + \frac{\partial^2\varphi^a}{\partial x^f\partial x^e}\right)V^fV^e\right]\tilde{\mathbf{e}}_a +$$

$$\left[\frac{\partial\xi^a}{\partial t} + \xi^b\left(\tilde{\gamma}^a_{bc}\xi^c + \frac{\partial\xi^a}{\partial\tilde{x}^b}\right) + 2\left(\tilde{\gamma}^a_{bc}\xi^b + \frac{\partial\xi^a}{\partial\tilde{x}^b}\right)\frac{\partial\varphi^b}{\partial x^e}V^e\right]\tilde{\mathbf{e}}_a.$$

Due to the transformation behaviour of the *Christoffel symbols* [32] we substitute $\tilde{\gamma}^a_{bc}\frac{\partial\varphi^c}{\partial x^f}\frac{\partial\varphi^b}{\partial x^e} + \frac{\partial^2\varphi^a}{\partial x^f\partial x^e} = \frac{\partial\varphi^a}{\partial x^d}\gamma^d_{ef}$. Note, that $\frac{\partial\varphi^a}{\partial x^d}\left(\frac{dV^d}{dt} + \gamma^d_{ef}V^eV^f\right) = \frac{\partial\varphi^a}{\partial x^d}A^d = (\boldsymbol{\varphi}_*\mathbf{A})^a$ due to (2.18) and (2.12). According to (2.4) we introduce $(\boldsymbol{\varphi}_*\mathbf{V})^c\left(\frac{\partial\xi^a}{\partial\tilde{x}^c} + \tilde{\gamma}^a_{bc}\xi^b\right) = \left(\widetilde{grad}_{(\boldsymbol{\varphi}_*\mathbf{V})}\boldsymbol{\xi}\right)^a$.
Finally we get

$$\widetilde{\mathbf{A}} = \boldsymbol{\varphi}_*\mathbf{A} + \frac{\partial\boldsymbol{\xi}}{\partial t} + \widetilde{grad}_{\boldsymbol{\xi}}\boldsymbol{\xi} + 2\,\widetilde{grad}_{(\boldsymbol{\varphi}_*\mathbf{V})}\boldsymbol{\xi}\,. \tag{9.14}$$

Proof of equation (5.10)

We have $\frac{d}{dt}e = \frac{\partial e}{\partial t} + \frac{\partial e}{\partial x^a}v^a + \frac{\partial e}{\partial g_{ab}}\frac{\partial g_{ab}}{\partial x^c}v^c$, since the metric doesn't depend explicitly on time. Starting from (5.9) we find for the *material time derivative* of the *internal energy* in the transformed state

$$\left(\frac{d}{dt}\tilde{e}\right)_{|t_0} = \left(\frac{\partial e}{\partial t} + \frac{\partial e}{\partial x^a}v^a + \frac{\partial e}{\partial g_{ab}}\,\frac{d}{dt}\left(\frac{\partial\varphi^c}{\partial x^a}\frac{\partial\varphi^d}{\partial x^b}\,\tilde{g}_{cd}\right)\right)_{|t_0} = \frac{d}{dt}e - \frac{\partial e}{\partial g_{ab}}\frac{\partial g_{ab}}{\partial x^c}v^c +$$

$$\frac{\partial e}{\partial g_{ab}}\left(\frac{\partial\varphi^c}{\partial x^a}\frac{\partial\varphi^d}{\partial x^b}\frac{\partial\tilde{g}_{cd}}{\partial\tilde{x}^e}\tilde{v}^e + \tilde{g}_{cd}\left[\frac{\partial\varphi^c}{\partial x^a}\left(\frac{\partial\xi^d}{\partial x^b} + \frac{\partial^2\varphi^d}{\partial x^b\partial x^e}v^e\right) + \frac{\partial\varphi^d}{\partial x^b}\left(\frac{\partial\xi^c}{\partial x^a} + \frac{\partial^2\varphi^c}{\partial x^a\partial x^e}v^e\right)\right]\right)_{|t_0} =$$

$$\frac{d}{dt}e + \frac{\partial e}{\partial g_{ab}}\left(\delta^c_a\delta^d_b\frac{\partial g_{cd}}{\partial x^e}(v^e + \xi^e) - \frac{\partial g_{ab}}{\partial x^c}v^c + g_{cd}\left[\delta^c_a\frac{\partial\xi^d}{\partial x^b} + \delta^d_b\frac{\partial\xi^c}{\partial x^a}\right]\right) = \frac{d}{dt}e + \frac{\partial e}{\partial g_{ab}}(\mathcal{L}_\xi\mathbf{g})_{ab}\,,$$

cf.(2.27).

Proof of equation (5.11)

First we show identity (9.15):

$$\frac{1}{2}\;\frac{d}{dt}\langle\mathbf{v},\mathbf{v}\rangle = \langle\mathbf{a},\mathbf{v}\rangle \tag{9.15}$$

Using (3.5) we get $\langle\mathbf{a},\mathbf{v}\rangle := \mathbf{g}(\mathbf{a},\mathbf{v}) = g_{ab}a^av^b = g_{ab}v^b\left(\frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^c}v^c + \gamma^a_{cd}v^dv^c\right)$.
Since $\frac{1}{2}\frac{d}{dt}\langle\mathbf{v},\mathbf{v}\rangle = \left(\frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^c}v^c\right)v^bg_{ab} + \frac{1}{2}v^av^bv^c\frac{\partial g_{ab}}{\partial x^c}$ holds, it remains to check, that $g_{ab}\gamma^a_{cd}v^bv^cv^d = \frac{1}{2}\frac{\partial g_{db}}{\partial x^c}v^bv^cv^d$, and this can be done by means of (2.10). Next, we compare

[32] $\tilde{\Gamma}^A_{BC} = \frac{\partial\tilde{X}^A}{\partial X^D}\left[\Gamma^D_{EF}\frac{\partial X^E}{\partial\tilde{X}^B}\frac{\partial X^F}{\partial\tilde{X}^C} + \frac{\partial^2X^D}{\partial\tilde{X}^B\partial\tilde{X}^C}\right]$

the balance of energy on $\mathbf{\Phi}_t(\mathcal{A})$ and on $\boldsymbol{\varphi}_t(\mathbf{\Phi}_t(\mathcal{A}))$ at $t = t_0$. On $\mathbf{\Phi}_t(\mathcal{A})$ equation (5.1) combined with (9.11) and (9.15) leads to

$$\int\limits_{\Phi_t(\mathcal{A})} \left[\left(e+\frac{1}{2}\langle \mathbf{v},\mathbf{v}\rangle\right)\left(\frac{d}{dt}\,\rho+\rho\; div\mathbf{v}\right)+\rho\left(\frac{d}{dt}e+\langle \mathbf{a}-\mathbf{l},\mathbf{v}\rangle\right)\right] dv = \int\limits_{\partial\Phi_t(\mathcal{A})} \langle \mathbf{r},\mathbf{v}\rangle\; da\;. \qquad (9.16)$$

The analogon to (9.16) for the superposed motion is

$$\int\limits_{\Phi_t(\mathcal{A})} \left[\left(\tilde{e}+\frac{1}{2}\langle \tilde{\mathbf{v}},\tilde{\mathbf{v}}\rangle\right)\left(\frac{d}{dt}\,\tilde{\rho}+\tilde{\rho}\; div\mathbf{v}\right)+\tilde{\rho}\left(\frac{d}{dt}\tilde{e}+\langle \tilde{\mathbf{a}}-\tilde{\mathbf{l}},\tilde{\mathbf{v}}\rangle\right)\right] dv = \int\limits_{\partial\Phi_t(\mathcal{A})} \langle \tilde{\mathbf{r}},\tilde{\mathbf{v}}\rangle\; da\;. \qquad (9.17)$$

To prove (9.17), we formulate (5.1) on $\boldsymbol{\varphi}_t(\mathbf{\Phi}_t(\mathcal{A}))$:

$$\frac{d}{dt}\int\limits_{\varphi(\Phi_t(\mathcal{A}))} \tilde{\rho}\left(\tilde{e}+\frac{1}{2}\langle \tilde{\mathbf{v}},\tilde{\mathbf{v}}\rangle\right) d\tilde{v} \;=\; \int\limits_{\varphi(\Phi_t(\mathcal{A}))} \tilde{\rho}\langle \tilde{\mathbf{l}},\tilde{\mathbf{v}}\rangle d\tilde{v} \;+\; \int\limits_{\partial\varphi(\Phi_t(\mathcal{A}))} \langle \tilde{\mathbf{r}},\tilde{\mathbf{v}}\rangle d\tilde{a}\;. \qquad (9.18)$$

Due to (9.12), the first term on the right of (9.18) is equal to $\int\limits_{\Phi_t(\mathcal{A})}\tilde{\rho}\langle\tilde{\mathbf{l}},\tilde{\mathbf{v}}\rangle dv$. To transform the second term on the right we apply *Gaussian formula* (cf. page 31) getting an integral over $\boldsymbol{\varphi}(\mathbf{\Phi}_t(\mathcal{A}))$, use (9.12) and apply *Gaussian formula* again on $\mathbf{\Phi}_t(\mathcal{A})$ to get $\int\limits_{\partial\Phi_t(\mathcal{A})}\langle\tilde{\mathbf{r}},\tilde{\mathbf{v}}\rangle da$. After applying the transport theorem (9.13) to the integral on the left of (9.18) and using (9.15), taken in the transformed state, the equation (9.17) is proved by recombining the araising expressions. Due to (5.7)–(5.9) the balance of energy (9.17) for the superposed motion at $t = t_0$ reads

$$\int\limits_{\Phi_t(\mathcal{A})}\left[\left(e+\frac{1}{2}\langle \mathbf{v}+\boldsymbol{\xi},\mathbf{v}+\boldsymbol{\xi}\rangle\right)\left(\frac{d}{dt}\rho+\rho div\mathbf{v}\right)+\rho\left(\frac{d}{dt}\tilde{e}+\langle \mathbf{a}-\mathbf{l},\mathbf{v}+\boldsymbol{\xi}\rangle\right)\right] dv = \int\limits_{\partial\Phi_t(\mathcal{A})}\langle \mathbf{r},\mathbf{v}+\boldsymbol{\xi}\rangle da\;. \quad (9.19)$$

Finally we subtract (9.16) from (9.19), regard (5.10) and get

$$\int\limits_{\Phi_t(\mathcal{A})}\left[\left(\frac{d}{dt}\rho+\rho\; div\mathbf{v}\right)\left(\frac{1}{2}\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\langle\mathbf{v},\boldsymbol{\xi}\rangle\right)+\rho\left(\frac{\partial e}{\partial \mathbf{g}}:\mathcal{L}_\xi\mathbf{g}+\langle\mathbf{a}-\mathbf{l},\boldsymbol{\xi}\rangle\right)\right] dv = \int\limits_{\partial\Phi_t(\mathcal{A})}\langle\mathbf{r},\boldsymbol{\xi}\rangle da\;. \quad (9.20)$$

Proof of equations (5.16) and (5.17)

Equation (5.1) with $\mathbf{r} = \langle\boldsymbol{\sigma},\mathbf{n}\rangle$ reads

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})}\rho\left[e+\frac{1}{2}\langle\mathbf{v},\mathbf{v}\rangle\right] dv = \int\limits_{\Phi_t(\mathcal{A})}\rho\langle\mathbf{l},\mathbf{v}\rangle\; dv \;+\; \int\limits_{\partial\Phi_t(\mathcal{A})}\langle\langle\boldsymbol{\sigma},\mathbf{v}\rangle,\mathbf{n}\rangle\; da,$$

where we used, that $\langle\langle\boldsymbol{\sigma},\mathbf{n}\rangle,\mathbf{v}\rangle = \langle\langle\boldsymbol{\sigma},\mathbf{v}\rangle,\mathbf{n}\rangle$, what can be understood from simple computation. Applying (9.5) and (9.6) to this equation we get

$$\frac{d}{dt}\int_{\mathcal{A}} \rho_{_{Ref}}\left[e+\frac{1}{2}\langle\mathbf{v},\mathbf{v}\rangle\right]\Big|_{x\,=\,\boldsymbol{\Phi}_t(\mathbf{X})} dV = \int_{\mathcal{A}} \rho_{_{Ref}}\left[\langle\mathbf{l},\mathbf{v}\rangle\right]\Big|_{x\,=\,\boldsymbol{\Phi}_t(\mathbf{X})} dV + \int_{\partial\mathcal{A}} \langle\boldsymbol{\Phi}^*\langle\boldsymbol{\sigma},\mathbf{v}\rangle,\mathbf{N}\rangle J\,dA$$

with $\rho(\boldsymbol{\Phi}_t(\mathbf{X}))J = \rho_{_{Ref}}$ from (4.6), J from (9.8) and

$$E := \boldsymbol{\Phi}^* e(\mathbf{x},t,\mathbf{g}) = e(\boldsymbol{\Phi}_t(\mathbf{X}),t,\boldsymbol{\Phi}_*(\boldsymbol{\Phi}^*\mathbf{g})) = e(\boldsymbol{\Phi}_t(\mathbf{X}),t,\boldsymbol{\Phi}_*\mathbf{C}^\flat) = E(\mathbf{X},t,\mathbf{C}^\flat)$$

due to (5.8), (5.9) and (3.17). Since $\mathbf{v}\big|_{\mathbf{x}\,=\,\boldsymbol{\Phi}_t(\mathbf{X})} = \mathbf{V}$ (section 3) and $\mathbf{l}(\boldsymbol{\Phi}_t(\mathbf{X}),t) = \mathbf{L}(\mathbf{X},t)$ as in (4.6), this equation is equivalent to

$$\frac{d}{dt}\int_{\mathcal{A}} \rho_{_{Ref}}\left[E+\frac{1}{2}\langle\mathbf{V},\mathbf{V}\rangle\right] dV = \int_{\mathcal{A}} \rho_{_{Ref}}\langle\mathbf{L},\mathbf{V}\rangle\; dV + \int_{\partial\mathcal{A}} \langle J\boldsymbol{\Phi}^*\langle\boldsymbol{\sigma},\mathbf{v}\rangle,\mathbf{N}\rangle\; dA.$$

Equation (2.13) combined with (4.3) and (4.4), gives $J\,(\boldsymbol{\Phi}^*\langle\boldsymbol{\sigma},\mathbf{v}\rangle)^A = J\dfrac{\partial(\Phi_t^{-1})^A}{\partial x^a}(\langle\boldsymbol{\sigma},\mathbf{v}\rangle)^a = J(F^{-1})^A_a\sigma^{ac}g_{bc}v^b = P^{cA}g_{bc}V^b = (\langle\mathbf{P},\mathbf{V}\rangle)^A$. So, the equation

$$\frac{d}{dt}\int_{\mathcal{A}} \rho_{_{Ref}}\left[E+\frac{1}{2}\langle\mathbf{V},\mathbf{V}\rangle\right] dV = \int_{\mathcal{A}} \rho_{_{Ref}}\langle\mathbf{L},\mathbf{V}\rangle\; dV + \int_{\partial\mathcal{A}} \langle\langle\mathbf{P},\mathbf{V}\rangle,\mathbf{N}\rangle\; dA \tag{9.21}$$

with $\langle\langle\mathbf{P},\mathbf{V}\rangle,\mathbf{N}\rangle = \langle\langle\mathbf{P},\mathbf{N}\rangle,\mathbf{V}\rangle$ is obtained, and this is exactly (5.16). To prove (5.17), we permute differentiation and integration[33], include (9.15) and are led to

$$\int_{\mathcal{A}} \rho_{_{Ref}}\left(\frac{d}{dt}E + \langle\mathbf{A}-\mathbf{L},\mathbf{V}\rangle\right) dV = \int_{\partial\mathcal{A}} \langle\langle\mathbf{P},\mathbf{N}\rangle,\mathbf{V}\rangle\;\; dA. \tag{9.22}$$

As on page 34, we formulate the analogon to (9.22) for an arbitrary superposed motion with the *material velocity* $\boldsymbol{\Xi} := \dfrac{\partial\boldsymbol{\varphi}}{\partial t} = \boldsymbol{\xi}$:
From (5.3) and (5.4) we use $\tilde{\mathbf{V}} = \boldsymbol{\varphi}_*\mathbf{V} + \boldsymbol{\Xi}$. Like above, we have $\tilde{E}(\mathbf{X},t) = \tilde{\boldsymbol{\Phi}}^*\tilde{e}(\tilde{\mathbf{x}},t,\tilde{\mathbf{g}})$ $= \tilde{e}(\tilde{\boldsymbol{\Phi}}_t(\mathbf{X}),t,\tilde{\boldsymbol{\Phi}}_*(\tilde{\boldsymbol{\Phi}}^*\tilde{\mathbf{g}})) = \tilde{e}(\tilde{\boldsymbol{\Phi}}_t(\mathbf{X}),t,\tilde{\boldsymbol{\Phi}}_*\tilde{\mathbf{C}}^\flat) = \tilde{E}(\mathbf{X},t,\tilde{\mathbf{C}}^\flat)$ with $\tilde{\mathbf{C}}^\flat := \tilde{\boldsymbol{\Phi}}^*\tilde{\mathbf{g}}$, cf. (3.17) and $\tilde{\mathbf{L}}(\mathbf{X},t) = \tilde{\mathbf{l}}(\tilde{\boldsymbol{\Phi}}_t(\mathbf{X}),t)$. The definition (3.3) gives $\tilde{\mathbf{A}}(\mathbf{X},t) = \tilde{\mathbf{a}}(\tilde{\boldsymbol{\Phi}}_t(\mathbf{X}),t)$, and from (5.7) we deduce $\tilde{\mathbf{L}}-\tilde{\mathbf{A}} = \tilde{\mathbf{l}}-\tilde{\mathbf{a}} = \boldsymbol{\varphi}_*(\mathbf{l}-\mathbf{a}) = \boldsymbol{\varphi}_*(\mathbf{L}-\mathbf{A})$. With (4.4) we have $\mathbf{P} = J\boldsymbol{\Phi}^*\boldsymbol{\sigma}$ and $\tilde{\mathbf{P}} = \tilde{J}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\sigma}}$ with $\tilde{J} = J$ as shown near (9.12). With those preliminaries we note (9.22) for the superposed motion as

$$\int_{A} \rho_{_{Ref}}\left(\frac{d}{dt}\tilde{E} + \langle\tilde{\mathbf{A}}-\tilde{\mathbf{L}},\tilde{\mathbf{V}}\rangle\right) dV = \int_{\partial A} \langle\langle\tilde{\mathbf{P}},\mathbf{N}\rangle,\tilde{\mathbf{V}}\rangle dA \tag{9.23}$$

[33]In contrast to (5.1), here is no need to heed (9.11), since $\mathcal{A}$ does not depend on the time.

Subtracting (9.22) from (9.23) and selecting the time t_0 for which $\tilde{\mathbf{\Phi}}_{t_0} = \mathbf{\Phi}_{t_0}$ holds, we get

$$\int\limits_{A} \rho_{_{Ref}} \left(\left(\frac{d}{dt}(\tilde{E}-E) \right)_{|t_0} + \langle \mathbf{A}-\mathbf{L}, \mathbf{\Xi} \rangle \right) dV = \int\limits_{\partial A} \langle \langle \mathbf{P}, \mathbf{N} \rangle, \mathbf{\Xi} \rangle dA \ , \tag{9.24}$$

since $(\tilde{\mathbf{A}}-\tilde{\mathbf{L}})_{|t_0} = \mathbf{A}-\mathbf{L}, \tilde{\mathbf{V}}_{|t_0} = \mathbf{V}+\mathbf{\Xi}$, and from (4.2), (5.8) we get $\tilde{\boldsymbol{\sigma}}_{|t_0} = \boldsymbol{\sigma}$, delivering $\tilde{\mathbf{P}}_{|t_0} = \mathbf{P}$. For the term $\frac{d}{dt}(\tilde{E}-E) = \frac{\partial \tilde{E}}{\partial t} - \frac{\partial E}{\partial t} + \frac{\partial \tilde{E}}{\partial C_{AB}} \frac{d}{dt}\tilde{C}_{AB} - \frac{\partial E}{\partial C_{AB}} \frac{d}{dt} C_{AB}$ we see, that $\frac{\partial \tilde{E}}{\partial t} = \frac{\partial E}{\partial t}$ and $\frac{\partial \tilde{E}}{\partial C_{AB}} = \frac{\partial E}{\partial C_{AB}}$ for $t = t_0$ holds.
So we get $\frac{d}{dt}(\tilde{E}-E)_{|t_0} = \frac{\partial E}{\partial C_{AB}} \frac{d}{dt}(\tilde{C}_{AB}{}_{|t_0} - C_{AB}) = 2\frac{\partial E}{\partial C_{AB}}(\tilde{D}_{AB}{}_{|t_0} - D_{AB})$. Due to (3.19) we have $\tilde{\mathbf{D}}^\flat{}_{|t_0} - \mathbf{D}^\flat = \frac{1}{2}\mathbf{\Phi}^* \left((\mathcal{L}_{\tilde{v}}\tilde{\mathbf{g}})_{|t_0} - \mathcal{L}_v \mathbf{g} \right) = \frac{1}{2}\mathbf{\Phi}^* \mathcal{L}_\xi \mathbf{g} =: \mathbf{D}^\flat_\Xi$, cf. (2.29). So the equation (9.25) is proven:

$$\int\limits_{\mathcal{A}} \rho_{_{Ref}} (2\frac{\partial E}{\partial \mathbf{C}^\flat} : \mathbf{D}^\flat_\Xi + \langle \mathbf{A} - \mathbf{L}, \mathbf{\Xi} \rangle) \, dV = \int\limits_{\partial \mathcal{A}} \langle \langle \mathbf{P}, \mathbf{N} \rangle, \mathbf{\Xi} \rangle \ dA \ . \tag{9.25}$$

To continue, we need an analogon to the divergency theorem (5.12) formulated in the reference configuration. This reads

$$DIV\langle \mathbf{P}, \mathbf{\Xi} \rangle = \langle DIV\mathbf{P}, \mathbf{\Xi} \rangle + \left(\mathbf{P}\cdot\mathbf{F}^{-1} \right) : \mathbf{\Omega}^\flat_\Xi + \left(\mathbf{P}\cdot\mathbf{F}^{-1} \right) : \mathbf{D}^\flat_\Xi \tag{9.26}$$

with the spin $\mathbf{\Omega}^\flat_\Xi$ defined by

$$\Omega_{\Xi AB} := \frac{1}{2}\left((g_{bc}\Xi^c)_{|A} F^b_B - (g_{ac}\Xi^c)_{|B} F^a_A \right) = \frac{1}{2}\left(g_{bc}\Xi^c_{|A} F^b_B - g_{ac}\Xi^c_{|B} F^a_A \right) \ , \tag{9.27}$$

the *rate of deformation* $\mathbf{D}^\flat_\Xi$ with components

$$D_{\Xi AB} =: \frac{1}{2}\left(g_{bc}\Xi^c_{|A} F^b_B + g_{ac}\Xi^c_{|B} F^a_A \right) \ , \tag{9.28}$$

and $(\mathbf{P}\mathbf{F}^{-1})^{AB} = P^{aA}(F^{-1})^B_a = T^{AB}$ with T^{AB} from (4.8). This shall be proved in the sequal:
Following the proof on page 31, including (4.7) and introducing the temporary substitution $\mathcal{N}_a := g_{ab}\Xi^b$ we get
$DIV\langle \mathbf{P}, \mathbf{\Xi} \rangle = \mathcal{N}_a \left(\frac{\partial P^{aA}}{\partial X^A} + \Gamma^A_{AB} P^{aB} \right) + P^{aA} \frac{\partial \mathcal{N}_a}{\partial X^A} = \mathcal{N}_a \left(\frac{\partial P^{aA}}{\partial X^A} + P^{aA}\Gamma^B_{AB} + \gamma^b_{bc} F^c_A P^{aA} \right) + P^{bA}(\frac{\partial \mathcal{N}_b}{\partial X^A} - \mathcal{N}_a F^c_A \gamma^a_{bc}) = \mathcal{N}_a \left(DIV\mathbf{P} \right)^a + P^{bA}\mathcal{N}_{b|A} = \langle DIV\mathbf{P}, \mathbf{\Xi} \rangle + P^{bA}\mathcal{N}_{b|A}$ with $\mathcal{N}_{b|A} := \frac{\partial \mathcal{N}_b}{\partial X^A} - \gamma^a_{bc} F^c_A \mathcal{N}_a = \frac{\partial \mathcal{N}_b}{\partial x^c}\frac{\partial x^c}{\partial X^A} - \gamma^a_{bc} F^c_A \mathcal{N}_a = \frac{\partial \mathcal{N}_b}{\partial x^c} F^c_A - \gamma^a_{bc} F^c_A \mathcal{N}_a = F^c_A \left(\frac{\partial \mathcal{N}_b}{\partial x^c} - \gamma^a_{bc}\mathcal{N}_a \right) = F^c_A \mathcal{N}_{b|c}$, and therefore $DIV\langle \mathbf{P}, \mathbf{\Xi} \rangle = \langle DIV\mathbf{P}, \mathbf{\Xi} \rangle + P^{bA} F^c_A \mathcal{N}_{b|c}$. To verify (9.26), it remains to show, that $P^{bA} F^c_A \mathcal{N}_{b|c} = \left(\mathbf{P}\cdot\mathbf{F}^{-1} \right) : \left(\mathbf{\Omega}^\flat_\Xi + \mathbf{D}^\flat_\Xi \right)$ and that the identity inside (9.27)

is true.
To start with the latter, we state, that $\Xi^c_{|A} = F^b_A \Xi^c_{|b}$ follows from simple computations. So, rearranging the terms, we get $g_{bc}\Xi^c_{|A}F^b_B - g_{ac}\Xi^c_{|B}F^a_A = g_{ab}\Xi^b_{|c}(F^c_A F^a_B - F^c_B F^a_A)$. In a similar manner the following transformations $(g_{bc}\Xi^c)_{|A}F^b_B - (g_{ac}\Xi^c)_{|B}F^a_A = \mathcal{N}_{b|A}F^b_B - \mathcal{N}_{a|B}F^a_A = F^c_A\mathcal{N}_{b|c}F^b_B - F^c_B\mathcal{N}_{a|c}F^a_A = (F^c_A F^a_B - F^c_B F^a_A)(\Xi^b \frac{\partial g_{ab}}{\partial x^c} + g_{ab}\frac{\partial \Xi^b}{\partial x^c} - \gamma^d_{ac}g_{de}\Xi^e) = (F^c_A F^a_B - F^c_B F^a_A)(\Xi^b \frac{\partial g_{ab}}{\partial x^c} + g_{ab}\Xi^b_{|c} - \gamma^b_{cd}g_{ab}\Xi^d - \gamma^d_{ac}g_{de}\Xi^e)$ are obtained and it remains to verify, that $\Xi^b\left(\frac{\partial g_{ab}}{\partial x^c} - g_{ad}\gamma^d_{cb} - g_{db}\gamma^d_{ac}\right) = 0$, what is a straight consequence of (2.10).
To complete the proof to (9.26) we compute $\left(\mathbf{\Omega}^\flat_\Xi + \mathbf{D}^\flat_\Xi\right)_{AB} = g_{bc}\Xi^c_{|A}F^b_B$ and $\left(\mathbf{P}\cdot\mathbf{F}^{-1}\right):\left(\mathbf{\Omega}^\flat_\Xi + \mathbf{D}^\flat_\Xi\right) = P^{aA}g_{bc}\Xi^c_{|A}F^b_B\,(F^{-1})^B_a = P^{aA}g_{bc}\Xi^c_{|A}\delta^b_a = P^{aA}g_{ac}\Xi^c_{|A} = P^{aA}g_{ac}\Xi^c_{|d}F^d_A$ and compare it to $P^{bA}F^c_A\mathcal{N}_{b|c} = P^{bA}F^c_A\left(\frac{\partial \mathcal{N}_b}{\partial x^c} - \gamma^a_{bc}\mathcal{N}_a\right) = P^{bA}F^c_A\left(\frac{\partial g_{ab}}{\partial x^c}\Xi^a + g_{ab}\frac{\partial \Xi^a}{\partial x^c} - \gamma^a_{bc}g_{ad}\Xi^d\right) = P^{bA}F^c_A\left(g_{ab}\Xi^a_{|c} + \Xi^d\left(\frac{\partial g_{db}}{\partial x^c} - \gamma^a_{bc}g_{ad} - \gamma^a_{cd}g_{ab}\right)\right) = P^{bA}F^c_A g_{ab}\Xi^a_{|c}$. Applying Gauss' formula to $\int\limits_{\partial\mathcal{A}} \langle\langle \mathbf{P},\mathbf{N}\rangle, \mathbf{\Xi}\rangle dA = \int\limits_{\partial\mathcal{A}} \langle\langle \mathbf{P},\mathbf{\Xi}\rangle, \mathbf{N}\rangle dA = \int\limits_{\mathcal{A}} DIV\langle \mathbf{P},\mathbf{\Xi}\rangle dV$ and the divergency theorem (9.26) to (9.25), we end at (5.17).

Proof of equations (6.3), (6.4)

Equation (6.1) with $h(\mathbf{x},t,\mathbf{n}) = -\langle \mathbf{q}(\mathbf{x},t),\mathbf{n}\rangle$, reads

$$\frac{d}{dt}\int\limits_{\Phi_t(\mathcal{A})} \rho\eta\, dv \geq \int\limits_{\Phi_t(\mathcal{A})} \frac{\rho s}{\vartheta}\, dv - \int\limits_{\partial\Phi_t(\mathcal{A})} \frac{\langle \mathbf{q}(\mathbf{x},t),\mathbf{n}\rangle}{\vartheta}\, da\ . \tag{9.29}$$

Applying the *transport theorem* (9.11) to the left of (9.29) and the *Gauss theorem* (9.7) to the second term on the right, we get

$$\int\limits_{\Phi_t(\mathcal{A})} \left\{\rho\frac{d}{dt}\eta - \frac{\rho s}{\vartheta} + div\left(\frac{\mathbf{q}}{\vartheta}\right) + \eta\left[\frac{d}{dt}\rho + \rho div\mathbf{v}\right]\right\} dv \geq 0.$$

with $\frac{d}{dt}\rho + \rho div\mathbf{v}$ vanishing due to the supposed *conservation of mass*. Taking into account the arbitraryness of $\mathcal{A}$, the first part of (6.3) is found. The second part of (6.3) follows from simple calculus.
In a similar way, using (4.2), the *divergency theorem* (5.12), the *conservation of mass* (5.14), (9.7), (9.11) and (9.15), from (6.2) we get

$$\int\limits_{\Phi_t(\mathcal{A})} \left\{\rho\frac{d}{dt}e + \langle \rho(\mathbf{a}-\mathbf{l}) - div\ \boldsymbol{\sigma}\ ,\mathbf{v}\rangle - \frac{1}{2}\boldsymbol{\sigma}:\mathcal{L}_{\mathbf{v}}\mathbf{g} - \boldsymbol{\sigma}:\boldsymbol{\omega}^\flat_{\mathbf{v}} - \rho s + div\ \mathbf{q}\right\} dv = 0.$$

According to the symmetry of $\boldsymbol{\sigma}$ we have $\boldsymbol{\sigma}:\boldsymbol{\omega}^\flat_{\mathbf{v}} = 0$, and the *conservation of momentum* from (5.14) gives $\langle \rho(\mathbf{a}-\mathbf{l}) - div\ \boldsymbol{\sigma}\ ,\mathbf{v}\rangle = 0$. Inserting the definition of the *spatial rate of* *deformation tensor* $\mathbf{d}^\flat$ from (3.20), we find

$$\int\limits_{\Phi_t(\mathcal{A})} \left[\rho \frac{d}{dt} e - \boldsymbol{\sigma} : \mathbf{d}^\flat - \rho s + div\ \mathbf{q}\right]\ dv = 0\ ,$$

and the arbitraryness of $\mathcal{A}$ supplies (6.4).

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