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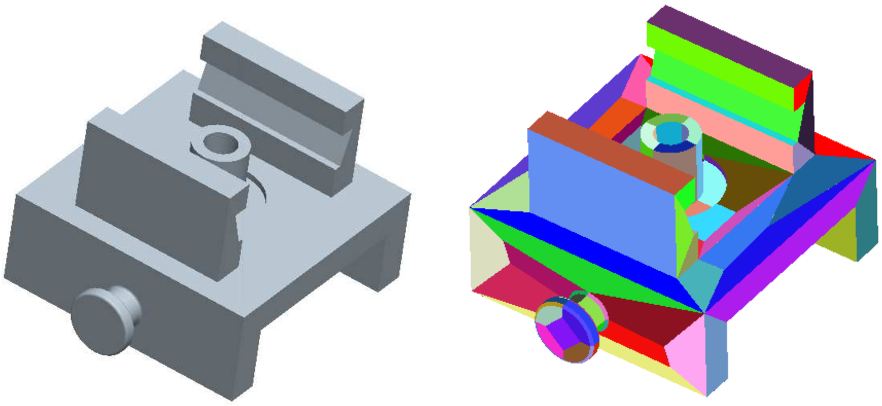
Abstract

We aim at splitting a given multiply connected polygon into a set of convex quadrilaterals. Our approach consists in repeatedly removing quadrilaterals from the given polygon. We will first derive theoretical results pertaining to quadrangulation of simple polygons from the usual 2-ear theorem. Unfortunately, that theorem holds only for simply connected polygons while we are interested in quadrangulation of polygons with holes. As a consequence, we will introduce a special type of polygons which serve as intermediate steps to derive results for multiply connected polygons. After generalizing the 2-ear theorem to those special auxiliary polygons, we will state theories about our quadrangulation method. Numerical results will substantiate our theoretical predictions.

1 Introduction

There are generally two categories of numerical methods for quadrilating a polygon. The indirect method (see [11, 12] and the references there) consists in generating an initial triangulation which is then converted into a quadrangulation. On the opposite, the second approach generates a quadrangulation directly from the initial polygon [9, 7, 6]. The methods that are discussed in this paper fall in the second category. In simple words, the fundamental algorithms of our approach consist in removing a quadrilateral Q from the polygon by using the constructive proofs of our theorems. We repeat the same process to the remaining polygon recursively until the whole polygon is quadrilated. The process of removing a quadrilateral amounts to introducing some cuts within the polygon. We will try to insert very few cuts in order to eventually obtain few quadrilaterals.
In the next section, we will describe the problem setting more accurately and we will introduce some important terminology. Our main contributions are found in sections 3, 4 and 5 where we prove several theoretical results pertaining to removal of boundary convex



(a) (b)

Figure 1: Application in splitting of CAD surfaces.

or nonconvex quadrilaterals by introducing only internal Steiner points (additional nodes not belonging to the initial polygon). First, we will discuss about chopping procedure for simply connected polygons in order to gain insight about the type of theoretical results that are developed in this document. Unfortunately, many results which are correct for simply connected polygons cannot be directly generalized to multiply connected polygons. Even the 2-ear theorem will fail for multiply connected polygons without additional assumption as we will show in the next discussion. As a consequence, we will introduce in section 4 the notion of admissible polygons in order to carry the results over to multiply connected polygons. Those auxiliary polygons may contain double nodes and double edges which are traversed in opposite directions.
Although we show a few numerical results at the end of this paper, our main aim is to consider theoretical aspects of quadrilateral removals. In most theoretical results that we will meet, quadrangulations may contain initially nonconvex quadrilaterals. Since having a quadrilateral decomposition having only convex members is very desired in practical cases, we will describe in section 6 a method of converting a nonconvex quadrangulation into a convex one.

2 Problem setting

Suppose we have a polygon P which has an even number of boundary vertices $\{\mathbf{x}_i\}$ and which might have internal boundaries. Our objective is to find a list of convex quadrilaterals Q_i such that

$$P = \bigcup_i Q_i \tag{1}$$

with the following properties:

(P1) Either the vertices of Q_i are taken from among $\{\mathbf{x}_i\}$ or they are generated strictly in the interior domain of P. In other words, we do not allow boundary Steiner points.

(P2) The intersection $Q_i \cap Q_j$ of two quadrilaterals is either empty or a complete edge.

Since the number of boundary nodes is even, the positive solvability of that problem is discussed in several literatures (see [11, 3, 9] and their references) related to quadrangulation. Such a polygon decomposition has many interesting applications in practical situations. Among others, we can state the splitting of a CAD objects into four-sided patches [13], as in Fig. 1, which are needed in mesh-free numerical solvers of integral equations [14, 5, 2]. Let us now introduce a few definitions which are necessary in the understanding of the subsequent discussions.

Definition 1 Consider a polygon P having vertices $\mathbf{x}_i$. A cut is a segment $[\mathbf{x}_t, \mathbf{x}_s]$ having endpoints from the vertices of P such that the open segment $]\mathbf{x}_t, \mathbf{x}_s[$ is located completely inside P. An ear E of the polygon P is a triangle such that its apices are three consecutive vertices of P and that one edge of E is a cut.

Definition 2 For a simply connected polygon P, the kernel is the set of points from which all vertices of P are visible.

Before giving any details about our methodology, let us note that in our graphical illustrations throughout this paper, nodes which are very close to one another such as those of Fig. 2(a) are supposed to have the same coordinates. We only shift some a bit in order to have better descriptions.

Definition 3 Consider a periodic list of points $\{\mathbf{x}_s\}$ in the plane. We say that we have a double edge if there exist $i \neq j$ such that:

$$\begin{cases} \text{coordinates}(\mathbf{x}_{i+1}) & = & \text{coordinates}(\mathbf{x}_j) \\ \text{coordinates}(\mathbf{x}_i) & = & \text{coordinates}(\mathbf{x}_{j+1}). \end{cases} \tag{2}$$

As a consequence, the edge is traversed in two opposite directions as illustrated in Fig. 2(a).

Definition 4 For any polygon P, we will denote by $\mathcal{I}(P)$ its internal domain which is defined as follows.

$$\begin{array}{rcl} \mathcal{I}(P) & := & \{\mathbf{x} \in \mathbf{R}^2 \text{ not on edge of } P \text{ : the half-line } [\mathbf{x}, \vec{u}) \\ & & \text{intersects } P \text{ an odd number of times for any } \vec{u}\} \end{array} \tag{3}$$

As an illustration, we show in Fig. 2(b) the internal domain by the shaded area.

Since our theoretical development is partially based on the 2-ear theorem [8], let us first recall its statement:

Theorem 1 (Meister, 1975) Every simple (simply connected, without intersecting edges) polygon having at least four vertices has two nonoverlapping ears.

Both the statement and the proof of the 2-ear theorem are very simple. It is yet a very powerful tool for both theoretical and programming purposes. This theorem is very useful in practice when we want to decompose a given polygon into a set of triangles. A very fast algorithm can be easily devised to chop off recursively a triangle from the polygon in order to form a triangulation. In this paper, this theorem will mainly serve as theoretical tool to decrement the number of vertices in proofs using induction.

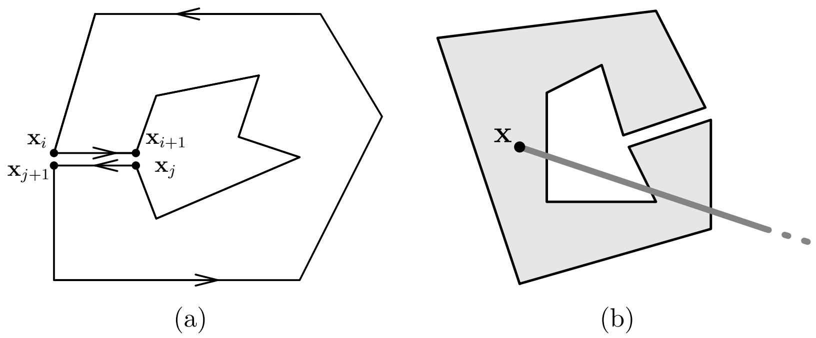


Figure 2: (a) Double edge (b) Internal domain of a polygon.

3 Results for simple polygons

With the above definitions in mind, we are now ready to state our first fundamental result about quadrilateral removal. The following statement can be used to decompose a simple polygon into quadrilaterals which are not necessarily convex.

Theorem 2 From a simple polygon P having at least four vertices, one may remove one quadrilateral by using one of the following operations:

(Op1) Insert a single internal cut (Fig. 3(a)). In this case, the quadrilateral is not necessarily convex.

(Op2) Introducing two cuts and one internal Steiner node (Fig. 3(b)). This always generates a convex quadrilateral.

Proof
We will proceed by induction with respect to the number n of the vertices $\mathbf{x}_i$ of the polygon. For $n = 4$, the theorem is evident. As hypothesis of induction, we suppose that the claim holds for every polygon having n vertices.
Consider now a polygon P having $n + 1$ vertices and let us show that we may chop a quadrilateral Q from P. First, let us apply the 2-ear theorem in order to chop off a triangle $E = [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ from P and let us denote by $\tilde{P}$ the remaining polygon which must have n vertices. That is,

$$\tilde{P} := [\mathbf{x}_0, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_n]. \tag{4}$$

After applying the hypothesis of induction to the new polygon $\tilde{P}$, we obtain a quadrilateral $\tilde{Q}$. Let us now consider three cases according to the incidence of $\tilde{Q}$ upon the ear E and to the existence of Steiner point in $\tilde{Q}$.
Case 1: If the quadrilateral $\tilde{Q}$ is not incident upon the edge $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ inside the polygon $\tilde{P}$, then we simply need to take $Q := \tilde{Q}$.

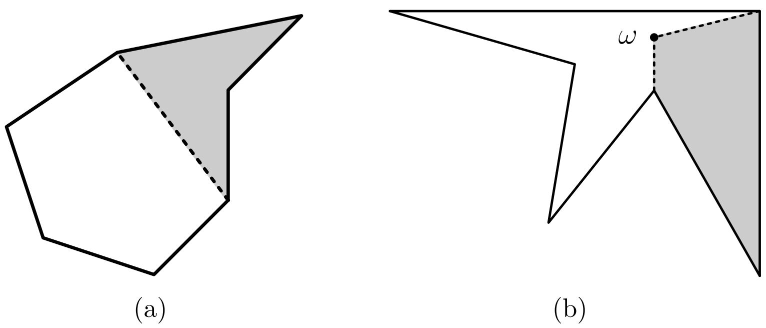


Figure 3: (a) Chop off a quadrilateral with one cut (b) Remove a quadrilateral with two cuts and one internal node.

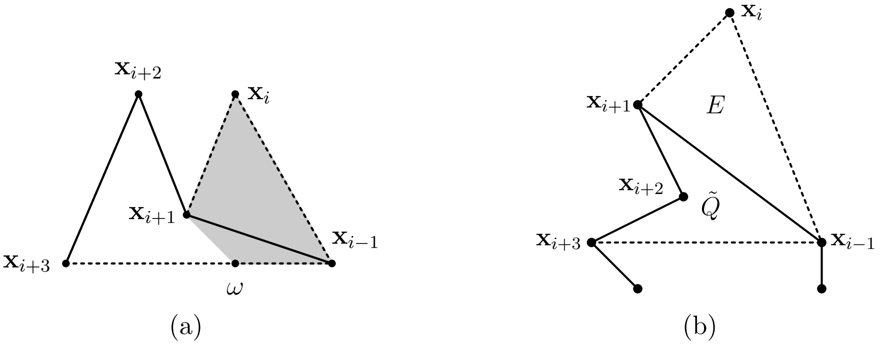


Figure 4: (a) Generate ω (b) $\mathbf{x}_{i+2}$ is reflex in $\tilde{Q}$.

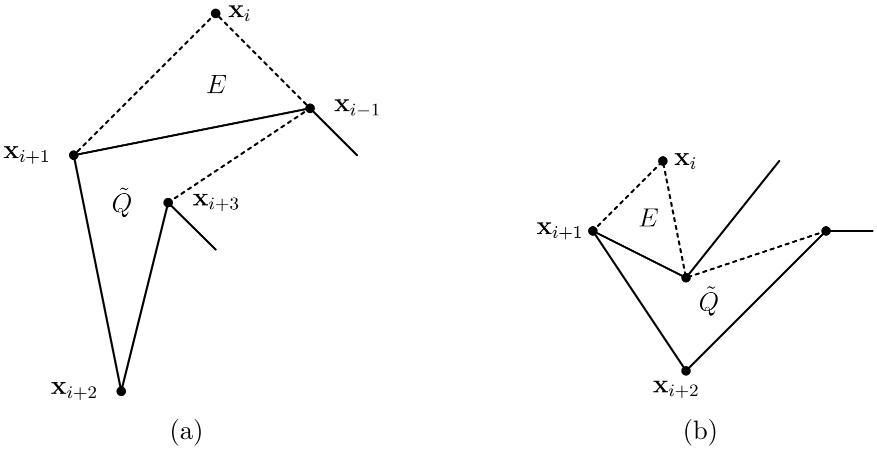


Figure 5: (a)$\mathbf{x}_{i+3}$ is reflex in $\tilde{Q}$ (b)$\mathbf{x}_{i-1}$ is reflex in $\tilde{Q}$.

Case 2: Suppose now that $\tilde{Q}$ has $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ as an edge and $\tilde{Q}$ has no internal Steiner node as illustrated in Fig. 6. We will investigate a few subcases.
Case 2.a: $\tilde{Q} = [\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$.
Inside the quadrilateral $\tilde{Q}$, $\mathbf{x}_{i-1}$ is visible from $\mathbf{x}_{i+2}$ or $\mathbf{x}_{i+1}$ is visible from $\mathbf{x}_{i-2}$ (indeed, you can apply a simple 2-ear theorem inside $\tilde{Q}$). In the former situation, define $Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]$. In the latter situation, define $Q := [\mathbf{x}_{i+1}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i]$. That is, we have just applied operation (Op1) to P in case 2.a.
Case 2.b: $\tilde{Q} = [\mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}]$.
If the quadrilateral $\tilde{Q}$ is convex, then we need simply to apply (Op1) by defining

$$Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]. \tag{5}$$

In the situation that $\tilde{Q}$ is nonconvex, we investigate four events according to the position of its reflex vertex.
If the vertex $\mathbf{x}_{i+2}$ (resp. $\mathbf{x}_{i+3}$) (resp. $\mathbf{x}_{i-1}$) is its reflex vertex as depicted in Fig. 4(b) (resp. Fig. 5(a)) (resp. Fig. 5(b)), we define $Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$ (resp. $Q := [\mathbf{x}_{i+3}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$) (resp. $Q := [\mathbf{x}_{i+2}, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$).

In the case that $\mathbf{x}_{i+1}$ is the reflex vertex of $\tilde{Q}$, take any node ω on the open segment $]\mathbf{x}_{i+3}, \mathbf{x}_{i-1}[$ in the wedge of $\mathbf{x}_i$ as illustrated in Fig. 4(a) and apply (Op2) by defining

$$Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}]. \tag{6}$$

Note that the quadrilateral Q in relation (6) must be convex.

Case 2.c: $\tilde{Q} = [\mathbf{x}_{i-3}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$.

Proceed as in case 2.b and obtain a quadrilateral Q incident upon the edge $[\mathbf{x}_i, \mathbf{x}_{i-1}]$.

Case 3: In the event that the segment $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{Q}$ which has a Steiner point and which is convex, take any point ω strictly inside $\tilde{Q}$. On account of the convexity of $\tilde{Q}$, both $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ are visible from the node ω. Therefore we may define

$$Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}] \tag{7}$$

as a quadrilateral which can be removed from P.

Remark 1 One can observe that the number of vertices of the remaining polygon is $(n-2)$ after applying operation (Op1) to a polygon having n vertices. On the opposite, applying operation (Op2) as illustrated in Fig. 3(b) does not reduce the number of vertices. At first sight, it may therefore come to one's mind that recursively applying the above theorem does not split a polygon into a set of quadrilaterals (convex or not). In other words, one may wonder if the following algorithm of quadrilateral decomposition is guaranteed to terminate when applied to a polygon having an even number of vertices.

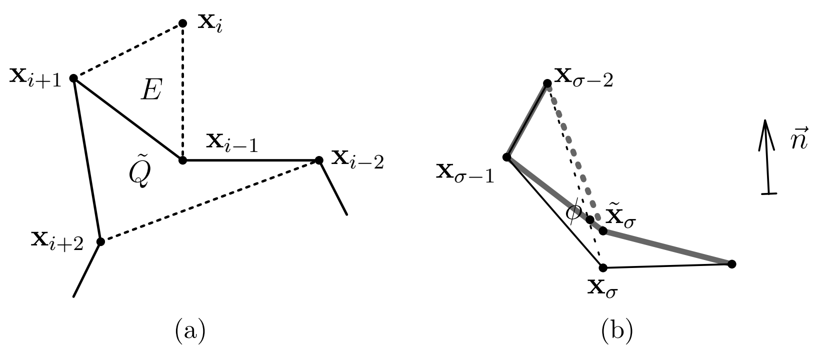


Figure 6: (a) The ear $[\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ and the removed quadrilateral are adjacent (b)$[\mathbf{x}_\sigma, \mathbf{x}_{\sigma-2}, \mathbf{x}_{\sigma-1}]$ can be chopped off.

Algorithm

step 0	:	Initialize $P_0 := P$ and $k = 0$.
step 1	:	Test if operation (Op1) can be applied to the polygon P_k. If operation (Op1) applies, chop off a quadrilateral. Else apply operation (Op2). In both cases, let Q be the resulting quadrilateral.
step 2	:	Define $P_{k+1} := P_k \setminus Q$.
step 3	:	If the number of vertices of $P_{k+1} = 4$, terminate. Otherwise, set $k := k + 1$ and go to step 1.

If we inspect the above proof more closely, we notice in our constructions of the removed quadrilaterals that if we apply operation (Op2), then a neighboring quadrilateral can be chopped off in the next iteration by applying operation (Op1). As illustrated in Fig. 4(a), after removing the shaded quadrilateral, one may chop the quadrilateral $[\mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}, \omega]$ off by applying operation (Op1). That is to say, after at most two iterations of the above algorithm, the number of vertices must decrement twice.

4 Toward multiply connected polygons

Since we are interested in the quadrangulation of multiply connected polygons, we will present in this section theoretical results which may be used as intermediate steps to derive statements pertaining to internal boundaries for the next section. Before going any further, let us notice that the above proof cannot be directly generalized for multiply connected polygons because of the following observation.

Remark 2 (Counter-example for multiply connected polygons) The 2-ear theorem does not hold true for general multiply connected polygons. As a counter example, we simply need to consider a doubly connected polygon whose exterior and interior boundaries are squares as illustrated in Fig. 9(a). As one can clearly observe, it is impossible to chop off a triangle by inserting a single cut.

Definition 5 A polygon P is admissible if it fulfills the following criteria:

(C1) It is simply connected in the sense that it is given as a single sequence of vertices $\mathbf{x}_0$, $\mathbf{x}_1$, ... , $\mathbf{x}_{N-1}$.

(C2) We may have double edges but we do not allow consecutive double edges:

$$\begin{cases} \text{coordinates}(\mathbf{x}_i) & = & \text{coordinates}(\mathbf{x}_{j+2}) \\ \text{coordinates}(\mathbf{x}_{i+1}) & = & \text{coordinates}(\mathbf{x}_{j+1}) \\ \text{coordinates}(\mathbf{x}_{i+2}) & = & \text{coordinates}(\mathbf{x}_j). \end{cases} \tag{8}$$

(C3) No three vertices (apart from those on double edges) are located on one line.

(C4) We do not allow zero internal angle as in Fig. 8(b).

(C5) The internal domain $\mathcal{I}(P)$ is connected.

Remark 3 In Fig. 8(a), we have a polygon which violates the criterion (C5) because the internal domain has two connected components. Since criterion (C3) is too restrictive to treat interesting practical situations, we are going to relax it later on.

Theorem 3 From every admissible polygon P having more than four vertices, one may chop two triangles E_1 and E_2 off.

Proof
We proceed by induction with respect to the number m of double edges. First of all, let us consider the case where $m = 1$ and suppose that the edges $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma+1}]$ and $[\mathbf{x}_\rho, \mathbf{x}_{\rho+1}]$ coincide as displayed in Fig. 7. Define

$$\vec{u} := \overrightarrow{\mathbf{x}_\sigma \mathbf{x}}_{\sigma+1} / \|\overrightarrow{\mathbf{x}_\sigma \mathbf{x}}_{\sigma+1}\| \tag{9}$$

and denote by $\vec{n}$ the unit vector normal to the segment $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma+1}]$ such that $\det(\vec{u}, \vec{n}) = 1$. Denote by $[\kappa(\mathbf{x}), \lambda(\mathbf{x})]$ the coordinates of a point $\mathbf{x}$ in the reference $[\mathbf{0}, \vec{u}, \vec{n}]$. Let us introduce $\mu_0 := \alpha.\lambda_0$ where

$$\lambda_0 := \min\{\lambda(\mathbf{x}_r) > \lambda(\mathbf{x}_\sigma) \ : \ \mathbf{x}_r \text{ vertex of } P\}. \tag{10}$$

Now we may use an auxiliary polygon $\tilde{P}$ having the following vertices

$$\begin{cases} \tilde{\mathbf{x}}_\sigma & := & \mathbf{x}_\sigma + \mu_0 \vec{n} \\ \tilde{\mathbf{x}}_{\sigma+1} & := & \mathbf{x}_{\sigma+1} + \mu_0 \vec{n} \\ \tilde{\mathbf{x}}_r & := & \mathbf{x}_r \quad \text{if} \quad r \notin \{\sigma, \sigma+1\}. \end{cases} \tag{11}$$

Since the auxiliary polygon $\tilde{P}$ does not contain any double edge, we may apply the usual 2-ear theorem in order to obtain two triangles $\tilde{E}_1$ and $\tilde{E}_2$ from $\tilde{P}$. For every triangle $\tilde{E}_i$ $(i = 1, 2)$, we distinguish three cases according to the incidence of $\tilde{E}_i$ upon the nodes $\tilde{\mathbf{x}}_\sigma$ and $\tilde{\mathbf{x}}_{\sigma+1}$.
Case 1: If $\tilde{E}_i$ does not have any of the nodes $\tilde{\mathbf{x}}_\sigma$ and $\tilde{\mathbf{x}}_{\sigma+1}$ as apex, we simply define $E_i := \tilde{E}_i$.
Case 2: Let us now assume that the segment $[\tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}]$ is an edge of $\tilde{E}_i$. Without loss of generality we may suppose that the third node of $\tilde{E}_i$ is $\mathbf{x}_{\sigma+2}$ i.e. $\tilde{E}_i = [\tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}, \mathbf{x}_{\sigma+2}]$ (the case where $\tilde{E}_i = [\mathbf{x}_\sigma, \mathbf{x}_{\sigma+1}, \mathbf{x}_{\sigma-1}]$ can be treated exactly in the same manner).
Since no three vertices are located on one staight line as specified by criterion (C3), we may introduce

$$\begin{array}{rcl} \delta_1 & := & \min\left\{\text{dist}\left([\mathbf{x}_\sigma, \mathbf{x}_{\sigma+2}], \mathbf{x}_r\right), \ r \neq \sigma \qquad r \neq \sigma+2\right\} \\ \delta_2 & := & \min\left\{\text{dist}\left([\mathbf{x}_{\sigma+1}, \mathbf{x}_{\sigma-1}], \mathbf{x}_r\right), \ r \neq \sigma - 1 \qquad r \neq \sigma+1\right\}. \end{array} \tag{12}$$

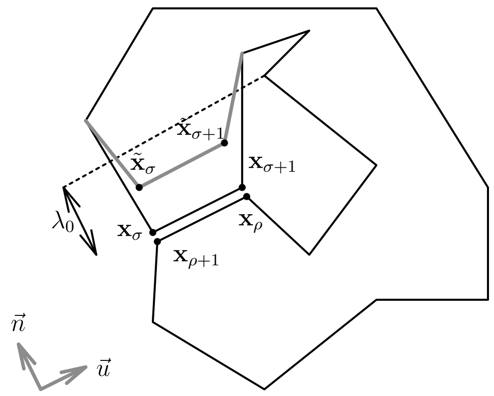


Figure 7: Original and auxiliary polygons.

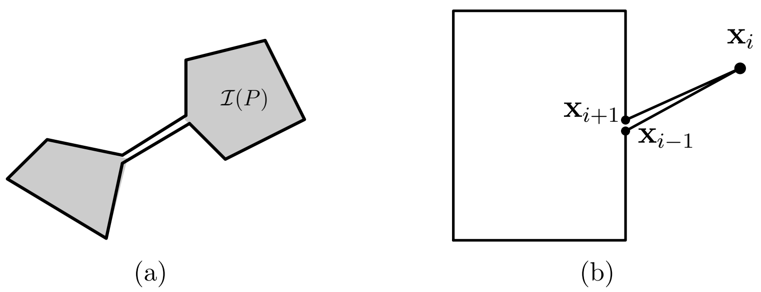
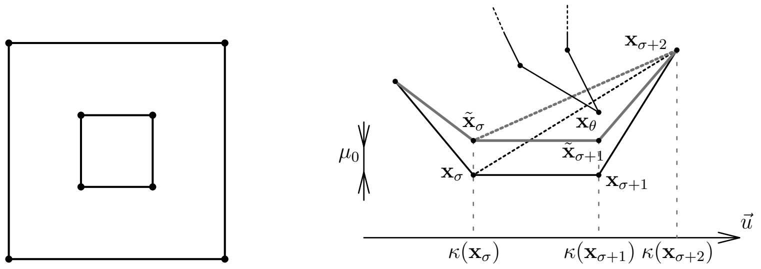


Figure 8: (a) The interior domain is disconnected (b) Zero angle at $\mathbf{x}_i$.



(a) (b)

Figure 9: (a) A doubly connected polygon from which one may not chop any triangle off. (b) $\kappa(\mathbf{x}_{\sigma+1}) \leq \kappa(\mathbf{x}_{\sigma+2})$.

Define $\delta := \min\{\delta_1, \delta_2\}$. Since dist $([\mathbf{x}_\sigma, \mathbf{x}_{\sigma+2}], \tilde{\mathbf{x}}_\sigma)$ tends to zero as $\alpha \to 0$, there must exist α_0 sufficiently small such that

$$\text{if} \quad \alpha < \alpha_0 \quad \text{then} \quad \text{dist}\,([\mathbf{x}_\sigma, \mathbf{x}_{\sigma+2}], \tilde{\mathbf{x}}_\sigma) < \delta. \tag{13}$$

Let us introduce the regions

$$S := \left\{\mathbf{x} \in \mathbf{R}^2 : \lambda(\mathbf{x}_\sigma) < \lambda(\mathbf{x}) < \mu_0\right\}, \tag{14}$$

$$R := \left\{\mathbf{x} \in \mathbf{R}^2 : \lambda(\mathbf{x}) > \mu_0\right\}. \tag{15}$$

According to the above definition of $\vec{u}$ and κ, we must have $\kappa(\mathbf{x}_\sigma) < \kappa(\mathbf{x}_{\sigma+1})$. We will need three subcases according to the position of $\kappa(\mathbf{x}_{\sigma+2})$ with respect to $\kappa(\mathbf{x}_\sigma)$ and $\kappa(\mathbf{x}_{\sigma+1})$.
Case 2.a: if $\kappa(\mathbf{x}_\sigma) < \kappa(\mathbf{x}_{\sigma+2}) < \kappa(\mathbf{x}_{\sigma+1})$
In this case as illustrated in Fig. 10(b), the segment $[\tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}]$ must intersect the two edges $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma+2}]$ and $[\mathbf{x}_{\sigma+1}, \mathbf{x}_{\sigma+2}]$ of the triangle E_i. Because of the definition of μ_0, there could not exist any vertex in the stripe S. On the other hand, we have $(E_i \cap R) \subset \tilde{E}_i$. As a consequence, the triangle E_i cannot contain any vertex of P.
Case 2.b: if $\kappa(\mathbf{x}_{\sigma+1}) \leq \kappa(\mathbf{x}_{\sigma+2})$
In $E_i \cap S$, there could not exist any node. If there was a node $\mathbf{x}_\theta$ in $E_i \cap R$ as illustrated in Fig. 9(b), then there would exist an edge traversing the cut $[\tilde{x}_\sigma, \tilde{x}_{\sigma+2}]$ which is in contradiction with the fact that $\tilde{E}_i$ can be chopped off from $\tilde{P}$.
Case 2.c: if $\kappa(\mathbf{x}_{\sigma+2}) \leq \kappa(\mathbf{x}_\sigma)$
As a consequence of the choice of α in (13), there could not exist any vertex of P within the region $[\tau, \tilde{\mathbf{x}}_\sigma, \mathbf{x}_{\sigma+2}]$. Therefore, the triangle E_i can be chopped from the polygon P off.
Case 3: Let us now assume that $\tilde{E}_i$ is not incident upon the edge $[\tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}]$ but it has $\tilde{\mathbf{x}}_\sigma$ or $\tilde{\mathbf{x}}_{\sigma+1}$ as apex. Let us only treat the situation where $\tilde{E}_i = [\tilde{\mathbf{x}}_\sigma, \mathbf{x}_{\sigma-2}, \mathbf{x}_{\sigma-1}]$ because the other case where $\tilde{E}_i = [\tilde{\mathbf{x}}_{\sigma+1}, \mathbf{x}_{\sigma+2}, \mathbf{x}_{\sigma+3}]$ can be treated in a similar manner.

$$\begin{aligned} \delta_3 &:= \min\{\text{dist}\,([\mathbf{x}_{\sigma-2}, \mathbf{x}_\sigma], \mathbf{x}_r)\,,\ r \neq \sigma \qquad r \neq \sigma - 2\} \\ \delta_4 &:= \min\{\text{dist}\,([\mathbf{x}_{\sigma+1}, \mathbf{x}_{\sigma+3}], \mathbf{x}_r)\,,\ r \neq \sigma + 1 \qquad r \neq \sigma + 3\}\,. \end{aligned} \tag{16}$$

By introducing $\delta' := \min\{\delta_3, \delta_4\}$, we proceed the way we have done in (13) in order to obtain an $\alpha_1 > 0$ sufficiently small such that

$$\text{if} \quad \alpha < \alpha_1 \quad \text{then} \quad \text{dist}\left([\mathbf{x}_\sigma, \mathbf{x}_{\sigma-2}], \tilde{\mathbf{x}}_\sigma\right) < \delta'. \tag{17}$$

Thus, the open thin triangle $[\tilde{\mathbf{x}}_\sigma, \mathbf{x}_{\sigma-2}, \phi]$ does not contain any node where ϕ is the intersection of $[\tilde{\mathbf{x}}_\sigma, \mathbf{x}_{\sigma-1}]$ and $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma-2}]$ as displayed in Fig. 6(b). As a consequence, the triangle $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma-2}, \mathbf{x}_{\sigma-1}]$ can be chopped off from the polygon P.
That is, if α is chosen so that $\alpha < \min\{\alpha_0, \alpha_1\}$ where α_0 and α_1 are obtained from relations (13) and (17) then the claim is true for $m = 1$. As hypothesis of induction, we suppose that the claim holds for any m. For a polygon having $m + 1$ double edges, we apply the same procedure as above in order to have an auxiliary polygon $\tilde{P}$ having m double edges and apply the hypothesis of induction to $\tilde{P}$.
If we read the above proof well, we see that the above theorem holds even for polygons in which the condition (C3) is relaxed. More accurately we have the following result.

Theorem 4 Suppose that we have a polygon P fulfilling conditions (C1), (C2), (C4) and (C5). Furthermore, we suppose that we have the following condition:

($\widetilde{C3}$) The eight open segments $]\mathbf{x}_s, \mathbf{x}_{s+2}[$, $s = \sigma,\ \sigma-1,\ \rho,\ \rho-1,\ \sigma-2,\ \sigma+1,\ \rho-2,\ \rho+1$ do not contain any node.

Then one can remove two ears from the polygon P.

Remark 4 The proof can fundamentally remain unchanged because the parameters δ_1, δ_2, δ_3, δ_4 from relations (12) and (16) are still strictly positive. Please note that we do not penalize the existence of nodes inside the triangles which are next to the double edge such as $[\mathbf{x}_\sigma, \mathbf{x}_{\sigma+1}, \mathbf{x}_{\sigma+2}]$ in theorem 4. We can even remove the condition ($\widetilde{C3}$) completely as in the following statement. But we still need the former theorems as intermediate steps in the proof.

Theorem 5 Suppose that a polygon P meets the four conditions (C1), (C2), (C4), (C5). Then one can remove two ears from the polygon P.

Proof
Adopt the definition of the vectors $\vec{n}$ and $\vec{u}$ as in the previous proof. Now we would like to shift some vertices in the direction of $\vec{u}$. Let us define:

$$\delta_s := \min\left\{\text{dist}([\mathbf{x}_s, \mathbf{x}_{s+2}], \mathbf{x}_r), \quad \mathbf{x}_r \notin [\mathbf{x}_s, \mathbf{x}_{s+2}]\right\} > 0. \tag{18}$$

$$\gamma := 0.5 \cdot \min\{\delta_s : s = \sigma, \sigma-1, \rho, \rho-1, \sigma-2, \sigma+1, \rho-2, \rho+1\}.$$

Let us introduce also an auxiliary polygon $\tilde{P}$ which has the following vertices in place of $\mathbf{x}_\sigma$, $\mathbf{x}_{\rho+1}$ $\mathbf{x}_{\sigma+1}$, $\mathbf{x}_\rho$ (the other vertices are the same as those of P):

$$\begin{array}{lcl} \tilde{\mathbf{x}}_\sigma & := & \mathbf{x}_\sigma - \gamma.\vec{u} \\ \tilde{\mathbf{x}}_{\rho+1} & := & \mathbf{x}_{\rho+1} - \gamma.\vec{u} \\ \tilde{\mathbf{x}}_{\sigma+1} & := & \mathbf{x}_{\sigma+1} + \gamma.\vec{u} \\ \tilde{\mathbf{x}}_\rho & := & \mathbf{x}_\rho + \gamma.\vec{u}. \end{array} \tag{19}$$

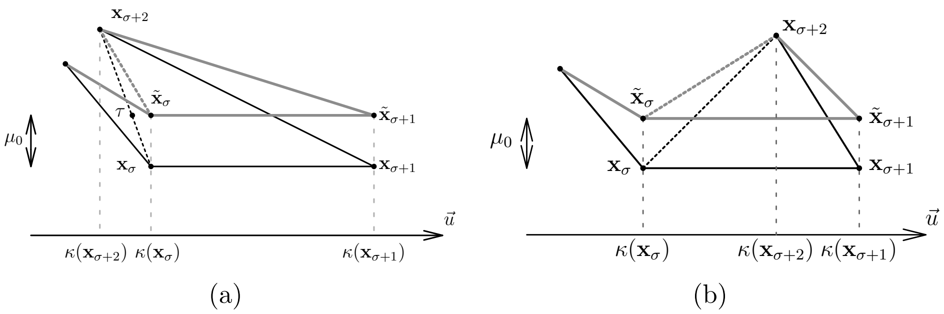


Figure 10: (a) $\kappa(\mathbf{x}_{\sigma+2}) \le \kappa(\mathbf{x}_\sigma)$ (b) $\kappa(\mathbf{x}_\sigma) < \kappa(\mathbf{x}_{\sigma+2}) < \kappa(\mathbf{x}_{\sigma+1})$.

As a consequence, there could not exist any polygon vertex on the eight open segment $]\tilde{\mathbf{x}}_s, \tilde{\mathbf{x}}_{s+2}[$, $s = \sigma$, $\sigma-1$, ρ, $\rho-1$, $\sigma-2$, $\sigma+1$, $\rho-2$, $\rho+1$ of the auxiliary polygon $\tilde{P}$.
Apply theorem 4 to the auxiliary polygon $\tilde{P}$ so that we can remove two triangles $\tilde{E}_1$ and $\tilde{E}_2$. In the easy case that $\tilde{E}_i$ $(i = 1, 2)$ is not incident upon any double node, we simply take $E_i := \tilde{E}_i$. For the situation that $\tilde{E}_i$ is incident upon the nodes $\tilde{\mathbf{x}}_\sigma$ or $\tilde{\mathbf{x}}_{\sigma+1}$, we will consider the four next cases.
Case 1: $\tilde{E}_i = [\mathbf{x}_{\sigma-1}, \tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}]$
In this case as illustrated in Fig. 11(a), the triangle $E_i := [\mathbf{x}_{\sigma-1}, \mathbf{x}_\sigma, \mathbf{x}_{\sigma+1}]$ can be chopped from the polygon P off because E_i must be included in $\tilde{E}_i$.
Case 2: $\tilde{E}_i = [\mathbf{x}_{\sigma-2}, \mathbf{x}_{\sigma-1}, \tilde{\mathbf{x}}_\sigma]$.
Let us define by τ the intersection of the segment $[\tilde{\mathbf{x}}_\sigma, \mathbf{x}_{\sigma-2}]$ and $[\mathbf{x}_{\sigma-1}, \mathbf{x}_\sigma]$ as illustrated by Fig. 11(b). According to the choice of δ from relation (18), there cannot exist any node within the triangle $[\mathbf{x}_{\sigma-2}, \tau, \mathbf{x}_\sigma]$.
Case 3: If $\tilde{E}_i = [\tilde{\mathbf{x}}_\sigma, \tilde{\mathbf{x}}_{\sigma+1}, \mathbf{x}_{\sigma+2}]$, proceed as in case 1.
Case 4: If $\tilde{E}_i = [\tilde{\mathbf{x}}_{\sigma+1}, \mathbf{x}_{\sigma+2}, \mathbf{x}_{\sigma+3}]$, proceed as in case 2.
The cases where $\tilde{E}_i$ is incident upon $\tilde{\mathbf{x}}_\rho$ or $\tilde{\mathbf{x}}_{\rho+1}$ are treated exactly in the same fashion.

Lemma 1 From a simply connected polygon P which has more that four vertices and which might have double nodes but without double edges such that the internal domain is connected, one can chop two triangles E_1 and E_2 off.

Proof [sketch]

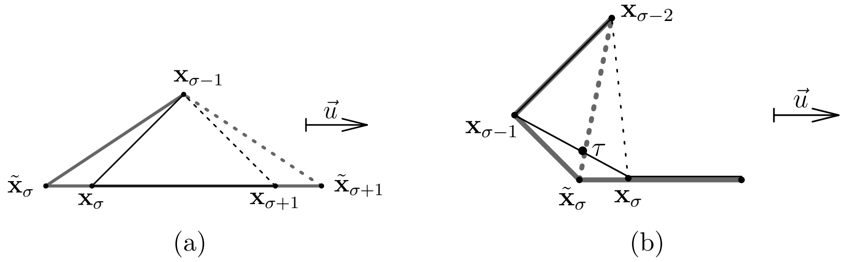


Figure 11: Chopping an ear off next to a shifted node.

We will only sketch the proof in which we suppose that we have only one double node $\mathbf{x}_i$ and $\mathbf{x}_j$ as illustrated in Fig. 12. The general case can be demonstrated by induction. Let us denote by α_s the internal angle made by $\mathbf{x}_{s-1}$, $\mathbf{x}_s$, $\mathbf{x}_{s+1}$. At least one of the internal angles α_i and α_j must be smaller than π because it is impossible that two reflex vertices coincide (Fig. 12). Thus, let us suppose $\alpha_i < \pi$.
Let $\vec{w}$ be the normalized form of

$$\overrightarrow{\mathbf{x}_i\mathbf{x}}_{i-1} + \overrightarrow{\mathbf{x}_i\mathbf{x}}_{i+1}. \tag{20}$$

Choose a positive value μ so that the shifted vertex

$$\tilde{\mathbf{x}}_i := \mathbf{x}_i + \mu.\vec{w} \tag{21}$$

does not interfere with any other vertex. Apply the usual 2-ear theorem to the new polygon with $\tilde{\mathbf{x}}_i$ in place of $\mathbf{x}_i$ in order to chop two triangles $\tilde{E}_i$. Then, consider the following three cases. First, $\tilde{E}_i$ is not incident upon $\tilde{\mathbf{x}}_i$. Second, the segment $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{E}_i$. Last, the segment $[\mathbf{x}_i, \mathbf{x}_{i+2}]$ or $[\mathbf{x}_i, \mathbf{x}_{i-2}]$ is an edge of $\tilde{E}_i$.

Remark 5 By using a similar argument as in the above proof, one can easily see that the claim in theorem 5 holds true even for polygons having double nodes which are not incident to double edges.

5 Quadrilating multiply connected polygons

Lemma 2 Consider a doubly connected polygon P (i.e. with one hole) having $\mathcal{E}$ as exterior boundary and $\mathcal{J}$ as interior one. Denote by the $\mathbf{s}_i$ the vertices of $\mathcal{E}$ and by $\mathbf{t}_i$ those of $\mathcal{J}$. There must exist two vertices $\mathbf{s}_l$ and $\mathbf{t}_k$ which are mutually visible. Thus, the segment $[\mathbf{s}_l, \mathbf{t}_k]$ is a cut inside the polygon P.

Proof

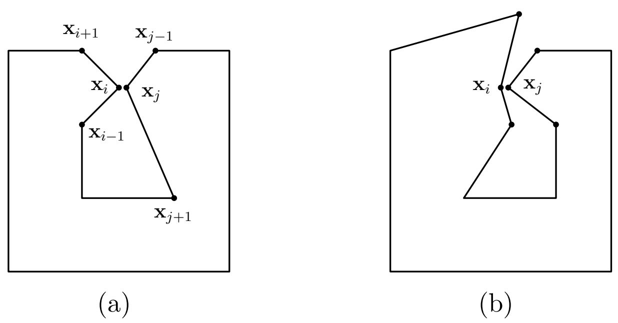


Figure 12: Double nodes: (a) $\mathbf{x}_i$ and $\mathbf{x}_j$ are convex (b) $\mathbf{x}_i$ is reflex.

Consider any direction $\vec{u}$ such that $\|\vec{u}\| = 1$ and consider the axis $[\mathbf{0}, \vec{u})$ centered at the origin and directed by $\vec{u}$. Denote by $\lambda(\mathbf{x})$ the coordinate of the projection of a point $\mathbf{x}$ on the axis $[\mathbf{0}, \vec{u})$.
Define $\mathbf{t}_k$ the vertex of $\mathcal{J}$ having the largest λ value:

$$\mathbf{t}_k := \text{argmax}\{\lambda(\mathbf{t}_i) \,:\, \mathbf{t}_i \,\text{vertex of}\, \mathcal{J}\}. \tag{22}$$

We would like not to introduce the set

$$\kappa := \{\mathbf{s}_i \text{ vertex of } \mathcal{E} : \lambda(\mathbf{s}_i) > \lambda(\mathbf{t}_k)\} \tag{23}$$

which must be nonempty because the polygonal boundary $\mathcal{E}$ is outside $\mathcal{J}$. Therefore, we may define

$$\mathbf{s}_l := \text{argmin}\{\lambda(\mathbf{s}_i) \,:\, \mathbf{s}_i \in \kappa\}. \tag{24}$$

We note that no edge of the polygon P can intersect the cut $[\mathbf{s}_l, \mathbf{t}_k]$ because there is no node of $\mathcal{J}$ in the region

$$\{\mathbf{x} \in \mathbf{R}^2 : \lambda(\mathbf{x}) > \lambda(\mathbf{t}_k)\}. \tag{25}$$

Corollary 1 From a doubly connected polygon, you may discard one quadrilateral which may be nonconvex by inserting at most three cuts.

Proof [Sketch]
Let us adopt the notations $\mathbf{s}_i$ and $\mathbf{t}_j$ of lemma 2 to denote the vertices on the exterior and interior boundaries respectively. Let m and n be the respective numbers of vertices on the exterior and interior boundaries. According to lemma 2, we may insert a cut $[\mathbf{s}_l, \mathbf{t}_k]$ inside the polygon. In order to simplify the next notation, let us extend the indices in the following periodic manner:

$$\begin{aligned} \mathbf{s}_r &:= \mathbf{s}_{r-m} \quad \text{if } r \geq m & (26) \\ \mathbf{t}_r &:= \mathbf{t}_{r-n} \quad \text{if } r \geq n. & (27) \end{aligned}$$

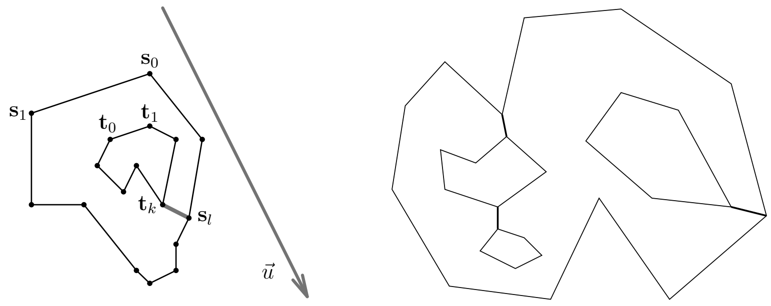


Figure 13: (a) $[\mathbf{s}_l, \mathbf{t}_k]$ is a cut joining the interior and the exterior boundaries (b) cuts inside a multiply connected polygon.

Let us now introduce the polygon $\tilde{P}$ whose vertices are defined by

$$\begin{array}{lll} \mathbf{y}_i & := & \mathbf{s}_{l+i} \quad \text{for} \quad i = 0, ..., m \\ \mathbf{y}_{m+i} & := & \mathbf{t}_{k+i-1} \quad \text{for} \quad i = 1, ..., n \\ \mathbf{y}_{m+n} & := & \mathbf{t}_{k+n}. \end{array} \tag{28}$$

As one can easily notice, the polygon $\tilde{P}$ having $(n+m+2)$ vertices is an admissible polygon in which the edge $e := [\mathbf{s}_l, \mathbf{t}_k]$ is traversed twice in opposite direction.
As a consequence, one may chop one ear E from $\tilde{P}$. By using an argument similar to that in the proof of Theorem 2, one may chop a quadrilateral Q from $\tilde{P}$ by using at most two edges f_1 and f_2. As a conclusion, the quadrilateral Q may be removed from P by using at most three edges e, f_1 and f_2.

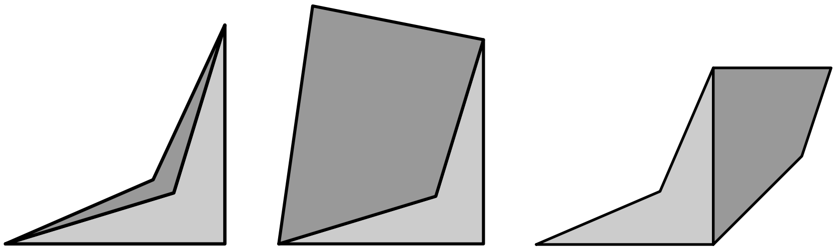
Remark 6 For a general multiply connected polygon with h holes, you can easily generalize the proof by adding h internal cuts as illustrated in Fig. 13(b). Please note that by chopping off a quadrilateral from a polygon satisfying the properties discussed in remark 5, we obtain a polygon with the same properties. As a consequence, we may repeat the chopping off process recursively.

6 Convertion into convex quadrangulation

The purpose of this section is to describe (see also [1, 12]) how to convert a quadrangulation which has non-convex quadrilaterals, such as those obtained from former sections, into another quadrangulation whose members are exclusively convex quadrilaterals. Before giving details about the convertion, let us consider the following remark. Two adjacent quadrilaterals q and p form a single quadrilateral or a hexagon. In the former case, the quadrilaterals q and p share two edges and it is possible that the union $q \cup p$ is a nonconvex or a convex quadrilateral as illustrated in Figs. 14(a) and 14(b). In the latter case, only one edge is shared by q and p as illustrated in Fig. 14(c). Having understood those simple facts, let us recall the following result about hexagon quadrangulations which are needed in the description of the convertion algorithm.

Theorem 6 (Bremner *et. al***)** Every hexagon (which may include reflex vertices) can be decomposed into a set of convex quadrilaterals by using at most three internal Steiner points.

With those facts in mind, we may describe the conversion algorithm in two steps:
Step1: For every nonconvex quadrilateral p having a neighboring quadrilateral q such that $p \cup q$ is a quadrilateral, replace p by $p \cup q$ and remove the quadrilateral q from the quadrangulation. We repeat this step until such a union does not exist any more. After this step, there can only exist nonconvex quadrilaterals whose union with a neighboring quadrilateral forms a hexagon.
Step2: We merge a nonconvex quadrilateral q of Q with a neighboring quadrilateral p in order to have a hexagon $(q \cup p)$. If we have the choice then we select a nonconvex neighbor



(a) (b) (c)

Figure 14: The union is (a) a nonconvex quadrilateral (b) a convex quadrilateral (c) a hexagon.

p. Then, we re-quadrilate the resulting hexagon by using the hexagon quadrangulation method from theorem 6 in order to obtain a local *convex* quadrangulation Q_{loc}. Afterwards, we substitute the union $(q \cup p)$ by Q_{loc} in the quadrangulation Q.

7 Cleanup

In several decomposition techniques [6, 10, 4], *cleanup* is the process of generating a splitting by improving an available one according to some quality measurement. Some cleanup procedures aim at having quadrilateral quality enhancement by introducing new nodes and edges. In the opposite, we are going to keep the numbers of nodes and edges unchanged as we will only focus on the modification of their coordinates and postures.

7.1 Quality control

Before describing the cleanup operations, let us review the way qualities of a quadrilateral or a node or an edge can be evaluated. There are two techniques of obtaining quality assessment of quadrilaterals. The first one requires the introduction of the following distortion coefficient [7] of any triangle $[A, B, C]$:

$$\alpha := 2\sqrt{3}\frac{\|\overrightarrow{CA} \times \overrightarrow{CB}\|}{\|\overrightarrow{CA}\|^2 + \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2} \in [0, 1]. \tag{29}$$

In fact, we can easily notice that the triangular distortion α is unity if the triangle $[A, B, C]$ is equilateral. From a convex quadrilateral $[A, B, C, D]$, we may derive four triangles $[A, B, C]$, $[A, C, D]$, $[A, B, D]$ and $[D, B, C]$. We can denote by α_i the triangular distortions of those four triangles such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. We define the first quality measurement of the quadrilateral $[A, B, C, D]$ to be

$$\beta := (\alpha_3\alpha_4)/(\alpha_1\alpha_2) \in [0, 1]. \tag{30}$$

Effectively, the value of β is unity for rectangles and it approaches zero as a quadrilateral becomes triangular shaped. The second way of measuring qualities of quadrilaterals is by

means of the μ-value which is the smallest internal angle in the four possible triangles. In the next discussion, we will denote by $\mu(q)$ the quality of a quadrilateral q by using one of those two methods.
We would like now to consider quality measurements of a node and an edge inside a quadrangulation. To that end, let us consider an internal node ω which is shared by the quadrilaterals q_i $i \in \mathcal{J}$. Thus, we evaluate the quality of this node by

$$\mu(\omega) := \frac{1}{\mathrm{card}(\mathcal{J})} \sum_{i \in \mathcal{J}} \mu(q_i). \tag{31}$$

Similarly, for an internal edge e upon which two quadrilaterals q_1 and q_2 are incident, its quality can be measured by

$$\mu(e) := \frac{1}{2}[\mu(q_1) + \mu(q_2)]. \tag{32}$$

7.2 Cleanup operations

We will treat two types of cleanup operations: *node repositioning* and *edge flipping*. The first one consists in shifting an internal node to another position in order to improve the quality of the neighboring quadrilaterals. In the course of node shifting, we have to make sure that all incident quadrilaterals remain convex. The second operation consists in modifying the endpoints of an internal edge.
Let us first show how to find the region inside which a node ω can be shifted. Let us denote by E_ω the set of edges which emanate from the node ω. The first method consists in taking the shortest edge $\tilde{e}$ from among E_ω and in considering a circle centered at the node ω and having radius $\rho := \lambda \cdot \mathrm{length}(\tilde{e})$ where λ is a user defined parameter from $[0, 1[$. The new position of ω is then searched inside this circle.
Another alternative is to consider a region $\mathcal{R}$ which is formed by the kernel of the union of the surrounding quadrilaterals. On account of the convexity of the incident quadrilaterals, we must note that $\mathcal{R}$ is nonempty because at least ω itself is a member of this region. The node repositioning consists in shifting ω inside a scaled version of the region $\mathcal{R}$ according to value of $\mu(\omega)$.
The case of edge flipping is treated in a similar manner. We flip an internal edge e to a position which keeps the two incident quadrilaterals convex and which improves the value of $\mu(e)$.

8 Numerical results and future works

As we have stated in the begining, we want mainly to focus on theoretical aspect in this paper. In this section, we present anyway a few results of the formerly discussed theories when applied to real polygons simply in order to address additional features that have to be considered in practice. We investigate four sets of polygons whose quadrangulations are to be found in Fig. 15 through Fig. 17. We display there different polygons with

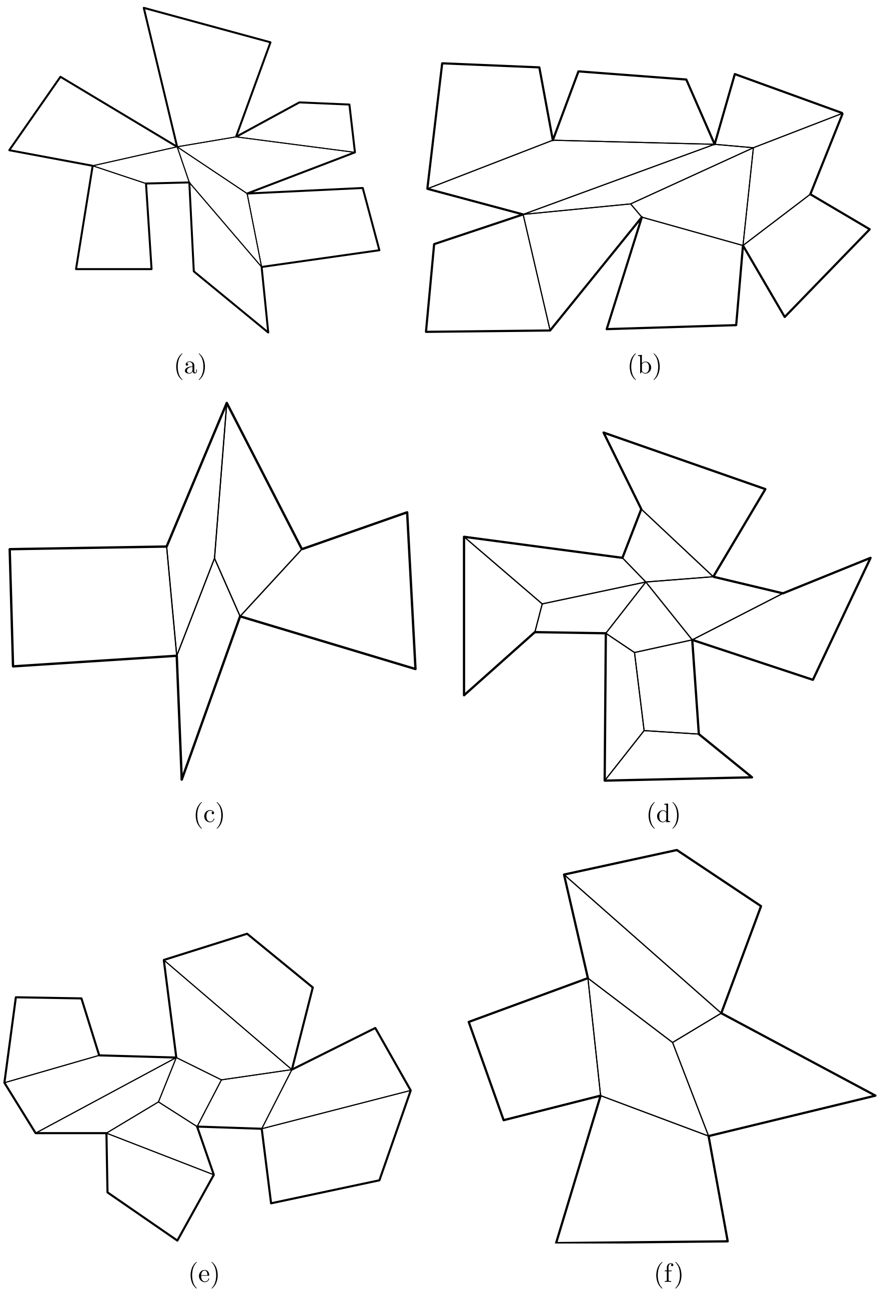
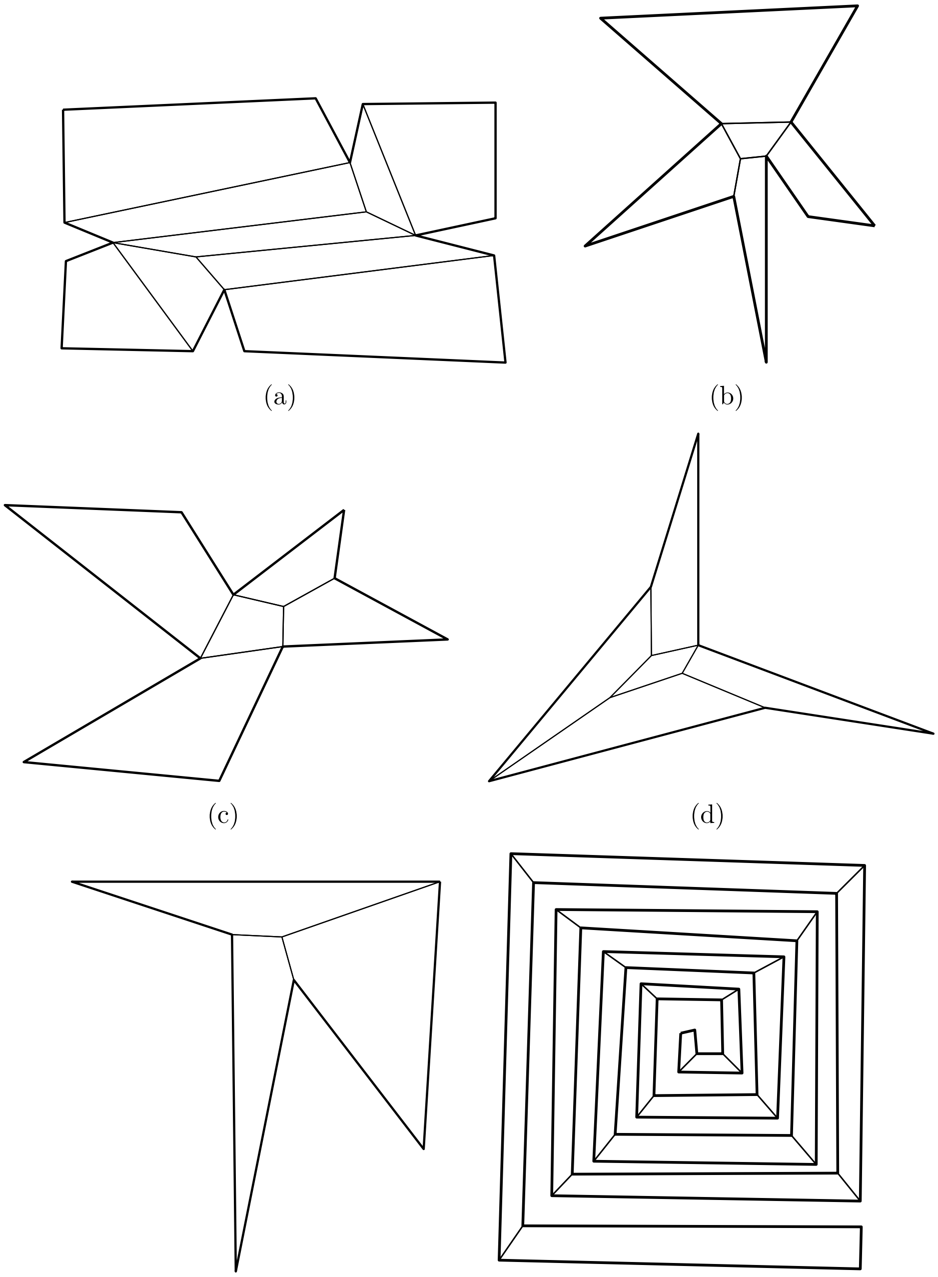


Figure 15: First set of results.



(a) (b) (c) (d) (e) (f)

Figure 16: Second set of results.

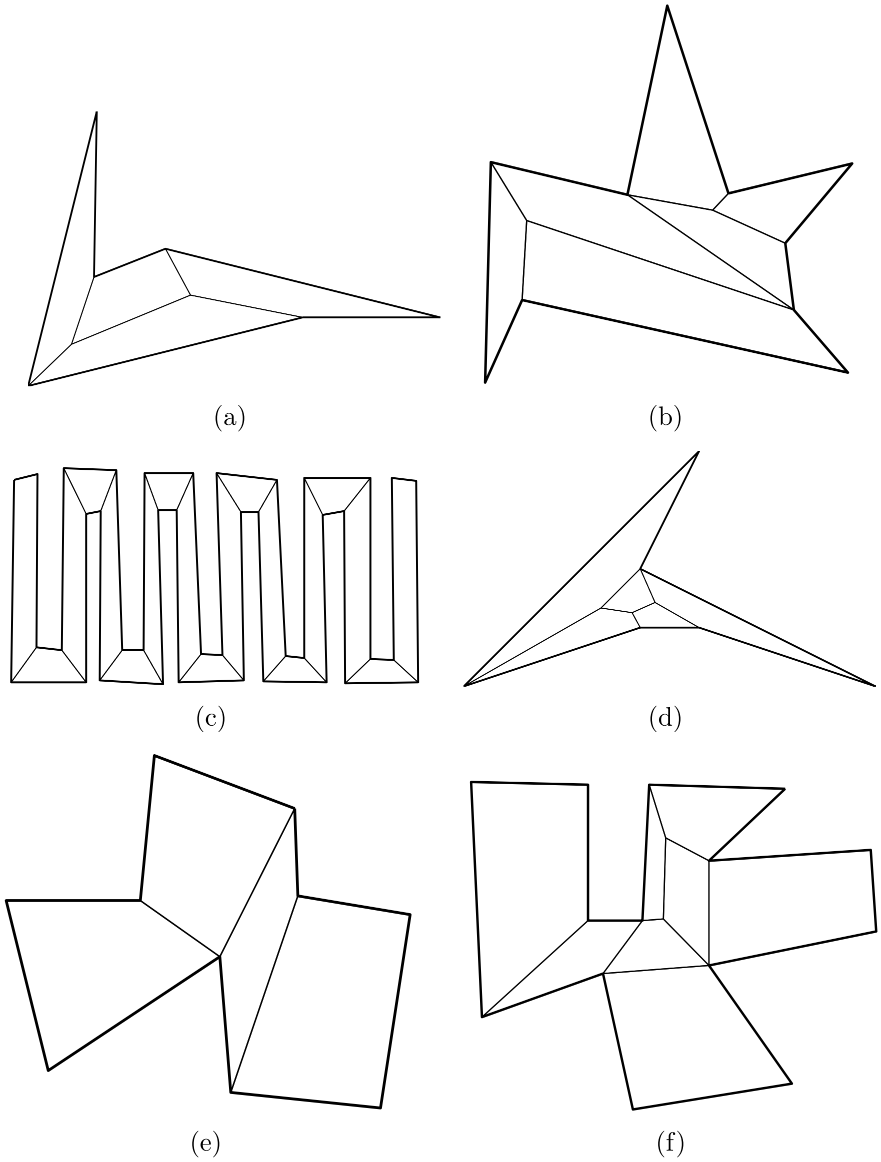


Figure 17: Third set of results.

various complexities (not in terms of number of vertices but in function of convexity). Some polygons are highly nonconvex while others are only slightly nonconvex. In fact, we display only results about simply connected polygons for now.
The only main difficulty if we want to implement the former techniques for multiply connected polygons is the cut search that we have seen in section 5 in order to reduce the problem into admissible polygons. In lemma 2 and corollary 1, we have only shown theoretically that there exists at least a cut from the exterior boundary to the interior one. In practice, there might exist many internal cuts while the choice of such a cut might affect the quality of the ultimate quadrangulation of the multiply connected polygon. To be more specific, the following four items have to be investigated more carefully in practical situations for polygons with holes:

- How to choose the cuts which are the best ones.
- Specification of "*best*" cuts.
- Efficient ways of finding the cuts.
- If we have domains with many holes, how to test visibility.

In the near future, we intend to analyze those issues more closely with the accomplishment of efficient approaches in mind.

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