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Numerische Simulation auf massiv parallelen Rechnern

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The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges

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Abstract. This paper is concerned with a specific finite element strategy for solving elliptic boundary value problems in domains with corners and edges. First, the anisotropic singular behaviour of the solution is described. Then the finite element method with anisotropic, graded meshes and piecewise linear shape functions is investigated for such problems; the schemes exhibit optimal convergence rates with decreasing mesh size. For the proof, new local interpolation error estimates for functions from anisotropically weighted spaces are derived. Finally, a numerical experiment is described, that shows a good agreement of the calculated approximation orders with the theoretically predicted ones.

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Contents

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1 Introduction

Consider the Dirichlet problem for a second order elliptic equation in a three-dimensional polyhedral domain Ω,

$$-\sum_{i,j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where the coefficients $a_{ij} = a_{ji}$ are constant, $\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \geq C_0 > 0$ for all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ such that $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, and the right hand side f satisfies

$$f \in L^p(\Omega) \text{ for some } p > 2. \tag{1.2}$$

If Ω is not convex then the solution has in general singular behaviour near edges and corners. It is well known that these singularities lead to a low approximation order of the standard finite element method.

Two-dimensional problems with corner singularities can be treated with a certain mesh refinement near these corners in order to improve the approximation order [6, 7, 26, 28]. This approach has been generalized to the three-dimensional case in [2, 6, 22]. The isotropic mesh refinement as described in these papers seems to be appropriate near corners, however, it leads to overrefinement near edges.

The anisotropic structure of an edge is reflected by an anisotropic behaviour of the solution near the edge: The singular part of the solution can be represented by a convolution of some two-dimensional singularity functions with a regular function in the third direction [23, 24]. Thus it seems to be natural to treat edge singularities with meshes which are graded perpendicularly to the edge and quasi-uniform in the edge direction. However, such meshes are anisotropic in the sense that elements in the refinement region have an aspect ratio which is growing to infinity for $h \to 0$, h is the global mesh size. In [1, 5] it is shown for the Poisson equation that this strategy is successful. But in these papers problems with only edge singularities were considered, corner singularities were excluded.

Our aim is to treat more general operators and both corner and edge singularities. The idea is quite obvious, we want to combine anisotropic mesh refinement near singular edges with isotropic refinement near corners. The main difficulty is to describe and to construct the meshes in the transition from anisotropy to isotropy. A complication is that corner singularities can be stronger or weaker than edge singularities. In [6], where isotropic mesh refinement was considered, this was circumvented by controlling the refinement with the strongest singularity appearing in the problem under consideration. We try to avoid this by allowing different refinement parameters in different regions. Moreover, in the previous paper [1] on anisotropic mesh refinement, prismatic domains were considered only. The tensor product character of such domains was used to describe the mesh. But these orthogonalities are no longer available because we want to treat general polyhedral domains. Finally, we want to assume data with low regularity. We use right hand sides $f \in L^p(\Omega)$ with some $p > 2$, see (1.2). The case $p = 2$ did not work but (1.2) is considerably weaker than the assumption in [1].

To explain our approach subdivide Ω into a finite number of disjoint tetrahedral subdomains, $\overline{\Omega} = \bigcup_{\ell=1}^{L} \overline{\Lambda_\ell}$, such that each subdomain contains at most one singular edge and at most one singular corner. In this way we localize the problem and reduce all considerations to few standard cases. Note that the singularities are of local nature only. In Section 2, we describe the properties of the solution u in suitable weighted Sobolev spaces.

Section 3 is devoted to the meshing in the subdomains and the proof that the sub-meshes fit together. We describe the mesh by a set of properties which are suited for both proving the optimal approximation order and constructing such meshes for general domains Ω. We tried to keep the properties as simple as possible; therefore we did not try to give a minimal set of conditions and we allowed some kind of overrefinement.

Local interpolation error estimates are derived in Section 4. In the following section they are fitted together to interpolation error estimates in the subdomains Λ_ℓ and then in the domain Ω. Via the Céa lemma we can conclude the estimate of the finite element error in the $W^{1,2}(\Omega)$-norm. With a numerical test we complete our paper.

To refer to some more literature we mention that there are several approaches to cope with singularities. Regularity investigations go back to the pioneering work of Kondrat'ev [18]. The theory has been developed then in two ways, namely the characterization of the solution by weighted Sobolev spaces it belongs to, and by representation formulae. For an introduction and overview on this topic see for example [15, 16, 19].

In any case the crucial point is the knowledge of the singularity exponents; they are also of interest in the paper at hand because they determine the mesh grading. For edges the exponents can in general be given analytically, but for corners an eigenvalue problem for the Laplace-Beltrami operator has to be solved numerically, see for example [10, 20, 29].

In our paper we study the numerical solution of the boundary value problem (1.1), (1.2), by a finite element method using anisotropic mesh refinement. Another method is the boundary element method with anisotropic mesh refinement, see for example [27, 29]. The singular function method is well developed for two-dimensional problems [12, 30], but it is hard to handle in the case of edge singularities [9, 21]. Some authors calculate the leading singular part of the solution explicitly. Additionally to the solution of the eigenvalue problem mentioned above this includes the computation of the corresponding coefficient, the so-called stress intensity factor [8, 13].

Finally, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of $\mathcal{T}_h$ and of the function under consideration) such that $a \le C_2 b$ and $C_1 b \le a \le C_2 b$, respectively.

2 Regularity results

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain whose boundary Γ consists of plain faces. Over this domain Ω, we consider the boundary value problem (1.1) whose variational formulation is given by:

$$\textit{Find } u \in \overset{o}{H}{}^1(\Omega) \textit{ such that } a_\Omega(u,v) = (f,v) \textit{ for all } v \in \overset{o}{H}{}^1(\Omega). \tag{2.1}$$

The bilinear form $a_\Omega(.,.)$ and the linear form $(f,.)$ are defined by

$$a_\Omega(u,v) := \int_\Omega \sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v \, d\underline{x}, \tag{2.2}$$

$$(f,v) := \int_\Omega f v \, d\underline{x}. \tag{2.3}$$

We use the abbreviations ∂_i for $\frac{\partial}{\partial x_i}$ and ∂_{ij} for $\partial_i\partial_j$. The space $\overset{o}{H}{}^1(\Omega)$ is defined, as usual, by $\overset{o}{H}{}^1(\Omega) := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. The datum f is assumed to be in $L^p(\Omega)$ for some $p \ge 2$. The restriction $p > 2$, see (1.2), is necessary only in Section 5. $L^p(.)$ $(1 \le p \le \infty)$ are the usual Lebesgue spaces, $W^{s,p}(.)$ $(s \ge 0,\ 1 \le p \le \infty)$ the Sobolev(-Slobodetskiĭ) spaces (sometimes we write $W^{0,p}(.)$ for $L^p(.)$), and $H^s(.) := W^{s,2}(.)$. — Note that the conditions of the Lax–Milgram lemma are satisfied; thus the solution $u \in \overset{o}{H}{}^1(\Omega)$ of problem (2.1) exists and is unique.

It is well known [15, 16, 18, 23] that u contains edge and/or vertex singularities. Since there exists a linear change of variables $\underline{y} = B\underline{x}$ which transforms the problem (1.1) into the Laplace operator with Dirichlet boundary conditions in another polyhedral domains Ω', it suffices to describe the singularities of the Laplace operator. Moreover in view of their local character, we simply need to describe them in a neighbourhood of one vertex S of Ω. Without loss of generality, we may suppose that S is the origin of our Cartesian system of coordinates. Let C_S be the infinite polyhedral cone of $\mathbb{R}^3$ which coincides with Ω in a neighbourhood of S; we set $G_S = C_S \cap S^2(S)$, the intersection of C_S with the unit sphere centered at S. Then the vertex singular exponent related to S is given by $\lambda_{v,S} = -\frac{1}{2} + \sqrt{\lambda_{S,1} + \frac{1}{4}}$, where $\lambda_{S,k} > 0$, $k \in \mathbb{N}^* = \{1, 2, \ldots\}$, are the eigenvalues (in increasing order) of the (positive) Laplace-Beltrami operator Δ' on G_S with Dirichlet boundary conditions (the associated eigenvector will be denoted by $\varphi_{S,k}$). The vertex S will be called singular if $\lambda_{v,S} < 2 - \frac{3}{p}$ (note that we always have $\lambda_{v,S} > 0$). On the other hand, for any edge $A_{S,j}$ adjacent to S, $1 \leq j \leq J_S$ (J_S denotes the number of such edges), the edge singular exponent is simply $\lambda_{e,S,j} = \frac{\pi}{\omega_{S,j}}$, where $\omega_{S,j}$ is the interior angle between the two faces containing $A_{S,j}$. Similarly the edge $A_{S,j}$ will be called singular if $\lambda_{e,S,j} < 2 - \frac{2}{p}$ (remark that $\lambda_{e,S,j} > \frac{1}{2}$). Recall that for the general system (1.1), we need first to perform the above change of variables which is changing the cone C_S and the angles $\omega_{S,j}$.

Recall from the introduction that Ω is supposed to be divided into a finite number of disjoint tetrahedral subdomains: $\overline{\Omega} = \bigcup_{\ell=1}^{L} \overline{\Lambda_\ell}$, such that each subdomain contains at most one singular edge and at most one singular corner. For any $\ell = 1, \ldots, L$, we set $\lambda_v^{(\ell)} = \lambda_{S,v}$ if Λ_ℓ contains one singular vertex S of Ω, otherwise we take $\lambda_v^{(\ell)} = +\infty$ and $\lambda_e^{(\ell)} = \lambda_{e,S,j}$ if Λ_ℓ contains one singular edge $A_{S,j}$ of Ω, otherwise we take $\lambda_e^{(\ell)} = +\infty$.

Further, define in each subdomain Λ_ℓ ($\ell = 1, \ldots, L$) a Cartesian coordinate system $(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)})$ with the following properties:

- One vertex of Λ_ℓ is located in the origin. In particular, if Λ_ℓ possesses a refinement vertex, then this one is chosen.
- One edge of Λ_ℓ is contained in the $x_3^{(\ell)}$-axis. In particular, if Λ_ℓ possesses a refinement edge, then this one is used.

In order to describe anisotropic regularities of the solution $u \in \overset{\circ}{H}{}^1(\Omega)$ of problem (2.1), we need to introduce some weighted Sobolev space of Kondrat'ev type defined as follows and already introduced in [14, 25] (see also [22] for slightly different spaces):

Definition 2.1 *Let Λ be a fixed subdomain of Ω. For a integer $k \geq 0$, $1 \leq p \leq \infty$ and two real numbers β, δ, we set*

$$V_{\beta,\delta}^{k,p}(\Lambda) := \{v \in \mathcal{D}'(\Lambda) : \ R^{\beta-k+|\alpha|}\theta^{\delta-k+|\alpha|}D^\alpha v \in L^p(\Lambda), \forall \alpha \in \mathbb{N}^3 : |\alpha| \leq k\},$$

where $R(x)$ is the distance from x to the vertices of Ω, $r(x)$ is the distance from x to the edges of Ω and $\theta(x) := \frac{r(x)}{R(x)}$ is the "angular" distance from x to the edges of Ω. It is a Banach space for the norm

$$\|v; V_{\beta,\delta}^{k}(\Lambda)\| := \left\{ \sum_{|\alpha|\leq k} \|R^{\beta-k+|\alpha|}\theta^{\delta-k+|\alpha|}D^\alpha v; L^p(\Lambda)\|^p \right\}^{\frac{1}{p}}. \tag{2.4}$$

Theorem 2 of [25] (see also Theorem 2.3 of [6]) implies that the solution $u \in \overset{\circ}{H}{}^1(\Omega)$ of problem (2.1) with a datum in $L^p(\Omega)$ has the regularity $u \in V_{\beta,\delta}^{2,p}(\Lambda_\ell)$ for any $\beta, \delta \geq 0$ such that

$$\beta > 2 - \frac{3}{p} - \lambda_v^{(\ell)}, \quad \delta > 2 - \frac{2}{p} - \lambda_e^{(\ell)}.$$

Unfortunately, we have no extra information for the derivatives in the direction of one singular edge (if Λ_ℓ contains it). In other words, the above result gives no anisotropic regularities. Therefore our goal is to improve such results in order to get them.

As already explained before we are reduced to consider the Laplace operator in Ω, so that, until further notice, we suppose that $a_{ij} = \delta_{ij}$.

Since we are working with data in $L^p(\Omega)$ with p not necessarily equal to 2 and since Ω may have singular edges, it is not direct that $\Delta(\eta u)$ belongs to $L^p(\Omega)$ for any cut-off function η. Therefore we first solve this localization problem along the edges.

Lemma 2.2 *Let ξ be a fixed interior point of one edge A of Ω and let η be a cut-off function such that $\eta \equiv 1$ in a neighbourhood of ξ and $\eta \equiv 0$ in a neighbourhood of the vertices and the other edges. Take the x_3-axis parallel to the edge A. Then*

$$\eta \partial_3 u \in L^{p'}(\Omega) \quad \forall p' < 6. \tag{2.5}$$

Proof Set $V = \operatorname{supp} \eta \cap \Omega$. Let us consider a certain $v \in \mathcal{D}(V)$ and fix the unique solution $y \in \overset{\circ}{H}{}^1(V)$ of

$$a_V(y, w) = \int_V vw \, d\underline{x} \quad \forall w \in \overset{\circ}{H}{}^1(V). \tag{2.6}$$

Introduce for $h > 0$ the finite difference operator

$$\delta_h v(x_1, x_2, x_3) = \frac{v(x_1, x_2, x_3 + h) - v(x_1, x_2, x_3)}{h}.$$

Then for $h > 0$ small enough, we clearly have $\delta_h(\eta u) \in \overset{\circ}{H}{}^1(V)$. Applying (2.6) with $w = \delta_h(\eta u)$, we get

$$\int_V v\delta_h(\eta u) \, d\underline{x} = a_\Omega(\tilde{y}, \delta_h(\eta u)),$$

where $\tilde{y}$ means the extension of y by 0 outside V which is still in $\overset{\circ}{H}{}^1(\Omega)$, because V has a Lipschitz boundary. Using a change of variable and the symmetry of a_Ω, we arrive at

$$\int_V v\delta_h(\eta u) \, d\underline{x} = a_\Omega(\delta_h(\tilde{y}), \eta u) = \int_\Omega g\delta_h(\tilde{y}) \, d\underline{x},$$

because ηu is solution of (1.1) with the datum $g = -\Delta(\eta u) := \eta f - 2\nabla\eta \cdot \nabla u - u\Delta\eta \in L^2(\Omega)$. Using the Cauchy-Schwarz inequality and Lemma 2.2.2.2 of [16], we get

$$\left| \int_V v\delta_h(\eta u) \, d\underline{x} \right| \lesssim \|g; L^2(\Omega)\| \|\tilde{y}; H^1(\Omega)\|. \tag{2.7}$$

Finally as y is solution of (2.6), we have

$$\|y; H^1(V)\| \lesssim \|v; H^{-1}(V)\| \lesssim \|v; L^{q'}(V)\|,$$

for $q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$, since the Sobolev embedding theorem yields $\overset{\circ}{H}{}^1(V) \hookrightarrow L^{p'}(V)$, for all $p' < 6$. Inserting this estimate into (2.7), we obtain

$$\left| \int_V \delta_h(\eta u) v \, d\underline{x} \right| \lesssim \|g; L^2(\Omega)\| \|v; L^{q'}(V)\|. \tag{2.8}$$

Because $L^{q'}(V)$ is the dual space of $L^{p'}(V)$ this means that

$$\|\delta_h(\eta u); L^{p'}(V)\| \lesssim \|g; L^2(\Omega)\|,$$

because (2.8) holds for all $v \in \mathcal{D}(V)$. Finally since $\delta_h(\eta u) \to \partial_3(\eta u)$ in $\mathcal{D}'(V)$ as $h \to 0$, we get the conclusion. $\square$

Corollary 2.3 *Let the assumption of Lemma 2.2 be satisfied, and take the cut-off function η introduced in Lemma 2.2 in the tensorial form $\eta(x_1, x_2, x_3) = \eta_{1,2}(x_1, x_2)\eta_3(x_3)$. Then*

$$\Delta(\eta u) \in L^p(\Omega), \text{ if } p < 6. \tag{2.9}$$

Proof Using the Leibniz rule we have $\Delta(\eta u) = u\Delta\eta + 2\nabla\eta \cdot \nabla u + \eta\Delta u$ and it remains to show that $\nabla\eta \cdot \nabla u \in L^p(\Omega)$. In the interesting strip where $\nabla\eta$ does not vanish we are either far from the edge or $\partial_1\eta = \partial_2\eta = 0$. Thus only $\partial_3\eta\partial_3 u$ plays a role, and the previous lemma gives the assertion. □

The previous corollary shows that for $p < 6$, we can always localize our problem in a neighbourhood of an edge. Therefore, we can apply Theorem 2.2 of [5] to ηu to get the (anisotropic) edge regularity of ηu and then of u. This is summarized in the next theorem. Note that $\theta \sim r$ in V.

Theorem 2.4 *Let A be a fixed edge of Ω and let η be a cut-off function as introduced in Corollary 2.3. Denote by λ_e the edge singular exponent associated with A. Then for $p < 6$, one has*

$$\eta u \in V^{2,p}_{0,\delta}(\Omega), \tag{2.10}$$

for any $\delta \geq 0$ such that $\delta > 2 - \frac{2}{p} - \lambda_e$. If moreover $1 - \frac{2}{p} < \lambda_e$, then

$$\partial_3(\eta u) \in V^{1,p}_{0,0}(\Omega). \tag{2.11}$$

This theorem gives the desired regularity near the edges. We now attack the same problem near a fixed corner S. For the sake of simplicity, we drop the dependence with respect to S if no confusion is possible. Let us now fix a cut-off function χ such that $\chi \equiv 1$ in a neighbourhood of S and $\chi \equiv 0$ in a neighbourhood of the other vertices. We further suppose that $\chi \equiv \chi(R)$, that means, χ depends only on R (here R means the distance to S).

Lemma 2.5 *Assume that $1 - \frac{2}{p} < \lambda_{e,S,j}$, for all $j \in \{1, \ldots, J_S\}$ and $p < 6$. Then*

$$\Delta(\chi u) \in L^p(C_S). \tag{2.12}$$

Proof Direct consequence of Theorem 2.4 using the Leibniz rule (the hypothesis on p implying that we may choose $\delta < 1$). □

We are now in position to apply the results of section 7 of [17] to χu, the solution of

$$-\Delta(\chi u) = g \in L^p(C_S), \qquad \chi u = 0 \quad \text{on } \partial C_S, \tag{2.13}$$

in the cone C_S. Using these results in our framework, we get the following decomposition:

Theorem 2.6 *Assume that $\lambda_{e,S,j} := \frac{k\pi}{\omega_{S,j}} \neq 2 - \frac{2}{p}$, for all $k \in \mathbb{N}^*$, $j \in \{1, \ldots, J_S\}$ and $\lambda_{S,k} \neq 2 - \frac{3}{p}$, for all $k \in \mathbb{N}^*$. Then the solution χu of (2.13) admits the decomposition*

$$\chi u = u_r + u_e + u_v, \tag{2.14}$$

where $u_r \in W^{2,p}(C_S)$ is the regular part, u_e is the edge singularity given by

$$u_e = R^{2-\frac{3}{p}} v(\ln R, \omega), \tag{2.15}$$

with $v \in \overset{o}{H}{}^1(\mathbb{R} \times G_S)$ being a function satisfying ($t = \ln R$)

$$-\frac{\partial^2 v}{\partial t^2} + \Delta' v \in L^p(\mathbb{R} \times G_S). \tag{2.16}$$

Finally, u_v is the usual vertex singularity given by

$$u_v = \sum_{-\frac{1}{2}<\lambda_{S,k}<2-\frac{3}{p}} c_k R^{\lambda_{S,k}} \varphi_{S,k}(\omega), \tag{2.17}$$

where $c_k \in \mathbb{R}$.

We first look at the regularity of the edge singularity. We introduce the following notation (see [22]): For any $j \in \{1, \dots, J_S\}$, denote by $\theta_{S,j}$ the angle between a point x of C_S and the edge $A_{S,j}$ and for a fixed (sufficiently small) constant $\nu > 0$, set $C_{S,j} = \{x \in C_S : \theta_{S,j} < \nu\}$ ($\nu > 0$ is chosen sufficiently small such that $C_{S,j}$ does not contain the other edges of C_S).

Theorem 2.7 *Let the assumptions of Theorem 2.6 be satisfied. Fix $j \in \{1, \dots, J_S\}$ and a Cartesian system of coordinates (x_1, x_2, x_3) such that the x_3-axis contains the edge $A_{S,j}$. If $1 - \frac{2}{p} < \lambda_{e,S,j}$, then we have*

$$\partial_k u_e \in V^{1,p}_{0,\delta}(C_{S,j}), \quad k = 1, 2, \qquad \partial_3 u_e \in V^{1,p}_{0,0}(C_{S,j}), \tag{2.18}$$

for any $\delta \geq 0$ such that

$$\delta > 2 - \frac{2}{p} - \lambda_{e,S,j}. \tag{2.19}$$

Proof Since $v \in \overset{o}{H}{}^1(\mathbb{R} \times G_S)$ is a function satisfying (2.16) in the cylinder $\mathbb{R} \times G_S$, we can proceed as in the proof of Theorem 2.2 of [5]. (Here, the operator has variable coefficients with a principal part frozen at $\theta_{S,j} = 0$ equal to the Laplace operator, consequently in the arguments of [5] we use Theorems 10.2 and 10.3 of [23] instead of Theorems 4.1 and 7.2 of [23], respectively.) Using the coordinates $(t, \theta_{S,j}, \varphi_{S,j})$ (near $A_{S,j}$, $(\theta_{S,j}, \varphi_{S,j})$ plays the role of polar coordinates on G_S) and fixing a cut-off function $\chi_{S,j}$ on G_S which is equal to 1 near $A_{S,j}$ and equal to 0 near the other vertices of G_S, this yields

$$\begin{aligned}
\chi_{S,j} \theta_{S,j}^{\delta+k-2} \frac{\partial^{k+\ell} v}{\partial \theta_{S,j}^k \partial \varphi_{S,j}^\ell} &\in L^p(\mathbb{R} \times G_S), \quad \forall 1 \leq k + \ell \leq 2, \\
\chi_{S,j} \theta_{S,j}^{\delta-\ell} \frac{\partial^2 v}{\partial t \partial \theta_{S,j}^k \partial \varphi_{S,j}^\ell} &\in L^p(\mathbb{R} \times G_S), \quad \forall k + \ell = 1, \\
\chi_{S,j} \theta_{S,j}^{-1} \frac{\partial v}{\partial t} &\in L^p(\mathbb{R} \times G_S), \\
\chi_{S,j} \frac{\partial^2 v}{\partial t^2} &\in L^p(\mathbb{R} \times G_S),
\end{aligned}$$

for any δ satisfying the assumption of the theorem. Performing the change of variable $R = e^t$, and going back to the Cartesian system of coordinates introduced in the theorem, we get

$$\begin{aligned}
\theta_{S,j}^{\delta} \partial_{k\ell} u_e &\in L^p(C_{S,j}), \quad \forall k, \ell = 1, 2, \\
\partial_{k3} u_e &\in L^p(C_{S,j}), \quad \forall k = 1, 2, 3, \\
\theta_{S,j}^{\delta-1} R^{-1} \partial_k u_e &\in L^p(C_{S,j}), \quad \forall k = 1, 2, \\
\theta_{S,j}^{-1} R^{-1} \partial_3 u_e &\in L^p(C_{S,j}).
\end{aligned}$$

This leads to (2.18) since $\theta_{S,j}^{\delta}$ is bounded on $C_{S,j}$. □

Theorem 2.8 *Let the assumptions of Theorem 2.6 be satisfied. Fix $j \in \{1,\dots,J_S\}$ and a Cartesian system of coordinates (x_1,x_2,x_3) such that the x_3-axis contains the edge $A_{S,j}$. Then we have*

$$\chi\partial_j u_v \in V^{1,p}_{\beta,\delta}(C_{S,j}), \quad j=1,2, \tag{2.20}$$

$$\chi\partial_3 u_v \in V^{1,p}_{\beta,0}(C_{S,j}), \tag{2.21}$$

for any $\delta \geq 0$ satisfying (2.19) and any $\beta \geq 0$ such that

$$\beta > 2 - \frac{3}{p} - \lambda_{v,S,j}. \tag{2.22}$$

Proof Direct calculations using the fact that in $C_{S,j}$, the vertex singularity u_v behaves near S like $R^{\lambda_{v,S,j}}\theta_{S,j}^{\lambda_{e,S,j}}$. □

In summary, we have obtained the following regularity result near the corner S:

Theorem 2.9 *Let the assumptions of Theorem 2.7 be satisfied. Then, for $j \in \{1,\dots,J_S\}$, the solution u of problem (1.1) admits the following decomposition near S:*

$$u = u_r + u_s,$$

where $u_r \in W^{2,p}(C_S)$ and

$$\chi\partial_j u_s \in V^{1,p}_{\beta,\delta}(C_{S,j}), \quad j=1,2, \tag{2.23}$$

$$\chi\partial_3 u_s \in V^{1,p}_{\beta,0}(C_{S,j}), \tag{2.24}$$

for any $\delta \geq 0$ satisfying (2.19) and any $\beta \geq 0$ satisfying (2.22).

Proof We have simply set $u_s = u_e + u_v$ and we remark that u_e also satisfies (2.23) and (2.24) since R^β is bounded on the support of χ. □

Remark now that Theorems 2.4 and 2.9 are also valid for the general problem (1.1) using a linear change of variables (note that one edge of Ω is transformed into one edge of Ω'). As a consequence of that results and the definition of the subdomains Λ_ℓ as well as the Cartesian system of coordinates $(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)})$, we clearly have the

Theorem 2.10 *Let the assumptions of Theorem 2.6 be satisfied. Fix $\ell \in \{1,\dots,L\}$ and assume that $1-\frac{2}{p} < \lambda_e^{(\ell)}$ and $p<6$. Then the solution u of the general problem (1.1) admits the following decomposition in Λ_ℓ:*

$$u = u_r + u_s, \tag{2.25}$$

where $u_r \in W^{2,p}(\Lambda_\ell)$ and

$$\frac{\partial u_s}{\partial x_j^{(\ell)}} \in V^{1,p}_{\beta,\delta}(\Lambda_\ell), \quad j=1,2, \tag{2.26}$$

$$\frac{\partial u_s}{\partial x_3^{(\ell)}} \in V^{1,p}_{\beta,0}(\Lambda_\ell), \tag{2.27}$$

for any $\beta,\delta \geq 0$ satisfying

$$\beta > 2 - \frac{3}{p} - \lambda_v^{(\ell)}, \quad \delta > 2 - \frac{2}{p} - \lambda_e^{(\ell)}.$$

Proof If Λ_ℓ does not contain a vertex of Ω this is a consequence of Theorem 2.4. On the other hand if Λ_ℓ contains a vertex of Ω this follows from Theorem 2.9 (recall that if Λ_ℓ contains a singular edge of Ω then the $x_3^{(\ell)}$-axis is chosen parallel to this edge, otherwise it does not matter). □

3 The finite element mesh

The freedom in the choice of the finite element mesh is restricted by the following three needs:

A. general admissibility conditions arising from the finite element theory and the subdomain approach,

B. refinement conditions, such that the global error estimates can be proven,

C. geometrical conditions on the elements such that anisotropic local interpolation error estimates can be proven.

We will now elaborate a set of conditions that satisfies all the needs. Afterwards we give simple examples how one can construct such a mesh. We point out that we do not attempt to give a minimal set of conditions. Rather, we want to describe a set of conditions that is both sufficient for our error estimates and simple to be verified for our examples. We also admit (but do not request) overrefinement in certain regions if the mesh generation algorithm can be kept simple then.

The general conditions on the triangulation $\mathcal{T}_h = \{\Omega_i\}_{i=1}^m$ are:

A1. Ω is exactly triangulated by tetrahedra Ω_i, $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$.

A2. The triangulation is such that the subdomains Λ_ℓ are resolved exactly, $\overline{\Lambda_\ell} = \bigcup_{i\in L_\ell} \overline{\Omega_i}$, $\ell = 1,\ldots,L$, where $L_\ell \subset \{1,\ldots,m\}$ is an index set.

A3. The elements are disjoint, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$.

A4. Any face of any element Ω_i is either a face of another element Ω_j or part of the boundary.

A5. The number m of elements is related to the global mesh parameter h by $m \sim h^{-3}$.

To describe the refinement conditions we need some further notation. Recall from Section 2 that we introduced in each Λ_ℓ a Cartesian coordinate system $(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)})$. For each finite element $\Omega_i \subset \Lambda_\ell$ we denote by

$$\begin{aligned} r_i &:= \inf_{x\in\Omega_i} \left[(x_1^{(\ell)})^2 + (x_2^{(\ell)})^2\right]^{\frac{1}{2}}, \quad i = 1,\ldots,m, \\ R_i &:= \inf_{x\in\Omega_i} \left[(x_1^{(\ell)})^2 + (x_2^{(\ell)})^2 + (x_3^{(\ell)})^2\right]^{\frac{1}{2}}, \quad i = 1,\ldots,m, \end{aligned}$$

the distances of Ω_i to the $x_3^{(\ell)}$-axis and the origin. Note that $R_i \geq r_i$. Moreover, we introduce in each Λ_ℓ refinement parameters $\mu_\ell, \nu_\ell \in (0,1]$ corresponding to the refinement edge/vertex, respectively. If there is no refinement edge/vertex we let $\mu_\ell = 1$ or $\nu_\ell = 1$, respectively.

As mentioned above we want to admit overrefinement. Therefore we distinguish between size parameters h_i, H_i $(i = 1,\ldots,m)$,

$$h_i := \begin{cases} h^{\frac{1}{\mu_\ell}} & \text{if } r_i = 0, \\ h r_i^{1-\mu_\ell} & \text{if } r_i > 0, \end{cases} \qquad H_i := \begin{cases} h^{\frac{1}{\nu_\ell}} & \text{if } 0 \leq R_i \lesssim h^{\frac{1}{\nu_\ell}}, \\ h R_i^{1-\nu_\ell} & \text{if } R_i \gtrsim h^{\frac{1}{\nu_\ell}}, \end{cases}$$

and actual mesh sizes $\tilde{h}_{1,i}$, $\tilde{h}_{2,i}$, $\tilde{h}_{3,i}$ which are defined as the lengths of the projections of $\Omega_i \subset \Lambda_\ell$ on the $x_1^{(\ell)}$-, $x_2^{(\ell)}$-, or $x_3^{(\ell)}$-axis, respectively. (The tilde is used because this definition is different from the mesh sizes $h_{1,i}$, $h_{2,i}$, $h_{3,i}$ as used for example in [3].) Note that $h^{\frac{1}{\nu_\ell}} \sim h R_i^{1-\nu_\ell}$ for $R_i \sim h^{\frac{1}{\nu_\ell}}$.

The relation between these sizes is given by condition B1:

B1. If $\mu_\ell < 1$ then $\tilde{h}_{1,i} \sim h_i$, $\tilde{h}_{2,i} \sim h_i$, $\tilde{h}_{3,i} \lesssim H_i$ $(i = 1, \dots, m)$. But in particular we demand that $\tilde{h}_{3,i} \sim H_i$ if $r_i = 0$.

If $\mu_\ell = 1$ then $\tilde{h}_{j,i} \lesssim H_i$ $(i = 1, \dots, m,\ j = 1, 2, 3)$ and in particular $\tilde{h}_{j,i} \sim H_i$ if $R_i = 0$.

Note that Assumption A5 is indeed a condition but not a consequence of B1. That was different in our previous paper [5] where overrefinement was not allowed. In this sense we will also demand two similar conditions:

B2. The number of elements $\Omega_i \subset \Lambda_\ell$ with $r_i = 0$ is of order h^{-1}.

B3. The number of elements $\Omega_i \subset \Lambda_\ell$ such that $0 \leq R_i \lesssim h^{\frac{1}{\nu_\ell}}$ is bounded by $h^{\frac{2\mu_\ell}{\nu_\ell} - 2}$. In particular, there is only one element Ω_i with $R_i = 0$.

Though further conditions on the parameters μ_ℓ and ν_ℓ are imposed in the following section, we want to ensure a priori that $h_i \lesssim H_i$ for $\mu_\ell < 1$:

B4. If $\mu_\ell < 1$ then $\mu_\ell \leq \nu_\ell$ $(\ell = 1, \dots, L)$.

The next set of conditions is imposed to prove anisotropic local interpolation error estimates which are needed in subdomains with a refinement edge. Such estimates are usually proven on a reference element Ω_0 (or a finite number of reference elements) and then transformed on the finite element Ω_i via a linear coordinate transformation

$$\underline{x}^{(i,\ell)} = F^{(i)}(\hat{\underline{x}}) = B^{(i)} \hat{\underline{x}}, \tag{3.1}$$

$\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, $B^{(i)} = (b^{(i)}_{j,k})^3_{j,k=1} \in \mathbb{R}^{3\times 3}$, and $\underline{x}^{(i,\ell)} = (x_1^{(i,\ell)}, x_2^{(i,\ell)}, x_3^{(i,\ell)})$ is a local Cartesian system of coordinates. For our application we need that the elements $b^{(i)}_{j,k}$ and the elements $b^{(i,-1)}_{j,k}$ of $(B^{(i)})^{-1}$ satisfy the relations

$$|b^{(i)}_{j,k}| \lesssim \begin{cases} H_i & \text{if } j = k = 3, \\ h_i & \text{else,} \end{cases} \tag{3.2}$$

$$|b^{(i)}_{j,k}| \cdot |b^{(i,-1)}_{j',k}| \lesssim 1 \qquad \forall j, j', k = 1, 2, 3. \tag{3.3}$$

We will show in Lemma 4.8 that these rather abstract relations can be concluded from B1 and the following conditions.

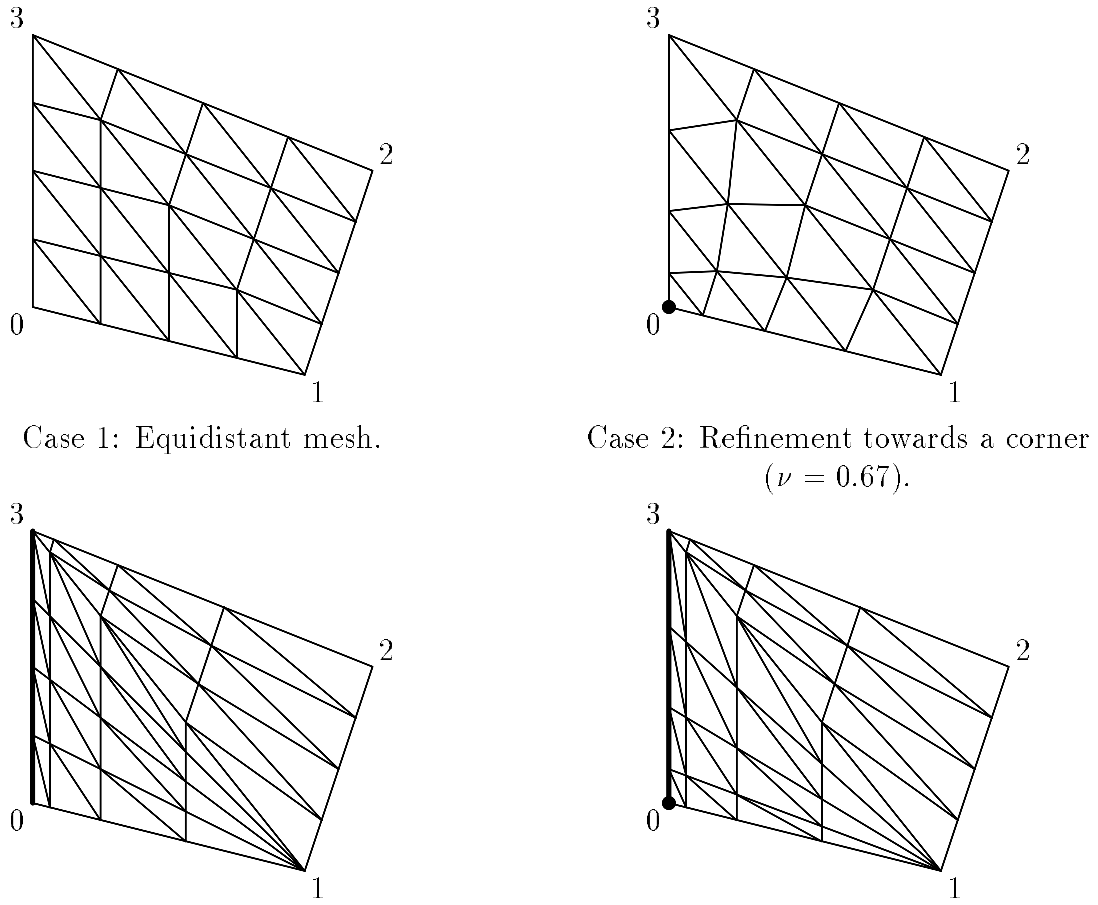
C1. The finite elements Ω_i must satisfy the maximal angle condition [1]: The maximal interior angle $\gamma_{F,i}$ of the four faces as well as the maximal angle $\gamma_{E,i}$ between any two faces of Ω_i is bounded by a constant $\gamma_\star < \pi$: $\gamma_{F,i} < \gamma_\star$, $\gamma_{E,i} < \gamma_\star$, $i = 1, \dots, m$.

C2. If Λ_ℓ contains a refinement edge then all elements $\Omega_i \subset \Lambda_\ell$ have two vertices such that the straight line through them is parallel to the $x_3^{(\ell)}$-axis.

C3. If Λ_ℓ does not contain a refinement edge then all elements are isotropic, that means, they have bounded aspect ratio.

Note that we proved (3.2), (3.3) in [3] under the maximal angle condition C1 and a coordinate system condition which is very technical.

To keep notation short we will omit from now on the indices i and ℓ, if the considerations are local in one element Ω_i or one subdomain Λ_ℓ, respectively.



Case 1: Equidistant mesh.

Case 2: Refinement towards a corner ($\nu = 0.67$).

Case 3: Refinement towards an edge ($\mu = 0.5$).

Case 4: Refinement towards a corner and an edge ($\nu = 0.67$, $\mu = 0.5$).

Figure 3.1: Illustration of the meshing of the subdomains ($n = 4$).

We will now give a constructive proof that one can always generate meshes which satisfy all the assumptions made. To do this we will start with the meshing of one subdomain Λ_ℓ and then we discuss the satisfaction of Condition A4 after gluing together the meshes of the subdomains. We distinguish four cases: 1) Λ_ℓ contains neither a singular corner nor a singular edge, 2) Λ_ℓ contains a singular corner but no singular edge, 3) Λ_ℓ contains a singular edge but no singular corner, 4) Λ_ℓ contains both a singular corner and a singular edge.

The meshing in these four situations is illustrated in Figure 3.1. A mathematical description of this mesh generation procedure can be given as follows: Introduce barycentric coordinates $\lambda_0, \ldots, \lambda_3$ ($\lambda_i > 0$, $\sum_{i=0}^{4} \lambda_i = 1$) in Λ_ℓ such that the refinement vertex has the coordinate $\lambda_0 = 1$ and the refinement edge is described by $\lambda_1 = \lambda_2 = 0$. Let $n \in \mathbb{N}$ be an integer such that $h \sim n^{-1}$.

Case 1: The vertices $P_{i,j,k}$ have the coordinates

$$\lambda_1 = \tfrac{i}{n}, \quad \lambda_2 = \tfrac{j}{n}, \quad \lambda_3 = \tfrac{k}{n}, \quad 0 \le i + j + k \le n.$$

The tetrahedra are described as quadruples of vertices; they are

$$\begin{array}{ll}
(P_{i,j,k}, P_{i+1,j,k}, P_{i,j+1,k}, P_{i,j,k+1}), & 0 \le i+j+k \le n-1, \\
(P_{i+1,j,k}, P_{i,j+1,k}, P_{i,j,k+1}, P_{i+1,j,k+1}), & 0 \le i+j+k \le n-2, \\
(P_{i,j+1,k}, P_{i,j,k+1}, P_{i+1,j,k+1}, P_{i,j+1,k+1}), & 0 \le i+j+k \le n-2, \\
(P_{i+1,j,k}, P_{i,j+1,k}, P_{i+1,j+1,k}, P_{i+1,j,k+1}), & 0 \le i+j+k \le n-2, \\
(P_{i,j+1,k}, P_{i+1,j+1,k}, P_{i+1,j,k+1}, P_{i,j+1,k+1}), & 0 \le i+j+k \le n-2, \\
(P_{i+1,j+1,k}, P_{i+1,j,k+1}, P_{i,j+1,k+1}, P_{i+1,j+1,k+1}), & 0 \le i+j+k \le n-3.
\end{array}$$

Case 2: The topology is as in Case 1 but the coordinates of the vertices $P_{i,j,k}$ change to

$$\lambda_1 = \tfrac{i}{n}\left(\tfrac{i+j+k}{n}\right)^{-1+\frac{1}{\nu}}, \quad \lambda_2 = \tfrac{j}{n}\left(\tfrac{i+j+k}{n}\right)^{-1+\frac{1}{\nu}}, \quad \lambda_3 = \tfrac{k}{n}\left(\tfrac{i+j+k}{n}\right)^{-1+\frac{1}{\nu}},$$

$0 \le i+j+k \le n$.

Case 3: We introduce here a larger set of nodes $P_{i,j,k}$

$$0 \le i+j \le n, \quad 0 \le k \le n \quad \text{if} \quad i+j<n, \quad k=0 \quad \text{if} \quad i+j=n,$$

with the coordinates

$$\lambda_1 = \tfrac{i}{n}\left(\tfrac{i+j}{n}\right)^{-1+\frac{1}{\mu}}, \quad \lambda_2 = \tfrac{j}{n}\left(\tfrac{i+j}{n}\right)^{-1+\frac{1}{\mu}}, \quad \lambda_3 = \tfrac{k}{n}(1-\lambda_1-\lambda_2).$$

The tetrahedra are described in three cases:

Subdivision of pentahedra:

$$\begin{array}{ll}
(P_{i,j,k}, P_{i+1,j,k}, P_{i,j+1,k}, P_{i,j,k+1}), & 0 \le i+j \le n-2, \\
(P_{i+1,j,k}, P_{i,j+1,k}, P_{i,j,k+1}, P_{i+1,j,k+1}), & 0 \le i+j \le n-2, \\
(P_{i,j+1,k}, P_{i,j,k+1}, P_{i+1,j,k+1}, P_{i,j+1,k+1}), & 0 \le i+j \le n-2, \\
(P_{i+1,j,k}, P_{i,j+1,k}, P_{i+1,j+1,k}, P_{i+1,j,k+1}), & 0 \le i+j \le n-3, \\
(P_{i,j+1,k}, P_{i+1,j+1,k}, P_{i+1,j,k+1}, P_{i,j+1,k+1}), & 0 \le i+j \le n-3, \\
(P_{i+1,j+1,k}, P_{i+1,j,k+1}, P_{i,j+1,k+1}, P_{i+1,j+1,k+1}), & 0 \le i+j \le n-3,
\end{array}$$

$0 \le k \le n-1$ in all cases.

Subdivision of pyramids:

$$\begin{array}{ll}
(P_{i+1,j,k}, P_{i,j+1,k}, P_{i+1,j,k+1}, P_{i+1,j+1,0}), & n-2 \le i+j \le n-2, \\
(P_{i,j+1,k}, P_{i+1,j,k+1}, P_{i,j+1,k+1}, P_{i+1,j+1,0}), & n-2 \le i+j \le n-2,
\end{array}$$

$0 \le k \le n-1$ in both cases.

Remaining tetrahedra:

$$(P_{i,j,k}, P_{i,j,k+1}, P_{i+1,j,0}, P_{i,j+1,0}), \quad n-1 \le i+j \le n-1, \quad 0 \le k \le n-1.$$

Case 4: The topology is as in Case 3 but the λ_3-coordinate of the points $P_{i,j,k}$ changes to

$$\lambda_3 = \left(\tfrac{k}{n}\right)^{\frac{1}{\nu}}(1-\lambda_1-\lambda_2).$$

We have now to prove that such a mesh satisfies all conditions: A1, A2, A3, and A5 are obvious. Assumption A4 is equivalent to the necessity that faces $\overline{\Lambda_\ell} \cap \overline{\Lambda_{\ell'}}$ are meshed in the same way. This leads in general to some cascade effect: Let $M \subset \partial\Omega$ be a connected set of singular edges and vertices (edges are considered as closed sets), then we have to choose

$$\mu_\ell = \nu_\ell = \mu_M \quad \text{for all } \ell : \overline{\Lambda_\ell} \cap M \ne \emptyset.$$

That means that the refinement is determined by the strongest singularity in this region. An exception is the case when the face $\lambda_3 = 0$ is part of the boundary $\partial\Omega$. Then ν_ℓ can be chosen larger than μ_ℓ. We remark that the cascade effect could be avoided by using mortar elements [11].

The coordinate transformation $\lambda_0, \dots, \lambda_3 \mapsto x_1, \dots, x_3$ is independent of h. Therefore, Assumption B1 can easily be verified noting that

$$\begin{aligned}(s+h)^{\frac{1}{\mu}} - s^{\frac{1}{\mu}} &\sim hs^{1-\mu},\\ \lambda_1 + \lambda_2 + \lambda_3 &\sim R,\\ \lambda_1 + \lambda_2 &\sim r.\end{aligned}$$

Indeed, in Case 2 all elements are isotropic, that means $\tilde{h}_m$ is of the size of the distance of the two planes $\lambda_4 = (\frac{i+j+k+1}{n})^{\frac{1}{\nu}}$ and $\lambda_4 = (\frac{i+j+k}{n})^{\frac{1}{\nu}}$,

$$\tilde{h}_m \sim \left(\tfrac{i+j+k+1}{n}\right)^{\frac{1}{\nu}} - \left(\tfrac{i+j+k}{n}\right)^{\frac{1}{\nu}} \sim hR^{1-\nu} \quad (m = 1, 2, 3).$$

In cases 3 and 4, the projection of the element into the λ_1, λ_2-plane is isotropic, that means

$$\tilde{h}_m \sim \left(\tfrac{i+j+1}{n}\right)^{\frac{1}{\mu}} - \left(\tfrac{i+j}{n}\right)^{\frac{1}{\mu}} \sim hr^{1-\mu} \quad (m = 1, 2).$$

Finally, we see in Case 4 that

$$\begin{aligned}\tilde{h}_3 &\lesssim \lambda_3(P_{.,.,k+1}) - \lambda_3(P_{.,.,k}) + (\tilde{h}_1 + \tilde{h}_2)\\ &\lesssim \left(\tfrac{k+1}{n}\right)^{\frac{1}{\nu}} - \left(\tfrac{k}{n}\right)^{\frac{1}{\nu}} + hr^{1-\mu}\\ &\lesssim hz^{1-\nu} + hr^{1-\nu}\\ &\lesssim hR^{1-\nu},\end{aligned}$$

because $\nu \geq \mu$.

Condition B2 is satisfied by construction. B3 is checked by realizing that the number of elements is of order i^2 where i satisfies $(\frac{i}{n})^{\frac{1}{\mu}} \lesssim (\frac{1}{n})^{\frac{1}{\nu}}$, that means $i \lesssim n^{1-\frac{\mu}{\nu}}$. Condition B4 is independent of our meshing strategy. Conditions C1–C3 are also satisfied by construction. Note that overrefinement is accepted in Cases 3 and 4 near the edge $\lambda_0 = \lambda_4 = 0$ and due to the cascade effect described above.

Note that the number of elements is n^3 for Cases 1 and 2, and $3n^3 - 3n^2 + n$ for Cases 3 and 4. We introduced the richer topology in the latter cases to ensure the maximal angle condition C1. However, we can use the topology of Cases 1/2 if $\mu = \nu < 1$, compare Figure 3.2. The vertices $P_{i,j,k}$ have then the coordinates

$$\lambda_1 = \tfrac{i}{n}\left(\tfrac{i+j}{n}\right)^{-1+\frac{1}{\mu}}, \quad \lambda_2 = \tfrac{j}{n}\left(\tfrac{i+j}{n}\right)^{-1+\frac{1}{\mu}}, \quad \lambda_3 = \left(\tfrac{i+j+k}{n}\right)^{\frac{1}{\mu}} - \lambda_1 - \lambda_2.$$

$0 \leq i + j + k \leq n$.

We point out that also simpler meshing strategies can be applied where overrefinement takes place in more regions. Figure 6.1 shows an example where artificial refinement edges are introduced. Moreover, we introduced the Assumptions A1-C3 in order to allow other refinement strategies which are not based on the domain decomposition approach, see Figure 3.3 for an example with a coordinate transformation.

We introduce now the finite element space V_h of all continuous functions whose restriction to any Ω_i $(i = 1, \dots, m)$ is a polynomial of first degree. Furthermore, we let V_{0h} be defined by $V_{0h} := \{v_h \in V_h : v_h|_{\partial\Omega} = 0\}$. Note that $V_h \subset H^1(\Omega)$ and $V_{0h} \subset \overset{o}{H}{}^1(\Omega)$. The finite element solution of problem (1.1) is defined by:

$$\textit{Find } u_h \in V_{0h} \textit{ such that } a_\Omega(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_{0h}, \tag{3.4}$$

Since the assumptions of the Lax–Milgram lemma are fulfilled this problem has a unique solution.

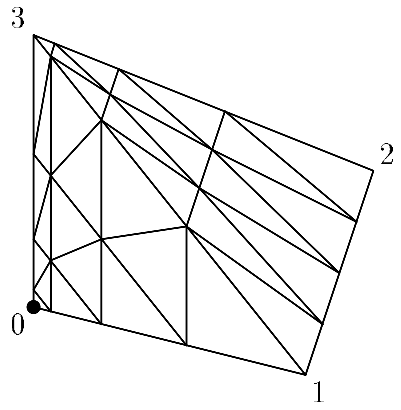


Figure 3.2: Modification of Case 4 for $\mu = \nu < 1$.

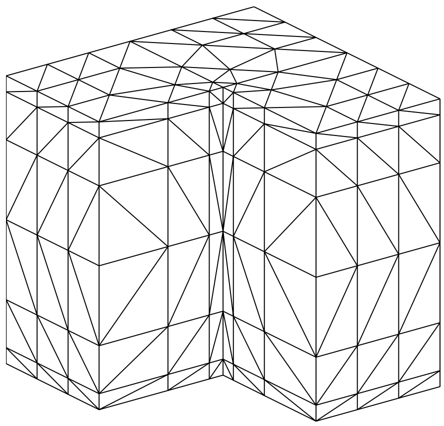


Figure 3.3: Tensor product domain with mesh refinement near the singular edge and the corners.

4 Local interpolation error estimates

As motivated in the Introduction, we are interested in local interpolation error estimates for anisotropic elements. In [1], the case of classical Sobolev spaces was treated, while in [5], the case of weighted Sobolev spaces with a weight which is the distance to one edge was considered. The first case is useful far from the edge and the corners, and the second one far from the corners, but both cannot be applied for the tetrahedra along one (singular) edge and hitting the corners. In this section, we shall extend these results to weighted Sobolev spaces with two weights: one is the distance to the corner and the other one the angular distance to the edge. For two-dimensional interpolation error estimates in weighted Sobolev spaces, we refer to [28].

We consider first estimates on a reference element $\Omega_0 \in \mathcal{R}$ where $\mathcal{R} = \{\Omega_a, \Omega_b\}$ is the set of reference elements discussed later, see Figure 4.1.

Using a similar notation as in [1, §2] we denote by P a space of polynomials, and since each monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ can be identified with the multi-index $\alpha \in \mathbb{N}^3$, we also identify P with the corresponding set of multi-indices. The hull $\overline{P}$ of P is the set $\overline{P} := P \cup \{\alpha + e_i : \alpha \in P, i = 1, 2, 3\}$ ($\{e_i\}_{i=1}^3$ denotes the canonical basis of $\mathbb{R}^3$) and the boundary $\partial\overline{P}$ of P is the set $\overline{P} \setminus P$. Note that $\max_{\alpha \in \overline{P}} |\alpha| = 1 + \max_{\alpha \in P} |\alpha|$.

We introduce now weighted Sobolev spaces on Ω_0: For a finite set $P \subset \mathbb{N}^3$ with $0 \in P$

and for $\beta, \delta \in \mathbb{R}$ we set $V^{P,p}_{\beta,\delta}(\Omega_0) := \{v \in \mathcal{D}'(\Omega_0) : \|v; V^{P,p}_{\beta,\delta}(\Omega_0)\| < \infty\}$, where

$$\|v; V^{P,p}_{\beta,\delta}(\Omega_0)\|^p := \sum_{\alpha\in P} \int_{\Omega_0} |\hat{R}^{\beta-k+|\alpha|}\hat{\theta}^{\delta-k+|\alpha|} D^\alpha v|^p \, d\underline{\hat{x}},$$

$k := \max_{\alpha\in P} |\alpha|$, $D^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, $\hat{R}(\underline{\hat{x}}) := (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)^{\frac{1}{2}}$, $\hat{r}(\underline{\hat{x}}) := (\hat{x}_1^2 + \hat{x}_2^2)^{\frac{1}{2}}$, and $\hat{\theta} = \frac{\hat{r}}{\hat{R}}$ is the angular distance to the edge $\hat{x}_1 = \hat{x}_2 = 0$. (Note that the case when there is no weight with respect to $\hat{\theta}$ is also available by simply putting $\hat{\theta} = 1$ in the rest of this section.) For $v \in V^{\overline{P},p}_{\beta,\delta}(\Omega_0)$ we also introduce the seminorm

$$|v; V^{\overline{P},p}_{\beta,\delta}(\Omega_0)|^p := \sum_{\alpha\in\partial\overline{P}} \int_{\Omega_0} |\hat{R}^{\beta-k-1+|\alpha|}\hat{\theta}^{\delta-k-1+|\alpha|} D^\alpha v|^p \, d\underline{\hat{x}}.$$

For $P = \{(0,0,0)\} \cup \{e_i\}_{i=1,2,3}$, we denote $V^{P,p}_{\beta,\delta}(\Omega_0)$ by $V^{1,p}_{\beta,\delta}(\Omega_0)$. The space $W^{P,p}(\Omega_0)$ is introduced in analogy to $V^{P,p}_{\beta,\delta}(\Omega_0)$ by omitting the weights. Note also that the space $V^{P,p}_{\beta,\beta}(\Omega_0)$ coincides with the space $V^{P,p}_{\beta}(\Omega_0)$ introduced in [5].

Lemma 4.1 *Let $P \subset \mathbb{N}^3$, P finite with $0 \in P$. Then we have the compact embedding*

$$V^{\overline{P},p}_{\beta,\delta}(\Omega_0) \overset{c}{\hookrightarrow} V^{P,p}_{\beta,\delta}(\Omega_0).$$

Proof The proof of that Lemma is analogous to that of Lemma 3.1 of [5] using spherical coordinates $(\hat{R}, \hat{\theta}, \hat{\varphi})$ and the compact embedding $W^{1,p}(\Omega_0) \overset{c}{\hookrightarrow} L^p(\Omega_0)$ (Rellich–Kondrašov theorem). □

Now using the Hölder inequality and again spherical coordinates, we can show the following result (see Lemma 3.2 of [5]).

Lemma 4.2 *Let $P \subset \mathbb{N}^3$, P finite, such that $0 \in P$. If $\beta < 3 - \frac{3}{p}$, and $\delta < 2 - \frac{2}{p}$, then for all $v \in V^{P,p}_{\beta,\delta}(\Omega_0)$ the following relation holds:*

$$D^\alpha v \in L^1(\Omega_0) \text{ for all } \alpha \in P. \tag{4.1}$$

From Lemmas 4.1 and 4.2 and using the same arguments as in [1, Lemma 2], we obtain the following lemma.

Lemma 4.3 *Let $P \subset \mathbb{N}^3$ be a finite set of multi-indices with $0 \in P$. If $\beta < 3 - \frac{3}{p}$ and $\delta < 2 - \frac{2}{p}$, then*

$$\|v; V^{\overline{P},p}_{\beta,\delta}(\Omega_0)\| \lesssim |v; V^{P,p}_{\beta,\delta}(\Omega_0)|, \tag{4.2}$$

for all $v \in V^{P,p}_{\beta,\delta}(\Omega_0)$ satisfying $\int_{\Omega_0} D^\alpha v \, d\underline{\hat{x}} = 0$ for $\alpha \in P$.

We are now ready to give the interpolation estimate, first in a very general form, then especially for our purposes.

Lemma 4.4 *Let $\beta < 3 - \frac{3}{p}$ and $\delta < 2 - \frac{2}{p}$, and let $P, Q \subset \mathbb{N}^3$ and $\gamma \in \mathbb{N}^3$ be such that $0 \in Q$ and $Q + \gamma \subset P$. Further introduce a linear operator $I : C^\mu(\Omega_0) \to P$, $\mu \in \mathbb{N}$, and assume that there are linear functionals $F_i \in \left(V^{\overline{Q},p}_{\beta,\delta}(\Omega_0)\right)'$, $i = 1, \ldots, j$, $j = \dim D^\gamma P$, satisfying*

$$\begin{aligned} &F_i(D^\gamma I v) = F_i(D^\gamma v) \quad (i = 1, \ldots, j) \text{ for all } v \in C^\mu(\Omega_0) \cap V^{\overline{Q}+\gamma,p}_{\beta,\delta}(\Omega_0), \\ &F_i(D^\gamma q) = 0 \text{ for all } i = 1, \ldots, j \Longrightarrow D^\gamma q = 0 \text{ for all } q \in P. \end{aligned} \tag{4.3}$$

Then

$$\|D^\gamma(v - Iv); V^{\overline{Q},p}_{\beta,\delta}(\Omega_0)\| \lesssim |D^\gamma v; V^{\overline{Q},p}_{\beta,\delta}(\Omega_0)|$$

for all $v \in C^\mu(\Omega_0) \cap V^{\overline{Q}+\gamma,p}_{\beta,\delta}(\Omega_0)$.

Proof We follow the proof of Lemma 3 of [1], since Lemma 1 of [1] can be extended to the spaces $V^{P,p}_{\beta,\delta}(\Omega_0)$ (owing to Lemma 4.2), while Lemma 2 of [1] is replaced by Lemma 4.3. $\square$

Theorem 4.5 *Suppose that* $0 \leq \beta < 1 - \frac{1}{p}$, $0 \leq \delta < 1 - \frac{1}{p}$, $p > 2$, *and let* Iv *be the linear Lagrangian interpolant of* v *with respect to the vertices. Then for all* $v \in C(\bar{\Omega}_0)$ *such that* $\partial_i v \in V^{1,p}_{\beta,\delta}(\Omega_0)$ *for* $i = 1, 2$ *and* $\partial_3 v \in V^{1,p}_{\beta,0}(\Omega_0)$, *then we have*

$$\left\|\hat{R}^{\beta-1}\hat{\theta}^{\delta-1}\partial_i(v - Iv); L^p(\Omega_0)\right\|^p \lesssim \int_{\Omega_0} \hat{R}^{p\beta}\hat{\theta}^{p\delta}\left[|\partial_{1i}v|^p + |\partial_{2i}v|^p + |\partial_{3i}v|^p\right] d\underline{\hat{x}},$$

$$\left\|\hat{R}^{\beta-1}\hat{\theta}^{-1}\partial_3(v - Iv); L^p(\Omega_0)\right\|^p \lesssim \int_{\Omega_0} \hat{R}^{p\beta}\left[|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right] d\underline{\hat{x}}.$$

Proof We set $Q := \{(0,0,0)\}$, $\overline{Q} := \{(0,0,0)\} \cup \{e_i\}_{i=1,2,3}$ and remark that the assumptions are simply that $\partial_i v \in V^{\overline{Q},p}_{\beta,\delta}(\Omega_0)$ $(i = 1, 2)$ and $\partial_3 v \in V^{\overline{Q},p}_{\beta,0}(\Omega_0)$. To prove the assertion we apply Lemma 4.4 with $P = \overline{Q}$, $\gamma := e_i$ and $F_1(v) := \int_{E_i} v\, dx_i$, where E_i is that edge of Ω_0 which is parallel to the x_i-axis, which exists due to the choice of the reference elements. It remains to prove the continuity of F_1.

For $i = 1, 2$ we use that $v \in V^{1,p}_{\beta,\delta}(\Omega_0)$ implies

$$\hat{R}^{\beta}\hat{\theta}^{\delta} v \in W^{1,p}(\Omega_0) \hookrightarrow W^{1-2/p,p}(E_i) \hookrightarrow L^p(E_i),\ i = 1, 2.$$

Using the Hölder inequality, we conclude for $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\int_{E_i} |v|\, dx_i \leq \|\hat{R}^{-\beta}\hat{\theta}^{-\delta}; L^q(E_i)\|\ \|\hat{R}^{\beta} v; L^p(E_i)\| \lesssim \|\hat{R}^{-\beta}\hat{\theta}^{-\delta}; L^q(E_i)\|\ \|v; V^{1,p}_{\beta,\delta}(\Omega_0)\|.$$

Using that $\hat{R}^{-\beta}\hat{\theta}^{-\delta} \in L^q(E_i)$ for $\beta < \frac{1}{q} = 1 - \frac{1}{p}$ and $\delta < \frac{1}{q}$, we get the conclusion.

The case $i = 3$ is treated in the same way by replacing δ by 0. Note that $\delta = 0$ is essential here because for both reference elements $\hat{\theta} = 0$ on E_3. $\square$

Remark 4.6 In our application, we have $\beta = 2 - \frac{2}{p} - \lambda_v + \varepsilon$ and $\delta = 2 - \frac{3}{p} - \lambda_e + \varepsilon$ with an arbitrarily small positive real ε. That means $\beta < 1 - \frac{1}{p}$, $\delta < 1 - \frac{1}{p}$ are equivalent to $1 - \frac{2}{p} < \lambda_v$ and $1 - \frac{1}{p} < \lambda_e$, respectively, so that for p close to 2 this condition always holds because $\lambda_v > 0$ and $\lambda_e > \frac{1}{2}$.

Corollary 4.7 *Under the assumptions of Theorem 4.5, the next estimates hold:*

$$\|\partial_i(v - Iv); L^p(\Omega_0)\|^p \lesssim \int_{\Omega_0} \hat{R}^{p\beta}\hat{\theta}^{p\delta}\left[|\partial_{1i}v|^p + |\partial_{2i}v|^p + |\partial_{3i}v|^p\right] d\underline{\hat{x}}, \tag{4.4}$$

$$\|\partial_3(v - Iv); L^p(\Omega_0)\|^p \lesssim \int_{\Omega_0} \hat{R}^{p\beta}\left[|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right] d\underline{\hat{x}}. \tag{4.5}$$

Proof The assertion follows from the two estimates of Theorem 4.5, since the weights on the left hand side are bounded from below by some constant $C > 0$. $\square$

Now we are going to transform these estimates to the actual finite elements Ω_i of any subdomain Λ_ℓ. As usual, we use a linear transformation (3.1) such that $\Omega_i = F^{(i)}(\Omega_0)$. In our case we consider two reference elements Ω_a and Ω_b as given in Figure 4.1 (see [3] for a similar point of view).

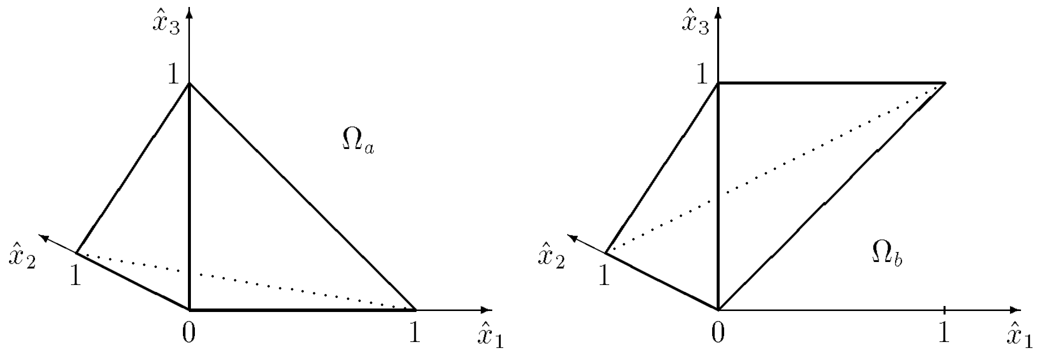


Figure 4.1: Basic reference elements for anisotropic interpolation error estimates in the three-dimensional case.

We first give a sufficient condition on Ω_i, fulfilled by the elements Ω_i such that the edge parallel to the $x_3^{(\ell)}$-axis is of length $\gtrsim h_i$, and more flexible that the coordinate system condition in [3], ensuring that the relations (3.2)–(3.3) hold.

Lemma 4.8 *Assume that $\mu_\ell < 1$. Let Ω_i be a finite element of Λ_ℓ such that its edge $e_{3,i}$ parallel to the x_3-axis satisfies $|e_{3,i}| \gtrsim h_i$. Then there exist two other edges $e_{1,i}$, $e_{2,i}$ such that $e_{j,i} \cap e_{3,i} \neq \emptyset$ and $|e_{j,i}| \sim h_i$, $j = 1, 2$.*

Moreover, there exists a local Cartesian system of coordinates $\underline{x}^{(i,\ell)} = F^{(i)}(\underline{\hat{x}}) = B^{(i)}\underline{\hat{x}}$ such that 1) there exists an $\Omega_0 \in \mathcal{R}$ such that $\Omega_i = F^{(i)}(\Omega_0)$, 2) the $x_3^{(i,\ell)}$-axis is parallel to the $x_3^{(\ell)}$-axis, and 3) the estimates (3.2)–(3.3) hold.

Proof The first assertion follows from the conditions B1 and C1. For the second assertion, we define the local Cartesian system of coordinates $\underline{x}^{(i,\ell)}$ as follows: let the $x_3^{(i,\ell)}$-axis contain the edge $e_{3,i}$; the $x_2^{(i,\ell)}$-axis is fixed so that the $x_2^{(i,\ell)}, x_3^{(i,\ell)}$-plane is the plane induced by $e_{2,i}$ and $e_{3,i}$, with the origin at their intersection; the $x_1^{(i,\ell)}$-axis is consequently determined to have a direct orthogonal system. We take Ω_a as the reference element if $e_{j,i}$, $j = 1, 2, 3$, meet in one vertex, and Ω_b if $e_{1,i} \cap e_{3,i} \neq e_{2,i} \cap e_{3,i}$. Then the matrix $B^{(i)}$ appearing in the above transformation takes the form

$$\begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where each column j corresponds to $e_{j,i}$ (considered as a vector).

Let us now show that

$$|b_{jk}| \lesssim \min\{h^\star_{j,i}, h^\star_{k,i}\}, \quad |b^{(-1)}_{jk}| \lesssim \min\{(h^\star_{j,i})^{-1}, (h^\star_{k,i})^{-1}\}, \tag{4.6}$$

where $h^\star_{j,i} = |e_{j,i}|$. Indeed, $|b_{jk}| \leq \sqrt{\sum_{n=1}^3 b_{nk}^2} = h^\star_{k,i}$, which yields the first estimate since we have $h^\star_{1,i} \sim h^\star_{2,i} \sim h_i \lesssim h^\star_{3,i}$. Denoting by T_i the projection of Ω_i in the plane $x_3^{(\ell)} = 0$, owing to B1 and C1, we have meas $T_i = \frac{1}{2} b_{11} b_{22} \sim h_i^2 \sim h^\star_{1,i} h^\star_{2,i}$. Using this last equivalences, we obtain $b_{kk} \sim h^\star_{k,i}$ and the second estimate of (4.6) is then direct.

The two estimates of (4.6) directly yield (3.3), while (3.2) follows from the fact that $h^\star_{1,i} \sim h^\star_{2,i} \sim h_i, h^\star_{3,i} \lesssim H_i$ □

Theorem 4.9 *Consider that element $\Omega_i \subset \Lambda_\ell$ which has one vertex in the origin of the local coordinates $(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)})$. Let $I_h v$ be the linear Lagrangian interpolant of $v \in C(\bar{\Omega}_i)$ with* *respect to the vertices. Suppose that $\mu_\ell < 1$. Assume further that $\partial_j v \in V^{1,p}_{\beta,\delta}(\Omega_i)$, $j = 1,2$ and $\partial_3 v \in V^{1,p}_{\beta,0}(\Omega_i)$, for $0 \leq \beta < 1-\frac{1}{p}$, $0 \leq \delta < 1-\frac{1}{p}$, $p > 2$, then the norm of the derivatives of the interpolation error can be estimated by*

$$\begin{aligned} |v - I_h v; W^{1,p}(\Omega_i)| \quad &\lesssim \quad h^{\frac{1-\beta}{\mu_\ell}+(\frac{1}{\nu_\ell}-\frac{1}{\mu_\ell})\delta} \sum_{j,k=1}^{2} \|R^\beta \theta^\delta \partial_{jk} v; L^p(\Omega_i)\| \\ &\quad + h^{\frac{-\beta}{\mu_\ell}+\frac{1}{\nu_\ell}} \sum_{k=1}^{3} \|R^\beta \partial_{k3} v; L^p(\Omega_i)\|. \qquad (4.7) \end{aligned}$$

Proof By our assumptions made on the mesh, the edge included into the $x_3^{(\ell)}$-axis is of length of order $h^{\frac{1}{\nu_\ell}}$ and the two other edges containing the origin are of length $h^{\frac{1}{\mu_\ell}}$. Therefore, Ω_i satisfies the assumption of Lemma 4.8 with $e_{3,i}$ equal to the edge included in the $x_3^{(\ell)}$-axis and $e_{1,i}$, $e_{2,i}$ the two other edges containing the origin. Consequently, this Lemma yields a transformation

$$\underline{x}^{(i,\ell)} = B^{(i)} \underline{\hat{x}}, \qquad (4.8)$$

which maps Ω_a to Ω_i and such that (3.2)–(3.3) hold. Moreover, we easily check the following estimates:

$$\begin{aligned} h^{-\frac{1}{\nu_\ell}} R \ \leq \ \hat{R} \ &\leq \ h^{-\frac{1}{\mu_\ell}} R, \\ \hat{r} \ &\leq \ h^{-\frac{1}{\mu_\ell}} r, \\ \hat{\theta} \ &\leq \ h^{\frac{1}{\nu_\ell}-\frac{1}{\mu_\ell}} \theta. \end{aligned}$$

The assertion is now a consequence of Corollary 4.7 using the transformation (4.8) with (3.2)–(3.3), the above estimates, the fact that for $k = 1,2$, $x_k^{(i,\ell)}$ is a linear combination of $x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)}$ and since $\mu_\ell \leq \nu_\ell$. □

To finish this section, we give two more error estimates: the first one (Theorem 4.10) concerns the elements Ω_i, which are far from the singular edges, while the second one (Theorem 4.11) concerns the elements along the singular edges but far from the singular vertices. Note that an estimate similar to Theorem 4.10 can be found in [1] but there it was used another definition of the mesh sizes. An estimate similar to Theorem 4.11 was given in [5] but there it was proved using a coordinate system condition which is here replaced by the more practical condition B1. Therefore we prove both estimates here.

Theorem 4.10 *For every $v \in W^{2,p}(\Omega_i)$ and for $p > 2$ one has*

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim h_i \sum_{k=1}^{2} |\partial_k v; W^{1,p}(\Omega_i)| + H_i |\partial_3 v; W^{1,p}(\Omega_i)|. \qquad (4.9)$$

in the coordinate system related to the subdomain Λ_ℓ containing Ω_i. The index ℓ is dropped for the sake of shortness.

Proof Let us denote by $h^\star_{3,i}$ the length of the edge parallel to the x_3-axis. We distinguish the cases $h^\star_{3,i} \lesssim h_i$ or not.

If $h^\star_{3,i} \lesssim h_i$, then by the conditions C1 and B1, we deduce that $\operatorname{diam}\Omega_i \lesssim h_i$. Therefore, applying directly the isotropic local error estimate of Jamet type (Corollary 4.5 in [3]), we get

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim h_i \sum_{k=1}^{3} |\partial_k v; W^{1,p}(\Omega_i)|,$$

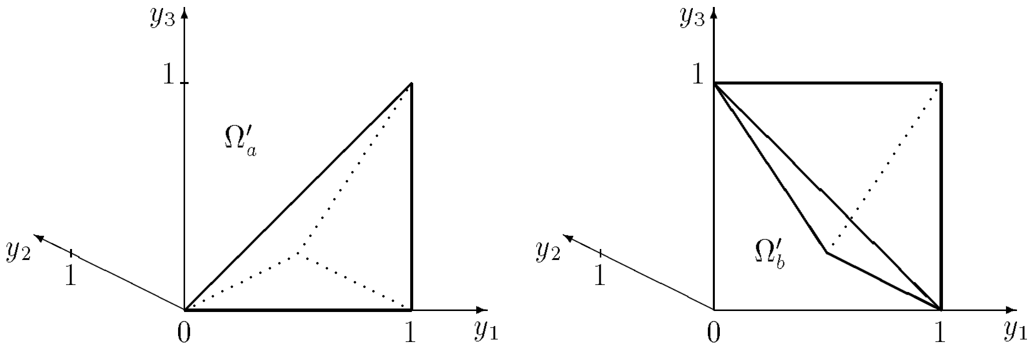


Figure 4.2: Additional reference elements for interpolation error estimates in weighted Sobolev spaces.

leading to the estimate (4.9) since $h_i \lesssim H_i$.

On the other hand, if $h^\star_{3,i} \gtrsim h_i$, then by Corollary 4.7 with $\beta = \delta = 0$ and Lemma 4.8, we get

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim \sum_{k=1}^{3} h^\star_{k,i} \left| \frac{\partial v}{\partial x_k^{(i,\ell)}}; W^{1,p}(\Omega_i) \right|.$$

As $\frac{\partial v}{\partial x_3^{(i,\ell)}} = \pm \frac{\partial v}{\partial x_3^{(\ell)}}$, $h^\star_{j,i} \lesssim h_i$, for $j = 1, 2$ and $h^\star_{3,i} \lesssim H_i$, the above estimate becomes

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim \sum_{k=1}^{2} h_i \left| \frac{\partial v}{\partial x_k^{(i,\ell)}}; W^{1,p}(\Omega_i) \right| + H_i |\partial_3 v; W^{1,p}(\Omega_i)|.$$

The desired estimate follows because $x_k^{(i,\ell)}$, $k = 1, 2$, is a linear combination of x_1, x_2, x_3 and the fact that $h_i \lesssim H_i$. $\square$

Theorem 4.11 *If $\Omega_i \subset \Lambda_\ell$ contains a singular edge but is far from the singular corner, then for every $v \in L^p(\Omega_i)$, for $p > 2$, such that $\partial_k v \in V^{1,p}_{\delta,\delta}(\Omega_i)$, $k = 1, 2$ with $0 < \delta < 1 - \frac{1}{p}$ and $\partial_3 v \in V^{1,p}_{0,0}(\Omega_i)$. Then one has*

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim h_i^{1-\delta} \sum_{k=1}^{2} |\partial_k v; V^{1,p}_{\delta,\delta}(\Omega_i)| + H_i |\partial_3 v; V^{1,p}_{0,0}(\Omega_i)|. \tag{4.10}$$

Proof Let us denote by $h^\star_{3,i}$ the length of the edge $e_{3,i}$ parallel to the x_3-axis. Consider first the case that $e_{3,i}$ is included in the singular edge. By the condition B1, we always have $h^\star_{3,i} \gtrsim h_i$, then applying Corollary 4.7 with $\beta = \delta$ and Lemma 4.8, we obtain

$$|v - I_h v; W^{1,p}(\Omega_i)| \lesssim \sum_{k=1}^{2} h_i^{1-\delta} \left| \frac{\partial v}{\partial x_k^{(i,\ell)}}; V^{1,p}_{\delta,\delta}(\Omega_i) \right| + h^\star_{3,i} |\partial_3 v; V^{1,p}_{0,0}(\Omega_i)|.$$

We conclude as before due to the choice of the $\underline{x}^{(i,\ell)}$-system of coordinates.

In the case that only one vertex of Ω_i is contained in the singular edge we proceed as above using the reference elements Ω'_a and Ω'_b, see Figure 4.2. $\square$

5 Global error estimates

In this section, we investigate first the global interpolation error, that is the difference between the solution u of our boundary value problem (2.1) and its piecewise linear interpolant $I_h u$ on the family of anisotropically graded meshes introduced in Section 3. The difficulty is that we are interested on one hand in an estimate in the energy norm which is equivalent to $|\,.\,; W^{1,2}(\Omega)|$, in order to apply the Céa lemma for the finite element error. But on the other hand, the above local interpolation error estimates are valid for $|\,.\,; W^{1,p}(\Omega_i)|$ with $p > 2$ only. — We secondly derive the global finite element error estimate via the Céa lemma.

Theorem 5.1 *Let u be the solution of the boundary value problem (2.1) with $f \in L_p(\Omega)$, $2 < p < p_+$,*

$$p_+ := \min_{\ell} \left\{ 6; \frac{2}{1-\lambda_v^{(\ell)}}; \frac{1}{1-\lambda_e^{(\ell)}} \right\}. \tag{5.1}$$

In addition to the condition B4, assume that the refinement parameters μ_ℓ, ν_ℓ satisfy the following conditions for all ℓ (see Remarks 5.4 and 5.5 below for a discussion of these conditions):

$$\mu_\ell < \lambda_e^{(\ell)} \frac{p}{2p-2}, \tag{5.2}$$

$$\nu_\ell < \left(\lambda_v^{(\ell)} + \frac{1}{2} \right) \frac{2p}{5p-6}, \tag{5.3}$$

$$\frac{1}{\nu_\ell} \left(\frac{5}{2} - \frac{3}{p} \right) + \frac{1}{\mu_\ell} \left(\lambda_v^{(\ell)} - 2 + \frac{3}{p} \right) > 1. \tag{5.4}$$

Then for the interpolation error $u - I_h u$ the following estimate holds:

$$|u - I_h u; W^{1,2}(\Omega)| \lesssim h \|f; L^p(\Omega)\|. \tag{5.5}$$

Proof We reduce the estimation of the global error to the evaluation of the global error on one subdomain Λ_ℓ with one singular edge and one singular corner, the other cases being treated in an even simpler way, so we can omit the index ℓ.

In the sequel, we shall make use of the decomposition (2.25) of u obtained in Theorem 2.10, therefore we normally need that p satisfies the assumptions of that theorem. The condition $1 - \frac{2}{p} < \lambda_e$ follows from the assumption (5.1). On the contrary, the assumptions of Theorem 2.6 can be avoided by possibly replacing p by $p' = p - \epsilon$, with a fixed $\epsilon > 0$ small enough, in the case when $2 - \frac{2}{p} = \frac{k\pi}{\omega_{S,j}}$ or $2 - \frac{3}{p} = \lambda_{S,k'}$ (this is possible because the set of $\frac{k\pi}{\omega_{S,j}}$, $\lambda_{S,k'}$ is discrete). Note further that this choice does not disturb the conditions (5.2)–(5.4) for ϵ small enough. In that case, all the arguments below are made with p' instead of p and the conclusion still holds because $\|f; L^{p'}(\Omega)\| \lesssim \|f; L^p(\Omega)\|$.

Since u admits the decomposition (2.25), we need to estimate the regular part and the singular one. In both cases, we reduce this global error into local errors.

For the regular part, using the local error estimate (5.5) of [1] (see also (4.9))

$$|u_r - I_h u_r; W^{1,p}(\Omega_i)| \lesssim h \sum_{k=1}^{3} |\partial_k u_r; W^{1,p}(\Omega_i)|,$$

which holds for any element Ω_i, summing up these estimates and using the Hölder inequality, we easily get that

$$|u_r - I_h u_r; W^{1,2}(\Lambda)| \lesssim h |u_r; W^{2,p}(\Lambda)|. \tag{5.6}$$

For the singular part u_s, we distinguish between the elements whose closure has at least one common point with the singular edge or the singular corner, and the elements away from these singular part of the boundary.

For the elements Ω_i far from the singular edge and the singular corner such that $r_i \gtrsim h^{\frac{1}{\mu}}$ and $R_i \gtrsim h^{\frac{1}{\nu}}$, written in short $i \in I_R$ (R for regular), we apply the local estimate (4.9). Using the Hölder inequality, we have

$$\begin{aligned}|u_s - I_h u_s; W^{1,2}(\Omega_i)|^p &\leq (\text{meas}\Omega_i)^{-1+\frac{p}{2}} |u_s - I_h u_s; W^{1,p}(\Omega_i)|^p \\ &\lesssim (h_i^2 H_i)^{-1+\frac{p}{2}} \left(\sum_{k=1,2} h_i^p |\partial_k u_s; W^{1,p}(\Omega_i)|^p + H_i^p |\partial_3 u_s; W^{1,p}(\Omega_i)|^p \right). \qquad (5.7)\end{aligned}$$

Since here $h_i = h r_i^{1-\mu}$ and $H_i = h R_i^{1-\nu}$, one gets

$$|u_s - I_h u_s; W^{1,2}(\Omega_i)|^p \lesssim h^{\frac{5p-6}{2}} \left(\sum_{k=1,2} |\partial_k u_s; V^{1,p}_{\beta_1,\delta_1}(\Omega_i)|^p + |\partial_3 u_s; V^{1,p}_{\beta_2,0}(\Omega_i)|^p \right), \qquad (5.8)$$

with $\delta_1 = (1-\mu)(2-\frac{2}{p})$, $\beta_1 = \delta_1 + (1-\nu)(\frac{1}{2}-\frac{1}{p})$, and $\beta_2 = (1-\nu)(\frac{3}{2}-\frac{1}{p}) + (1-\mu)(1-\frac{2}{p})$. Therefore in view of the regularity result (Theorem 2.10), we need to check that

$$\delta_1 > 2 - \frac{2}{p} - \lambda_e, \quad \beta_1 > 2 - \frac{3}{p} - \lambda_v, \quad \beta_2 > 2 - \frac{3}{p} - \lambda_v.$$

The first inequality is equivalent to (5.2). As $\mu \leq \nu \leq 1$, for the second one, we get $\beta_1 \geq (1-\nu)(\frac{5}{2}-\frac{3}{p}) > 2 - \frac{3}{p} - \lambda_v$ via (5.3). For the third inequality, as $\mu \leq \nu$, we deduce that $\beta_2 \geq (1-\nu)(\frac{5}{2}-\frac{3}{p})$, and the conclusion follows.

Summing up the estimates (5.8) for all $i \in I_R$, and using again the Hölder inequality, we obtain

$$\begin{aligned}\sum_{i\in I_R} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 &\leq \left(\sum_{i\in I_R} 1\right)^{1-\frac{2}{p}} \left(\sum_{i\in I_R} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^p\right)^{\frac{2}{p}} \\ &\lesssim h^{-3(1-\frac{2}{p})} h^{\frac{5p-6}{p}} \left(\sum_{k=1,2} |\partial_k u_s; V^{1,p}_{\beta_1,\delta_1}(\Lambda)| + |\partial_3 u_s; V^{1,p}_{\beta_2,0}(\Lambda)| \right)^2,\end{aligned}$$

due to Assumption A5. By Theorem 2.10, we conclude that

$$\sum_{i\in I_R} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 \lesssim h^2 \|f; L^p(\Omega)\|^2. \qquad (5.9)$$

For the elements Ω_i far from the singular edge, $r_i \gtrsim h^{\frac{1}{\mu}}$, and close to but away from the singular corner, $0 < R_i \lesssim h^{\frac{1}{\nu}}$, written in short $i \in I_{RS}$ (RS for regular but under the influence of the singular vertex), (5.7) still holds but here $h_i = h r_i^{1-\mu}$ and $H_i = h^{\frac{1}{\nu}}$. This yields

$$\begin{aligned}|u_s - I_h u_s; W^{1,2}(\Omega_i)|^p \quad &\lesssim \quad h^{-2+2p+\frac{p-2}{2\nu}} r_i^{(1-\mu)(2p-2)} \sum_{k=1,2} |\partial_k u_s; W^{1,p}(\Omega_i)|^p \\ &\quad + h^{-2+p+\frac{3p-2}{2\nu}} r_i^{(1-\mu)(p-2)} |\partial_3 u_s; W^{1,p}(\Omega_i)|^p. \qquad (5.10)\end{aligned}$$

In order to obtain an estimate like (5.8), we use the fact that in Ω_i, one has $h^{\frac{1}{\mu}} \lesssim R \lesssim h^{\frac{1}{\nu}}$ and that $r \sim R\theta$, leading to the estimates

$$\begin{aligned}r^{(1-\mu)(2p-2)} \quad &\lesssim \quad h^{[\frac{1-\mu}{\nu}(2-\frac{2}{p})-\frac{\beta}{\mu}]p} R^{\beta p} \theta^{\delta_1 p}, \\ r^{(1-\mu)(p-2)} \quad &\lesssim \quad h^{[\frac{1-\mu}{\nu}(1-\frac{2}{p})-\frac{\beta}{\mu}]p} R^{\beta p},\end{aligned}$$

with δ_1 as before and any $\beta \geq 0$. Inserting these inequalities into (5.10), we obtain

$$\begin{aligned} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^p &\lesssim h^{s_1} \sum_{k=1,2} |\partial_k u_s; V^{1,p}_{\beta,\delta_1}(\Omega_i)|^p + h^{s_2} |\partial_3 u_s; V^{1,p}_{\beta,0}(\Omega_i)|^p \\ &\lesssim h^{s_2} \left(\sum_{k=1,2} |\partial_k u_s; V^{1,p}_{\beta,\delta_1}(\Omega_i)|^p + |\partial_3 u_s; V^{1,p}_{\beta,0}(\Omega_i)|^p \right), \end{aligned}$$

with $s_1 := 2p - 2 + (\frac{p}{2} - 1)\frac{1}{\nu} + [\frac{1-\mu}{\nu}(2 - \frac{2}{p}) - \frac{\beta}{\mu}]p$, $s_2 := p - 2 + \frac{1}{\nu}(\frac{3p}{2} - 1) + [\frac{1-\mu}{\nu}(1 - \frac{2}{p}) - \frac{\beta}{\mu}]p = p(1 - \frac{2}{p})(1 - \frac{\mu}{\nu}) + \frac{p}{\nu}(\frac{5}{2} - \frac{3}{p}) - p\frac{\beta}{\mu}$, where $s_1 - s_2 = p(1 - \frac{\mu}{\nu}) \geq 0$ due to the condition $\mu \leq \nu$. Summing up these estimates for all $i \in I_{RS}$, using the Hölder inequality and Assumption B3, we get

$$\sum_{i \in I_{RS}} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 \lesssim h^{2s_0} \left(\sum_{k=1,2} |\partial_k u_s; V^{1,p}_{\beta,\delta_1}(\Lambda)| + |\partial_3 u_s; V^{1,p}_{\beta,0}(\Lambda)| \right)^2,$$

with $s_0 = \frac{1}{p}s_2 + \frac{1}{2}(\frac{2\mu}{\nu} - 2)(1 - \frac{2}{p}) = \frac{1}{\nu}(\frac{5}{2} - \frac{3}{p}) - \frac{\beta}{\mu}$. Therefore taking $\beta = 2 - \frac{3}{p} - \lambda_v + \varepsilon$ with $\varepsilon > 0$ small enough, we see that the condition (5.4) implies that $s_0 \geq 1$ and by Theorem 2.10, we get

$$\sum_{i \in I_{RS}} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 \lesssim h^2 \|f; L^p(\Omega)\|^2. \tag{5.11}$$

For the elements Ω_i far from the singular corner, $R_i \gtrsim h^{\frac{1}{\nu}}$, but along the singular edge, $r_i = 0$ (written later on $i \in I_E$), we can use the local error estimate (4.10). Together with the regularity results, this yields

$$|u_s - I_h u_s; W^{1,p}(\Omega_i)|^p \lesssim \sum_{j,k=1,2} h_i^{(1-\beta')p} \|r^{\beta'} \partial_{jk} u_s; L^p(\Omega_i)\|^p + H_i^p \sum_{k=1}^{3} \|\partial_{3k} u_s; L^p(\Omega_i)\|^p,$$

for any $\beta' \geq 0$ such that

$$2 - \frac{2}{p} - \lambda_e < \beta' < 1 - \frac{1}{p}. \tag{5.12}$$

With the Hölder inequality as above, the fact that $h_i = h^{\frac{1}{\mu}}$ and $H_i = hR_i^{1-\nu}$, and since the number of elements along the edge is of order h^{-1} (Assumption B2), we get

$$\begin{aligned} \sum_{i \in I_E} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 &\lesssim (h^{-1})^{1-\frac{2}{p}} \left(\sum_{i \in I_E} |u_s - I_h u_s; W^{1,2}(\Omega_i)|^p \right)^{\frac{2}{p}} \\ &\lesssim (h^{-1})^{1-\frac{2}{p}} \left(\sum_{i \in I_E} (h_i^2 H_i)^{-1+\frac{p}{2}} |u_s - I_h u_s; W^{1,p}(\Omega_i)|^p \right)^{\frac{2}{p}} \\ &\lesssim \left(\sum_{i \in I_E} h^{\frac{p-2}{\mu}} R_i^{(1-\nu)(\frac{1}{2}-\frac{1}{p})p} |u_s - I_h u_s; W^{1,p}(\Omega_i)|^p \right)^{\frac{2}{p}} \\ &\lesssim \Bigg(h^{\frac{p-2}{\mu} + \frac{p(1-\beta')}{\mu}} \sum_{j,k=1,2} \|R^{\beta_1} \theta^{\delta_1} \partial_{jk} u_s; L^p(\Lambda)\|^p \\ &\qquad + \sum_{i \in I_E} \sum_{k=1}^{3} h^{p+\frac{p-2}{\mu}} R_i^{p\kappa} \|\partial_{3k} u_s; L^p(\Omega_i)\|^p \Bigg)^{\frac{2}{p}}, \end{aligned} \tag{5.13}$$

where β_1 and δ_1 are as before and $\kappa = (1 - \nu)(\frac{3}{2} - \frac{1}{p})$. The first term in this right-hand side is treated as in the first case since we can show that $\beta' = \delta_1$ satisfies the two above conditions

(5.12). We treat now the second term. As on each Ω_i, with $i \in I_E$, one has $h^{\frac{1}{\mu}} \lesssim hR_i^{1-\nu}$, we deduce that the second term can be estimated by

$$\sum_{i\in I_E}\sum_{k=1}^{3} h^{p+\frac{p-2}{\mu}} R_i^{p\kappa}\|\partial_{3k}u_s; L^p(\Omega_i)\|^p \lesssim h^{2p-2}\sum_{k=1}^{3}\|R^{\beta_3}\partial_{3k}u_s; L^p(\Lambda)\|^p,$$

with $\beta_3 = (1-\nu)(\frac{5}{2}-\frac{3}{p})$. Since the condition (5.3) is equivalent to $\beta_3 > 2-\frac{3}{p}-\lambda_v$, by Theorem 2.10, we get

$$\sum_{i\in I_E}\sum_{k=1}^{3} h^{p+\frac{p-2}{\mu}} R_i^{p\kappa}\|\partial_{3k}u_s; L^p(\Lambda)\|^p \lesssim h^p\|f; L^p(\Omega)\|^p.$$

Consequently, we arrive at

$$\sum_{i\in I_E}|u_s - I_h u_s; W^{1,2}(\Omega_i)|^2 \lesssim h^2\|f; L^p(\Omega)\|^2. \tag{5.14}$$

For the element Ω_i meeting the singular corner, we directly use Theorem 4.9. Namely by Estimate (4.7) and the Hölder inequality, we have

$$|u_s - I_h u_s; W^{1,2}(\Omega_i)| \lesssim h^{s_3}\sum_{j,k=1,2}\|R^\beta\theta^\delta\partial_{jk}u_s; L^p(\Omega_i)\| + h^{s_4}\sum_{k=1}^{3}\|R^\beta\partial_{k3}u_s; L^p(\Omega_i)\|, \tag{5.15}$$

with $s_3 = \frac{1}{\mu}(2-\frac{2}{p}-\beta-\delta)+\frac{1}{\nu}(\frac{1}{2}-\frac{1}{p}+\delta)$, $s_4 = \frac{1}{\mu}(1-\frac{2}{p}-\beta)+\frac{1}{\nu}(\frac{3}{2}-\frac{1}{p})$, and β, δ meeting both conditions of Theorems 4.9 and 2.10. The appropriate choice is $\beta = 2-\frac{3}{p}-\lambda_v+\varepsilon$ and $\delta = 2-\frac{2}{p}-\lambda_e+\varepsilon$, with $\varepsilon > 0$ small enough, since they satisfy the conditions of Theorems 4.9 and 2.10, owing to the assumption $2 < p < p_+$ and because the condition (5.4) implies that $s_3 \geq 1$, $s_4 \geq 1$ (since $\mu \leq \nu$). In order words, with this choice and Theorem 2.10, Estimate (5.15) yields

$$|u_s - I_h u_s; W^{1,2}(\Omega_i)| \lesssim h\|f; L^p(\Omega)\|. \tag{5.16}$$

From (5.6), (5.9), (5.11), (5.14) and (5.16), we get the assertion. □

Corollary 5.2 *Let u be the solution of the boundary value problem (2.1) with $f \in L_p(\Omega)$, $2 < p < p_+$, p_+ from (5.1), and let u_h be the finite element solution of (3.4). Then under the assumptions of Theorem 5.1, the following error estimate holds:*

$$\|u-u_h; W^{1,2}(\Omega)\| \lesssim |u-u_h; W^{1,2}(\Omega)| \lesssim h\|f; L^p(\Omega)\|.$$

Remark 5.3 Note that the restriction $p < p_+$ is not essential for this estimate, because $f \in L^p(\Omega)$ yields $f \in L^q(\Omega)$ for $q \leq p$ and $\|f; L^q(\Omega)\| \lesssim \|f; L^p(\Omega)\|$. We can apply Theorem 5.1 for $q < p_+$. Nevertheless, we have to replace p in the conditions of the above theorem by $\min\{p;\ p_+ - \kappa\}$, $\kappa > 0$ arbitrary.

Remark 5.4 In order to use meshes which are not too much refined, the estimates are most favourable for p close to 2. For $p = 2+\delta$ (δ is an arbitrarily small real number), the refinement conditions reduce to

$$\begin{aligned}
\mu_\ell &< \lambda_e^{(\ell)}\left(1-\frac{\delta}{2+2\delta}\right),\\
\nu_\ell &< \left(\lambda_v^{(\ell)}+\frac{1}{2}\right)\left(1-\frac{3\delta}{4+5\delta}\right),\\
\frac{1}{\nu_\ell}+\frac{1}{\mu_\ell}\left(\lambda_v^{(\ell)}-\frac{1}{2}\right) &> 1+\frac{3\delta}{4+2\delta}\left(\frac{1}{\mu_\ell}-\frac{1}{\nu_\ell}\right).
\end{aligned}$$

On the other hand it is not clear in which way the constant C in the error estimate depends on p; we suspect that $C \to \infty$ for $p \to 2$.

Remark 5.5 The conditions (5.2) and (5.3) are the edge and vertex refinement conditions, respectively. They are natural because they balance the edge and vertex singularities (compare with [5, 6, 22]). On the contrary, the condition (5.4) seems to be artificial but it actually comes from the anisotropy of the mesh near the corner. Indeed, (5.4) follows from (5.3) and $p > 2$ in the case $\mu_\ell = \nu_\ell$. In the case $\mu_\ell \neq \nu_\ell$, it imposes a condition between μ_ℓ and ν_ℓ, this means that the mesh cannot be too much anisotropic. For the Fichera example treated in Section 6, we have $\lambda_v \approx 0.45$ and $\lambda_e = \frac{2}{3}$. We then see that for p close to 2, the condition (5.4) holds for $\mu = 0.6$ and $\nu = 0.9$.

6 A test example

We consider the Poisson equation with a specific right hand side, together with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= R^{-1} \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

The domain $\Omega := (-1,1)^3 \setminus [0,1]^3$ has three edges with interior angle $\omega_0 = \frac{3}{2}\pi$, which meet in the center of coordinates; we denote by R the distance to this point. Sometimes such a corner is called a Fichera corner. Note that the right hand side is contained in $L_p(\Omega)$ for $p < 3$.

In order to determine the regularity of the solution, we consider first the corner singularity and find that $\lambda_v \approx 0.45$ [29]. The edge singularities are described by $\lambda_e = \frac{\pi}{\omega_0} = \frac{2}{3}$.

This problem was solved first with ungraded meshes and mesh sizes $h_i = \frac{1}{i}$ ($i = 2, 3, \ldots, 48$). We compare this with three refinement strategies. The first one is obtained by a simple coordinate transformation

$$x_j := x_j \cdot |x_j|^{-1+\frac{1}{\mu}}, \quad j = 1, 2, 3$$

for all vertices (x_1, x_2, x_3). It leads to overrefinement near the coordinate planes, see Figure 6.1. The second one was described by our constructive proof of the existence of meshes satisfying all the conditions posed in Section 3, see pages 9–12. The corresponding mesh is illustrated in Figure 6.2. The optically bad elements near the diagonals can be avoided by using the strategy of Case 4a instead of Case 4, compare the remark at the end of Section 3 and Figure 6.3. For all ℓ we used the parameters $\mu^{(\ell)} = \nu^{(\ell)} = 0.6$.

The calculations were done using the code *FEMPS3D*, details are described in [4]. We remark only that the energy of the finite element error was estimated with an error estimator of residual type which was tuned for treating anisotropic meshes. The norms are given in form of a diagram in Figure 6.4.

We see that the theoretical approximation order $h \sim N^{-\frac{1}{3}}$, N is the number of nodes, can be verified in the practical calculation for all three refinement strategies. The error is the smallest in the third refinement strategy, however, the difference between the strategies is small.

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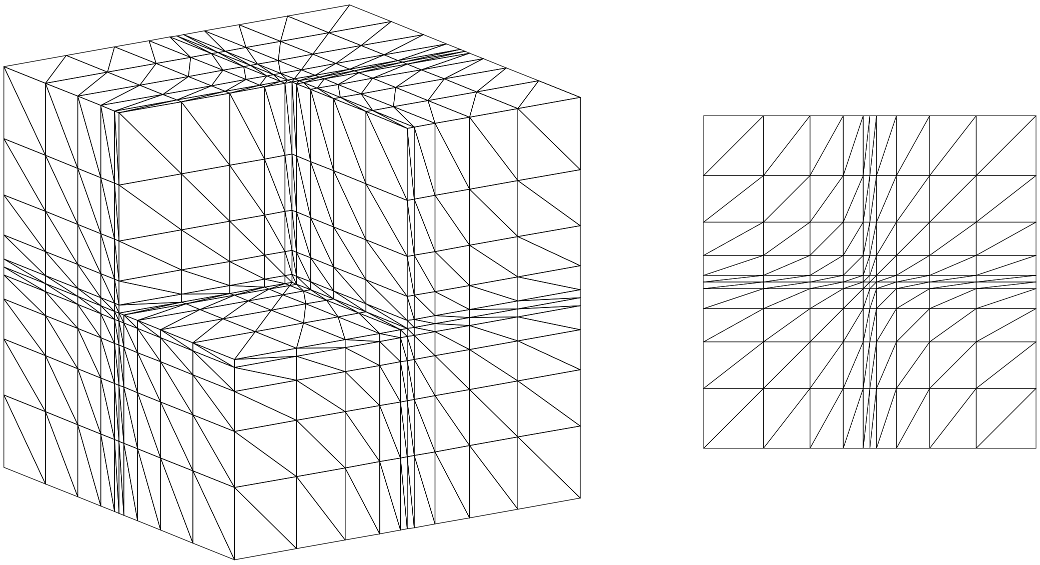

Figure 6.1: Strategy 1: Simple coordinate transformation. Left: perspective view. Right: cut at $x_3 = 0$.

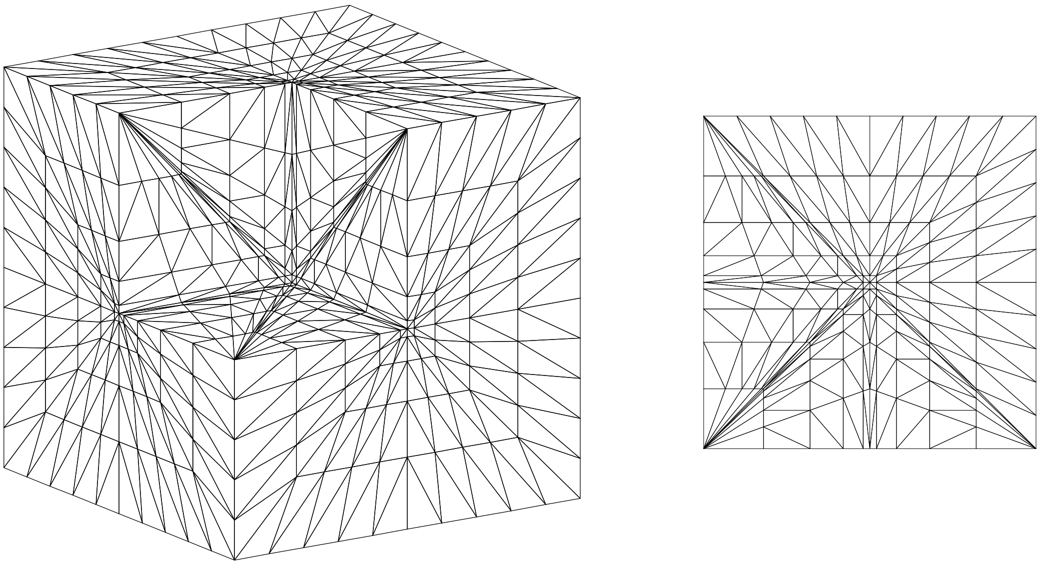

Figure 6.2: Strategy 2: Refinement according to Cases 1–4. Left: perspective view. Right: cut at $x_3 = 0$.

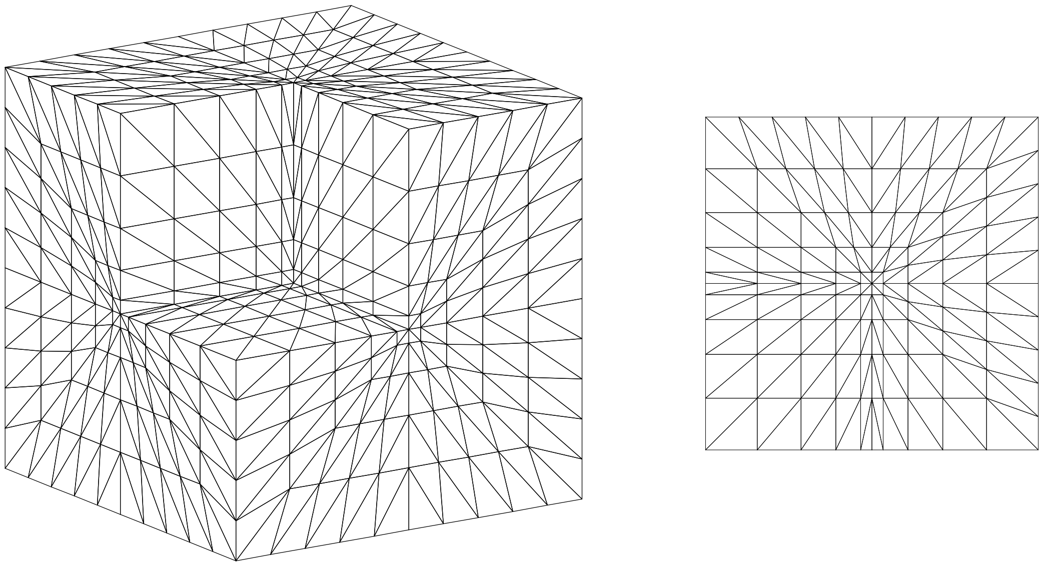

Figure 6.3: Strategy 3: Refinement with Case 4a instead of Case 4. Left: perspective view. Right: cut at $x_3 = 0$.

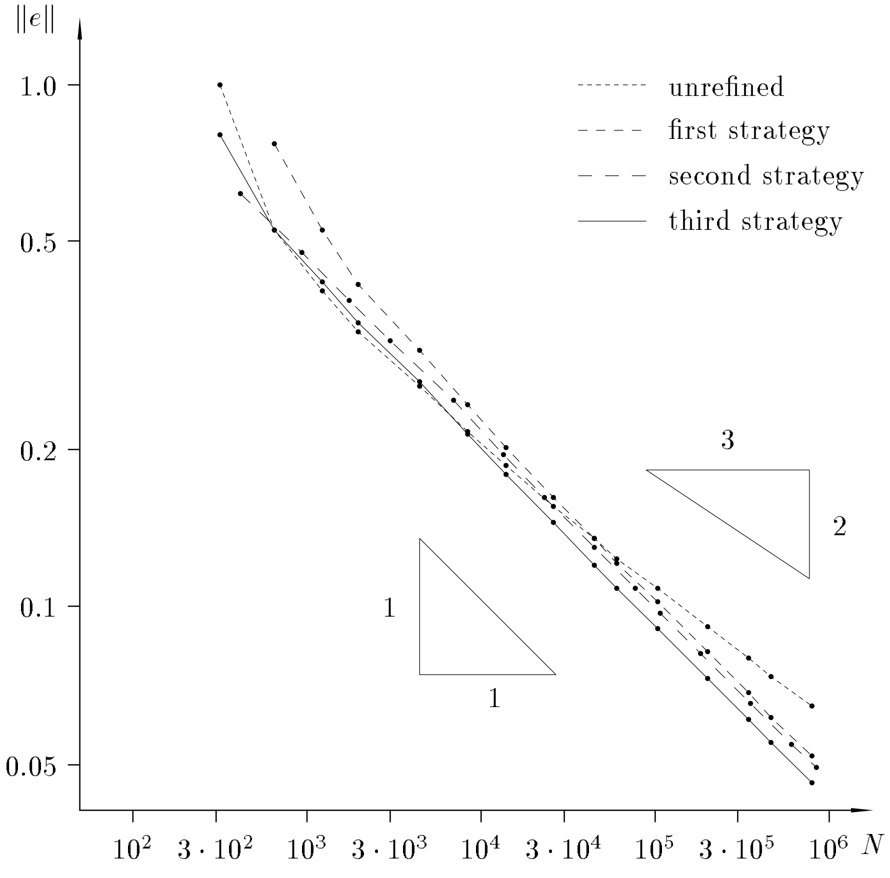

Figure 6.4: Estimated error η in the energy norm for various mesh sizes.

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