TECHNISCHE UNIVERSITÄT CHEMNITZ

Sonderforschungsbereich 393

Parallele Numerische Simulation für Physik und Kontinuumsmechanik

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The Laplace and the linear elasticity problems near polyhedral corners and associated eigenvalue problems

Preprint SFB393/04-12

Abstract The solutions to certain elliptic boundary value problems have singularities with a typical structure near polyhedral corners. This structure can be exploited to devise an eigenvalue problem whose solution can be used to quantify the singularities of the given boundary value problem.

It is necessary to parametrize a ball centered at the corner. There are different possibilities for a suitable parametrization; from the numerical point of view, spherical coordinates are not necessarily the best choice. This is why we do not specify a parametrization in this paper but present all results in a rather general form. We derive the eigenvalue problems that are associated with the Laplace and the linear elasticity problems and show interesting spectral properties.

Finally, we discuss the necessity of widely accepted symmetry properties of the elasticity tensor. We show in an example that some of these properties are not only dispensable, but even invalid, although claimed in many standard books on linear elasticity.

Key Words corner singularities, elliptic boundary value problems, Laplace-Beltrami operator, linear elasticity problem, eigenvalue problems

AMS subject classification 35J05, 58J50, 65N25

Preprintreihe des Chemnitzer SFB 393

ISSN 1619-7178 (Print) ISSN 1619-7186 (Internet)

SFB393/04-12 December 2004

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1 Introduction

The analysis of conical domains with polyhedral corners reveals that the solution to a given boundary value problem with smooth data might not be smooth in the sense of Sobolev regularity; it is composed of a *singular* and a *regular* part. The specific structure of the solution to elliptic boundary value problems was studied by Kondrat'ev (1967). He showed that, under certain conditions, the solution can be expressed by an asymptotic series with members of the form

$$|x|^\alpha \sum_{k=0}^{s} \frac{1}{k!}(\log|x|)^k u_{s-k}(x/|x|)$$

in a neighbourhood of the conical point, where the so-called *singular exponent* α is an eigenvalue of an operator pencil, u_0 is the corresponding eigenelement and $u_1, \ldots, u_s$ are the generalized eigenelements corresponding to α. The terms $r := |x|$ and $x/|x|$ denote the distance to the corner and the projection of x to the unit sphere, respectively.

We are, in particular, interested in the boundary value problem for the Laplace operator and the linear elasticity problem near polyhedral corners and search solutions of the form

$$U = |x|^\alpha u(x/|x|). \tag{1}$$

It is well-known that the determination of α and u leads to a quadratic eigenvalue problem. In the case of the Laplace problem, a substitution allows to reduce the corresponding eigenvalue problem to a linear eigenvalue problem for the Laplace-Beltrami operator with a symmetric spectrum. For the linear elasticity problem, the reduction to a linear eigenvalue problem is not possible, but the spectrum of the corresponding eigenvalue problem has a considerable structure, see Leguillon (1995); Kozlov, Maz'ya, and Roßmann (2000); Apel, Sändig, and Solov'ev (2002b). These spectral properties allow an efficient computation of eigenpairs and therefore of the corner singularities. Engineers use the knowledge of the corner singularities, for instance, to predict the onset of cracks in brittle material, see Leguillon (1995); Leguillon and Sanchez-Palencia (1999); Leguillon (2002).

In the analysis community, it is common to apply a Mellin transformation for the deduction of the eigenvalue problem, see Kufner and Sändig (1987). Alternatively, the structure (1) of the solution can be directly inserted into the given problem. A good overview is given by Kozlov, Maz'ya, and Roßmann (2000). We concentrate on the second way and follow the ideas of Leguillon (1995). We could not find a reference, where the detailed deduction of the eigenvalue problems is given; usually the main ideas are merely outlined, see Beagles (1987); Leguillon (1995); Apel, Sändig, and Solov'ev (2002b). This is why we develop the eigenvalue problems corresponding to the Laplace and the linear elasticity problems step by step in this paper. But this is not the only purpose of this paper.

To use the approach (1) for the solution U, we have to parametrize a ball centered at the corner with the radial variable r (distance from the origin) and spherical parameters ξ_1, ξ_2. The question is, how to choose ξ_1 and ξ_2. It lies in the nature of spherical domains to use spherical coordinates, as it was done, for instance, by Leguillon (1995) and Apel, Sändig, and Solov'ev (2002b). The difficulties caused by the singularity of this parametrization are known and require a careful usage, see Mu (1996); Layton (2002); Apel and Pester (2004). From the numerical point of view, other parametrizations might be preferable, for example, a stereographic projection (Fichera (1975); Steger (1983)) or the projection of a refined icosahedron onto the sphere (Baumgardner and Frederickson (1985); Mu (1996)). Unfortunately, they also produce difficulties somewhere; the stereographic projection produces an infinite parameter domain and it is not clear how a subdomain of an icosahedron has to look like so that arbitrary subdomains of the sphere can be treated. We refer to Apel and Pester (2004) for a more detailed discussion of this problem.

The essence of this paper is that we leave the specific choice of the parametrization open to the user and use a general approach to present the deduction of the eigenvalue problems associated with the Laplace and the linear elasticity problems. This allows for a continuation of the (numerical) analysis of the eigenvalue problems in the same generality. Furthermore, we dealt with the necessity of the symmetry properties of the elasticity tensor. As a byproduct we detected that some of these properties – although accepted among engineers and claimed in many standard books on continuum mechanics and linear elasticity – are not only unnecessary, but also sometimes invalid or should at least be handled with care. We demonstrate this in an example.

Section 2 is of introductory nature; we give an overview of nomenclature in differential geometry and tensor calculus. We summarize the terms that are essential for our purposes. Details can be found in any standard book on differential geometry (e.g. Peschl (1973); Schöne (1987)) or tensor calculus (e.g. Leipholz (1968); de Boer (1982)). In Sections 3 and 4, we derive the eigenvalue problems that are associated with the Laplace and the linear elasticity problems. The spectra of both eigenvalue problems have a symmetric structure which can be exploited in numerical algorithms for a fast computation of the eigenvalues. We show this for the linear elasticity problem by proving important properties of the sesquilinear forms that define the corresponding eigenvalue problem. In Section 4.2, we analyse the assumptions that are used for the analysis of the linear elasticity problem and discuss their necessity and validity.

Throughout the paper, we write scalars in italic type, vector functions in underlined italic type and special vectors (or points) in boldface roman type.

2 Spherical nomenclature and tensor calculus

2.1 Covariant and contravariant tensor bases

We consider the three-dimensional space $\mathbb{R}^3$. Each point $\underline{\mathbf{X}} \in \mathbb{R}^3$ can be represented by three parameters ξ_1, ξ_2, ξ_3:

$$\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, \xi_3).$$

The *covariant* and the *contravariant tensor bases* are given by $\mathbf{g}_i$, $\mathbf{g}^i$, respectively:

$$\mathbf{g}_i := \frac{\partial}{\partial \xi_i}\underline{\mathbf{X}} \quad \text{and} \quad \mathbf{g}^i \text{ with } \mathbf{g}_i \cdot \mathbf{g}^j = \delta_{ij}, \quad i = 1, 2, 3.$$

The *gradient* and the *Laplace operator* are given by

$$\nabla = \sum_{i=1}^{3} \mathbf{g}^i \frac{\partial}{\partial \xi_i} \quad \text{and} \quad \Delta = \nabla \cdot \nabla.$$

2.2 Parametrization of the unit ball

Let $\underline{\mathbf{x}}(\xi_1, \xi_2)$ be any parametrization of a subset of the unit sphere $\mathcal{S}^2 \subset \mathbb{R}^3$, i.e., let $\|\underline{\mathbf{x}}(\xi_1, \xi_2)\| = 1$ for all (ξ_1, ξ_2) in a given parameter domain $\mathcal{G} \subset \mathbb{R}^2$. Any vector $\underline{\mathbf{X}} \in \mathbb{R}^3$ is then represented by the three parameters ξ_1, ξ_2, ξ_3, where $\xi_3 = r$ describes the radial variable (or distance from the origin). Hence, $\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, \xi_3) = r \cdot \underline{\mathbf{x}}(\xi_1, \xi_2)$.

For abbreviation, we write $\underline{\mathbf{x}}_{,i}$ instead of $\partial \underline{\mathbf{x}} / \partial \xi_i$. The covariant tensor basis reads

$$\mathbf{g}_i = r \underline{\mathbf{x}}_{,i} \quad (i = 1, 2) \qquad \text{and} \qquad \mathbf{g}_3 = \underline{\mathbf{x}}.$$

Note that $\underline{\mathbf{x}} \perp \underline{\mathbf{x}}_{,i}$ (that is $\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}_{,i} = 0$) for $i = 1, 2$, since $\|\underline{\mathbf{x}}\|^2 = \underline{\mathbf{x}} \cdot \underline{\mathbf{x}} = 1$.

The contravariant tensor basis $\{\mathbf{g}^i\}_{i=1}^3$ satisfies

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_{ij} \quad \text{and} \quad \mathbf{g}^3 = \mathbf{g}_3 = \underline{\mathbf{x}}. \tag{2}$$

We use the index $\mathcal{S}$ to indicate that we restrict our considerations to the unit sphere $(r = 1)$:

$$\mathbf{g}^i_{\mathcal{S}} := \mathbf{g}^i(\xi_1, \xi_2, 1) = r\mathbf{g}^i(\xi_1, \xi_2, r), \quad i = 1, 2.$$

For a point $\underline{\mathbf{x}} = \underline{\mathbf{x}}(\xi_1, \xi_2) \in \mathcal{S}^2$, the (spherical) gradient of u is given by

$$\nabla_{\mathcal{S}} u := \sum_{i=1}^{2} \frac{\partial u}{\partial \xi_i} \mathbf{g}^i_{\mathcal{S}}. \tag{3}$$

Let $g_{ij} := \mathbf{g}_i \cdot \mathbf{g}_j$ be the so called first metric fundamental terms and let $g^{ij} := \mathbf{g}^i \cdot \mathbf{g}^j$, $i, j = 1, 2, 3$. We define the 2×2-matrices

$$G := (g_{ij})_{i,j=1}^2 \quad \text{and} \quad \tilde{G} := (g^{ij})_{i,j=1}^2.$$

We conclude that $G = r^2 (\underline{\mathbf{x}}_{,i} \cdot \underline{\mathbf{x}}_{,j})_{i,j=1}^2 = r^2 G_{\mathcal{S}}$, where

$$G_{\mathcal{S}} := (\underline{\mathbf{x}}_{,i} \cdot \underline{\mathbf{x}}_{,j})_{i,j=1}^2$$

is the matrix containing the first metric fundamental terms of the unit sphere. It turns out that $\tilde{G} = G^{-1} = \frac{1}{r^2} G_{\mathcal{S}}^{-1}$. We denote the elements of $G_{\mathcal{S}}^{-1}$ by $g^{ij}_{\mathcal{S}}$.

2.3 Volume and surface elements

The volume element $\mathrm{d}\Omega$ is given by $\mathrm{d}\Omega = |[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]| \,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$, where $[\cdot,\cdot,\cdot]$ denotes the scalar triple product. Since $\mathbf{g}_3 \| \mathbf{g}_1 \times \mathbf{g}_2$ and $\|\mathbf{g}_3\| = 1$, we have $\mathbf{g}_3 = \pm(\mathbf{g}_1 \times \mathbf{g}_2)/\|\mathbf{g}_1 \times \mathbf{g}_2\|$. Hence,

$$|[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]| = |(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3| = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \frac{(\mathbf{g}_1 \times \mathbf{g}_2)}{\|\mathbf{g}_1 \times \mathbf{g}_2\|} = \|\mathbf{g}_1 \times \mathbf{g}_2\| = r^2 \|\underline{\mathbf{x}}_{,1} \times \underline{\mathbf{x}}_{,2}\|,$$

and therefore

$$\mathrm{d}\Omega = r^2 \,\mathrm{d}\mathcal{S}\,\mathrm{d}r, \quad \text{where} \quad \mathrm{d}\mathcal{S} := \|\underline{\mathbf{x}}_{,1} \times \underline{\mathbf{x}}_{,2}\| \,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

is the surface element of the unit sphere.

2.4 Tensor calculus

2.4.1 First- and second-order tensors

A tensor is a multilinear form over a vector space $\mathcal{V}$ over $\mathbb{R}$ equipped with fixed bases. We consider only three-dimensional vector spaces.

The *first-order tensors* are the vectors themselves, once a basis $\{\mathbf{b}_i\}_{i=1}^3$ in $\mathcal{V}$ is chosen. They can be written as linear combinations of the basis vectors $\mathbf{b}_i$:

$$\tau = \sum_{i=1}^{3} \tau_i \mathbf{b}_i.$$

Usually, real coefficients τ_i are sufficient. For the application in linear elasticity, however, complex coefficients are necessary, see Section 4. The (inner) product of two first-order tensors is given by

$$\tau \cdot \sigma = \Big(\sum_{i=1}^{3} \tau_i \mathbf{b}_i\Big) \cdot \Big(\sum_{j=1}^{3} \sigma_j \mathbf{c}_j\Big) := \sum_{i,j=1}^{3} \tau_i \sigma_j (\mathbf{b}_i \cdot \mathbf{c}_j),$$

where $\cdot$ in the last sum denotes the usual inner product of vectors in $\mathbb{R}^3$.

Second-order tensors are linear combinations of pairs of basis vectors, in general

$$\tau = \sum_{i,j=1}^{3} \tau_{ij} \mathbf{b}_i \mathbf{c}_j,$$

where $\{\mathbf{b}_i\}_{i=1}^3$ and $\{\mathbf{c}_i\}_{i=1}^3$ are two fixed bases in the vector space $\mathcal{V}$. A bilinear form over $\mathcal{V}$ is given by

$$u \cdot \tau \cdot v = \Big(\sum_{i=1}^{3} u_i \mathbf{b}_i\Big) \cdot \Big(\sum_{k,h=1}^{3} \tau_{kh} \mathbf{c}_k \mathbf{d}_h\Big) \cdot \Big(\sum_{j=1}^{3} v_j \mathbf{e}_j\Big) := \sum_{i,j,k,h=1}^{3} u_i v_j \tau_{kh} (\mathbf{b}_i \cdot \mathbf{c}_k)(\mathbf{d}_h \cdot \mathbf{e}_j),$$

where u, v are first-order tensors and where τ is a second-order tensor. For second-order tensors, we define the binary operator $:$ by

$$\tau : \sigma = \Big(\sum_{i,j=1}^{3} \tau_{ij} \mathbf{b}_i \mathbf{c}_j \Big) : \Big(\sum_{k,h=1}^{3} \sigma_{kh} \mathbf{d}_k \mathbf{e}_h \Big) := \sum_{i,j,k,h=1}^{3} \tau_{ij} \sigma_{kh} (\mathbf{b}_i \cdot \mathbf{e}_h)(\mathbf{c}_j \cdot \mathbf{d}_k).$$

By analogy, tensors and products of higher order can be introduced.

2.4.2 The vector product of tensors

The 'vector' product for tensors is well-defined, see, for example, de Boer (1982). For our purposes, we need only the vector product of first-order tensors which equals the usual vector product, possibly with complex coefficients:

$$\tau \times \sigma = \Big(\sum_{i=1}^{3} \tau_i \mathbf{b}_i \Big) \times \Big(\sum_{j=1}^{3} \sigma_j \mathbf{c}_j \Big) = \sum_{i,j=1}^{3} \tau_i \sigma_j (\mathbf{b}_i \times \mathbf{c}_j),$$

where $\{\mathbf{b}_i\}_{i=1}^3$ and $\{\mathbf{c}_i\}_{i=1}^3$ are bases in $\mathbb{R}^3$. Obviously, the properties

$$\tau \times \sigma = -(\sigma \times \tau), \qquad (c\,\tau) \times \sigma = \tau \times (c\,\sigma) = c\,(\tau \times \sigma)$$

are satisfied also for complex tensors τ, σ.

2.4.3 The transposed tensor

Let $\tau = \sum_{i,j=1}^{3} \tau_{ij} \mathbf{b}_i \mathbf{c}_j$ be a second-order tensor. Such as for matrices, the transposed tensor $\tau^\top$ should fulfil the equality

$$u \cdot \tau \cdot v = v \cdot \tau^\top \cdot u$$

for all first-order tensors u, v. One easily checks that this is true, if

$$\tau^\top = \sum_{i,j=1}^{3} \tau_{ij} \mathbf{c}_j \mathbf{b}_i.$$

Using the definition of the product of second-order tensors, one readily verifies that

$$\tau : \sigma = \sigma : \tau = \tau^\top : \sigma^\top \qquad \text{and} \qquad \tau^\top : \sigma = \sigma^\top : \tau. \tag{4}$$

We denote the transposed conjugate complex tensor by τ^H:

$$\tau^H := \overline{\tau}^\top = \sum_{i,j=1}^{3} \overline{\tau}_{ij} \mathbf{c}_j \mathbf{b}_i.$$

We say that the tensor τ is *real*, if $\tau_{ij} = \overline{\tau}_{ij}$ for any real bases $\{\mathbf{b}_k\}$, $\{\mathbf{c}_k\}$ of $\mathbb{R}^3$. Then $\tau^H = \tau^\top$.

3 The Laplace problem

Let $\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, \xi_3)$ be a point in $\mathbb{R}^3$ with the radial variable $\xi_3 = r = \|\underline{\mathbf{X}}\|$ (distance to the origin). We consider a conical domain $\mathcal{K} \subset \mathbb{R}^3$ and its intersection $\mathcal{S}_\mathcal{K}$ with the unit sphere, so that

$$\mathcal{K} = \{\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, r) \in \mathbb{R}^3 \mid r > 0,\ \underline{\mathbf{x}} = \underline{\mathbf{x}}(\xi_1, \xi_2) = \underline{\mathbf{X}}/\|\underline{\mathbf{X}}\| \in \mathcal{S}_\mathcal{K}\}.$$

We consider the homogeneous Dirichlet problem for the Laplace operator

$$\Delta U = 0 \quad \text{in } \mathcal{K}, \qquad U = 0 \quad \text{on } \partial\mathcal{K} \setminus \{0\}. \tag{5}$$

Note that $U \equiv 0$ is not the only solution, because we consider an infinite cone. If $\partial\mathcal{K} = \emptyset$, that is, $\mathcal{S}_\mathcal{K} = \mathcal{S}^2$, we omit the boundary condition. The function U is a real-valued scalar function mapping from $\mathcal{K}$ to $\mathbb{R}$. Its gradient is a first-order tensor. We write the function $U = U(\underline{\mathbf{X}})$ as $U = U(\xi_1, \xi_2, r)$, where we omit the introduction of a new symbol for U. As stated in the introduction, the singular part of the solution has the form

$$U(\underline{\mathbf{X}}) = U(\xi_1, \xi_2, r) = r^\alpha u(\xi_1, \xi_2), \tag{6}$$

see Kondrat'ev (1967) or Kozlov, Maz'ya, and Roßmann (2000). The function U has not to be an L^2-function in the common sense. For a discussion of the regularity of such functions, see Kufner and Sändig (1987) and Kozlov, Maz'ya, and Roßmann (2000) for details.

Recall $\mathbf{g}^i_\mathcal{S}(\xi_1, \xi_2) = \mathbf{g}^i(\xi_1, \xi_2, r) = r^{-1}\mathbf{g}^i(\xi_1, \xi_2, 1)$ for $i = 1, 2$ and

$$\nabla_\mathcal{S} u := \sum_{i=1}^{2} \frac{\partial u}{\partial \xi_i} \mathbf{g}^i_\mathcal{S},$$

where the symbol $\nabla_\mathcal{S}$ is used to emphasize that this gradient lives only on the unit sphere.

We require that the function u and its spherical gradient $\nabla_\mathcal{S} u$ are quadratically integrable with respect to the surface element $\mathrm{d}\mathcal{S}$ and denote the space of all such functions by $H^1(\mathcal{S}_\mathcal{K})$,

$$H^1(\mathcal{S}_\mathcal{K}) := \Big\{u : \mathcal{S}_\mathcal{K} \to \mathbb{R} \,\Big|\, \int_{\mathcal{S}_\mathcal{K}} u^2 + \nabla_\mathcal{S} u \cdot \nabla_\mathcal{S} u \,\mathrm{d}\mathcal{S} < \infty\Big\}.$$

We define the bilinear form

$$a(U, V) := \int_\mathcal{K} \nabla U \cdot \nabla V \,\mathrm{d}\Omega$$

for $U, V \in \mathcal{V}_0$, where the functions in $\mathcal{V}_0$ vanish on $\partial\mathcal{K}$ and are smooth enough that the integral in the bilinear form exists. We consider the weak formulation of problem (5): Find $U \in \mathcal{V}_0$, such that for all $V \in \mathcal{V}_0$

$$a(U, V) = 0.$$

For the validity of the divergence theorem on the sphere, we refer to work on spherical calculus including Malvern (1969) and Freeden, Gervens, and Schreiner (1998).

We search for solutions U of the form (6) and use a similar approach for the test functions V:

$$V(\underline{\mathbf{X}}) = \Phi(r)v(\xi_1, \xi_2),$$

where $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ is a scalar function with bounded support. We conclude that

$$\nabla V(\underline{\mathbf{X}}) = \sum_{i=1}^{2} \Phi(r) v_{,i} \mathbf{g}^i + \Phi'(r) v \mathbf{g}^3 = \Phi(r) r^{-1} \nabla_{\mathcal{S}} v + \Phi'(r) v(\xi_1, \xi_2) \underline{\mathbf{x}}.$$

Consequently, using r^α instead of $\Phi(r)$ and u instead of v, we obtain that

$$\nabla U = r^{\alpha-1} \nabla_{\mathcal{S}} u + \alpha r^{\alpha-1} u(\xi_1, \xi_2) \underline{\mathbf{x}}$$

and therefore with (2)

$$\nabla U \cdot \nabla V = r^{\alpha-2} \Phi(r) (\nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v) + \alpha r^{\alpha-1} \Phi'(r) uv.$$

Finally, we have that

$$\begin{aligned} a(U, V) &= \int_0^\infty \int_{\mathcal{S}_\mathcal{K}} \Big[r^{\alpha-2} \Phi(r) (\nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v) + \alpha r^{\alpha-1} \Phi'(r) uv \Big] r^2 \, \mathrm{d}r \, \mathrm{d}\mathcal{S} \\ &= \Big(\int_0^\infty r^\alpha \Phi(r) \, \mathrm{d}r \Big) \Big(\int_{\mathcal{S}_\mathcal{K}} \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v \, \mathrm{d}\mathcal{S} \Big) + \Big(\int_0^\infty r^{\alpha+1} \Phi'(r) \, \mathrm{d}r \Big) \Big(\int_{\mathcal{S}_\mathcal{K}} \alpha uv \, \mathrm{d}\mathcal{S} \Big). \end{aligned}$$

Partial integration and the assumption $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ yield

$$\int_0^\infty r^{\alpha+1} \Phi'(r) \, \mathrm{d}r = -(\alpha+1) \int_0^\infty r^\alpha \Phi(r) \, \mathrm{d}r \tag{7}$$

and consequently

$$a(U, V) = \Big(\int_0^\infty r^\alpha \Phi(r) \, \mathrm{d}r \Big) \cdot \Big(\int_{\mathcal{S}_\mathcal{K}} \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v - \alpha(\alpha+1) uv \, \mathrm{d}\mathcal{S} \Big).$$

Since $a(U, V) = 0$ in $\mathcal{K}$, we divide by the integral over r and obtain

$$\int_{\mathcal{S}_\mathcal{K}} \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v - \alpha(\alpha+1) uv \, \mathrm{d}\mathcal{S} = 0.$$

This is the weak formulation of the eigenvalue problem

$$-\Delta_{\mathcal{S}} u = \alpha(\alpha+1) u \quad \text{in } \mathcal{S}_\mathcal{K}, \qquad u = 0 \quad \text{on } \partial\mathcal{S}_\mathcal{K}, \tag{8}$$

where $\Delta_{\mathcal{S}} = \nabla_{\mathcal{S}} \cdot \nabla_{\mathcal{S}}$ denotes the Laplace operator of the unit sphere (also known as Laplace-Beltrami operator). The substitution $\lambda = \alpha(\alpha+1)$ shows that this problem can be reduced to a linear eigenvalue problem on the sphere.

Remark 3.1. *Another interesting substitution is* $\lambda = \alpha + \frac{1}{2}$ *although it seems to be a bit peculiar. The weak formulation of the eigenvalue problem reads now*

$$\lambda^2 \int_{\mathcal{S}_\mathcal{K}} uv \,\mathrm{d}\mathcal{S} = \int_{\mathcal{S}_\mathcal{K}} \nabla_\mathcal{S} u \cdot \nabla_\mathcal{S} v + \frac{1}{4} uv \,\mathrm{d}\mathcal{S}$$

and satisfies all the assumptions summarized by Pester (2004) which ensure a symmetric structure of the spectrum. This means, in particular, that with λ *also* $-\lambda$ *is an eigenvalue, and therefore, the spectrum of problem (8) is symmetric with respect to the axis* $\operatorname{Re}\alpha = -\frac{1}{2}$.

4 The linear elasticity problem

4.1 The formulation with tensors

We consider a conical domain $\mathcal{K} \subset \mathbb{R}^3$ and its intersection $\mathcal{S}_\mathcal{K}$ with the unit sphere $\mathcal{S}^2$ as described in the previous section. But now, we study complex-valued vector functions $\underline{U}$ and $\underline{V}$ which map from $\mathcal{K}$ to $\mathbb{C}^3$. The gradient of a vector function is a second-order tensor.

We define the sesquilinear form

$$a(\underline{U}, \underline{V}) := \int_\mathcal{K} \varepsilon(\overline{\underline{V}}) : A : \varepsilon(\underline{U}) \,\mathrm{d}\Omega,$$

where A is a constant tensor of order four (the so-called *elasticity tensor*) and $\varepsilon(\cdot)$ is the *Green* strain tensor,

$$\varepsilon(\underline{U}) = \frac{1}{2}\Big(\nabla \underline{U} + (\nabla \underline{U})^\top\Big) \tag{9}$$

based on the displacement function $\underline{U}$. The term $\overline{\underline{V}}$ denotes the conjugate complex vector of $\underline{V}$. We consider the generalized form of the homogeneous elasticity problem: Find $\underline{U} \in \mathcal{V}_0$, such that for all $\underline{V} \in \mathcal{V}_0$

$$a(\underline{U}, \underline{V}) = 0,$$

where the space $\mathcal{V}_0$ consists of vector functions that vanish on $\partial K \setminus \{0\}$ and that are smooth enough so that the integral in the bilinear form exists. As in the previous section, it is not required that $\underline{U}$ is an element of $[L^2(\mathcal{K})]^3$.

We require that the tensor A is real and defines an inner product on the space of second-order tensors, that is, in particular,

$$\tau^\top : A : \sigma = \sigma^\top : A : \tau \tag{10}$$

and

$$M_1 \,(\tau^H : \tau) \le \tau^H : A : \tau \le M_2 \,(\tau^H : \tau) \tag{11}$$

for all (real or complex) second-order tensors τ, σ and for some positive constants M_1, M_2.

Remark 4.1. *It is sufficient to require the boundedness and ellipticity condition (11) only for real second-order tensors, since in combination with (10) the complex version is a consequence of the real one. This can be proven by applying the real version of (11) to the real and imaginary parts of* $\tau = \tau_R + \mathrm{i}\,\tau_I$*: We have*

$$\begin{aligned}\tau^H : A : \tau &= \tau_R^\top : A : \tau_R + \tau_I^\top : A : \tau_I + \mathrm{i}\,(\tau_R^\top : A : \tau_I - \tau_I^\top : A : \tau_R)\\ &= \tau_R^\top : A : \tau_R + \tau_I^\top : A : \tau_I,\end{aligned}$$

which is, by assumption, bounded by $\tau_R^\top : \tau_R + \tau_I^\top : \tau_I = \tau^H : \tau$*.*

Note that the restriction of (10) to real tensors τ*,* σ *is also sufficient to imply the validity of (10) for complex tensors.*

Using (10) and (11), one easily verifies that the form

$$\langle \tau, \sigma \rangle_A := \int_{\mathcal{S}_\mathcal{K}} \sigma^H : A : \tau \,\mathrm{d}\mathcal{S}$$

defines an inner product on the space of all functions whose images are second order tensors (and for which the integral exists).

As in the previous section, we consider $\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with the radial variable $\xi_3 = r$ (distance from the origin) and write

$$\underline{U} = \underline{U}(\underline{\mathbf{X}}) = \underline{U}(\xi_1, \xi_2, \xi_3).$$

The solutions $\underline{U}$ of physical interest can be expanded in terms $r^\alpha \underline{u}(\xi_1, \xi_2)$ with the singular exponent $\alpha \in \mathbb{C}$, see the argumentation of Section 1, Kondrat'ev (1967) or Kozlov, Maz'ya, and Roßmann (2000). We use a similar approach for $\underline{V}$:

$$\underline{U}(\xi_1, \xi_2, \xi_3) = r^\alpha \underline{u}(\xi_1, \xi_2), \qquad \underline{V}(\xi_1, \xi_2, \xi_3) = \Phi(r) \underline{v}(\xi_1, \xi_2),$$

where $\underline{u}, \underline{v} \in [H^1(\mathcal{S}_\mathcal{K})]^3$ are vector functions on the sphere and where $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ is a scalar function with bounded support.

With similar arguments as in the previous section, one obtains that

$$\nabla \underline{U} = r^{\alpha-1} \nabla_\mathcal{S} \underline{u} + \alpha r^{\alpha-1} \underline{\mathbf{x}} \underline{u} \qquad \text{and} \qquad \nabla \underline{V} = \Phi(r) r^{-1} \nabla_\mathcal{S} \underline{v} + \Phi'(r) \underline{\mathbf{x}} \underline{v}.$$

Here, $\nabla_\mathcal{S} \underline{u} = \sum_{i=1}^2 \mathbf{g}_\mathcal{S}^i \partial \underline{u} / \partial \xi_i$ and $\nabla_\mathcal{S} \underline{v} = \sum_{i=1}^2 \mathbf{g}_\mathcal{S}^i \partial \underline{v} / \partial \xi_i$ are special second-order tensors, defined on the unit sphere.

For abbreviation, we define

$$\varepsilon_\mathcal{S}(\underline{u}) := \frac{1}{2}(\nabla_\mathcal{S} \underline{u} + (\nabla_\mathcal{S} \underline{u})^\top), \qquad \varepsilon_3(\underline{u}) := \frac{1}{2}(\underline{\mathbf{x}} \underline{u} + (\underline{\mathbf{x}} \underline{u})^\top). \tag{12}$$

Then, we get from (9) that

$$\varepsilon(\overline{\underline{V}}) : A : \varepsilon(\underline{U}) \;=\; \Big(r^{-1} \Phi(r) \varepsilon_\mathcal{S}(\overline{\underline{v}}) + \Phi'(r) \varepsilon_3(\overline{\underline{v}})\Big) : A : \Big(r^{\alpha-1} \varepsilon_\mathcal{S}(\underline{u}) + \alpha r^{\alpha-1} \varepsilon_3(\underline{u})\Big)$$

$$
\begin{aligned}
&= \Big(r^{-1}\Phi(r)\varepsilon_{\mathcal{S}}(\underline{\bar{v}})\Big) : A : \Big(r^{\alpha-1}\varepsilon_{\mathcal{S}}(\underline{u})\Big)\\
&\quad + \Big(\Phi'(r)\varepsilon_3(\underline{\bar{v}})\Big) : A : \Big(r^{\alpha-1}\varepsilon_{\mathcal{S}}(\underline{u})\Big)\\
&\quad + \Big(r^{-1}\Phi(r)\varepsilon_{\mathcal{S}}(\underline{\bar{v}})\Big) : A : \Big(\alpha r^{\alpha-1}\varepsilon_3(\underline{u})\Big)\\
&\quad + \Big(\Phi'(r)\varepsilon_3(\underline{\bar{v}})\Big) : A : \Big(\alpha r^{\alpha-1}\varepsilon_3(\underline{u})\Big)\\
&= r^{\alpha-2}\Phi(r)\Big(\varepsilon_{\mathcal{S}}(\underline{\bar{v}}) : A : \varepsilon_{\mathcal{S}}(\underline{u}) + \varepsilon_{\mathcal{S}}(\underline{\bar{v}}) : A : \alpha\varepsilon_3(\underline{u})\Big)\\
&+ r^{\alpha-1}\Phi'(r)\Big(\varepsilon_3(\underline{\bar{v}}) : A : \varepsilon_{\mathcal{S}}(\underline{u}) + \varepsilon_3(\underline{\bar{v}}) : A : \alpha\varepsilon_3(\underline{u})\Big).
\end{aligned}
$$

We introduce the sesquilinear forms

$$
\begin{aligned}
a_{\mathcal{SS}}(\underline{u},\underline{v}) &:= \int_{\mathcal{S}_{\mathcal{K}}} \varepsilon_{\mathcal{S}}(\underline{\bar{v}}) : A : \varepsilon_{\mathcal{S}}(\underline{u})\ \mathrm{d}\mathcal{S}, \qquad a_{\mathcal{S}3}(\underline{u},\underline{v}) := \int_{\mathcal{S}_{\mathcal{K}}} \varepsilon_3(\underline{\bar{v}}) : A : \varepsilon_{\mathcal{S}}(\underline{u})\ \mathrm{d}\mathcal{S},\\
a_{3\mathcal{S}}(\underline{u},\underline{v}) &:= \int_{\mathcal{S}_{\mathcal{K}}} \varepsilon_{\mathcal{S}}(\underline{\bar{v}}) : A : \varepsilon_3(\underline{u})\ \mathrm{d}\mathcal{S}, \qquad a_{33}(\underline{u},\underline{v}) := \int_{\mathcal{S}_{\mathcal{K}}} \varepsilon_3(\underline{\bar{v}}) : A : \varepsilon_3(\underline{u})\ \mathrm{d}\mathcal{S}.
\end{aligned}
$$

With relation (7), one obtains

$$
\begin{aligned}
a(\underline{U},\underline{V}) &= \Big(\int_0^\infty r^\alpha\Phi(r)\ \mathrm{d}r\Big)\cdot\\
&\cdot\Big(a_{\mathcal{SS}}(\underline{u},\underline{v}) + \alpha a_{3\mathcal{S}}(\underline{u},\underline{v}) - (\alpha+1)a_{\mathcal{S}3}(\underline{u},\underline{v}) - \alpha(\alpha+1)a_{33}(\underline{u},\underline{v})\Big).
\end{aligned}
$$

Since $a(\underline{U},\underline{V}) = 0$ in $\mathcal{K}$, the division by the integral over r yields

$$
a_{\mathcal{SS}}(\underline{u},\underline{v}) + \alpha a_{3\mathcal{S}}(\underline{u},\underline{v}) - (\alpha+1)a_{\mathcal{S}3}(\underline{u},\underline{v}) - \alpha(\alpha+1)a_{33}(\underline{u},\underline{v}) = 0.
$$

This is an operator eigenvalue problem for α and $\underline{u}$. The analysis of the spectral properties of this eigenvalue problem reveals that it is useful to substitute α by $\lambda - \frac{1}{2}$. Then, under certain conditions, the spectrum is symmetric with respect to the real an the imaginary axes; see Remark 4.8. We obtain the quadratic operator eigenvalue problem for $\lambda, \underline{u}$: Find $(\lambda,\underline{u}) \in \mathbb{C}\times[H^1(\mathcal{S}_{\mathcal{K}})]^3$, such that for all $\underline{v} \in [H^1(\mathcal{S}_{\mathcal{K}})]^3$

$$
\begin{aligned}
&\lambda^2 a_{33}(\underline{u},\underline{v}) - \lambda(a_{3\mathcal{S}}(\underline{u},\underline{v}) - a_{\mathcal{S}3}(\underline{u},\underline{v}))\\
&= a_{\mathcal{SS}}(\underline{u},\underline{v}) - \frac{1}{2}(a_{3\mathcal{S}}(\underline{u},\underline{v}) + a_{\mathcal{S}3}(\underline{u},\underline{v})) + \frac{1}{4}a_{33}(\underline{u},\underline{v}).
\end{aligned}
$$

Substituting

$$
\begin{aligned}
m(\underline{u},\underline{v}) &:= a_{33}(\underline{u},\underline{v}),\\
g(\underline{u},\underline{v}) &:= a_{\mathcal{S}3}(\underline{u},\underline{v}) - a_{3\mathcal{S}}(\underline{u},\underline{v}),\\
k(\underline{u},\underline{v}) &:= a_{\mathcal{SS}}(\underline{u},\underline{v}) - \frac{1}{2}(a_{3\mathcal{S}}(\underline{u},\underline{v}) + a_{\mathcal{S}3}(\underline{u},\underline{v})) + \frac{1}{4}a_{33}(\underline{u},\underline{v}),
\end{aligned}
$$

we obtain the quadratic operator eigenvalue problem

$$
\lambda^2 m(\underline{u},\underline{v}) + \lambda g(\underline{u},\underline{v}) = k(\underline{u},\underline{v}). \tag{13}
$$

Definition 4.2. *Let $H^0(\mathcal{S}_\mathcal{K})$ be the space of all functions over $\mathcal{S}_\mathcal{K}$ that are quadratically integrable with the spherical surface element. Recall that $H^1(\mathcal{S}_\mathcal{K})$ was the space of all functions whose derivatives are quadratically integrable over $\mathcal{S}_\mathcal{K}$ as well.*

We set $H := [H^0(\mathcal{S}_\mathcal{K})]^3$ and $V := [H^1(\mathcal{S}_\mathcal{K})]^3$. These spaces are equipped with the norms

$$\begin{aligned} \|\underline{u}\|_0 &:= \Big(\int_{\mathcal{S}_\mathcal{K}} (\mathbf{x}\underline{u})^H : (\mathbf{x}\underline{u})\,\mathrm{d}\mathcal{S}\Big)^{1/2} = \Big(\int_{\mathcal{S}_\mathcal{K}} \overline{\underline{u}}\cdot\underline{u}\,\mathrm{d}\mathcal{S}\Big)^{1/2} \qquad \textit{and} \\ \|\underline{u}\|_1 &:= = \Big(\frac{1}{4}\|\underline{u}\|_0^2 + |\underline{u}|_1^2\Big)^{1/2} \\ \textit{with} \qquad |\underline{u}|_1 &:= \Big(\int_{\mathcal{S}_\mathcal{K}} (\nabla_\mathcal{S}\underline{u})^H : (\nabla_\mathcal{S}\underline{u})\,\mathrm{d}\mathcal{S}\Big)^{1/2}. \end{aligned}$$

In addition, we introduce the norms

$$\|\underline{u}\|_H := \Big(\int_{\mathcal{S}_\mathcal{K}} \varepsilon_3(\overline{\underline{u}}) : \varepsilon_3(\underline{u})\,\mathrm{d}\mathcal{S}\Big)^{1/2} \qquad \textit{and}$$

$$\|\underline{u}\|_V := \Big(\int_{\mathcal{S}_\mathcal{K}} (\varepsilon_\mathcal{S}(\overline{\underline{u}}) - \frac{1}{2}\varepsilon_3(\overline{\underline{u}})) : (\varepsilon_\mathcal{S}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u}))\,\mathrm{d}\mathcal{S}\Big)^{1/2}$$

for $\underline{u}$ in H or V, respectively.

The factor 1/4 in the 1-norm was introduced for convenience by Apel, Sändig, and Solov'ev (2002b) (although, accidentally placed in front of the 1-seminorm instead of the 0-norm). See also Kozlov, Maz'ya, and Roßmann (2000) for a definition of the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on curved surfaces. Due to (4), the H^1-norm can also be written as

$$\|\underline{u}\|_1 = \Big(\int_{\mathcal{S}_\mathcal{K}} (\nabla_\mathcal{S}\underline{u} - \frac{1}{2}(\mathbf{x}\underline{u}))^H : (\nabla_\mathcal{S}\underline{u} - \frac{1}{2}(\mathbf{x}\underline{u}))\,\mathrm{d}\mathcal{S}\Big)^{1/2}, \tag{14}$$

since $(\nabla_\mathcal{S}\underline{u})^H : (\mathbf{x}\underline{u}) = 0$ (or $\mathbf{g}_\mathcal{S}^i \cdot \mathbf{g}^3 = 0$ for $i = 1, 2$) and therefore the terms with the minus sign vanish.

For our purposes, the norms $\|\cdot\|_H$ and $\|\cdot\|_V$ are useful, too, as they allow the proof of important properties of our sesquilinear forms m, g and k. It turns out that the norms $\|\cdot\|_H$ and $\|\cdot\|_V$ are equivalent to the norms $\|\cdot\|_0$ and $\|\cdot\|_1$. Before we verify this, we will show that the norms that we introduced are rightly called norms, this means that they satisfy the norm properties. At least the positive definiteness of the V-norm is not obvious.

Lemma 4.3. *The expressions $\|\underline{u}\|_H$ and $\|\underline{v}\|_V$ defined in Definition 4.2 satisfy the norm properties.*

Proof. The form $\langle\tau,\sigma\rangle := \int_{\mathcal{S}_\mathcal{K}} \sigma^H : \tau\,\mathrm{d}\mathcal{S}$ defines an inner product and $\sqrt{\langle\tau,\tau\rangle}$ is therefore a norm. Together with the linearity of $\varepsilon_\mathcal{S}(\underline{u})$ and $\varepsilon_3(\underline{u})$, we can conclude all the norm properties from this, apart from the positive definiteness concerning $\underline{u}$.

It remains to show that $\|\underline{u}\|_H = 0$ or $\|\underline{u}\|_V = 0$ if and only if $\underline{u} \equiv 0$. With (4), this can be done easily for the H-norm:

$$
\begin{aligned}
\|\underline{u}\|_H^2 &= \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} (\underline{\mathbf{x}u})^H : (\underline{\mathbf{x}u}) + (\underline{\mathbf{x}\overline{u}}) : (\underline{\mathbf{x}u})\,\mathrm{d}\mathcal{S} \\
&= \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} (\underline{\mathbf{x}}\cdot\underline{\mathbf{x}})(\underline{\overline{u}}\cdot\underline{u}) + (\underline{\mathbf{x}}\cdot\underline{\overline{u}})(\underline{\mathbf{x}}\cdot\underline{u})\,\mathrm{d}\mathcal{S} \\
&= \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} \underline{\overline{u}}\cdot\underline{u} + |\underline{\mathbf{x}}\cdot\underline{u}|^2\,\mathrm{d}\mathcal{S} \quad = 0 \qquad \Longleftrightarrow \qquad \underline{u} \equiv 0.
\end{aligned}
$$

Concerning the V-norm, we consider the second-order tensor $\nabla\underline{U} = r^{\alpha-1}\nabla_\mathcal{S}\underline{u} + \alpha r^{\alpha-1}\underline{\mathbf{x}u}$ for $\alpha = -\frac{1}{2}$ and obtain that

$$
\begin{aligned}
\varepsilon(\underline{\overline{U}}) : \varepsilon(\underline{U}) &= \frac{1}{4}(\nabla\underline{\overline{U}} + (\nabla\underline{\overline{U}})^\top) : (\nabla\underline{U} + (\nabla\underline{U})^\top) \\
&= \frac{1}{4}\Big(r^{-3/2}[\nabla_\mathcal{S}\underline{\overline{u}} + (\nabla_\mathcal{S}\underline{\overline{u}})^\top - \frac{1}{2}(\underline{\mathbf{x}\overline{u}} + (\underline{\mathbf{x}\overline{u}})^\top)]\Big) : \\
&\qquad\qquad : \Big(r^{-3/2}[\nabla_\mathcal{S}\underline{u} + (\nabla_\mathcal{S}\underline{u})^\top - \frac{1}{2}(\underline{\mathbf{x}u} + (\underline{\mathbf{x}u})^\top)]\Big) \\
&= r^{-3}(\varepsilon_\mathcal{S}(\underline{\overline{u}}) - \frac{1}{2}\varepsilon_3(\underline{\overline{u}})) : (\varepsilon_\mathcal{S}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u})).
\end{aligned}
$$

The right hand side (except for the factor r^{-3}) is contained in the V-norm and vanishes if and only if the V-norm vanishes. Furthermore, the right hand side vanishes if and only if the expression $\varepsilon(\underline{\overline{U}}) : \varepsilon(\underline{U})$ on the left hand side vanishes. We show that this happens if and only if $\underline{U} \equiv 0$, which is equivalent to $\underline{u} \equiv 0$.

We know that $\varepsilon(\underline{\overline{U}}) : \varepsilon(\underline{U}) = \varepsilon(\underline{U})^H : \varepsilon(\underline{U})$ vanishes if and only if $\varepsilon(\underline{U}) = 0$. This means that no deformations are applied, but only rigid body motions (translations and rotations) are performed. Consequently $\underline{U}$ has the form $\underline{U} = \underline{c}_0 + \underline{c}_1 \times \underline{\mathbf{X}}$, where $\underline{c}_0$ and $\underline{c}_1$ are constant vectors. Exploiting the structure of $\underline{U}$ and $\underline{\mathbf{X}}$, we obtain that

$$
r^{-1/2}\underline{u}(\underline{\mathbf{x}}) \;=\; \underline{U} \;=\; \underline{c}_0 + \underline{c}_1 \times r\underline{\mathbf{x}} \;=\; \underline{c}_0 + r(\underline{c}_1 \times \underline{\mathbf{x}}).
$$

The comparison of the coefficients corresponding to the r-terms yields for r^0 that $\underline{c}_0 = 0$, for r^1 that $\underline{c}_1 \times \underline{\mathbf{x}} = 0$ for all $\underline{\mathbf{x}} \in \mathcal{S}_\mathcal{K}$, which means $\underline{c}_1 = 0$, and for $r^{-1/2}$ that $\underline{u}(\underline{\mathbf{x}}) \equiv 0$. Consequently, the positive definiteness of $\|\underline{u}\|_V$ can be concluded. Note that this is only possible due to our special approach $\underline{U} = r^{-1/2}\underline{u}$. $\square$

Theorem 4.4. *The norms $\|\underline{u}\|_H$ and $\|\underline{u}\|_V$ are equivalent to the norms $\|\underline{u}\|_0$ and $\|\underline{u}\|_1$, respectively, i.e., there are positive constants c_0, C_0 and c_1, C_1 so that*

$$
\begin{aligned}
c_0\|\underline{u}\|_0 \;&\leq\; \|\underline{u}\|_H \;\leq\; C_0\|\underline{u}\|_0 \qquad \textit{and} \\
c_1\|\underline{u}\|_1 \;&\leq\; \|\underline{u}\|_V \;\leq\; C_1\|\underline{u}\|_1.
\end{aligned}
$$

Proof. The H-norm satisfies

$$\|\underline{u}\|_H^2 = \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} \overline{\underline{u}} \cdot \underline{u} + |\underline{\mathbf{x}} \cdot \underline{u}|^2 \,\mathrm{d}\mathcal{S} = \frac{1}{2}\|\underline{u}\|_0^2 + \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} |\underline{\mathbf{x}} \cdot \underline{u}|^2 \,\mathrm{d}\mathcal{S},$$

which we obtained already in the proof of Lemma 4.3. We know for the last addend that $0 \leq |\underline{\mathbf{x}} \cdot \underline{u}|^2 \leq (\underline{\mathbf{x}} \cdot \underline{\mathbf{x}})(\overline{\underline{u}} \cdot \underline{u}) = \overline{\underline{u}} \cdot \underline{u}$ (recall $\underline{\mathbf{x}} \cdot \underline{\mathbf{x}} = 1$). Hence, $\frac{1}{2}\|\underline{u}\|_0^2 \leq \|\underline{u}\|_H^2 \leq \|\underline{u}\|_0^2$.

Concerning the V-norm, we set $\gamma(\underline{u}) := \nabla_\mathcal{S}\underline{u} - \frac{1}{2}\underline{\mathbf{x}}\underline{u}$ and obtain from (4) that

$$\begin{aligned}
\|\underline{u}\|_V^2 &= \frac{1}{4}\int_{\mathcal{S}_\mathcal{K}} (\gamma(\overline{\underline{u}}) + \gamma(\overline{\underline{u}})^\top) : (\gamma(\underline{u}) + \gamma(\underline{u})^\top) \,\mathrm{d}\mathcal{S} \\
&= \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} \gamma(\overline{\underline{u}})^\top : \gamma(\underline{u}) + \gamma(\overline{\underline{u}}) : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S} \\
&= \frac{1}{2}\int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) + \gamma(\overline{\underline{u}}) : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}.
\end{aligned}$$

To estimate the second addend, we consider the inner product $\langle \tau, \sigma \rangle := \int_{\mathcal{S}_\mathcal{K}} \sigma^H : \tau \,\mathrm{d}\mathcal{S}$; in particular, we have that $\langle \tau, \sigma \rangle \leq |\langle \tau, \sigma \rangle| \leq \sqrt{\langle \tau, \tau \rangle}\sqrt{\langle \sigma, \sigma \rangle}$ (Cauchy-Schwarz). We set, in particular, $\tau := \gamma(\underline{u})$ and $\sigma := \gamma(\underline{u})^\top$. Note that $\langle \gamma(\underline{u}), \gamma(\underline{u})^\top \rangle = \int_{\mathcal{S}_\mathcal{K}} \gamma(\overline{\underline{u}}) : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}$ is always real. Using (4), we obtain that

$$\begin{aligned}
\int_{\mathcal{S}_\mathcal{K}} \gamma(\overline{\underline{u}}) : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S} &= \langle \gamma(\underline{u}), \gamma(\underline{u})^\top \rangle \leq \sqrt{\langle \gamma(\underline{u}), \gamma(\underline{u}) \rangle}\sqrt{\langle \gamma(\underline{u})^\top, \gamma(\underline{u})^\top \rangle} \\
&= \Big(\int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}\Big)^{1/2}\Big(\int_{\mathcal{S}_\mathcal{K}} (\gamma(\underline{u})^\top)^H : \gamma(\underline{u})^\top \,\mathrm{d}\mathcal{S}\Big)^{1/2} \\
&= \Big(\int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}\Big)^{1/2}\Big(\int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}\Big)^{1/2} \\
&= \int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S}.
\end{aligned}$$

Consequently, we conclude from (14) that

$$\|\underline{u}\|_V^2 \leq \int_{\mathcal{S}_\mathcal{K}} \gamma(\underline{u})^H : \gamma(\underline{u}) \,\mathrm{d}\mathcal{S} = \|\underline{u}\|_1^2.$$

For the estimate in the other direction, we employ Korn's second inequality which states that

$$\int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}) : \varepsilon(\underline{U}) \geq C_{\mathrm{Korn}}\|\underline{U}\|_{1,\Omega_\mathcal{K}}^2$$

for some open, bounded domain $\Omega_\mathcal{K} \subset \mathbb{R}^3$ *with Dirichlet boundary of positive measure*. In this case, we denote by $H^0(\Omega_\mathcal{K})$ and $H^1(\Omega_\mathcal{K})$ the usual Sobolev spaces equipped with the usual Sobolev norms in $[H^0(\Omega_\mathcal{K})]^3$ and $[H^1(\Omega_\mathcal{K})]^3$:

$$\|\underline{U}\|_0^2 = \int_{\Omega_\mathcal{K}} \overline{\underline{U}} \cdot \underline{U} \,\mathrm{d}\Omega, \qquad \|\underline{U}\|_1^2 = \|\underline{U}\|_0^2 + \int_{\Omega_\mathcal{K}} \nabla\underline{U}^H : \nabla\underline{U} \,\mathrm{d}\Omega.$$

Note the difference to Definition 4.2, where the norms were defined only on a two-dimensional manifold.

We show that in our case, where we assume that $\underline{U}$ has the structure $r^{-1/2}\underline{u}$, the Dirichlet boundary condition can be omitted. To this end, we follow the proof of the second Korn's inequality in the book of Braess (1997).

Assume, the inequality is wrong. This means that for all constants $c > 0$, there is a function $\underline{U} \neq 0$ (possibly depending on c) so that $0 \leq \int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}) : \varepsilon(\underline{U})\, \mathrm{d}\Omega < c\|\underline{U}\|^2_{1,\Omega_\mathcal{K}}$.

Without loss of generality, we can assume that $\|\underline{U}\|_{1,\Omega_\mathcal{K}} = 1$ (otherwise divide $\underline{U}$ and thereby the inequality by $\|\underline{U}\|_{1,\Omega_\mathcal{K}}$). This means that we can construct a sequence $\underline{U}_n$ with

$$\int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}_n) : \varepsilon(\underline{U}_n)\, \mathrm{d}\Omega \;<\; \frac{1}{n} \qquad \text{and} \qquad \|\underline{U}_n\|_1 = 1.$$

Obviously, the sequence $\{\underline{U}_n\}$ is bounded in the space $[H^1(\Omega_\mathcal{K})]^3$ and therefore possesses a subsequence that converges in $[H^0(\Omega_\mathcal{K})]^3$, since $H^1(\Omega_\mathcal{K})$ is compactly embedded in $H^0(\Omega_\mathcal{K})$ (Rellich's theorem). For convenience, we denote this subsequence again by $\{\underline{U}_n\}$ and obtain that $\|\underline{U}_n - \underline{U}_m\|_0^2$ becomes arbitrarily small for large enough m and n. Form Korn's first inequality, we conclude that

$$\begin{aligned}
&C_{1,\mathrm{Korn}}\|\underline{U}_n - \underline{U}_m\|_1^2 \\
&\leq \int_{\Omega_\mathcal{K}} (\varepsilon(\overline{\underline{U}}_n) - \varepsilon(\overline{\underline{U}}_m)) : (\varepsilon(\underline{U}_n) - \varepsilon(\underline{U}_m))\, \mathrm{d}\Omega + \|\underline{U}_n - \underline{U}_m\|_0^2 \\
&\leq 2\int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}_n) : \varepsilon(\underline{U}_n)\, \mathrm{d}\Omega + 2\int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}_m) : \varepsilon(\underline{U}_m)\, \mathrm{d}\Omega + \|\underline{U}_n - \underline{U}_m\|_0^2 \\
&\leq \frac{2}{n} + \frac{2}{m} + \|\underline{U}_n - \underline{U}_m\|_0^2.
\end{aligned}$$

Since $[H^1(\Omega_\mathcal{K})]^3$ is complete, the sequence $\underline{U}_n$ converges to some element $\underline{U}_* \in [H^1(\Omega_\mathcal{K})]^3$ with $\int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}_*) : \varepsilon(\underline{U}_*)\, \mathrm{d}\Omega = 0$ and $\|\underline{U}_*\|_1^2 = 1$.

We learnt already in the proof of Lemma 4.3 that $\varepsilon(\overline{\underline{U}}) : \varepsilon(\underline{U})$ vanishes if and only if $\underline{U} = \underline{c}_0 + \underline{c}_1 \times \underline{\mathbf{X}}$. We want to use Korn's second inequality only in that case, where the left hand side contains our V-norm, that is, where $\underline{U}$ has the structure $\underline{U} = r^{-1/2}\underline{u}$. Note that the integrals in the 0- and 1-norms exist for this structure, if we assume $0 \notin \Omega_\mathcal{K}$. In the proof of Lemma 4.3, we showed that $\underline{U} = \underline{c}_0 + \underline{c}_1 \times \underline{\mathbf{X}}$ finally yields $\underline{U} \equiv 0$ which is a contradiction to $\|\underline{U}\|_1 > 0$.

This means that for all functions $\underline{u} \in V$ and $\underline{U} = r^{-1/2}\underline{u}$ Korn's inequality is also valid, if we have no Dirichlet boundary. In order to combine this fact with a relation between the norms $\|\underline{u}\|_V$ and $\|\underline{u}\|_1$, we take a closer look on the terms in Korn's inequality.

Let $\Omega_\mathcal{K} \subset \mathcal{K}$ be the three-dimensional domain

$$\Omega_\mathcal{K} := \{\underline{\mathbf{X}} = \underline{\mathbf{X}}(\xi_1, \xi_2, r) \in \mathcal{K} \mid R_1 < r < R_2\}$$

with fixed positive constants R_1, R_2. We require $0 < R_1 < R_2 < \infty$ to ensure that the integral $C_{R_1,R_2} := \int_{R_1}^{R_2} r^{-1}\, \mathrm{d}r$ is finite.

Recall $\mathrm{d}\Omega = r^2\,\mathrm{d}\mathcal{S}\,\mathrm{d}r$. We conclude that

$$\begin{aligned}
C_{R_1,R_2}\|\underline{u}\|_V^2 &= \int_{R_1}^{R_2}\int_{\mathcal{S}_\mathcal{K}} r^{-3}(\varepsilon_\mathcal{S}(\overline{\underline{u}}) - \frac{1}{2}\varepsilon_3(\overline{\underline{u}})) : (\varepsilon_\mathcal{S}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u}))\, r^2\,\mathrm{d}\mathcal{S}\,\mathrm{d}r \\
&= \int_{\Omega_\mathcal{K}} \varepsilon(\overline{\underline{U}}) : \varepsilon(\underline{U})\,\mathrm{d}\Omega \;\geq\; C_{\mathrm{Korn}}\|\underline{U}\|^2_{1,\Omega_\mathcal{K}} \\
&\geq C_{\mathrm{Korn}}|\underline{U}|^2_{1,\Omega_\mathcal{K}} \;=\; C_{\mathrm{Korn}}\int_{\Omega_\mathcal{K}} \nabla\underline{U}^H : \nabla\underline{U}\,\mathrm{d}\Omega \\
&= C_{\mathrm{Korn}}C_{R_1,R_2}\int_{\mathcal{S}_\mathcal{K}} (\nabla_\mathcal{S}\underline{u} - \frac{1}{2}(\underline{\mathbf{x}u}))^H : (\nabla_\mathcal{S}\underline{u} - \frac{1}{2}(\underline{\mathbf{x}u}))\,\mathrm{d}\mathcal{S}.
\end{aligned}$$

Dividing by C_{R_1,R_2}, we get from (14) that

$$\|\underline{u}\|_V^2 \geq C_{\mathrm{Korn}}\|\underline{u}\|_1^2,$$

where C_{Korn} depends on R_1 and R_2. As we are free in the choice of R_1 and R_2, the constant C_{Korn} can be chosen as the supremum of all such constants for $R_1, R_2 \in \mathbb{R}_+$. □

It is a known result (or follows from Rellich's theorem) that the space V is compactly embedded into the space H, if these spaces are provided with the norms $\|\underline{u}\|_0$ and $\|\underline{u}\|_1$. Using Theorem 4.4, we obtain the following corollary.

Corollary 4.5. *Let the spaces H and V be equipped with the norms $\|\cdot\|_H$ and $\|\cdot\|_V$, respectively, as defined in Definition 4.2. Then, the space V is compactly embedded into the space H.*

We introduced the H- and the V-norms, because their structure goes better with the structure of the sesquilinear forms that define the eigenvalue problem (13). In the following theorem, we present important properties of the sesquilinear forms. Due to Theorem 4.4, the norms can be replaced by the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ with the effect that we cannot determine all the constants in the estimates below.

Theorem 4.6. *Let the spaces H and V be equipped with the norms $\|\cdot\|_H$ and $\|\cdot\|_V$, respectively. Then, the sesquilinear forms m, g, k satisfy the following properties.*

(i) The sesquilinear forms m and k are Hermitian, g is skew-Hermitian,

$$m(\underline{u},\underline{v}) = \overline{m(\underline{v},\underline{u})}, \qquad g(\underline{u},\underline{v}) = -\overline{g(\underline{v},\underline{u})}, \qquad k(\underline{u},\underline{v}) = \overline{k(\underline{v},\underline{u})}.$$

(ii) Let $d(\underline{u},\underline{v}) := a_{3\mathcal{S}}(\underline{u},\underline{v}) - \frac{1}{2}a_{33}(\underline{u},\underline{v})$. Then $g(\underline{u},\underline{v}) = \overline{d(\underline{v},\underline{u})} - d(\underline{u},\underline{v})$ and

$$\begin{aligned}
M_1\|\underline{u}\|_H^2 \;\leq\; m(\underline{u},\underline{u}) &\leq M_2\|\underline{u}\|_H^2 && \text{for } \underline{u}\in H,\\
|m(\underline{u},\underline{v})| &\leq M_2\|\underline{u}\|_H\|\underline{v}\|_H && \text{for } \underline{u},\underline{v}\in H,\\
M_1\|\underline{u}\|_V^2 \;\leq\; k(\underline{u},\underline{u}) &\leq M_2\|\underline{u}\|_V^2 && \text{for } \underline{u}\in V,\\
|k(\underline{u},\underline{v})| &\leq M_2\|\underline{u}\|_V\|\underline{v}\|_V && \text{for } \underline{u},\underline{v}\in V,\\
|d(\underline{u},\underline{v})| &\leq \sqrt{m(\underline{u},\underline{u})}\sqrt{k(\underline{v},\underline{v})} && \text{for } \underline{u}\in H,\ \underline{v}\in V,\\
|d(\underline{u},\underline{v})| &\leq M_2\|\underline{u}\|_H\|\underline{v}\|_V && \text{for } \underline{u}\in H,\ \underline{v}\in V,
\end{aligned}$$

where M_1 and M_2 are the constants in the positive definiteness assumption (11).

Proof. The relations

$$a_{\mathcal{S}\mathcal{S}}(\underline{u},\underline{v}) = \overline{a_{\mathcal{S}\mathcal{S}}(\underline{v},\underline{u})}, \qquad a_{\mathcal{S}3}(\underline{u},\underline{v}) = \overline{a_{3\mathcal{S}}(\underline{v},\underline{u})}, \qquad a_{33}(\underline{u},\underline{v}) = \overline{a_{33}(\underline{v},\underline{u})},$$

are obvious from (10) and the definitions of the forms $a_{\mathcal{S}\mathcal{S}}$, $a_{\mathcal{S}3}$, $a_{3\mathcal{S}}$, a_{33}, since ε_3 and $\varepsilon_{\mathcal{S}}$ are symmetric tensors. Assertion (i) and the representation of g in (ii) are simple consequences.

The first assertion for $m(u,u) = \int_{\mathcal{S}_\mathcal{K}} \varepsilon_3(\overline{\underline{u}}) : A : \varepsilon_3(\underline{u})\,\mathrm{d}\mathcal{S} = \int_{\mathcal{S}_\mathcal{K}} \varepsilon_3(\underline{u})^H : A : \varepsilon_3(\underline{u})\,\mathrm{d}\mathcal{S}$ in (ii) follows from (11). Consequently, m defines an inner product on H and the Cauchy-Schwarz inequality yields the second assertion for m.

We know for k that

$$\begin{aligned} k(u,u) &= \int_{\mathcal{S}_\mathcal{K}} \varepsilon_{\mathcal{S}}(\overline{\underline{u}}) : A : \varepsilon_{\mathcal{S}}(\underline{u}) \\ &\quad -\frac{1}{2}\Big(\varepsilon_3(\overline{\underline{u}}) : A : \varepsilon_{\mathcal{S}}(\underline{u}) + \varepsilon_{\mathcal{S}}(\overline{\underline{u}}) : A : \varepsilon_3(\underline{u})\Big) + \frac{1}{4}\varepsilon_3(\overline{\underline{u}}) : A : \varepsilon_3(\underline{u})\,\mathrm{d}\mathcal{S} \\ &= \int_{\mathcal{S}_\mathcal{K}} \Big(\varepsilon_{\mathcal{S}}(\overline{\underline{u}}) - \frac{1}{2}\varepsilon_3(\overline{\underline{u}})\Big) : A : \Big(\varepsilon_{\mathcal{S}}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u})\Big)\,\mathrm{d}\mathcal{S} \\ &= \int_{\mathcal{S}_\mathcal{K}} \Big(\varepsilon_{\mathcal{S}}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u})\Big)^H : A : \Big(\varepsilon_{\mathcal{S}}(\underline{u}) - \frac{1}{2}\varepsilon_3(\underline{u})\Big)\,\mathrm{d}\mathcal{S}. \end{aligned}$$

and obtain the boundedness and ellipticity properties for k from (11) and from the Cauchy-Schwarz inequality. On page 9, we defined the inner product $\langle \tau, \sigma\rangle_A = \int_{\mathcal{S}_\mathcal{K}} \sigma^H : A : \tau\,\mathrm{d}\mathcal{S}$. The Cauchy-Schwarz inequality implies

$$|\langle \tau, \sigma\rangle_A| \leq \sqrt{\langle \tau, \tau\rangle_A}\sqrt{\langle \sigma, \sigma\rangle_A}$$

We choose $\tau := \varepsilon_3(\underline{u})$ and $\sigma := \varepsilon_{\mathcal{S}}(\underline{v}) - \frac{1}{2}\varepsilon_3(\underline{v})$ and conclude from the symmetry properties of the tensors ε_3 and $\varepsilon_{\mathcal{S}}$ that

$$|d(\underline{u},\underline{v})| = |\langle \tau, \sigma\rangle_A| \leq \sqrt{m(\underline{u},\underline{u})}\sqrt{k(\underline{v},\underline{v})}.$$

The second estimate for d follows from the estimates for m and k. ☐

Remark 4.7. *Replacing $\|\underline{u}\|_H$ and $\|\underline{u}\|_V$ by $\|\underline{u}\|_0$ and $\|\underline{u}\|_1$, respectively, we obtain exactly the properties that were already written down for a special case (spherical coordinates) by Apel, Sändig, and Solov'ev (2002b). Nevertheless, there is a small discrepancy: the derivation in the cited paper is not correct; they involve sesquilinear forms whose structures differ from ours, but obtain the same constants as we do in spite of other norms, because invalid assumptions and conclusions were made. This mistake was caused by long standing wrong symmetry assumptions on the elasticity tensor A in the standard literature on the linear elasticity problem. We discuss details of this misapprehension in Section 4.2.*

Remark 4.8 (Spectral symmetry properties). *Considering the adjoint eigenvalue problem, one readily verifies that if λ is an eigenvalue of (13), then so is $-\overline{\lambda}$. If $-\lambda$ and*

$\overline{\lambda}$ *are eigenvalues, too, we speak about a* Hamiltonian eigenvalue symmetry *or a* Hamiltonian structure *of the eigenvalue problem, meaning that the eigenvalues are symmetric with respect to the real and imaginary axes as is the case for Hamiltonian matrices.*

The proof of the Hamiltonian structure of a given eigenvalue problem requires a closer consideration of spectral operator theory and embedding theorems. Sufficient conditions that guarantee the Hamiltonian structure were summarized by Pester (2004). A more general framework including an introduction to the spectral theory for operator pencils is given by Kozlov, Maz'ya, and Roßmann (2000). According to Pester (2004), the Hamiltonian structure of problem (13) follows from Theorem 4.6.

The advantage of an eigenvalue problem with Hamiltonian structure is that its discretized form can be transformed into an eigenvalue problem of a Hamiltonian, skew-Hamiltonian or symplectic matrix, so that adapted Arnoldi or Lanczos algorithms allow fast computations of the eigenvalues and eigenelements. For details, we refer to Freund (1994); Benner and Faßbender (1997, 2000); Mehrmann and Watkins (2001); Apel, Mehrmann, and Watkins (2002a); Watkins (2004).

4.2 The symmetry and boundedness conditions on A

Let $\{\mathbf{b}_k\}_{k=1}^3$ be a basis in $\mathbb{R}^3$ and let the fourth-order tensor A be developed four times into this basis so that

$$A = \sum_{i,j,k,h=1}^{3} a_{ijkh}\mathbf{b}_i\mathbf{b}_j\mathbf{b}_k\mathbf{b}_h.$$

We required that A is real, that is, $a_{ijkh} \in \mathbb{R}$. Let $\{\mathbf{c}_k\}_{k=1}^3$ and $\{\mathbf{d}_k\}_{k=1}^3$ be two further bases of $\mathbb{R}^3$ and $\tau = \sum_{i,j=1}^3 \tau_{ij}\mathbf{c}_i\mathbf{d}_j$, $\sigma = \sum_{i,j=1}^3 \sigma_{ij}\mathbf{c}_i\mathbf{d}_j$. The symmetry assumption (10) can be written as

$$a_{ijkh} = a_{hkji}, \tag{15}$$

since then, exchanging the order of summation, one verifies that

$$\begin{aligned}
\tau^\top : A : \sigma &= \sum_{p,q=1}^{3} \tau_{pq}\mathbf{d}_q\mathbf{c}_p : \sum_{i,j,k,h=1}^{3} a_{ijkh}\mathbf{b}_i\mathbf{b}_j\mathbf{b}_k\mathbf{b}_h : \sum_{s,t=1}^{3} \sigma_{st}\mathbf{c}_s\mathbf{d}_t \\
&= \sum_{p,q,i,j,k,h,s,t=1}^{3} \tau_{pq}a_{ijkh}\sigma_{st}(\mathbf{c}_p\cdot\mathbf{b}_i)(\mathbf{d}_q\cdot\mathbf{b}_j)(\mathbf{b}_h\cdot\mathbf{c}_s)(\mathbf{b}_k\cdot\mathbf{d}_t) \\
&= \sum_{p,q,i,j,k,h,s,t=1}^{3} \tau_{pq}a_{hkji}\sigma_{st}(\mathbf{c}_p\cdot\mathbf{b}_h)(\mathbf{d}_q\cdot\mathbf{b}_k)(\mathbf{b}_i\cdot\mathbf{c}_s)(\mathbf{b}_j\cdot\mathbf{d}_t) \\
&= \sum_{s,t=1}^{3} \sigma_{st}\mathbf{d}_t\mathbf{c}_s : \sum_{i,j,k,h=1}^{3} a_{ijkh}\mathbf{b}_i\mathbf{b}_j\mathbf{b}_k\mathbf{b}_h : \sum_{p,q=1}^{3} \tau_{pq}\mathbf{c}_p\mathbf{d}_q \\
&= \sigma^\top : A : \tau.
\end{aligned}$$

We can choose $\{\mathbf{c}_k\} = \{\mathbf{d}_k\}$ biorthogonal to $\{\mathbf{b}_k\}$ and τ and σ so that $\tau = \mathbf{c}_p\mathbf{d}_q$ and $\sigma = \mathbf{c}_s\mathbf{d}_t$ (i.e. $\tau_{ij} = \delta_{ip}\delta_{jq}$ and $\sigma_{ij} = \delta_{is}\delta_{jt}$) for fixed $p, q, s, t \in \{1, 2, 3\}$. Thus, we obtain that $\tau^\top : A : \sigma = a_{pqts}$ and $\sigma^\top : A : \tau = a_{stqp}$, which yields the equivalence of (15) and (10).

If we assume that $\{\mathbf{b}_k\} = \{\mathbf{c}_k\} = \{\mathbf{d}_k\}$ is an orthonormal basis in $\mathbb{R}^3$, the boundedness and ellipticity condition (11) can be written as

$$M_1 \sum_{i,j=1}^{3} |\tau_{ij}|^2 \leq \sum_{i,j,k,h=1}^{3} a_{ijkh}\overline{\tau}_{ij}\tau_{hk} \leq M_2 \sum_{i,j=1}^{3} |\tau_{ij}|^2 \qquad \forall \tau_{ij} \in \mathbb{C}.$$

These two conditions provide that $\langle \cdot, \cdot \rangle_A$ defined on page 9 is an inner product. Since in linear elasticity the generalized Hooke's law $\sigma = A : \varepsilon$ with the (symmetric) Cauchy stress tensor σ and the (symmetric) Green strain tensor ε is considered, it makes sense from the mechanical point of view to require in addition that A also operates within the subspace of *symmetric* second-order tensors, that is, symmetric tensors are mapped by A to symmetric tensors.

Usually, the additional but unnecessary conditions $a_{ijkh} = a_{jikh}$ and $a_{ijkh} = a_{ijhk}$ are required, or rather incorrectly concluded from the relation $\sigma = A : \varepsilon$, see Leipholz (1968); Malvern (1969). Both tensors are symmetric ($\sigma_{ij} = \sigma_{ji}$, $\varepsilon_{kh} = \varepsilon_{hk}$) and allow the conclusion

$$\begin{aligned} \sigma_{ij} &= \sum_{k,h=1}^{3} a_{ijkh}\varepsilon_{kh} = \sum_{k,h=1}^{3} a_{ijhk}\varepsilon_{hk} = \sum_{k,h=1}^{3} a_{ijhk}\varepsilon_{kh} \\ \overset{!}{=} \sigma_{ji} &= \sum_{k,h=1}^{3} a_{jikh}\varepsilon_{kh}. \end{aligned}$$

Note that this neither implies $a_{ijkh} = a_{jikh}$ nor $a_{ijkh} = a_{ijhk}$ although argued in several articles on linear elasticity including the literature cited above. We can choose $\varepsilon_{kh} = \varepsilon_{hk} = 1$ for fixed k, h and zero for all other combinations, so that we merely obtain that

$$a_{ijkh} + a_{ijhk} = a_{jikh} + a_{jihk}. \tag{16}$$

The latter relation is exactly what guarantees that symmetric tensors are mapped to symmetric tensors.

As correctly explained by Malvern (1969), we could assume that $a_{ijkh} = a_{ijhk} = 1/2(a_{ijkh} + a_{ijhk})$, since we have always the combination $a_{ijkh}\varepsilon_{kh} + a_{ijhk}\varepsilon_{hk} = (a_{ijkh} + a_{ijhk})\varepsilon_{kh}$ in the sum; but this assumption can only be made, if we never apply A to unsymmetric tensors. In particular, all skew-symmetric tensors would be mapped to the zero-tensor, which is a contradiction to the positive definiteness of A. The relation $a_{ijkh} = a_{jikh}$ would finally follow from $a_{ijkh} = a_{ijhk}$ with (15).

It is legitimate to assume these symmetry properties if we never apply A to unsymmetric tensors; but they are not true in general (see the example below) and might collide with other assumptions. We found that a mistake caused by this discrepancy occurred in the paper by Apel, Sändig, and Solov'ev (2002b)

We omitted these additional symmetry assumptions, because they are unnecessary and rather restrictive. Therefore, the sesquilinear forms defining the eigenvalue problem (13) have a more sophisticated structure than, for example, in the papers by Leguillon (1995) or Apel, Sändig, and Solov'ev (2002b).

Example 4.9. *We consider the St. Venant–Kirchhoff material. The relation of the stress and the strain tensors is then given by*

$$\sigma = A : \varepsilon = 2\mu\varepsilon + \lambda \mathrm{tr}\,(\varepsilon) I.$$

This means that

$$A = \sum_{i,j,k,h=1}^{3} (2\mu g^{ih} g^{jk} + \lambda g^{ij} g^{kh}) \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k \mathbf{g}_h$$

(recall $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = g^{ji}$*). Obviously,* A *is a fourth-order tensor which satisfies our symmetry property (15) and it also maps symmetric second-order tensors to themselves because*

$$a_{ijkh} + a_{ijhk} = 2\mu(g^{ih} g^{jk} + g^{ik} g^{jh}) + 2\lambda g^{ij} g^{kh} = a_{jikh} + a_{jihk}.$$

The spare symmetry properties $a_{ijkh} = a_{jikh}$ *and* $a_{ijkh} = a_{ijhk}$*, however, are not satisfied.*

5 Conclusion

We studied elliptic boundary value problems near polyhedral corners. The computation of the singularities of the corresponding solutions is related to the determination of the eigenpairs of an associated eigenvalue problem which is defined on the surface of a ball centered at the corner. Therefore, it is necessary to parametrize the ball. Each specific parametrization, however, causes trouble in some detail of possible further analysis. This is why we omitted a specification of the parametrization in our calculations so that all results are valid in their generality and for all parametrizations of the sphere.

For two examples (the Laplace and the linear elasticity problems), we derived the associated eigenvalue problems. Both have interesting spectral properties. Concerning the linear elasticity problem, we introduced new norms in the Sobolev spaces $[H^0]^3$ and $[H^1]^3$ that simplified the proof of important properties of the sesquilinear forms defining the corresponding eigenvalue problem. These properties imply a symmetric structure of the spectrum of this eigenvalue problem, so that adapted Lanczos and Arnoldi algorithms allow a fast computation of the singularities.

We were able to show a relation between our norms and a strain tensor corresponding to a carefully chosen displacement function. This trick allowed us to conclude that our norms are equivalent to the norms with which the spaces $[H^0]^3$ and $[H^1]^3$ are usually equipped. During the verification of our results, we revealed a long standing mistake in the literature on linear elasticity, where arguable properties of the elasticity tensor are claimed. Although acceptable in most situations, these properties are not always valid; we showed this in an example for the St. Venant–Kirchhoff material.

Acknowledgement

We thank Thomas Apel and Anton Heinen (Universität der Bundeswehr München) for valuable advice and the DFG (Deutsche Forschungsgemeinschaft) for financial support.

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