A Multi-Grid Method for Generalized Lyapunov Equations

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Abstract

We present a multi-grid method for a class of structured generalized Lyapunov matrix equations. Such equations need to be solved in each step of the Newton method for algebraic Riccati equations, which arise from linear-quadratic optimal control problems governed by partial differential equations. We prove the rate of convergence of the two-grid method to be bounded independent of the dimension of the problem under certain assumptions. The multi-grid method is based on matrix-matrix multiplications and thus it offers a great potential for a parallelization. The efficiency of the method is demonstrated by numerical experiments.

Key Words: algebraic Riccati equation, iterative methods, Lyapunov equation, multi-grid methods, multi-level methods, Newton method.

AMS Subject Classification: 65M55, 93C20.

1 Introduction

In this paper we present a multi-grid method for a class of generalized Lyapunov equations resulting from the discretization of a certain model problem. Essentially, our method is based on an adaptation of the principles used in multi-grid methods for systems of linear equations arising from the discretization of partial differential equations [Fed61, Fed64, Hac85] to linear matrix equations.

A multi-grid method for the approximate solution of the model problem considered here has already been proposed by Rosen and Wang [RW95]. The differences and advantages of our method compared to that by Rosen and Wang are highlighted in Section 9.

We consider the *linear-quadratic control problem* with the cost functional

$$\frac{1}{2}\int_0^\infty y(\tau)^2 + u(\tau)^2 d\tau \longrightarrow \text{Min !} \tag{1}$$

and the constraints

$$\begin{aligned}
\dot{\mathbf{x}}(\xi,\tau) &= \mathcal{A}\mathbf{x}(\xi,\tau) + \mathbf{b}(\xi)u(\tau) \qquad \text{for } \xi \in \Omega = (0,1) && (2)\\
y(\tau) &= \int_0^1 \mathbf{c}(\xi)\mathbf{x}(\xi,\tau)d\xi && (3)\\
\mathbf{x}(\xi,\tau) &= 0 \qquad \text{for } \xi \in \partial\Omega = \{0,1\} && (4)\\
\mathbf{x}(\xi,0) &= \mathbf{x_0}(\xi) \qquad \text{for } \xi \in \Omega = (0,1) && (5)
\end{aligned}$$

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with $\tau \in [0, \infty)$. The quantities $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, u, and y are real functions. Dotted quantities are the partial derivatives w.r.t. τ, e.g., $\dot{\mathbf{x}}(\xi, \tau) = \partial \mathbf{x}(\xi, \tau)/\partial \tau$. Here, $\mathcal{A}$ is the differential operator

$$\mathcal{A}\mathbf{x}(\xi, \tau) := \frac{\partial}{\partial \xi}\left(\alpha(\xi)\frac{\partial}{\partial \xi}\mathbf{x}(\xi, \tau)\right) \tag{6}$$

with a function $\alpha \in L_\infty(\Omega)$, for which a constant $\alpha_0 > 0$ exists such that $\alpha(\xi) \geq \alpha_0$ a.e. in Ω.

A physical problem corresponding to such a model problem is, for example, the optimal control of the distribution of heat in a thin rod. In this case, $\mathbf{x}$, ξ, and τ denote the temperature, the space component, and the time component, respectively.

We discretize the constraints (2)-(5) using the *finite element method* (FEM) with a partitioning of the interval Ω into $n+1$ subintervals (ξ_i, ξ_{i+1}) of length $h = 1/(n+1)$ with $\xi_i = ih$ for $i = 0, \ldots, n+1$. FEM ansatz functions $\mathbf{p}_i(\xi)$ defined as

$$\mathbf{p}_i(\xi) := \begin{cases} (n+1)(\xi - \xi_{i-1}) & : \ \xi \in (\xi_{i-1}, \xi_i) \\ (n+1)(\xi_{i+1} - \xi) & : \ \xi \in (\xi_i, \xi_{i+1}) \\ 0 & : \ \text{otherwise} \end{cases}$$

are utilized. We pursue the Galerkin approach with an ansatz

$$\mathbf{x}(\xi, \tau) = \sum_{j=1}^{n} x_j(\tau)\mathbf{p}_j(\xi),$$

which results in

$$\begin{aligned} \sum_{j=1}^{n} \underbrace{< \mathbf{p}_j(\xi), \mathbf{p}_i(\xi) >}_{=:(E)_{ij}} (\dot{x}(\tau))_j &= \sum_{j=1}^{n} \underbrace{< \mathcal{A}\mathbf{p}_j(\xi), \mathbf{p}_i(\xi) >}_{=:-(A)_{ij}} (x(\tau))_j \\ &\quad + \underbrace{< \mathbf{b}(\xi), \mathbf{p}_i(\xi) >}_{=:(B)_i} (u(\tau))_j \\ y(\tau) &= \sum_{j=1}^{n} \underbrace{< \mathbf{c}(\xi), \mathbf{p}_j(\xi) >}_{=:(C)_j} (x(\tau))_j \\ \sum_{j=1}^{n} \underbrace{< \mathbf{p}_j(\xi), \mathbf{p}_i(\xi) >}_{=:(E)_{ij}} (x(0))_j &= \underbrace{< \mathbf{x}_0(\xi), \mathbf{p}_i(\xi) >}_{=:(x_0)_i} \end{aligned} \tag{7}$$

for $i = 1, \ldots, n$. Here $< \cdot, \cdot >$ designates the $L_2(\Omega)$ scalar product. Obviously, the discretization generates a descriptor system

$$\begin{aligned} E\dot{x}(\tau) &= -Ax(\tau) + Bu(\tau) \\ y(\tau) &= Cx(\tau) \\ Ex(0) &= x_0 \end{aligned} \tag{8}$$

with a column vector B, a row vector C, and the symmetric positive definite matrices A and E. In fact, A is the stiffness matrix and E is the mass matrix, which usually result from the discretization of the one-dimensional Poisson equation.

The solution of the discretized optimal control problem (1) and (8) is determined by the linear feedback

$$u(\tau) = -B^T P E x(\tau),$$

where P is the symmetric positive definite *stabilizing* solution of the *algebraic Riccati equation* (ARE)

$$0 = R(P) := C^T C - \mathcal{L}_{A,E} P - E^T P B B^T P E, \tag{9}$$

see [BL87, Meh91]. The generalized Lyapunov operator $\mathcal{L}_{A,E}$ forming the linear part of the Riccati equation is defined as

$$\mathcal{L}_{A,E} X := A^T X E + E^T X A.$$

A solution of the Riccati equation (9) is stabilizing iff each eigenvalue of the matrix $E^{-1}(-A - BB^T PE)$ has a negative real part. Under certain assumptions, a unique stabilizing solution of (9) exists, e.g., [LR95, Meh91].

One of the most popular methods for solving algebraic Riccati equations is the Newton method [Kle68, LR95, Meh91].

Algorithm 1 (Newton method for ARE (9))

FOR $i = 0, 1, 2, \ldots$

1. $K_i = B^T P_i E$
2. Solve $\mathcal{L}_{A+BK_i,E} X_i = R(P_i)$ for X_i.
3. $P_{i+1} = P_i + X_i$

END FOR

This method produces a sequence of stabilizing iterates P_i, provided that the initial guess P_0 is stabilizing, which is fulfilled when $P_0 = 0$. The convergence of the Newton method is quadratic. **After** the first step the method converges monotonically, see, e.g., [Ben97, LR95, Meh91].

In the following sections we focus on the solution of the structured generalized Lyapunov equation in Step 2 of Algorithm 1.

2 Analysis of the Operator $\mathcal{L}_{A,E}$

We study the eigenstructure of the operator $\mathcal{L}_{A,E}$ under the assumption $\alpha(\xi) \equiv 1$, which is valid throughout this section. In case of a constant function $\alpha(\xi) \not\equiv 1$, the eigenvalue problem can be reduced by scaling to the case $\alpha(\xi) \equiv 1$. In the general case, where α is a non-constant function, the matrices A and E do not commute and the following analysis is no longer applicable.

Under the above assumption, the matrices $A, E \in \mathbb{R}^{n,n}$ are given as:

$$\begin{aligned} A &= (n+1)\,\mathrm{tridiag}(-1, 2, -1), && (10) \\ E &= \frac{1}{6(n+1)}\,\mathrm{tridiag}(1, 4, 1). && (11) \end{aligned}$$

The matrix A possesses the orthogonal eigenvectors [Hac85]

$$v_k = \left(\sqrt{\frac{2}{n+1}} \sin \frac{ik\pi}{n+1} \right)_{i=1}^{n} \tag{12}$$

and the corresponding eigenvalues

$$\lambda_k^A = 4(n+1) s_k^2$$

with $s_k := \sin\frac{k\pi}{2(n+1)}$ for $k = 1,\ldots,n$. Since $E = \frac{1}{n+1}I - \frac{1}{6(n+1)^2}A$, the matrix E has the same eigenvectors and the eigenvalues

$$\lambda_k^E = \frac{1-\frac{2}{3}s_k^2}{n+1}.$$

Hence, the matrices $v_i v_j^T$ are "eigenmatrices" of the operator $\mathcal{L}_{A,E}$ corresponding to the eigenvalues

$$\lambda_{ij} := \lambda_{ij}^{\mathcal{L}_{A,E}} = \lambda_i^A \lambda_j^E + \lambda_j^A \lambda_i^E = 4s_i^2 + 4s_j^2 - \frac{16}{3}s_i^2 s_j^2 \tag{13}$$

for $i,j = 1,\ldots,n$. The following lemma is useful for a characterization of the eigenvalues.

Lemma 1 *Let the function*

$$r(\eta,\theta) = 4\eta + 4\theta - \frac{16}{3}\eta\theta$$

be defined on $[0,1]^2$. *Then* $r(\eta,\theta)$ *is monotonically (strictly monotonically) increasing in* η *if* $\theta \leq 3/4$ $(\theta < 3/4)$. *Otherwise,* $r(\eta,\theta)$ *is strictly monotonically (monotonically) decreasing in* η. *Moreover,* $r(\eta,\theta) = 0$ *iff* $\eta = \theta = 0$.

Proof. The first part is proved immediately by estimating the partial derivatives of r, e.g.,

$$\frac{\partial r(\eta,\theta)}{\partial\eta} = 4 - \frac{16}{3}\theta > 0 \quad\Longleftrightarrow\quad \frac{3}{4} > \theta.$$

From $r(1,1) > 0$ and the monotonicity of r it follows that $r(\eta,\theta) = 0$ iff $\eta = \theta = 0$.
■

Obviously, the first statement remains valid if the rolls of η and θ are exchanged. Taking account of the monotonicity of the sine function in the interval $[0,\pi/2]$, the substitution of η and θ by s_i^2 and s_j^2, respectively, leads to the following corollary.

Corollary 1 *Let* λ_{ij} $(i,j = 1,\ldots,n)$ *be the eigenvalues of the Lyapunov operator* $\mathcal{L}_{A,E}$ *according to (13). For* $j \leq 2(n+1)/3$ $(j < 2(n+1)/3)$ *the eigenvalues* λ_{ij} *increase monotonically (strictly monotonically) in* i. *Otherwise, the eigenvalues* λ_{ij} *decrease strictly monotonically (monotonically) in* i.

Again, Corollary 1 holds if the rolls of the indices i and j are exchanged. From the equation

$$s_n^2 = \sin^2\left(\frac{\pi}{2} - \frac{\pi}{2(n+1)}\right) = \cos^2\left(\frac{\pi}{2(n+1)}\right) = 1 - s_1^2,$$

the assumption $n \geq 2$, and Corollary 1 we gain the following characterization of the spectrum. The maximal eigenvalues are

$$\max_{i,j}\lambda_{ij} = \lambda_{1n} = \lambda_{n1} = 4 - \frac{16}{3}s_1^2 + \frac{16}{3}s_1^4 < 4. \tag{14}$$

Moreover, the estimates

$$0 < \lambda_{11} = 8s_1^2 - \frac{16}{3}s_1^4 < 8s_1^2 \tag{15}$$

and

$$\lambda_{nn} = \frac{8}{3} + \frac{8}{3}s_1^2 - \frac{16}{3}s_1^4 > \frac{8}{3}$$

result in

$$0 < \min_{i,j} \lambda_{ij} = \min\{\lambda_{11}, \lambda_{nn}\} = \lambda_{11}. \tag{16}$$

Hence, the operator $\mathcal{L}_{A,E}$ is nonsingular.

In context with multi-grid methods the limits of the "oscillating" subset of the spectrum

$$\sigma_{osc} := \{\lambda_{ij} : \max\{i,j\}\} \geq (n+1)/2\}$$

are of interest. Obviously,

$$\max \sigma_{osc} = \max \sigma = \lambda_{n1} < 4. \tag{17}$$

By Lemma 1 and Corollary 1,

$$\min \sigma_{osc} \geq \min \left\{ \lambda_{nn}, r(s^2_{(n+1)/2}, s^2_1) \right\}.$$

From $s^2_{(n+1)/2} = 1/2$ and $s^2_1 \leq 1/2$, we obtain the inequality

$$\min \sigma_{osc} \geq \min \left\{ \frac{8}{3} + \frac{8}{3} s^2_1 - \frac{16}{3} s^4_1, 2 + \frac{4}{3} s^2_1 \right\} = 2 + \frac{4}{3} s^2_1 > 2. \tag{18}$$

The bounds (14), (16), (17), and (18) are asymptotically sharp for $n \longrightarrow \infty$.

For estimating the speed of convergence of certain iterative methods for symmetric positive definite linear systems, the condition number and, of course, its dependence on the dimension of the problem are of importance, cf. [GV89, Saa96, Var62]. From (14), (15), and (16) we obtain for $n \gg 1$

$$\operatorname{cond} \mathcal{L}_{A,E} = \frac{\lambda_{n1}}{\lambda_{11}} = \frac{1}{2s^2_1} + \underbrace{\frac{2s^2_1 + 4s^4_1}{6s^2_1 - 4s^4_1}}_{\substack{\longrightarrow 1/3 \\ \text{for } n \longrightarrow \infty}} \approx \frac{2}{\pi} n^2 \tag{19}$$

as an estimate for the condition number.

3 Multi-Grid Components

In this section we introduce the *grid hierarchy*, the *smoothing method*, the *restriction operator*, and the *interpolation operator* which are utilized in our multi-grid method.

Grid hierarchy. The elements of the grid hierarchy are labelled by a level index l. They are produced by starting with an equidistant initial distribution of the grid points in the interval Ω (Level 1) and by successively halving the subintervals $l_{max} - 1$ times. The quantities corresponding to a certain level are provided with the superscript (l), e.g., $X^{(l)}$. As long as there is no danger of confusion, we neglect the level index for $l = l_{max}$.

The grid hierarchy is determined by the number of the inner grid points $n^{(1)}$ on the coarsest grid and by the number of levels l_{max}. By recursion we obtain $n^{(l+1)} = 2n^{(l)} + 1$ for $l = 1, \ldots, l_{max} - 1$. The spacing between neighbouring grid points is $h^{(l)} = 1/(n^{(l)} + 1)$. The matrices $A^{(l)}$, $B^{(l)}$, and $E^{(l)}$ are provided by (7) with $n = n^{(l)}$.

Smoothing method. Frequently, simple stationary iterative methods are used in multi-grid methods for "smoothing", or more precisely, for the reduction of the component in the error corresponding to σ_{osc}. In the sequel we utilize the Richardson iteration, which is the simplest of these methods. This choice is advantageous in view of the convergence analysis in Section 5. Nevertheless, better rates of convergence may be achieved using other smoothing methods. Algorithm 2 realizes ν steps of the Richardson iteration for $\mathcal{L}_{A,E}X = Y$ with the relaxation parameter ω

$$X_\nu \longleftarrow \text{RICHARDSON}(X_0, Y, A, E, \nu, \omega).$$

Algorithm 2 (Richardson iteration for $\mathcal{L}_{A,E}X = Y$)

FOR $k = 1, \ldots, \nu$

1. $X_k = X_{k-1} + \omega\,(Y - \mathcal{L}_{A,E}X_{k-1})$

END FOR

The following simple analysis gives an insight into the convergence and the smoothing properties of the Richardson iteration. A more detailed discussion can be found in [Hac85, Var62], for example. Since the solution $\mathcal{L}_{A,E}^{-1}Y$ is a stationary point of the mapping $X \longrightarrow X + \omega(Y - \mathcal{L}_{A,E}X)$, the equation

$$Z_k = (\mathcal{I} - \omega\mathcal{L}_{A,E})Z_{k-1} = \mathcal{S}Z_{k-1} \tag{20}$$

applies to the error $Z_k = \mathcal{L}_{A,E}^{-1}Y - X_k$, where $\mathcal{S}$ is the *error transfer operator*. Thus, the error component corresponding to $v_i v_j^T$, which is an eigenmatrix of S, is reduced by a factor $|1 - \omega\lambda_{ij}|$ per iteration step. For smoothing,

$$\omega = \underset{\omega\in\mathbb{R}}{\text{argmin}} \max_{\lambda_{ij}\in\sigma_{osc}} |1 - \omega\lambda_{ij}| = \frac{2}{\min\sigma_{osc} + \max\sigma_{osc}}$$

is a proper choice for the relaxation parameter ω. Because of the asymptotically sharp estimates (17) and (18), the choice $\omega = 1/3$ is asymptotically optimal for $n \longrightarrow \infty$. This guarantees the damping of the oscillatory error components to be independent of n, since

$$\max_{\lambda_{ij}\in\sigma_{osc}} |1 - \omega\lambda_{ij}| \le \frac{1}{3}$$

holds.

Restriction operator. Mappings from Level $l+1$ to Level l are realized by this operator. The restriction operator $\mathcal{I}_{(l+1)}^{(l)} : \mathbb{R}^{n^{(l+1)},n^{(l+1)}} \longrightarrow \mathbb{R}^{n^{(l)},n^{(l)}}$ is defined as

$$\mathcal{I}_{(l+1)}^{(l)} X^{(l+1)} = I_{(l+1)}^{(l)} X^{(l+1)} \left(I_{(l+1)}^{(l)}\right)^T,$$

where $I_{(l+1)}^{(l)}$ is the $n^{(l)} \times n^{(l+1)}$-matrix with the entries

$$\left(I_{(l+1)}^{(l)}\right)_{ij} = \begin{cases} 1 & : \ \text{if } j = 2i \\ \frac{1}{2} & : \ \text{if } j = 2i \pm 1 \\ 0 & : \ \text{otherwise} \end{cases}.$$

Interpolation operator. This operator maps matrices from Level l to Level $l+1$. The interpolation operator $\mathcal{I}_{(l)}^{(l+1)} : \mathbb{R}^{n^{(l)},n^{(l)}} \longrightarrow \mathbb{R}^{n^{(l+1)},n^{(l+1)}}$ is defined as

$$\mathcal{I}_{(l)}^{(l+1)} X^{(l)} = \left(I_{(l+1)}^{(l)}\right)^T X^{(l)} I_{(l+1)}^{(l)}.$$

4 A Two-Grid Method

In the sequel we present a two-grid method (i.e., $l_{max} = 2$) for the generalized Lyapunov equation

$$\mathcal{L}_{A,E}X = A^T X E + E^T X A = Y \tag{21}$$

with the selfadjoint Operator $\mathcal{L}_{A,E}$. This equation is to be solved in the first step of the Newton method with the initial guess $P_0 = 0$, but it also arises in other applications, e.g., [Fra87, LL61, SC89].

First, we give a brief motivation for the method. Multi-grid methods in the context of partial differential equations are motivated in more detail in [Bri87, Hac85, McC87], for example. The Richardson method greatly damps the error component related to the invariant subspace of $\mathcal{L}_{A,E}$ corresponding to σ_{osc} if the relaxation parameter ω is properly chosen (*presmoothing*). The eigenmatrices of eigenvalues not contained in σ_{osc} are outer products of vectors v_i and v_j which are **both** smooth. According to (12), these vectors are generated by low-frequent sine functions, which motivates the *coarse grid correction* for reducing of the smooth error component. The coarse grid correction consists of the projection (restriction) of the smoothed residual, the solution of the resulting error equation on the coarse grid, and the correction of the current iterate on the fine grid by the interpolated solution of the coarse grid equation. The coarse grid equation has a double advantage over the fine grid equation. It is reduced in size and its condition number is smaller, cf. (19). In a fifth step oscillations in the error caused by the interpolation are damped by *postsmoothing*.

Algorithm 3 realizes one iteration sweep of the two-grid method

$$X^{(2)} \longleftarrow \mathrm{TG}(X^{(2)}, Y^{(2)})$$

applied to equation (21).

Algorithm 3 (Two-grid cycle)

1. $X^{(2)} \longleftarrow \mathrm{RICHARDSON}(X^{(2)}, A^{(2)}, E^{(2)}, Y^{(2)}, \nu_1, \omega)$
   (Presmoothing)
2. $Y^{(1)} \longleftarrow \mathcal{I}_{(2)}^{(1)} \left(Y^{(2)} - \mathcal{L}_{A^{(2)},E^{(2)}} X^{(2)}\right)$
   (Restriction of the residual)
3. Solve $\mathcal{L}_{A^{(1)},E^{(1)}} X^{(1)} = Y^{(1)}$ for $X^{(1)}$.
   (Coarse grid equation)
4. $X^{(2)} \longleftarrow X^{(2)} + \mathcal{I}_{(1)}^{(2)} X^{(1)}$
   (Coarse grid correction)
5. $X^{(2)} \longleftarrow \mathrm{RICHARDSON}(X^{(2)}, A^{(2)}, E^{(2)}, Y^{(2)}, \nu_2, \omega)$
   (Postsmoothing)

Actually, the two-grid method is a sequence of two-grid cycles, i.e., $X_{i+1} = \mathrm{TG}(X_i, Y)$ for $i = 0, 1, 2, \ldots$.

5 Convergence of the Two-Grid Method

Assuming $\alpha(\xi) \equiv 1$ throughout this section we analyze the convergence of the sequence of approximations $\{X_i\}_{i=0}^{\infty}$ generated by Algorithm 3. For simplicity we partially neglect the level superscripts. Furthermore, we make use of the notation $\mathcal{L}^{(l)} = \mathcal{L}_{A^{(l)},E^{(l)}}$ and $\lambda_{ij}^{(l)} = \lambda_{ij}^{\mathcal{L}^{(l)}}$ for $l = 1, 2$.

The convergence of the method is characterized by the reduction of the error $Z_i = \left(\mathcal{L}^{(2)}\right)^{-1} Y - X_i$, which is governed by the error transfer operator $\mathcal{M}$ of the two-grid cycle:

$$Z_{i+1} = \mathcal{M} Z_i.$$

This operator can be split up into the error transfer operators for presmoothing $\mathcal{S}^{\nu_1}$ (cf. (20)), coarse grid correction (Steps 2–4 in Algorithm 3) $\mathcal{C}$, and postsmoothing

$\mathcal{S}^{\nu_2}$:

$$\mathcal{M} = \mathcal{S}^{\nu_2}\mathcal{C}\mathcal{S}^{\nu_1}. \tag{22}$$

First, we analyze the operator $\mathcal{C}$ and afterwards the operator $\mathcal{S}$. To this end, we consider the decomposition of the error into components related to the eigenmatrices $v_i^{(2)}{v_j^{(2)}}^T$ of the operator $\mathcal{L}^{(2)}$. We restrict ourself to studying the mapping of the single eigenmatrices by the operators $\mathcal{C}$ and $\mathcal{S}$.

Operator $\mathcal{C}$. Steps 2–4 of Algorithm 3 generate the mapping

$$X \longrightarrow X + \mathcal{I}_{(1)}^{(2)}\left(\mathcal{L}^{(1)}\right)^{-1}\mathcal{I}_{(2)}^{(1)}\left(Y - \mathcal{L}^{(2)}X\right)$$

with the corresponding error transfer operator

$$\mathcal{C} = \mathcal{I} - \mathcal{I}_{(1)}^{(2)}\left(\mathcal{L}^{(1)}\right)^{-1}\mathcal{I}_{(2)}^{(1)}\mathcal{L}^{(2)}. \tag{23}$$

It has been shown (e.g. [Hac85]) that

$$I_{(2)}^{(1)}v_i^{(2)} = \begin{cases} \alpha_i v_i^{(1)} & : \quad i < n^{(1)}+1 \\ 0 & : \quad i = n^{(1)}+1 \\ \beta_{n^{(2)}+1-i}v_{n^{(2)}+1-i}^{(1)} & : \quad i > n^{(1)}+1 \end{cases}$$

for $i = 1, \ldots, n^{(2)}$ and

$$I_{(1)}^{(2)}v_i^{(1)} = \alpha_i v_i^{(2)} + \beta_i v_{n^{(2)}+1-i}^{(2)}$$

for $i = 1, \ldots, n^{(1)}$, where

$$\begin{aligned} \alpha_i &= \sqrt{2}(1-s_i^2), \\ \beta_i &= -\sqrt{2}s_i^2, \end{aligned}$$

with $s_i := s_i^{(2)} = \sin\frac{i\pi}{2(n^{(2)}+1)}$. Setting

$$t_i := \begin{cases} i & : \quad i = 1, \ldots, n^{(1)} \\ n^{(2)}+1-i & : \quad i = n^{(1)}+2, \ldots, n^{(2)} \end{cases}, \tag{24}$$

$$\gamma_i := \begin{cases} \alpha_i & : \quad i = 1, \ldots, n^{(1)} \\ \beta_{n^{(2)}+1-i} & : \quad i = n^{(1)}+2, \ldots, n^{(2)} \end{cases} \tag{25}$$

we obtain from (23) the equations

$$\begin{aligned} & \mathcal{C}v_i^{(2)}{v_j^{(2)}}^T \\ = \; & v_i^{(2)}{v_j^{(2)}}^T - \lambda_{ij}^{(2)}\mathcal{I}_{(1)}^{(2)}\left(\mathcal{L}^{(1)}\right)^{-1}\mathcal{I}_{(2)}^{(1)}\left(v_i^{(2)}{v_j^{(2)}}^T\right) \\ = \; & v_i^{(2)}{v_j^{(2)}}^T - \lambda_{ij}^{(2)}\mathcal{I}_{(1)}^{(2)}\left(\mathcal{L}^{(1)}\right)^{-1}(I_{(2)}^{(1)}v_i^{(2)})(I_{(2)}^{(1)}v_j^{(2)})^T \qquad (26) \\ = \; & v_i^{(2)}{v_j^{(2)}}^T \\ & -\frac{\lambda_{ij}^{(2)}\gamma_i\gamma_j}{\lambda_{t_i,t_j}^{(1)}}\left(\alpha_{t_i}v_{t_i}^{(2)} + \beta_{t_i}v_{n^{(2)}+1-t_i}^{(2)}\right)\left(\alpha_{t_j}v_{t_j}^{(2)} + \beta_{t_j}v_{n^{(2)}+1-t_j}^{(2)}\right). \qquad (27) \end{aligned}$$

Here, the last equation is valid only for $i, j \leq n^{(1)}$, but it follows from (26) that $\mathcal{C}v_i^{(2)}{v_j^{(2)}}^T = v_i^{(2)}{v_j^{(2)}}^T$ holds for $\max\{i,j\} = n^{(1)}+1$. For $i, j \leq n^{(1)}$, the quadruples

$$V_{[ij]} := \left\{ v_i^{(2)}{v_j^{(2)}}^T, \; v_i^{(2)}{v_{n^{(2)}+1-j}^{(2)}}^T, \; v_{n^{(2)}+1-i}^{(2)}{v_j^{(2)}}^T, \; v_i^{(2)}{v_{n^{(2)}+1-j}^{(2)}}^T \right\}$$

span four-dimensional invariant subspaces of the operator $\mathcal{C}$. Completing $V_{[ij]} := \left\{ v_i^{(2)} v_j^{(2)^T} \right\}$ for $\max\{i,j\} = n^{(1)}+1$ we obtain an orthogonal basis in $\mathbb{R}^{n^{(2)},n^{(2)}}$ (w.r.t. the scalar product $< M, N >= \mathrm{trace}(N^T M)$), and we have

$$\mathcal{C} V_{[ij]} = V_{[ij]} C_{[ij]} \tag{28}$$

for $i,j = 1,\ldots,n^{(1)}+1$. Choosing the indices in (27) properly and taking account of (24) and (25) we get the columns of the 4×4-matrices $C_{[ij]}$ for $i,j \le n^{(1)}$. In summary we have

$$C_{[ij]} = \begin{cases} I_4 - \frac{1}{\lambda_{ij}^{(1)}} c_{[ij]} c_{[ij]}^T \Lambda_{[ij]} & : \quad i,j \le n^{(1)} \\ 1 & : \quad \max\{i,j\} = n^{(1)}+1 \end{cases}, \tag{29}$$

where

$$\begin{aligned} c_{[ij]} &= [\alpha_i\alpha_j,\ \alpha_i\beta_j,\ \beta_i\alpha_j,\ \beta_i\beta_j]^T, \\ \Lambda_{[ij]} &= \mathrm{diag}\left(\lambda_{ij}^{(2)},\ \lambda_{i,n^{(2)}+1-j}^{(2)},\ \lambda_{n^{(2)}+1-i,j}^{(2)},\ \lambda_{n^{(2)}+1-i,n^{(2)}+1-j}^{(2)}\right), \\ \lambda_{ij}^{(1)} &= 4s_{2i}^2 + 4s_{2j}^2 - \frac{16}{3} s_{2i}^2 s_{2j}^2 \\ &= 16 s_i^2(1-s_i^2) + 16 s_j^2(1-s_j^2) - \frac{256}{3} s_i^2(1-s_i^2) s_j^2(1-s_j^2). \end{aligned} \tag{30}$$

Operator $\mathcal{S}$. It follows from (20) that

$$\mathcal{S} v_i^{(2)} v_j^{(2)^T} = (1 - \omega\lambda_{ij}^{(2)}) v_i^{(2)} v_j^{(2)^T},$$

which leads to

$$\mathcal{S} V_{[ij]} = V_{[ij]} S_{[ij]} \tag{31}$$

with

$$S_{[ij]} = \begin{cases} I_4 - \omega\Lambda_{[ij]} & : \quad i,j \le n^{(1)} \\ 1 - \omega\lambda_{ij}^{(2)} & : \quad \max\{i,j\} = n^{(1)}+1 \end{cases}. \tag{32}$$

From (22), (28), and (31) we obtain the following lemma.

Lemma 2 *Let $\alpha(\xi) \equiv 1$ in (6). Then*

$$\mathcal{M} V_{[ij]} = V_{[ij]} M_{[ij]} \tag{33}$$

for $i,j = 1,\ldots,n^{(1)}+1$ with $M_{[ij]} = S_{[ij]}^{\nu_2} C_{[ij]} S_{[ij]}^{\nu_1}$, where $C_{[ij]}$ and $S_{[ij]}$ are given by (29) and (32), respectively.

As a consequence of Lemma 2, the operator $\mathcal{M}$ represents a block diagonal matrix $\mathrm{bdiag}(C_{[ij]})$ with block size at most 4×4 in the basis of the eigenmatrices of the operator $\mathcal{L}^{(2)}$.

The convergence of the two-grid method depends on the spectral radius $\rho(\mathcal{M})$ and the operator norm $\|\mathcal{M}\|$, which is generated by the Frobenius norm. Because of (33) these quantities are given by

$$\begin{aligned} \rho(\mathcal{M}) &= \max_{1\le i,j\le n^{(1)}+1} \rho(M_{[ij]}), \\ \|\mathcal{M}\| &= \max_{1\le i,j\le n^{(1)}+1} \left\| M_{[ij]} \right\|. \end{aligned} \tag{34}$$

Both depend on the parameters $n^{(2)}, \omega, \nu_1, \nu_2$. In the sequel we show that there are fixed values for ω, ν_1, ν_2 such that an upper bound for the norm and the spectral radius of the operator $\mathcal{M}$ exists, which is independent of $n^{(2)}$ and less than 1.

For simplicity we set $\omega = 1/3$, although this is not necessary for the following analysis. Since $\rho(\mathcal{M}) \leq \|\mathcal{M}\|$, we restrict ourself to estimating the norm of $\mathcal{M}$. We distinguish between two cases.

Case 1 ($\max\{i,j\} = n^{(1)}+1$). Assuming $\nu_1+\nu_2 \geq 1$ we conclude from $\omega = 1/3$ and $\lambda_{ij}^{(2)} \in (2,4)$, which results from (17) and (18), that

$$\|M_{[ij]}\| = \left|\left(1-\omega\lambda_{ij}^{(2)}\right)\right|^{\nu_1+\nu_2} \leq \frac{1}{3}.$$

Case 2 ($i,j \leq n^{(1)}$). In this case, we partition the matrices $S_{[ij]}$ and $C_{[ij]}$ such that $S_{[ij]11}$ and $C_{[ij]11}$ are scalars:

$$\begin{aligned} S_{[ij]} &= \begin{pmatrix} S_{[ij]11} & 0 \\ 0 & S_{[ij]22} \end{pmatrix}, \\ C_{[ij]} &= \begin{pmatrix} C_{[ij]11} & C_{[ij]12} \\ C_{[ij]21} & C_{[ij]22} \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \|M_{[ij]}\| &\leq |C_{[ij]11}| \left|1-\omega\lambda_{ij}^{(2)}\right|^{\nu_1+\nu_2} \\ &\quad + \left\| \begin{array}{cc} 0 & S_{[ij]11}^{\nu_2} C_{[ij]12} S_{[ij]22}^{\nu_1} \\ S_{[ij]22}^{\nu_2} C_{[ij]21} S_{[ij]11}^{\nu_1} & S_{[ij]22}^{\nu_2} C_{[ij]22} S_{[ij]22}^{\nu_1} \end{array} \right\|. \end{aligned} \tag{35}$$

What follows is the estimation of the terms on the right hand side. When considering the matrix $C_{[ij]}$, we make use of the fact that each of its entries is a function of s_i^2 and s_j^2, i.e., $C_{[ij]} =: C(s_i^2, s_j^2)$. This, taken together with $0 < s_i^2, s_j^2 < 1/2$, allows us to deduce estimates for $C(\eta,\theta)$ with $\eta,\theta \in (0,1/2)$ instead of $C_{[ij]}$ itself.

1. We show that

$$0 < C_{11}(\eta,\theta) = 1 + \frac{(1-\eta)^2(1-\theta)^2(4\eta\theta-3\eta-3\theta)}{3\eta(1-\eta)+3\theta(1-\theta)-16\eta(1-\eta)\theta(1-\theta)} < 3/4, \tag{36}$$

where the left inequality holds iff

$$\begin{aligned} 0 &< 3\eta(1-\eta)+3\theta(1-\theta)-16\eta(1-\eta)\theta(1-\theta) \\ &\quad +(1-\eta)^2(1-\theta)^2(-3\eta-3\theta+4\eta\theta) \\ &= \eta(1-\eta)\underbrace{\left(3-8\theta(1-\theta)+(1-\eta)(1-\theta)^2(2\theta-3)\right)}_{\text{(i)}} \\ &\quad +\theta(1-\theta)\underbrace{\left(3-8\eta(1-\eta)+(1-\theta)(1-\eta)^2(2\eta-3)\right)}_{\text{(ii)}}. \end{aligned} \tag{37}$$

For Term (i) we have

$$\begin{aligned} 3-8\theta(1-\theta)+(1-\eta)(1-\theta)^2(2\theta-3) &> 3-8\theta(1-\theta)+(1-\theta)^2(2\theta-3) \\ &= \theta^2(1+2\theta) > 0. \end{aligned} \tag{38}$$

The first part of the proposition follows from (38) and the analogous estimate of Term (ii).

Furthermore, $C_{11}(\eta,\theta) < 3/4$ iff

$$\begin{aligned} 0 &< 16\eta(1-\eta)\theta(1-\theta) + 4(1-\eta)^2(1-\theta)^2(3\eta+3\theta-4\eta\theta) \\ &\quad -3\eta(1-\eta) - 3\theta(1-\theta) \\ &= \eta(1-\eta)\underbrace{\left(8\theta(1-\theta)+4(1-\eta)(1-\theta)^2(3-2\theta)-3\right)}_{\text{(iii)}} \\ &\quad +\theta(1-\theta)\underbrace{\left(8\eta(1-\eta)+4(1-\theta)(1-\eta)^2(3-2\eta)-3\right)}_{\text{(iv)}}. \end{aligned}$$

Term (iii) is positive, since

$$8\theta(1-\theta)+4(1-\eta)(1-\theta)^2(3-2\theta)-3 > 8\theta(1-\theta)+(1-\theta)(3-2\theta)-3 = 3\theta(1-2\theta) > 0.$$

Similarly, Term (iv) can be proved to be positive. Therefore, the right inequality in (36) holds as well.

2. As a consequence of $\omega = 1/3$, $\nu_1+\nu_2 \geq 1$, and $\lambda_{ij}^{(2)} \in (0,4)$, where the latter results from (14) and (16), we obtain $\left|1-\omega\lambda_{ij}^{(2)}\right|^{\nu_1+\nu_2} < 1$.

3. We show that the entries of $C_{12}(\eta,\theta)$, $C_{21}(\eta,\theta)$, and $C_{22}(\eta,\theta)$ are bounded. Because of (29), we can write for $r,s=1,\ldots,4$, $(r,s)\neq(1,1)$

$$(C(\eta,\theta))_{rs} = \frac{\eta p_{rs1}(\eta,\theta)+\theta p_{rs2}(\eta,\theta)}{\lambda^{(1)}(\eta,\theta)} = \underbrace{\frac{p_{rs1}(\eta,\theta)}{\frac{1}{\eta}\lambda^{(1)}(\eta,\theta)}}_{\text{(v)}} + \underbrace{\frac{p_{rs2}(\eta,\theta)}{\frac{1}{\theta}\lambda^{(1)}(\eta,\theta)}}_{\text{(vi)}}$$

with certain polynomials $p_{rs1}(\eta,\theta)$ and $p_{rs2}(\eta,\theta)$. Due to Lemma 1 (applied to $r\left(4\eta(1-\eta),4\theta(1-\theta)\right)$),

$$\lambda^{(1)}(\eta,\theta) := 16\eta(1-\eta)+16\theta(1-\theta)-\frac{256}{3}\eta(1-\eta)\theta(1-\theta)$$

is non-negative on $[0,1/2]^2$ and equal to zero iff $\eta=\theta=0$. Splitting the denominator in Term (v)

$$\frac{1}{\eta}\lambda^{(1)}(\eta,\theta) = \underbrace{16(1-\eta)}_{\substack{\longrightarrow 16 \\ \text{for } \eta\longrightarrow +0}} + \underbrace{16\frac{\theta(1-\theta)}{\eta}}_{\substack{>0 \\ \text{for } (\eta,\theta)\in(0,1/2)^2}} - \underbrace{\frac{256}{3}(1-\eta)\theta(1-\theta)}_{\substack{\longrightarrow 0 \\ \text{for } \theta\longrightarrow +0}}$$

shows that a set $\mathcal{U}_\epsilon = \left\{(\eta,\theta):\ 0<\eta,\theta;\ \eta^2+\theta^2<\epsilon^2\right\}$ with $0<\epsilon<1/2$ and a constant $c_1\in(0,16)$ exist such that $\frac{1}{\eta}\lambda^{(1)}(\eta,\theta)\geq c_1$. The denominator is continuous and positive on the closed set $[0,1/2]^2\backslash\mathcal{U}_\epsilon$. Therefore, a constant c_2 exists such that $\frac{1}{\eta}\lambda^{(1)}(\eta,\theta)\geq c_2>0$. The numerator of Term (v) is continuous and thus bounded on $[0,1/2]^2$, and consequently, an upper bound for the absolute value of Term (v) exists. This and the analogous argumentation for Term (vi) proves the proposition.

4. It follows from $\omega=1/3$ and $\lambda_{ij}^{(2)}\in(0,4)$ that $S_{[ij]11} = \left|1-\omega\lambda_{ij}^{(2)}\right| < 1$.

5. The absolute values of the main diagonal elements of the matrix $S_{[ij]22}$ are less than $1/3$, since (30), (32), and $\lambda^{(2)}_{i,n^{(2)}+1-j}$, $\lambda^{(2)}_{n^{(2)}+1-i,j}$, $\lambda^{(2)}_{n^{(2)}+1-i,n^{(2)}+1-j}\in(2,4)$, where the latter results from (17) and (18).

As a consequence of Points 3–5, the right term on the right hand side in (35) is less than any small, but positive constant, if ν_1 and ν_2 are chosen sufficiently

$n^{(1)}$	$n^{(2)}$	$\nu_1 = 1$ $\nu_2 = 0$	$\nu_1 = 1$ $\nu_2 = 1$	$\nu_1 = 2$ $\nu_2 = 0$
31	63	0.333066	0.110933	0.110933
63	127	0.333266	0.111067	0.111067
127	255	0.333317	0.111100	0.111100
255	511	0.333329	0.111108	0.111108
511	1023	0.333332	0.111110	0.111110

Table 1: Spectral radius $\rho(\mathcal{M})$ (with $\omega = 1/3$).

$n^{(1)}$	$n^{(2)}$	$\nu_1 = 1$ $\nu_2 = 0$	$\nu_1 = 1$ $\nu_2 = 1$	$\nu_1 = 2$ $\nu_2 = 0$
31	63	0.437640	0.312489	0.156721
63	127	0.453964	0.331048	0.157033
127	255	0.462473	0.341048	0.157109
255	511	0.466885	0.346166	0.157128
511	1023	0.469129	0.348746	0.157133

Table 2: Norm $\|\mathcal{M}\|$ (with $\omega = 1/3$).

large. Notice that this estimate is independent of $n^{(2)}$. Points 1 and 2 imply that the absolute value of the left term on the right hand side in (35) is less than 3/4. Because of these two facts a constant c_3 independent of $n^{(2)}$ exists such that $\|M_{[ij]}\| \le c_3 < 1$ for $i, j \le n^{(1)}$.

Combining the results of Case 1 and 2, we obtain the following

Theorem 1 *Let $\alpha(\xi) \equiv 1$ in (6). Then there exist constants ω, ν_1, ν_2, and c, which are independent of $n^{(2)}$, such that*

$$\rho(\mathcal{M}(\omega, \nu_1, \nu_2)) \le \|\mathcal{M}(\omega, \nu_1, \nu_2)\| \le c < 1$$

for the error transfer operator $\mathcal{M}$ of the two-grid method (Algorithm 3).

This theorem reflects the convergence only qualitatively. Tables 1 and 2 suggest that the rate of convergence is bounded independently of $n^{(2)}$ even if only one presmoothing sweep and no postsmoothing sweeps are performed.

Of course, a quantitative convergence analysis seems to be desirable, but we expect such an analysis to be much more complicated compared to the qualitative analysis presented here. Whereas the first part of our analysis (up to Lemma 2) is formally similar to the Fourier analysis of the two-grid method for the **two**-dimensional Poisson equation on the unit square [Hac85], in the second part, which is essentially the analysis of the 4×4-blocks $M_{[ij]}$, serious problems are caused by the more complicated structure of the matrix entries. Estimating the spectral radius or the norm of $M_{[ij]}$ turns out to be much harder than deriving the corresponding estimates for the Poisson equation (cf. [Hac85]).

6 A Multi-Grid Method

The recursive application of the two-grid method leads to a multi-grid method. More precisely, the coarse grid equation in Step 3 of Algorithm 3 is solved approximately by a two-grid or a multi-grid method involving further coarse grids. In the end, the dimension of the Lyapunov equations on the coarsest grid is sufficiently small to solve these equations by direct methods [BS72, GLAM92, Ham82, Pen97]. Alternatively, these equations can be solved by standard iteration methods [Smi68, Sta89, SN91, Sta91, Wac92]. In this section we consider the type of equation

$$\mathcal{L}_{A+BK,E}X = (A+BK)^T XE + E^T X(A+BK) = Y, \tag{39}$$

which is to be solved in each sweep of the Newton method (Algorithm 1).

For simplicity we define the multi-grid method recursively. On Level l, the matrices $A^{(l)}$, $B^{(l)}$, and $E^{(l)}$ are provided by the discretization. The matrices $K^{(l)}$ are defined recursively as

$$K^{(l)} := K^{(l+1)} \left(I^{(l)}_{(l+1)} \right)^T \tag{40}$$

for $l = l_{max} - 1, \ldots, 1$. In contrast to the two-grid method, steering parameters $\mu^{(l)}$ are included, which determine the number of multi-grid sweeps starting on Level $l-1$ for solving the Lyapunov equation of this level approximately. If these numbers are equal on each level, i.e., $\mu^{(l)} = \mu$ for $l = 2, \ldots, l_{max}$, we get a *V-cycle* for $\mu = 1$ and a *W-cycle* for $\mu = 2$ [Hac85].

Algorithm 4 realizes one sweep of the multi-grid method

$$X^{(l)} \longleftarrow \mathrm{MG}^{(l)}(Y^{(l)}, X^{(l)})$$

starting on Level l.

Algorithm 4 (Multi-grid cycle on Level $l \geq 2$)

1. $X^{(l)} \longleftarrow$RICHARDSON$(X^{(l)}, Y^{(l)}, A^{(l)} + B^{(l)}K^{(l)}, E^{(l)}, \nu_1^{(l)}, \omega^{(l)})$

2. $Y^{(l-1)} \longleftarrow \mathcal{I}^{(l-1)}_{(l)}(Y^{(l)} - \mathcal{L}_{A^{(l)}+B^{(l)}K^{(l)},E^{(l)}}X^{(l)})$

IF $l - 1 = 1$

   3. $X^{(1)} \longleftarrow \mathcal{L}^{-1}_{A^{(1)}+B^{(1)}K^{(1)},E^{(1)}}Y^{(1)}$.
      (Exact solution on the coarsest grid)

ELSE

   4. $X^{(l-1)} \longleftarrow 0$

   FOR $i = 1, \ldots, \mu^{(l)}$

      5. $X^{(l-1)} \longleftarrow \mathrm{MG}^{(l-1)}(X^{(l-1)}, Y^{(l-1)})$

   END FOR
   (Approximate solution)

END IF

6. $X^{(l)} \longleftarrow X^{(l)} + \mathcal{I}^{(l)}_{(l-1)}X^{(l-1)}$

7. $X^{(l)} \longleftarrow$ RICHARDSON$(X^{(l)}, Y^{(l)}, A^{(l)} + B^{(l)}K^{(l)}, E^{(l)}, \nu_2^{(l)}, \omega^{(l)})$

7 Complexity of the Multi-Grid Cycle

We provide estimates for the computational cost and the storage cost of the multi-grid cycle. For simplicity we assume $\nu_1^{(l)} = \nu_1$, $\nu_2^{(l)} = \nu_2$, and $\mu^{(l)} = \mu \leq 3$ for $l = 2, \ldots, l_{max}$

Taking sparsity into account the computational cost of Algorithm 4 without Steps 4 and 5 is $c_4 \left(n^{(l)}\right)^2 + \mathcal{O}\left(n^{(l)}\right)$, where c_4 is a constant depending on ν_1 and

ν_2. Since $n^{(l+1)} \approx 2n^{(l)}$ for $n^{(l)} \gg 1$, the cost for performing one multi-grid cycle is about

$$c_4 n^2 \left(1 + \left(\frac{\mu}{4}\right) + \left(\frac{\mu}{4}\right)^2 + \dots\right) = \frac{4c_4}{4-\mu} n^2.$$

The memory requirement for the matrices on Level l is $c_5 \left(n^{(l)}\right)^2 + \mathcal{O}\left(n^{(l)}\right)$ with a constant c_5 depending on the implementation. This yields

$$c_5 n^2 \left(1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \dots\right) = \frac{4c_5}{3} n^2$$

as an estimate for the total memory cost.

In summary the computational cost and the memory requirement of the multi-grid method are of order the number of unknowns in the equation on the finest grid.

8 Numerical Experiments

The convergence of the two-grid method has been investigated in Section 4. Therefore, in this section we focus on the multi-grid method.

Concerning the choice of the parameters in the model problem (1)-(5) we consider the following two examples.

Example 1. $\alpha(\xi) \equiv 1$,

$$b(\xi) = \begin{cases} 100 & : \ \xi \in (1/6, 2/6) \\ 0 & : \ \text{otherwise} \end{cases}, \qquad \text{and} \qquad c(\xi) = \begin{cases} 10 & : \ \xi \in (4/6, 5/6) \\ 0 & : \ \text{otherwise} \end{cases}.$$

Example 2. Here,

$$\alpha(\xi) = \begin{cases} 1 & : \ \xi \in (0, 1/3) \\ 1/3 & : \ \text{otherwise} \end{cases},$$

whereas the other parameters coincide with Example 1.

We have performed numerical experiments for the following two problems.

Problem 1. Solution of the equation $\mathcal{L}_{A,E} X = Y$. The right hand side is chosen as $Y = (1/n) e e^T$ with $e = (1, \dots, 1)^T \in \mathbb{R}^n$, and the zero matrix is used as initial guess.

Problem 2. Solution of the sequence of equations $\mathcal{L}_{A+BK_i,E} X = Y$, which arise from the Newton method. The right hand side Y is provided by the outer Newton iteration. Again, the initial guesses are zero matrices.

The steering parameters of the multi-grid method are chosen independently of the level index, i.e., $\mu^{(l)} = \mu$, $\nu_1^{(l)} = \nu_1$, $\nu_2^{(l)} = \nu_2$, and $\omega^{(l)} = \omega$ for $l = 2, \dots, l_{max}$. If not mentioned otherwise, we set $\mu = 1$ (V-cycle) and $\nu_1 = \nu_2 = 1$. We set $\omega = 1/3$ for Example 1, which has been motivated heuristically in Section 3. For Example 2 an approximation $\omega = 0.4212$ for the optimal relaxation parameter is gained from Test 1. Depending on the kind of test, we use either the maximal number of iterations $maxit$ or the relative residual norm ($\|Y - \mathcal{L}_{A,E} X_i\|_F \,/\, \|Y\|_F \leq tol$) as stopping criterion.

In the tests described below several aspects of the multi-grid method are considered such as the influence of the relaxation parameter and the dependence of the number of iterations on the dimension of the problem.

In Test 1 we investigate the impact of the relaxation parameter ω on the convergence of the multi-grid method. We use the norm of the residual after a certain number $maxit$ of iterations to characterize the speed of convergence. This criterion is sensible, since the convergence is asymptotically linear, which can be explained by the fact that the multi-grid iteration operator is stationary, in contrast to, for example, CG-like methods.

Test 1 Problem 1; $n^{(1)} = 2$, $l_{max} = 7$, $n = 191$; stopping criterion: $maxit = 10$. See Figure 1.

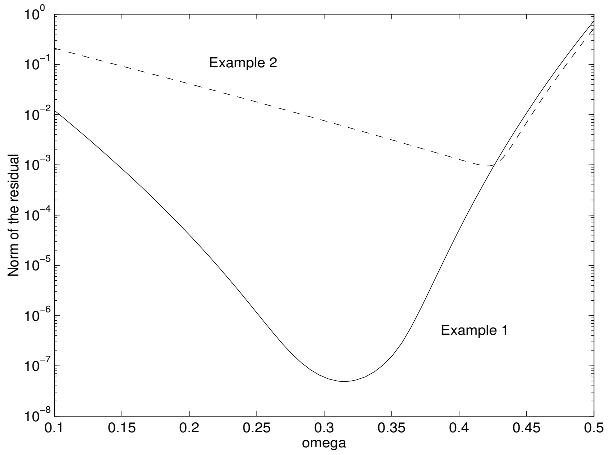

Figure 1: Test 1. Norm of the residual $\|Y - \mathcal{L}_{A,E}X_i\|_F$ after $i = 10$ iterations depending on ω.

Test 1 indicates that the multi-grid method converges for values of ω varying over a relatively wide range, although the choice of ω is crucial for attaining a (nearly) optimal speed of convergence.

Test 2 shows the dependence of the number of multi-grid iterations on the number of levels involved. We start with a fixed initial partitioning of the interval Ω and increase the number of levels l_{max} successively.

Test 2 Problem 1; $n^{(1)} = 2$; stopping criterion: $tol = 10^{-10}$. See Table 3.

			Ex.1			Ex.2	
l_{max}	n	$\nu_1 = 1$ $\nu_2 = 0$	$\nu_1 = 1$ $\nu_2 = 1$	$\nu_1 = 2$ $\nu_2 = 0$	$\nu_1 = 1$ $\nu_2 = 0$	$\nu_1 = 1$ $\nu_2 = 1$	$\nu_1 = 2$ $\nu_2 = 0$
4	23	21	13	15	65	33	34
5	47	22	14	16	68	34	36
6	95	23	14	17	70	35	37
7	191	24	14	18	72	35	38
8	383	25	15	19	74	36	39

Table 3: Test 2. Numbers of iterations depending on l_{max} for different numbers of smoothing steps.

Test 2 shows that the V-cycle of the multi-grid method converges, even if only one presmoothing step per multi-grid cycle is performed. Although the dimension of the problem is increasing relatively fast, the number of iterations remains nearly constant in each of the three cases. This indicates the existence of a bound for the rate of convergence that is independent of the dimension of the problem.

In Test 3 we consider the Lyapunov equations to be solved within the Newton method, where the dimension of the problem varies. We investigate the dependence of the number of multi-grid iterations on l_{max} in the Newton method. To this end, the number of levels is increased analogously to Test 2.

Test 3 Problem 2; $n^{(1)} = 2$; stopping criterion: $tol = 10^{-10}$. In each case, the norm of the Riccati residual $\|R(P_i)\|_F$ has been reduced by a factor 10^{-12} within 5 Newton steps (NS). See Table 4.

	l_{max}	n	NS 1	NS 2	NS 3	NS 4	NS 5	Total
Example 1	4	23	13	22	21	20	20	96
	5	47	14	22	20	20	20	96
	6	95	14	22	20	20	20	96
	7	191	14	22	20	20	20	96
	8	383	15	22	20	20	20	97
Example 2	4	23	34	41	39	38	37	189
	5	47	35	40	39	37	37	188
	6	95	36	40	39	37	37	189
	7	191	36	40	39	37	37	189
	8	383	37	40	39	37	37	190

Table 4: Test 3. Numbers of multi-grid iterations depending on l_{max} in Newton Steps 1–5.

Although the dimension of the problem is increasing relatively fast, the number of multi-grid iterations remains nearly constant in each Newton step. It might be possible to solve the Riccati equation with less multi-grid iterates using other stopping criteria for the multi-grid iterations within the Newton iteration. For example, we could solve the Lyapunov equations in the first Newton steps less accurate than those in the last steps. Unfortunately, this involves the danger of a convergence in the Newton method towards one of the non-stabilizing solutions of the Riccati equation. However, this has not been observed in our numerical experiments.

9 Comparison with the Method by Rosen and Wang

Algorithm 4 differs in a number of points from the multi-grid method due to Rosen and Wang [RW95].

The first essential difference is that our method is based on the generalized Lyapunov equation $\mathcal{L}_{A,E}X = Y$, which results immediately from the discretization. In contrast, this equation is transformed into a standard Lyapunov equation $\mathcal{L}_{AE^{-1},I_n}X = E^{-T}YE^{-1}$ in the Rosen-Wang method involving the matrix E^{-1}. Although this matrix is well-conditioned, this may cause problems for a parallelization and it produces fill-in when the method is extended to the two- or three-dimensional case. As a consequence, both multi-grid methods differ in the components used, e.g., the interpolation and restriction operators.

A second basic difference lies in the choice of the smoothing method. Whereas we make use of basic iterative methods for systems of linear equations, the Smith iteration [Smi68] is utilized in the Rosen-Wang method. The main drawback of the Smith iteration is that the inverses of sparse matrices are involved into the computation, which again leads to fill-in and problems for parallelization.

In contrast, our multi-grid method is free of matrix factorizations and fill-in. Its implementation is comparatively simple and it is based on matrix-matrix-products, which can be efficiently computed in parallel. The spectral properties of the Lyapunov operator and the convergence analysis of the two-grid method are strongly affected by the aforementioned differences in the choice of the components. Moreover, we have proved a convergence result for Algorithm 3, whereas the analysis for the Rosen-Wang method is restricted to the derivation of the structure of the two-grid error transfer operator, which is comparable to Lemma 2.

The rates of convergence of both methods are almost identical. This is shown in Table 5, which compares the spectral radii of the error transfer operators of both two-grid methods.

$n^{(2)}$	Rosen/Wang	Algorithm 3
19	0.31834	0.33060
39	0.32957	0.33265
59	0.33165	0.33303
79	0.33239	0.33316
99	0.33273	0.33322

Table 5: Spectral radii of the error transfer operators of the two-grid method by Rosen and Wang [RW95] and Algorithm 3 ($\omega = 1/3$, $\nu_1 = 1$, $\nu_2 = 0$).

10 Conclusions

In this paper we have proposed a multi-grid method for the solution of a certain class of structured Lyapunov equations. The efficiency of the method has been demonstrated by numerical experiments.

The essential advantage of our method is that no matrices are factorized. Except for the Lyapunov solver on the coarsest grid, the algorithm consists of additions and multiplications of matrices by (sparse) matrices and thus offers a great potential for a parallelization. Like for most other iterative methods, the computational cost per iteration is of order the number of unknowns in the equation. For the two-grid method we have proved that the number of iterations required to attain a prescribed accuracy is $\mathcal{O}(1)$. Numerical experiments (Tests 2 and 3 in Section 8) indicate that this is also the case for the multi-grid method. This is an important advantage over most other iterative methods, e.g., Richardson iteration [Var62] or CG method [HS52], where, as a consequence of (19), the order of the number of iterations depends on n.

An intrinsic disadvantage of multi-grid methods is their dependence on steering parameters. In particular, the choice of the relaxation parameter ω is crucial with respect to the speed of convergence. On the other hand, (nearly) optimal values can be gained analytically only for simple model problems, e.g., $\alpha(\xi) \equiv 1$, $K = 0$. Multi-grid methods are closely coupled to the structure of the underlying problem and thus they are applicable to a restricted class of problems. Moreover, its implementation is relatively complicated compared to other iterative methods, e.g., CG-like methods.

As we have mentioned, it is still an open question which stopping strategy for the multi-grid iterations within the outer Newton iteration is optimal. More precisely, it is not known which accuracy is required for the solution of the Lyapunov equation to make sure the Newton method converges to the *stabilizing* solution of the Riccati equation. Another problem, which may occur in context with the Newton method, is the dominance of the rank-1-term in the term $A+BK$ in (39). This may affect the symmetry and spectral properties of the operator $\mathcal{L}_{A,E}$ by a degree, which makes the multi-grid method converge slowly or even diverge. From the analytical point of view, the question for rigorous bounds for the rate of the convergence of the two-grid and multi-grid methods remains open.

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