Technische Universität Chemnitz-Zwickau

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Jörg Weickert

Navier-Stokes equations as a differential-algebraic system

Preprint SFB393/96-08

key words: differential-algebraic equations, strangeness index, Navier-Stokes equations, method of lines, Marker-and-Cell net

AMS classification: 65F20, 65M06, 65M15, 65M20, 65M60, 65N50

Author's adress:

Jörg Weickert
Fakultät für Mathematik
TU Chemnitz-Zwickau
D-09107 Chemnitz, FRG
e-mail: weickert@mathematik.tu-chemnitz.de

This work has been supported by the Deutsche Forschungsgemeinschaft under the grant no. Me 790/5-2 "Differentiell-algebraische Gleichungen."

Preprint-Reihe des Chemnitzer SFB 393

SFB393/96-08 **August 1996**

Abstract

Nonsteady Navier-Stokes equations represent a differential-algebraic system of strangeness index one after any spatial discretization. Since such systems are hard to treat in their original form, most approaches use some kind of index reduction. Processing this index reduction it is important to take care of the manifolds contained in the differential-algebraic equation (DAE). For several discretization schemes for the Navier-Stokes equations we investigate how the consideration of the manifolds is taken into account and propose a variant of solving these equations along the lines of the theoretically best index reduction. Applying this technique, the error of the time discretisation depends only on the method applied for solving the DAE.

Die Kunst besteht nicht darin, selbst wahnsinnig viele tolle Ideen zu haben. Wichtig ist, die wahnsinnig vielen tollen Ideen anderer zu erkennen, einzuordnen und zu einem Gesamtbild zusammenzufügen, das dann das Neue offenbart.

1 Introduction

Computational fluid dynamics (CFD) is a widely applied tool in modeling a lot of technical problems. A typical example are the equations of gas dynamics under the assumption of incompressibility. The resulting system is known as the Navier-Stokes equations. It consists of as many differential equations as the dimension of the model indicates and the condition of incompressibility, see e.g. [15]:

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \triangle \mathbf{u} - \nabla \mathbf{p} + \mathbf{f}. \tag{1}$$

$$0 = \nabla \cdot \mathbf{u} \tag{2}$$

These equations, together with appropriate initial and boundary conditions, are to be solved in $\Omega \times [0, T]$, where Ω is a bounded open domain in $\mathbf{R}^d$ ($d = 2$ or 3 the dimension of the model) and T the endpoint of the time interval. For reasons of simplification we will restrict our considerations to the two-dimensional case here. The results hold for a three-dimensional model as well. Besides, the domain of reference shall be rectangular. This is indeed

a restriction, but we will remark at the according places whether some technique may be generalized to other domains or not.
After applying the method of lines (MOL), i.e. carrying out a spatial discretization by finite difference or finite element techniques, these equations can be written as the differential-algebraic system

$$M\dot{\mathbf{u}}(t) = K(\mathbf{u})\mathbf{u}(\mathbf{t}) - Bp(t) + \mathbf{f}(t) \tag{3}$$
$$0 = B^T\mathbf{u}(t), \tag{4}$$

see [2]. Here $\mathbf{u}(t), p(t)$ and $\mathbf{f}(t)$ are approximations to the time- and space-dependent quantities $\mathbf{u}, p$ and $\mathbf{f}$ of (1), (2). The matrix M is symmetric and positive definite (in the case of finite differences or an at most bilinear finite element space, M is simply the identity). The quantity B stands for the discrete gradient operator, while $K(u)$ represents the linear and nonlinear velocity terms.
The DAE (3), (4) is of higher index (i.e. non-decoupled), since the pressure p does not appear in the algebraic condition. If we assume that B is of full column rank, then the differentiation index is two [2]. However, since p is only determined up to an additive constant, B has in general a rank deficiency which causes the undeterminedness of at least one solution component. The concept of the differentiation index [2] cannot be applied to such systems. Kunkel and Mehrmann [12] have generalized the index concept to the case of over- and underdetermined DAE's. Their so-called strangeness index (or s-index) μ is the number of additional block columns needed in the derivative array [10] to be able to filter out a strangeness free system by transformations from the left. This system then represents a DAE of differentiation index one with possibly undetermined components or a system of ordinary differential equations. Therefore μ is one lower than the differentiation index, if the system is a DAE of at least differentiation index one without undeterminedness. For ordinary differential equations (differentiation index zero), μ is defined as zero.
Within these settings, (3), (4) can be characterized as a DAE of strangeness index $\mu = 1$. Several difficulties appear, when solving this system numerically, which will be outlined in Section 2. It is common to reduce the index of (3), (4) aiming in the reformulation of the system as a strangeness free DAE. Many finite element (FE) and finite difference (FD) solution methods carry out such an index reduction, but not all of them take good care of the

manifold (4) and the so-called hidden manifold

$$0 = B^T M^{-1} K(\mathbf{u})\mathbf{u}(t) - B^T M^{-1} Bp(t) + B^T M^{-1}\mathbf{f}(t) \tag{5}$$

which arises in pre-multiplying (3) by $B^T M^{-1}$ and inserting the differentiated incompressibility condition (4). Both the manifold (4) and the hidden manifold (5) must be satisfied by the solution $(\mathbf{u}, p)$ in order to ensure that the solution reflects the properties of the DAE also after index reduction. In Section 3 we reveal this for a set of popular CFD solution techniques and make a proposal how the Navier-Stokes equations can be solved in the sense of a "correct" index reduction. This results in a system of s-index zero which preserves both manifolds as described in [12]. Section 4 deals with problems which must be taken into consideration when applying this particular index reduction. It will be shown there that only Marker-and-Cell (MAC) meshes are well-suited for forming the strangeness free system according to the proposal mentioned above. The advantage of this strategy over all the other ones is that the error of the time discretization is not influenced by the index reduction. The numerical solution then produces an error in time which is equal to the error of the time discretization method applied.

2 Problems in solving Navier-Stokes equations

An obvious, but not essential problem in solving the incompressible Navier-Stokes equations (1), (2) is the non-uniqueness of the solution caused by the pressure term which only appears as first derivative. Many different strategies have been developed to deal with this difficulty. They are designed to ensure the discrete analogue of the condition on p,

$$\int_\Omega p\, dx = 0. \tag{6}$$

Applying finite difference methods, this condition can be satisfied in the case of regular grids by claiming

$$\sum_{i \in \omega_h} p_i = 0. \tag{7}$$

Another possibility is to define the pressure in one single grid point explicitely. There are also methods which do not set pressure components at all in advance. They achieve a unique solution to the pressure equation by solving iteratively either a disturbed but regular pressure equation or the original

equation with suitable initial values.
In the case of finite element techniques, the condition (7) is posed on the space of the test functions for p.
Considering the differential-algebraic system (3), (4) after an arbitrary MOL discretization, the singularity of the solution may also be treated by solving in the least squares sense. However, since this usually leads to a global dependence of the solution on all time discretization points, other generalized inverses are often better here, see [11].
The discussion which spatial discretization technique is most appropriate for CFD is a more difficult problem than the one caused by the non-uniqueness of the solution. While FE methods became more and more popular during the last decades and have been accepted in many fields of mathematical modeling, it is not clear whether they will prove superior for the discretization of Navier-Stokes equations, too. This is particularly due to the opportunity of a straightforward finite difference discretization by means of the famous MAC net which was introduced by Harlow and Welch [6] in 1965, see Figure 1.

Since this technique requires different control volumina for each velocity

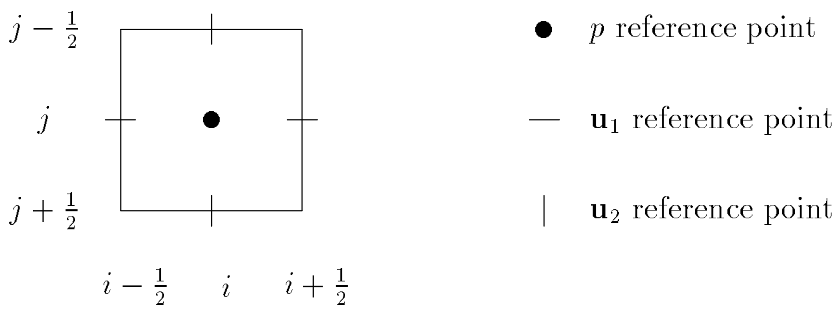

Figure 1: Location of variables in a staggered grid

component and another one for the pressure, it is separate from finite element approaches. The most obvious advantage of the MAC (or staggered) grid is that it works with a minimum of averaging operations [1] which is not the case for semi-staggerd and non-staggered grids. While semi-staggered grids have almost completely disappeared from practical CFD, the non-staggered (or collocated) grid may perform better with respect to non-rectangular do-

mains, and special techniques such as multigrid methods are easier to apply than for staggered grids. However, there are some problems with respect to pressure computation. Collocated grids require boundary conditions to the pressure, in contrast to MAC meshes. Besides, a straightforward construction of a Laplace operator for pressure computation out of the discrete divergence and gradient operator leads to a disintegration of the solution. In the case of two spatial dimensions, for instance, the solution vector p decouples into four independent pieces. A so-called selective interpolation can be used in order to avoid this unsatisfactory behavior. However, this produces a second-order error in the solution for p, see [19].
Turning to FEM, the element of lowest possible degree is the Q_1-P_0-element which corresponds to the semi-staggered grid. Thus all of the problems known from FDM appear: The kernel of the discrete gradient operator has two linear independent elements instead of the one caused by the nonuniqueness of p [5]. This raises so-called checkerboard instabilities, i.e. p shows an oscillating behavior. The inf-sup condition which is always important in FEM approaches to Navier-Stokes equations is not uniformly satisfied, but depends on the mesh size h [5] :

$$\sup_{\mathbf{v}\in X_h} \frac{1}{|\mathbf{v}|} \int_\Omega q \operatorname{div} v \, dx \geq C\, h \, \|q\|_{0,\Omega} \ \ \forall q \in M_h.$$

Here X_h and M_h are appropriate discrete spaces for the velocity and pressure vector, respectively.
When constructing the space V_h of divergence free trial functions for $\mathbf{u}$, the technique presented in [5] for equidistant grids is not applicable for rectangular discretizations with variable mesh size or non-regular grids. As the space V_h is important for index reduction preserving the manifolds (4) and (5), this non-transferability will be discussed in detail in Section 4.
A way to avoid the problematic Q_1-P_0-element is the use of trial functions of higher degree. The simplest variant is the Mini-FE which is investigated e.g. in [18], where a multigrid method is applied. However, this approach is not well-suited for the unsteady case, since it causes restrictions to the time discretization parameter.
In [4] a new finite element along the lines of the finite-volume strategy is presented which makes modeling with the Q_2-P_0-element possible. But the switch to trial spaces of higher degree creates additional difficulties which make FE schemes harder to handle than finite differences. For instance, the matrix M of (3), (4) is no longer the identity which gives the hidden manifold

(5) a more complicated form. Compared with this, the MAC discretization seems to be a practicable way. According to [14], [13], this technique can be generalized to other than rectangular domains as well.
Another difficulty in treating Navier-Stokes equations is the nonlinearity of the velocity term in (3). However, this problem is well understood today, and several strategies have been developed for the different discretization variants, e.g. upwind techniques, see [5]. With respect to a DAE approach, the nonlinear case will not influence the index (neither differentiation nor s-index), since we can linearize $K(\mathbf{u})$ so that the system (3), (4) yields the same structure.

3 Decoupling velocity and pressure computation by means of index reduction

As stated in Section 1, the system (3), (4) is of higher index, namely s-index 1. Solving such systems as they appear originally, one can get in difficulties because of the mingling of differential and algebraic components, the so-called "strangeness" in the terminology of [12]. It is useful to first remove this strangeness before solving the DAE. Most Navier-Stokes solution techniques do so although not explicitely mentioning that an index reduction is carried out. If the index reduction is omitted, the results may become unsatisfactory, especially in the nonsteady case. For example, in [20] examples are computed, where a steady state is reached, and it is stated that "satisfactory smoothing" is achieved "by choosing $\triangle t$ small enough." But this is completely unpractical if long time computations are carried out.
We have already outlined that the concept of the s-index guarantees a characterization also if no unique solution to the DAE exists. However, this is not the main advantage of this approach over the usual concept of the differentiation index. The biggest progress seems to be that [12] provides a way to reformulate the higher-index DAE as a strangeness free system of the same dimension and with the same solution structure as the original system. In other words, it is possible to rewrite a DAE of higher index in a so-called normal form of s-index zero. This form not only reflects the manifold included in the original system but also all of the hidden manifolds. Thus, using the strangeness free normal form, a consideration of all manifolds is ensured, which makes this approach superior over other index reduction vari-

ants. Moreover, the derivative term is not transformed, so that no errors in time are caused by the index reduction, as it is the case for any other known index reduction strategy for Navier-Stokes equations. This will be shown in the following.
A straightforward index reduction is e.g. the one described in [2]. A DAE of the original size arises replacing (4) by (5). But this leads to disregard of the mass balance expressed by (4) which may cause inexact solutions after numerical treatment.
Index reduction variants like the so-called penalty method are quite popular in the FEM framework [5]. This method, which is a singular perturbation approach, represents a regularization by adding a p-term to the incompressibility condition (4) leading to

$$\begin{aligned} M\dot{\mathbf{u}} &= K\mathbf{u} - Bp + \mathbf{f} \\ 0 &= B^T\mathbf{u} - \varepsilon p, \end{aligned}$$

which is strangeness free, since the derivative of the second condition with respect to p is nonsingular (see e.g. [2]). Rearranging this condition and inserting into the first one gives

$$M\dot{\mathbf{u}} = (K - \frac{1}{\varepsilon}BB^T)\mathbf{u} + \mathbf{f}.$$

The solution of this problem should differ from the one of the original system in the magnitude of $O(\varepsilon)$. As stated in [16], this is not true for time-dependent problems: Here we have a an error of $O(\sqrt{\varepsilon})$. Investigating this error in more detail, a dependence of ε and the time step according to $O(\tau+\varepsilon)$ is obtained. This implies restrictions for τ, such that the method is not suited for nonsteady problems.
In FDM approaches, a pressure correction method (also known as method of symmetrical approximation or operator splitting method) is often applied for decoupling $\mathbf{u}$- and p-computation [1]. Here a semi-implicit time discretization is carried out in advance so that the system can be written

$$\begin{aligned} \frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\tau} &= K\mathbf{u}^j - Bp^{j+1} + \mathbf{f}^{j+1} \\ B^T\mathbf{u}^{j+1} &= 0 \end{aligned}$$

where $j+1$ is the number of the current time step and τ the time discretization parameter. The momentum equation is then split into

$$\frac{\tilde{\mathbf{u}} - \mathbf{u}^j}{\tau} = K\mathbf{u}^j + \mathbf{f}^{j+1}, \tag{8}$$

$$\frac{\mathbf{u}^{j+1} - \tilde{\mathbf{u}}}{\tau} = -Bp^{j+1}. \tag{9}$$

Taking into account $B^T \mathbf{u}^{j+1} = 0$ (this will guarantee that the solution at the new time layer is divergence free), we obtain from (9) equations for p^{j+1} and $\mathbf{u}^{j+1}$:

$$\tau B^T B p^{j+1} = B^T \tilde{\mathbf{u}} \tag{10}$$

$$\mathbf{u}^{j+1} = \tilde{\mathbf{u}} - \tau B p^{j+1}. \tag{11}$$

The system is solved integrating first the perturbed momentum equation (8) where it is accepted that a non-divergence free solution $\tilde{\mathbf{u}}$ is obtained. After having integrated the Poisson equation (10), a re-projection to the manifold (4) is possible computing $\mathbf{u}^{j+1}$ by (11).
This strategy can be described by the system

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} K & 0 \\ -B^T & \tau B^T B \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} + \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

which is strangeness free as can be understood following the remarks of, e.g., [9], [12]. Also in this equation, a perturbation parameter (τ) occurs.
A disadvantage of this approach is that the accuracy of p depends not only on the spatial discretization, but also on the time discretization parameter τ, that means the decoupling is not complete. To be more specific, the time discretization error raised by the above decoupling is $O(\tau)$. Similiar variants are possible which do not omit the whole pressure term from the momentum equation. Then an error of $O(\tau^2)$ arises and therefore this method is more practical than (8), (10), (11). Both techniques are investigated in more detail in [1].
Hou and Wetton show in [8] that the pressure correction method is equivalent to the one of [2] described above.
In [17] a pressure correction method is applied to the time-dependent MOL-discretized Navier-Stokes equations which were obtained by a finite element spatial discretization. The results can be summarized, using our terminology, as follows:
Theoretically, splitting methods are more efficient than solving directly the s-index-one system. They also require a lower total expense, although the step sizes do not differ considerably in both cases. The resulting matrices are partially the same as for FDM/MAC discretizations, but FE techniques

are easier to generalize to other than regular kinds of grids, and allow the FEM error analysis. The solutions obtained from the s-index-one and the strangeness free system are almost the same, even in a pointwise sense.
These results are the ones to be expected from a DAE point of view. It is proposed in [17] to use the splitting technique for decoupling velocity and pressure computations and a non-staggered grid for spatial discretisation, but as we will see in the following, a better index reduction procedure is possible. Besides, if the MAC discretization is not applied, a loss of exactness occurs and the boundary conditions are harder to describe, see Section 2.
There are strategies which avoid perturbations as in the foregoing examples and therefore can take the manifolds into consideration more carefully. The Glowinsky-Pironneau scheme (see e.g. [5]) for example carries out an index reduction excluding (4) from the system, but a projection onto that manifold after each iteration is part of the method. The manifold is even included in the resulting system applying techniques like the FEM with divergence free trial and test spaces. Heywood and Rannacher [7] make use of this approach and determine error estimations for the Crank-Nicholson time discretization. They prove that under appropriate assumptions the error of $\mathbf{u}$ and p behave like $O(\tau^2)$ and $O(\tau)$, respectively. A disadvantage of this technique is that the error constants depend on t.
The analogous approach in the FDM case is a variant presented by Dobrowolski [3]. The momentum equation is multiplied from the left by a matrix P whose columns form an orthogonal basis of kernel B^T. Besides, a transformation of $\mathbf{u}$ according to $\mathbf{u} = P\mathbf{w}$ is carried out. Since $P^T B = 0$, the pressure term disappears, resulting in a condition to compute $\mathbf{w}$,

$$P^T P\dot{\mathbf{w}} = P^T K P\mathbf{w} + P^T\mathbf{f}. \tag{12}$$

Both approaches lead to a system consisting of the ODE (12) which contains less conditions than the momentum equation (3) and an algebraic equation to compute the pressure p. These conditions together form a strangeness free DAE. Note that the resulting system is not of the same dimension as the original system (3), (4).
A continuation of these ideas with only divergence free test space or only multiplication from the left by P^T supplies the strangeness free normal form as will be shown in Section 4. The resulting matrices loose their banded structure then, but the system reflects all manifolds in the right way. The great advantage of this approach is that no restrictions to τ occur, thus

allowing the application of any time discretization technique, e.g. Runge-Kutta or backward-differencing methods of any desirable order, see [12]. Summarizing these considerations, we can state that setting up a strangeness free normal form is essentially the only method to yield time discretizations of arbitrary order. In the following we will call this procedure the normal form approach. In Table 1 an overview over the error orders in time is given for the index reduction methods described in this section.

Method	Error order in time for $\mathbf{u}$
a) Penalty method	$\tau + \varepsilon$
b) Pressure correction methods	
• Using velocity equation (8)	τ
• Improved variant	τ^2
c) Divergence-free test and trial spaces	τ^2 with Crank-Nicholson
d) Normal form approach	τ^q, q the order of the time discretization method

4 The normal form approach

A linear differential-algebraic system of arbitrary s-index μ,

$$E(t)\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t),$$

can, under suitable assumptions (see [10]), be transformed into strangeness free normal form by means of at most $\mu \cdot 3 + 2$ rank decisions. The procedure is described in [12].

Consider the case of the MOL-discretized Navier-Stokes equations (3), (4). For a linear (Stokes equations) or linearized matrix K (see Section 2) we have

$$E = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} K & -B \\ B^T & 0 \end{bmatrix}, \tag{13}$$

which form a semi-explicit DAE if $M = I$ as it is the case when applying FDM or a FEM with Q_1-P_0-element. Here the index reduction can be carried out in an easy way. Defining P as in the previous section as a matrix whose columns form an orthogonal basis for kernel B^T and letting $\tilde{B}^T$ be obtained from B^T by leaving out as many rows as the rank defect of B indicates (one row at staggered and non-staggered grids), we can multiply the momentum equation by the nonsingular matrix

$$\begin{bmatrix} P^T \\ \tilde{B}^T \end{bmatrix}$$

without loss of information. This together with the hidden manifold (5) results in a strangeness free system

$$\begin{aligned} P^T\dot{\mathbf{u}} &= P^T K\mathbf{u} + P^T\mathbf{f} && (14)\\ 0 &= \tilde{B}^T\mathbf{u} && (15)\\ 0 &= B^T K\mathbf{u} + B^T Bp + B^T\mathbf{f}. && (16) \end{aligned}$$

The first two equations together form a strangeness free DAE and therefore can be used for the computation of $\mathbf{u}$ in a suitable way. In (16) we have returned to B^T instead of $\tilde{B}^T$ which is possible according to [12]. This equation then can serve for pressure computation. Thus, the system is completely decoupled into one part for velocity computation and another one for deriving the pressure. This means, that it is possible to compute just the velocity at each time step. The pressure may be determined by (16) at any optional point. The matrix $B^T K$ must be formed only once in the case of linear Navier-Stokes equations.
This approach is somehow like the one of Dobrowolski [3] with the difference that we have multiplication by P only from the left. It corresponds to a FEM with divergence free test functions $\mathbf{v} \in V_h$, but non-reduced trial space X_h. It is expressed by (15) that $\mathbf{u}$ is divergence free.
As the hidden manifold (5), which explicitely occurs in the strangeness free normal form, contains the matrix M^{-1}, it may be difficult to derive the system (14)-(16) in the case of finite elements with trial functions of higher degree. As outlined in [15], the matrix M is diagonalizable without disturbance of the method, at least for a uniform mesh of bilinear elements. In order to avoid the occurence of M, one could try the Q_1-P_0-element. Girault and Raviart [5] specify the construction of the divergence free space V_h in the case of an equidistant rectangular mesh. However, for meshes with varying step size, this technique fails:
The divergence is approximated at semi-staggered grids according to

$$(\mathrm{div}\mathbf{u})_{ij} = \frac{k_i}{2}[u_{or} + u_{ur} - (u_{ol} + u_{ul})] + \frac{h_j}{2}[v_{or} + v_{ol} - (v_{ur} + v_{ul})],$$

where u and v are the first and second component of $\mathbf{u}$ in a two-dimensional model, h_j and k_i stand for the horizontal and vertical mesh size of the mesh

(i,j), respectively, and o, u, r, l denote the upper, lower, right and left boundaries of the mesh. For instance, or is the upper right corner, see Figure 2.

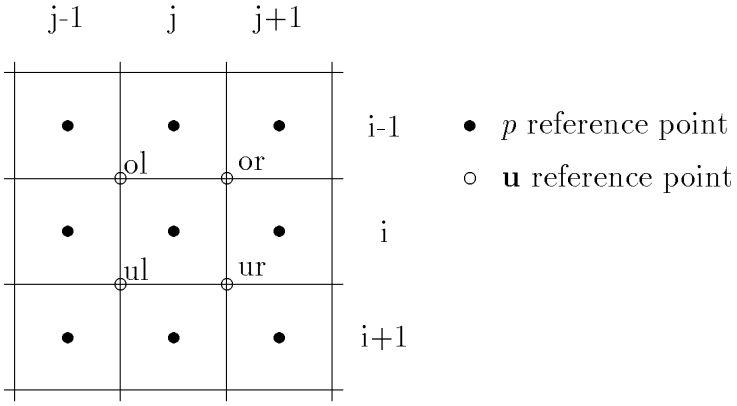

Figure 2: Semi-staggered grid with divergence free function $\mathbf{v}_{ij}$

For a non-equidistant spatial discretization, we define divergence free functions $\mathbf{v}$ according to [5]. The function located in the cell (i,j), e.g., takes the values $\begin{bmatrix} 2/k_i \\ 2/h_j \end{bmatrix}$ in ol, $\begin{bmatrix} -2/k_i \\ 2/h_j \end{bmatrix}$ in ul, $\begin{bmatrix} 2/k_i \\ -2/h_j \end{bmatrix}$ in or and $\begin{bmatrix} -2/k_i \\ -2/h_j \end{bmatrix}$ in ur. In all other points, it is defined to be zero. The divergence of this function indeed vanishes in the cell (i,j):

$$(\operatorname{div}\mathbf{v}_{ij})_{ij} = \frac{2k_i}{2k_i}[1-1-(1-1)] + \frac{2h_j}{2h_j}[1-1-(1-1)] = 0$$

For the neighbouring meshes, however, we have

$$(\operatorname{div}\mathbf{v}_{ij})_{i-1,j-1} = \frac{2k_{i-1}}{2k_i} - \frac{2h_{j-1}}{2h_j} = \frac{k_{i-1}}{k_i} - \frac{h_{j-1}}{h_j}$$

which only vanishes if $k_i = k_{i-1},\ h_j = h_{j-1}$. Forming the divergence in the other neighbouring meshes, similiar results are obtained which force the condition that all horizontal and vertical mesh sizes, respectively, must be equal in order to guarantee that the functions $\mathbf{v}$ are divergence free. That is, there is no straightforward approach for the construction of the matrix P in semi-staggered grids in the non-equidistant case.
It should be noted that a similiar result is obtained in the case of collocated

grids. Since we are not going to consider this type of grid in more detail, the proof is left out here.
Investigating the procedure for the MAC-net, one can see in an easy way at which places the divergence free elements must be located: The dimension of the space V_h is the number of columns of the matrix P which equals, according to (14)-(16), the length of the vector $\mathbf{u}$, lowered by the rank of B. Let m and n be the vertical and horizontal mesh numbers, respectively, in a two-dimensional model. We then obtain the dimension of V_h by

$$m(n-1)+n(m-1)-(mn-1) = mn-m-n+1 = (m-1)(n-1).$$

The result suggests, that each of the $(m-1)(n-1)$ points of intersection of the velocity grid is assigned to one divergence free function. The values of a vector $\mathbf{v} \in V_h$, which equals a column of P, are as shown in Figure 3 (zero in all other points). These functions are indeed divergence free which

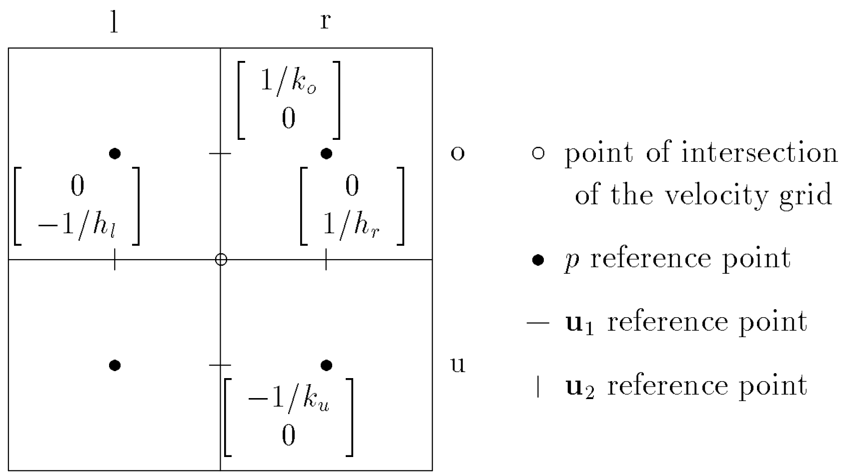

Figure 3: Marker-and-Cell net with divergence free function $\mathbf{v}$

is shown in [3] also for κ-connected grids with regular cells where κ is the number of "holes" in the domain. Thus, methods using staggered grids seem to be best suited for a normal form approach to solve nonsteady Navier-Stokes equations. So we can state that the use of the MAC mesh allows the index reduction (14)-(16) which takes all the manifolds into consideration and therefore makes the application of higher-order time discretization techniques convenient.

5 Conclusion

The cleanest index reduction techniques for the solution of Navier-Stokes equations are those that keep both the manifold (4) and the hidden manifold (5). Among them one can count the strategy of Dobrowolski [3] and FE techniques with divergence free test and trial spaces. The advantage of these methods is the banded structure of the determining matrices of the DAE which is lost in the strangeness free normal form. But error estimates of at most second order can be achieved here.

The index reduction process of [3] should be replaced by switching to the strangeness free normal form aiming in an explicit representation of both manifolds. This supplies a system of the same dimension as (3), (4) and with the same solution vector in contrast to the approach of [3] where a retransformation from $\mathbf{w}$ to $\mathbf{u}$ is necessary. Such a procedure yields an error of the solution which only depends on the time discretization method. Numerical solutions of higher than second order become possible then.

Since higher-order FE spaces require a higher effort because of the occurence of the matrix M^{-1} and semi- or non-staggered grids do not allow a suitable construction of the matrix P (among other difficulties), it is easier to use finite difference (or finite volume, respectively) methods for spatial discretization. Since the Marker-and-Cell scheme is, among other advantages, most appropriate for the normal form approach, we suggest to apply this technique in numerical CFD simulations.

References

[1] K. Bernert. Differenzenverfahren zur Lösung der Navier-Stokes-Gleichungen über orthogonalen Netzen. Wissenschaftliche Schriftenreihe 10/1990, Technische Universität Chemnitz, 1990.

[2] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations.* Elsevier, North Holland, New York, N. Y., 1989.

[3] M. Dobrowolski. A discrete solenoidal finite difference scheme for the numerical approximation of incompressible flows. *Numer. Math.*, 54:533–542, 1989.

[4] M. Farhloul and M. Fortin. A mixed finite element for the Stokes and elasticity problems. *SIAM J. Numer. Anal.*, 4:971–990, 1993.

[5] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms.* Springer, 1986.

[6] F. H. Harlow and J. E. Welch. Numerical calculation of time dependent viscous incompressible flow of fluid with free surfaces. *Phys. Fluids*, 8:2181–2189, 1965.

[7] J. G. Heywood and R. Rannacher. Finite element approximation for the nonstationary Navier-Stokes problem, Part IV: Error estimates for second order time discretizations. *SIAM J. Numer. Anal.*, 2:353–384, 1990.

[8] T. Y. Hou and B. T. R. Wetton. Second-order convergence of a projection scheme for the incompressible Navier-Stokes equations with boundaries. *SIAM J. Numer. Anal.*, 3:609–629, 1993.

[9] P. Kunkel and V. Mehrmann. Canonical forms for linear differential-algebraic equations with variable coefficients. *J. Comput. Appl. Math.*, 56:225–259, 1994.

[10] P. Kunkel and V. Mehrmann. Local and global invariants of linear differential-algebraic equations and their relation. Preprint SPC 95_25, TU Chemnitz-Zwickau, July 1995. Submitted for publication.

[11] P. Kunkel and V. Mehrmann. Generalized inverses of differential-algebraic operators. *SIAM J. Matrix. Anal. Appl.*, 17:426–442, 1996.

[12] P. Kunkel and V. Mehrmann. A new class of discretization methods for the solution of linear differential-algebraic equations. *SIAM J. Numer. Anal.*, 5, 1996. To appear.

[13] R. A. Nicolaides. Flow discretization by complementay volume techniques. In *Proceedings of the 9th AIAA CFD meeting, Buffalo, NY*, 1989.

[14] R. A. Nicolaides. Analysis and convergence of the MAC scheme. I. The linear problem. *SIAM J. Numer. Anal.*, 6:1579–1591, 1992.

[15] I. Quartapelle. *Numerical Solution of the incompressible Navier-Stokes Equations.* Birkhäuser, 1993.

[16] J. Shen. On error estimates for the penalty method for unsteady Navier-Stokes equations. *SIAM J. Numer. Anal.*, 2:386–403, 1995.

[17] S. Turek. *On discrete projection methods for the incompressible Navier-Stokes equations: An algorithmical approach.* Institute for Applied Mathematics, University of Heidelberg, 1995.

[18] R. Verfürth. Multilevel algorithms for mixed problems. II. Treatment of the mini-element. *SIAM J. Numer. Anal.*, 2:285–293, 1988.

[19] J. Weickert. Lösung der Navier-Stokes-Gleichungen mittels Finite-Volumen-Methode über nichtversetzten Gittern. Diplomarbeit, Fachbereich Mathematik, Technische Universität Chemnitz-Zwickau, 1993.

[20] S. Zeng and P. Wesseling. Multigrid solution of the incompressible Navier-Stokes equations in general coordinates. *SIAM J. Numer. Anal.*, 6:1764–1784, 1994.

Other titles in the SFB393 series:

96-01 V. Mehrmann, H. Xu. Chosing poles so that the single-input pole placement problem is well-conditioned. Januar 1996.

96-02 T. Penzl. Numerical solution of generalized Lyapunov equations. January 1996.

96-03 M. Scherzer, A. Meyer. Zur Berechnung von Spannungs- und Deformationsfeldern an Interface-Ecken im nichtlinearen Deformationsbereich auf Parallelrechnern. March 1996.

96-04 Th. Frank, E. Wassen. Parallel Solution Algorithms for Lagrangian Simulation of Disperse Multiphase Flows. Proc. of 2nd Int. Symposium on Numerical Methods for Multiphase Flows, ASME Fluids Engineering Division Summer Meeting, July 7-11, 1996, San Diego, CA, USA. June 1996.

96-05 P. Benner, V. Mehrmann, H. Xu. A numerically stable, structure preserving method for computing the eigenvalues of real Hamiltonian or symplectic pencils. April 1996.

96-06 P. Benner, R. Byers, E. Barth. HAMEV and SQRED: Fortran 77 Subroutines for Computing the Eigenvalues of Hamiltonian Matrices Using Van Loans's Square Reduced Method. May 1996.

96-07 W. Rehm (Ed.). Portierbare numerische Simulation auf parallelen Architekturen. April 1996.

96-08 J. Weickert. Navier-Stokes equations as a differential-algebraic system. August 1996.

96-09 R. Byers, C. He, V. Mehrmann. Where is the nearest non-regular pencil? August 1996.

The complete list of current and former preprints is available via
`http://www.tu-chemnitz.de/~pester/sfb/sfb96pr.html`.