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The Fourier-finite-element method with Nitsche-mortaring

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Abstract

The paper deals with a combination of the Fourier-finite-element method with the Nitsche-finite-element method (as a mortar method). The approach is applied to the Dirichlet problem of the Poisson equation in three-dimensional axisymmetric domains $\widehat{\Omega}$ with non-axisymmetric data. The approximating Fourier method yields a splitting of the 3D-problem into 2D-problems. For solving the 2D-problems on the meridian plane Ω_a of $\widehat{\Omega}$, the Nitsche-finite-element method with non-matching meshes is applied. Some important properties of the approximation scheme are derived and the rate of convergence in some H^1-like norm is proved to be of the type $\mathcal{O}(h + N^{-1})$ (h: mesh size on Ω_a, N: length of the Fourier sum) in case of a regular solution of the boundary value problem. Finally, some numerical results are presented.

Keywords. finite-element method, Fourier method, non-matching meshes, Nitsche-mortaring, Poisson equation

AMS subject classification. 65N30, 65N35

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1 Introduction

For the efficient numerical treatment of boundary value problems (BVPs) in 3D, domain decomposition methods as well as dimension decomposition methods are widely used in science and engineering. Both type of methods are convenient for the parallelization of the numerical solution of partial differential equations. In particular, nonconforming techniques like mortar methods provide a flexible approach in the framework of domain decomposition, see e.g. [1, 2, 3, 5, 11, 22].
In contrast to the domain decomposition, where a domain $\widehat{\Omega} \subset I\!R^d$ (here $d = 3$) is subdivided into a finite number of subdomains of the same dimension like $\widehat{\Omega}$, the partial decomposition of some differential operator in 3D employs the representation of this differential operator by a family of differential operators in 2D. For the numerical approximation of the BVP on $\widehat{\Omega} \subset I\!R^3$ only a finite set of problems in 2D is to be solved, cf. [4, 13, 14, 17, 18]. The combination of both methods would enable parallelization with respect to geometry and dimension of the BVP at the same time.
In this paper, we shall present such a combination as an approach for numerically solving the Dirichlet problem of the Poisson equation in some axisymmetric domain $\widehat{\Omega}$ in $I\!R^3$. In particular, we combine the Fourier-finite-element method with the Nitsche-finite-element method (as a mortar method). The domain $\widehat{\Omega}$ is generated by rotation (φ: rotational angle, $\varphi \in (-\pi, \pi]$) of some meridian domain Ω_a about the rotational axis, the x_3-axis. The data and the solution u of the BVP in 3D are non-axisymmetric. As an important method for the approximate solution of this BVP, we shall apply the so-called Fourier-finite-element method (FFEM), see [4, 7, 13, 14, 17, 21]. This method combines the approximating Fourier method (see, e.g. [6, 18]) with the finite-element method (FEM; cf. [8]). That is, trigonometric polynomials of degree $\leq N$ are used in one space direction, here with respect to the rotational angle φ. They yield an approximate splitting of the 3D-problem into a finite set of 2D-problems. The solutions u_k ($k = 0, \pm 1, ..., \pm N$) of the 2D-problems are the first $2N + 1$ Fourier coefficients of the solution u. For solving numerically the 2D-problems on the plane meridian domain Ω_a of $\widehat{\Omega}$, the FEM with piecewise polynomials (h-version of the FEM) is employed over a triangulation of Ω_a with mesh size h.
In the second step, we employ the Nitsche-finite-element discretization as a mortar method for solving numerically the 2D-problems on the meridian domain Ω_a, cf. [1, 11, 15, 16, 19]. For simplicity, the domain Ω_a is subdivided into two subdomains Ω_a^1 and Ω_a^2. Along the interface $\Gamma := \overline{\Omega}_a^1 \cap \overline{\Omega}_a^2$ of the domain decomposition, non-matching meshes as well as discontinuities of the approximated solutions are admitted.
The aim of this paper is to present the combined method, which seems to be new, and to give a rigorous justification of the approach. In particular, important properties of the approximation scheme are derived and, the convergence $u_{hN} \to u$ with respect to $N \to \infty$ and $h \to 0$ is proved. Here, N and h are independent from each other (anisotropic discretization).
The paper is organized as follows. First we present a derivation of the FFEM with Nitsche-mortaring for the Dirichlet problem of the Poisson equation. Some assumptions on the BVP and on the triangulation of the meridian domain Ω_a are given. Since the BVP is treated in cylindrical coordinates (r, φ, z) (where r is the distance of a point to the z-axis), we are also concerned with Hilbert spaces provided with power weights r^α (α real) and with

functions $u(r,\varphi,z)$ periodically with respect to the rotational angle $\varphi \in (-\pi,\pi]$. Then, some properties of the approximation schemes and a priori estimates are derived. Finally, error estimates and convergence rates with respect to the discretization parameters N and h are given (N: length of the Fourier sum, h: mesh size on Ω_a). In some H^1-like norm $\|\cdot\|_{1,h,\Omega}$ and for regular solutions u, the convergence rate is proved to be of the type $\mathcal{O}(h+N^{-1})$. The numerical example illustrates the approach and the rates of convergence.

2 Analytical preliminaries

Let $\widehat{\Omega} \subset I\!R^3$ be a bounded domain which is axisymmetric with respect to the x_3-axis. The part of the x_3-axis contained in $\widehat{\Omega}$ is denoted by Γ_0. Then the set $\widehat{\Omega} \setminus \Gamma_0$ is generated by rotation of the corresponding plane meridian domain Ω_a about the x_3-axis. The set Γ_a is defined by $\Gamma_a := \partial\Omega_a \setminus \overline{\Gamma}_0$, where $\partial\Omega_a$ is the boundary of Ω_a. In the following we assume that Ω_a is polygonally bounded. Further let R_i, $i=1,\dots,n$ (n: the total number of corners of $\overline{\Omega}_a$), denote the corners of the polygon $\overline{\Omega}_a$ such that $R_1,\ R_n \in \overline{\Gamma}_0 \cap \overline{\Gamma}_a$, cf. Figure 1. Then we require that for the interior angles γ_i at the corners R_i ($i=1,\dots,n$) holds: $\gamma_1, \gamma_n < 0.72616\pi$ (cf. [4]) and $\gamma_i < \pi$ for $i=2,\dots,n-1$. These assumptions are used to guarantee the regularity of the solution of the BVP considered subsequently. For functions defined on X, let $H^s(X)$ ($s \ge 0$, s real, $H^0 = L_2$) denote the usual Sobolev-Slobodetskiĭ space. Introduce cylindrical coordinates r,φ,z ($x_1 = r\cos\varphi, x_2 = r\sin\varphi, x_3 = z$), with $\varphi \in (-\pi,\pi]$. Then we get one-to-one mappings: $\widehat{\Omega}\setminus\Gamma_0 \to \Omega := \Omega_a \times (-\pi,\pi]$ and $\partial\widehat{\Omega}\setminus\overline{\Gamma}_0 \to \Gamma_a \times (-\pi,\pi]$. Consequently, for each function $\hat{v}(x)$ with $x \in \widehat{\Omega}\setminus\Gamma_0$, some function v on Ω is defined by

$$v(r,\varphi,z) := \hat{v}(r\cos\varphi, r\sin\varphi, z). \tag{1}$$

Using this, we can define spaces $X^l_{1/2}(\Omega)$ of Sobolev-type of functions periodic with respect to $\varphi \in (-\pi,\pi]$ as follows: $H^l(\widehat{\Omega}\setminus\Gamma_0) \to X^l_{1/2}(\Omega)$ ($l=0,1,2$). Since Γ_0 is one-dimensional, $H^l(\widehat{\Omega}\setminus\Gamma_0)$ and $H^l(\widehat{\Omega})$ can be identified. These spaces are equipped with the natural norms and seminorms given by the relations

$$|u|_{X^l_{1/2}(\Omega)} = |\hat{u}|_{H^l(\widehat{\Omega})}, \quad \|u\|_{X^l_{1/2}(\Omega)} = \|\hat{u}\|_{H^l(\widehat{\Omega})}, \quad l=0,1,2, \tag{2}$$

with u, $\hat{u}$ according to (1). In [12, 17, 21] the spaces $X^l_{1/2}(\Omega)$ are described in more detail.

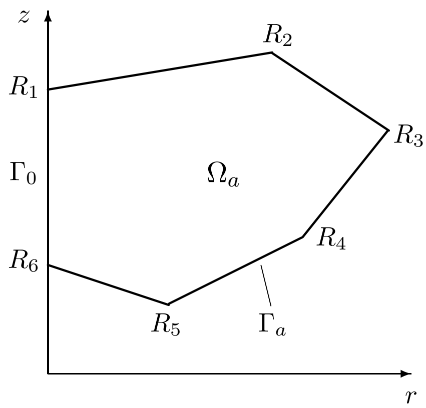

Figure 1

Further we shall need some spaces of functions defined on the meridian domain Ω_a and provided with power weights r^α (α real):

$$H^l_\alpha(\Omega_a) \;:=\; \{w = w(r,z) : r^\alpha D^\beta w \in L_2(\Omega_a),\ 0 \le |\beta| \le l\} \quad \text{for } l \in \{0,1,2\}; \tag{3}$$

$$D^\beta w := \frac{\partial^{|\beta|} w}{\partial r^{\beta_1} \partial z^{\beta_2}},\ \beta = (\beta_1, \beta_2),\ |\beta| = \beta_1 + \beta_2; \quad H^0_\alpha(\Omega_a) = L_{2,\alpha}(\Omega_a)\,.$$

The canonical scalar product in $L_{2,\alpha}(\Omega_a)$ is given by

$$(v,w)_{\alpha,\Omega_a} := \int\limits_{\Omega_a} v\bar{w}\, r^{2\alpha}\, dr dz \tag{4}$$

and the norms in the spaces $H^l_\alpha(\Omega_a)$ are defined as follows

$$\begin{aligned} \|w\|_{L_{2,\alpha}(\Omega_a)} \;&:=\; \Big\{ \int\limits_{\Omega_a} |r^\alpha\, w|^2\, dr dz \Big\}^{1/2},\ \ |w|_{H^l_\alpha(\Omega_a)} := \Big\{ \sum_{|\beta|=l} \|r^\alpha D^\beta w\|^2_{L_2(\Omega_a)} \Big\}^{1/2}, \\ \|w\|_{H^l_\alpha(\Omega_a)} \;&:=\; \Big\{ \|w\|^2_{H^{l-1}_\alpha(\Omega_a)} + |w|^2_{H^l_\alpha(\Omega_a)} \Big\}^{1/2} \quad \text{for } l \in \{1,2\}. \end{aligned} \tag{5}$$

Subsequently, these spaces, scalar products, and norms will also be used with Ω^i_a ($i = 1,2$) instead of Ω_a, where Ω^i_a are subdomains of Ω_a.

For $\hat{f} \in L_2(\widehat{\Omega})$, let us consider the Dirichlet problem for the Poisson equation on $\widehat{\Omega}$:

$$-\Delta_3 \hat{u} := -\sum_{i=1}^{3} \frac{\partial^2 \hat{u}}{\partial x_i^2} = \hat{f} \quad \text{in } \widehat{\Omega}, \quad \hat{u} = 0 \quad \text{on } \partial\widehat{\Omega}. \tag{6}$$

According to (1), we can write this problem in terms of cylindrical coordinates and obtain

$$-\Delta_{r,\varphi,z}\, u := -\Big\{ \frac{1}{r}\frac{\partial}{\partial r}\Big(r \frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} \Big\} = f \ \text{ in } \Omega, \quad u \;=\; 0 \ \text{ on } \Gamma_a \times (-\pi,\pi], \tag{7}$$

where u is periodic with respect to φ. The variational formulation of (6) in cylindrical coordinates is given as follows. Find $u \in V_0(\Omega) := \{u \in X^1_{1/2}(\Omega) : u|_{\Gamma_a \times (-\pi,\pi]} = 0\}$ such that

$$\begin{aligned} b(u,v) \;&=\; f(v) \quad \forall v \in V_0(\Omega), \\ \text{with} \quad b(u,v) \;&:=\; \int\limits_\Omega \Big\{ \frac{\partial u}{\partial r}\frac{\overline{\partial v}}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \varphi}\frac{\overline{\partial v}}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\overline{\partial v}}{\partial z} \Big\}\, r dr d\varphi dz, \quad f(v) := \int\limits_\Omega f\, \overline{v}\, r dr d\varphi dz. \end{aligned} \tag{8}$$

For $u(r,\varphi,z)$, $u \in X^1_{1/2}(\Omega)$, (and for $f(r,\varphi,z)$, $f \in X^0_{1/2}(\Omega)$, resp.) we employ partial Fourier analysis with respect to the rotational angle φ:

$$u(r,\varphi,z) = \sum_{k\in\mathbb{Z}} u_k(r,z)\, e^{ik\varphi}, \qquad u_k(r,z) := \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u(r,\varphi,z)\, e^{-ik\varphi} d\varphi \quad \text{for } k \in \mathbb{Z} \tag{9}$$

($\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$; $i^2 = -1$). Using the functionals

$$b_k(u_k, v_k) = \int\limits_{\Omega_a} \Big\{ \frac{\partial u_k}{\partial r} \overline{\frac{\partial v_k}{\partial r}} + \frac{\partial u_k}{\partial z} \overline{\frac{\partial v_k}{\partial z}} + \frac{k^2}{r^2} u_k \overline{v}_k \Big\} \, r dr dz, \quad f_k(v_k) = \int\limits_{\Omega_a} f_k \overline{v}_k \, r dr dz \ \ (k \in \mathbb{Z}),$$

the decomposition of the BVP (8) in the variational form can be written as follows:

$$\begin{aligned} k = 0 &: \text{find } u_0 \in V_0^a := \{ v \in H^1_{1/2}(\Omega_a) : v|_{\Gamma_a} = 0 \} : \ b_0(u_0, w) = f_0(w) \ \ \forall w \in V_0^a, \\ k \in \mathbb{Z} \backslash \{0\} &: \text{find } u_k \in W_0^a := \{ v \in V_0^a : v \in L_{2,-1/2}(\Omega_a) \} : \ b_k(u_k, w) = f_k(w) \ \ \forall w \in W_0^a. \end{aligned} \tag{10}$$

It is well-known (see e.g. [12, 17]) that the solutions u_k ($k \in \mathbb{Z}$) of (10) are the Fourier coefficients of u according to (9). If u_k ($k \in \mathbb{Z}$) is sufficiently regular, the following differential equations and boundary conditions for the Fourier coefficients u_k can be derived

$$\begin{aligned} -\Big\{ \frac{\partial^2 u_k}{\partial r^2} + \frac{\partial^2 u_k}{\partial z^2} + \frac{1}{r} \frac{\partial u_k}{\partial r} \Big\} + \frac{k^2}{r^2} u_k &= f_k \qquad \text{in } \Omega_a \\ u_k &= 0 \qquad \text{on } \Gamma_a \ \ \forall k \in \mathbb{Z} \\ u_k &= 0 \qquad \text{on } \Gamma_0 \ \ \forall k \in \mathbb{Z} \backslash \{0\}. \end{aligned} \tag{11}$$

The boundary condition for u_0 on Γ_0 is formulated in the context of the variational problem. Because of the assumptions on the geometry of Ω_a, the domain $\widehat{\Omega}$ has neither sharp conical vertices nor reentrant edges. Consequently, the solution of the 3D-BVP (8) has the regularity $u \in X^2_{1/2}(\Omega)$, and its Fourier coefficients u_k from (10) belong to the space $H^2_{1/2}(\Omega_a)$, cf. [4, 12, 17].

For the Nitsche-finite-element discretization we shall need a subdivision of Ω_a into subdomains. Throughout this paper we restrict ourselves to the case of two subdomains Ω_a^1, Ω_a^2 with

$$\overline{\Omega}_a = \overline{\Omega}_a^1 \cup \overline{\Omega}_a^2, \quad \Omega_a^1 \cap \Omega_a^2 = \emptyset, \quad \Gamma = \overline{\Omega}_a^1 \cap \overline{\Omega}_a^2.$$

Moreover, assume that the subdomains are polygonally bounded. There are different cases for the position of the two subdomains: Figure 2 shows the case $\partial\Omega_a^i \cap \Gamma_a \neq \emptyset$ for $i = 1, 2$, and in Figure 3 we have $\partial\Omega_a^2 \cap \Gamma_a = \emptyset$, $\Gamma = \partial\Omega_a^2$. Obviously, the decomposition of $\Omega_a \subset I\!\!R^2$ implies a decomposition of the three-dimensional domain Ω into two subdomains $\Omega^i = \Omega_a^i \times (-\pi, \pi]$, $i = 1, 2$.

In view of the subdivision of Ω_a we introduce the restrictions $v^i := v|_{\Omega_a^i}$ of some function v on Ω_a^i as well as the vectorized form $v = (v^1, v^2)$, i.e. $v^i(x) = v(x)$ holds for $x \in \Omega_a^i$ ($i = 1, 2$). It should be noted that for simplicity we use here the same symbol v for denoting the function on Ω_a as well as the vector (v^1, v^2).

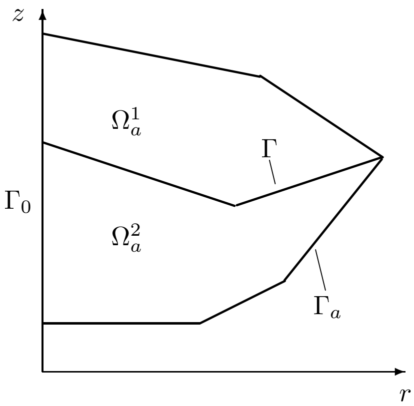

Figure 2

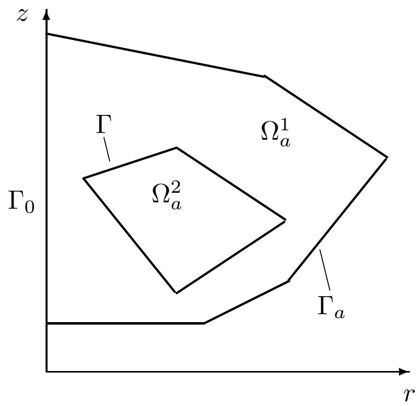

Figure 3

Using this notation we obtain that for each $k \in \mathbb{Z}$ the solution of the BVP (11) is equivalent to the solution of the following problem: Find (u_k^1, u_k^2) such that

$$\begin{aligned} -\Big\{\frac{\partial^2 u_k^i}{\partial r^2} + \frac{\partial^2 u_k^i}{\partial z^2} + \frac{1}{r}\frac{\partial u_k^i}{\partial r}\Big\} + \frac{k^2}{r^2}\, u_k^i \;&=\; f_k \qquad \text{in } \Omega_a^i, \quad i = 1, 2 \\ u_k^i \;&=\; 0 \qquad \text{on } \partial\Omega_a^i \cap \Gamma_a \\ u_k^i \;&=\; 0 \qquad \text{on } \partial\Omega_a^i \cap \Gamma_0 \;\; (\text{only for } k \in \mathbb{Z}\backslash\{0\}) \\ \frac{\partial u_k^1}{\partial n_1} + \frac{\partial u_k^2}{\partial n_2} \;&=\; 0 \qquad \text{on } \Gamma, \quad u_k^1 = u_k^2 \;\; \text{on } \Gamma \end{aligned} \tag{12}$$

are satisfied, where n_i $(i = 1, 2)$ denotes the outward normal to $\partial\Omega_a^i \cap \Gamma$. Introduce the spaces

$$\begin{aligned} &V_a^i = \{\, w \in H^1_{1/2}(\Omega_a^i) : w|_{\partial\Omega_a^i \cap \Gamma_a} = 0 \,\},\; W_a^i = \{\, w \in V_a^i : w \in L_{2,-1/2}(\Omega_a^i) \,\} \quad \text{for } i = 1, 2, \\ &V_a := V_a^1 \times V_a^2,\; W_a := W_a^1 \times W_a^2. \end{aligned} \tag{13}$$

Clearly, the BVPs (12) can be written in a variational form, where also the boundary condition of u_0^i on $\partial\Omega_a^i \cap \Gamma_0$ is specified. Then, for the Fourier coefficients u_k^i (in Ω_a^i) we have $u_0^i \in V_a^i$, $u_k^i \in W_a^i$ for $k \in \mathbb{Z}\backslash\{0\}$ as well as $u_0 = (u_0^1, u_0^2) \in V_a$, $u_k = (u_k^1, u_k^2) \in W_a$ for $k \in \mathbb{Z}\backslash\{0\}$. The continuity of the solution u_k and its normal derivative on Γ is to be required in the sense of the space $H^{1/2}_{1/2,*}(\Gamma)$ (the definition is given afterwards) and its dual space $[H^{1/2}_{1/2,*}(\Gamma)]'$, resp. Let the space $H^{1/2}_{1/2}(\partial\Omega_a^i \setminus \Gamma_0)$ be defined as the range of the trace operator: $v \to v|_{\partial\Omega_a^i \setminus \Gamma_0}$ for $v \in H^1_{1/2}(\Omega_a)$ (cf. [4, Section II.1.]). Then we use $H^{1/2}_{1/2,*}(\partial\Omega_a^i \setminus \Gamma_0) = H^{1/2}_{1/2}(\partial\Omega_a^i \setminus \Gamma_0)$ for $\partial\Omega_a^i \cap \Gamma_a = \emptyset$. In the case $\partial\Omega_a^i \cap \Gamma_a \neq \emptyset$ we identify $H^{1/2}_{1/2,*}(\partial\Omega_a^i \setminus \Gamma_0)$ with the space $H^{1/2}_{1/2,00}(\partial\Omega_a^i \setminus \partial\Omega_a)$ consisting of functions $v \in H^{1/2}_{1/2}(\partial\Omega_a^i \setminus \partial\Omega_a)$ for which the trivial extension $\tilde{v}$ by zero belongs to $H^{1/2}_{1/2}(\partial\Omega_a^i \setminus \Gamma_0)$.

3 The discretization method

The solutions $u_k = (u_k^1, u_k^2)$ $(k \in \mathbb{Z})$ of the 2D-BVP's (12) will be approximated by the Nitsche-finite-element method, cf. also [1, 11, 15, 16, 19].

First we describe the finite-element discretization with non-matching meshes. We cover Ω_a^i $(i = 1, 2)$ by a triangulation $\mathcal{T}_h^i$ $(i = 1, 2)$ consisting of triangles T $(T = \overline{T})$, where $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent of each other. Moreover, compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along the mortar interface $\Gamma = \partial\Omega_a^1 \cap \partial\Omega_a^2$ is not required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, with $0 < h \leq h_0$ and sufficiently small h_0. Take e.g. $h = \max\{h_T : T \in \mathcal{T}_h\}$, where h_T denotes the diameter of the triangle T. In the sequel, positive constants C occuring in the inequalities are generic constants.
Since the solutions of the 2D-BVP's do not have any singularities, it suffices to consider only quasi-uniform meshes in Ω_a^i $(i = 1, 2)$. Throughout this paper we suppose that the following assumption on the triangulations $\mathcal{T}_h^i$ $(i = 1, 2)$ is fulfilled.

Assumption 1

(i) For $i = 1, 2$, it holds $\overline{\Omega}_a^i = \cup_{T \in \mathcal{T}_h^i} T$, and two arbitrary triangles $T, T' \in \mathcal{T}_h^i$ $(T \neq T')$ are either disjoint or have a common vertex, or a common edge.

(ii) The mesh in $\overline{\Omega}_a^i$ $(i = 1, 2)$ is quasi-uniform, i.e. the relation

$$\frac{\max_{T \in \mathcal{T}_h^i} h_T}{\min_{T \in \mathcal{T}_h^i} \rho_T} \leq C \quad (i = 1, 2) \tag{14}$$

holds for $h \in (0, h_0]$, where ρ_T denotes the diameter of the largest inscribed sphere of T, and C is independent of h.

For $i = 1, 2$ and according to V_a^i, W_a^i from (13) introduce finite element spaces V_{ah}^i, W_{ah}^i of functions v_h^i on $\overline{\Omega}_a^i$ by

$$\begin{aligned} V_{ah}^i &:= \{\, v_h^i \in C(\overline{\Omega}_a^i) : \; v_h^i \in \mathbf{P}_1(T) \; \forall T \in \mathcal{T}_h^i, \; v_h^i|_{\partial\Omega_a^i \cap \Gamma_a} = 0 \,\}, \\ W_{ah}^i &:= \{\, v_h^i \in V_{ah}^i \quad \text{and} \quad v_h^i|_{\partial\Omega_a^i \cap \Gamma_0} = 0 \,\}, \end{aligned} \tag{15}$$

i.e. employ linear finite elements. It should be noted that $w \in W_a^i$ implies $w|_{\partial\Omega_a^i \cap \Gamma_0} = 0$ (cf. [17]) so that we require this also for the finite-element subspace. The finite element spaces V_{ah} and W_{ah} of vectorized functions v_h with components v_h^i on Ω_a^i are given by

$$\begin{aligned} V_{ah} &:= V_{ah}^1 \times V_{ah}^2 = \{v_h = (v_h^1, v_h^2) : \; v_h^1 \in V_{ah}^1, v_h^2 \in V_{ah}^2\} \\ W_{ah} &:= W_{ah}^1 \times W_{ah}^2 = \{v_h = (v_h^1, v_h^2) : \; v_h^1 \in W_{ah}^1, v_h^2 \in W_{ah}^2\}. \end{aligned} \tag{16}$$

It should be pointed out that the functions v_h in V_{ah} and in W_{ah} are in general not continuous across Γ.

Further we introduce some triangulation $\mathcal{E}_h$ of the mortar interface Γ by intervals E $(E = \overline{E})$, i.e., $\Gamma = \cup_{E \in \mathcal{E}_h} E$, where h_E denotes the diameter of E. We suppose that two segments E, E' are either disjoint or have a common endpoint. A natural choice for the triangulation $\mathcal{E}_h$ is $\mathcal{E}_h := \mathcal{E}_h^1$ or $\mathcal{E}_h := \mathcal{E}_h^2$ (cf. Figure 4), where $\mathcal{E}_h^1$ and $\mathcal{E}_h^2$ denote the triangulations of Γ defined by the traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ, resp.:

$$\mathcal{E}_h^i := \{E : E = \partial T \cap \Gamma, \text{ if } E \text{ is a segment}, T \in \mathcal{T}_h^i\,\} \quad \text{for } i = 1, 2.$$

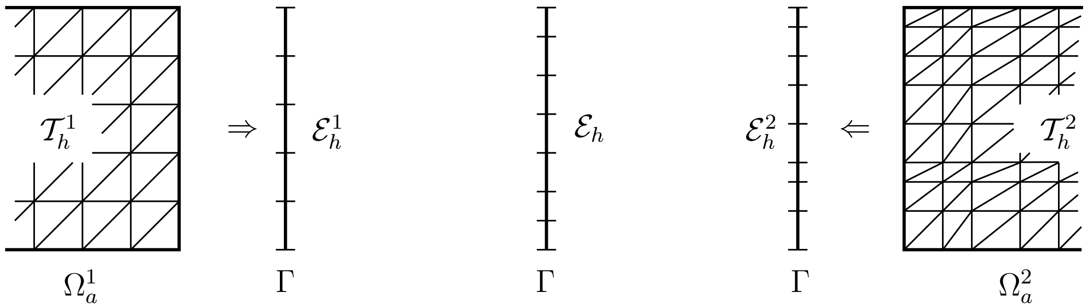

Figure 4

Subsequently we use real parameters α_1, α_2 with

$$0 \le \alpha_i \le 1 \quad (i = 1, 2), \quad \alpha_1 + \alpha_2 = 1. \tag{17}$$

The asymptotic behaviour of the triangulations $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and of $\mathcal{E}_h$ should be consistent on Γ in the sense of the following assumption.

Assumption 2

1. *For $E \in \mathcal{E}_h$ and $T \in \mathcal{T}_h^i$ with $\partial T \cap E \neq \emptyset$, $i = 1$ and $i = 2$, there are positive constants C_1 and C_2 independent of h_T, h_E and h $(0 < h \le h_0)$ such that the following condition is satisfied*
$$C_1 h_T \le h_E \le C_2 h_T. \tag{18}$$
2. *In the special case $\mathcal{E}_h := \mathcal{E}_h^i$ and $\alpha_i := 1$ (cf. (17)), where $i = 1$ or $i = 2$, for $E \in \mathcal{E}_h$ and $T \in \mathcal{T}_h^{3-i}$ with $\partial T \cap E \neq \emptyset$, instead of relation (18) the following condition is required:*
$$C_1 h_T \le h_E. \tag{19}$$

Relation (18) means that the diameter h_T of the triangle T touching the interface Γ at E is asymptotically equivalent to the diameter of the segment E, i.e. the equivalence of h_T, h_E is required only locally. In contrast, condition (19) is weaker and admits even locally at Γ different asymptotics of triangles $T_1 \in \mathcal{T}_h^1$, $T_2 \in \mathcal{T}_h^2$: $T_1 \cap T_2 \neq \emptyset$.

In order to define the Nitsche-finite-element approximation of the solutions of the BVP's (12), we now introduce bilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$ and linear forms $\mathcal{F}_{h,k}(\cdot)$, $k \in \mathbb{Z}$. We follow the ideas as given e.g. in [1, 11, 15, 16, 19] which are here to be adapted to the situation of weighted spaces. Moreover, a new term containing the parameter $k \in \mathbb{Z}$ occurs now in the bilinear form.
For $k \in \mathbb{Z}\backslash\{0\}$ and $u_h, v_h \in W_{ah}$ as well as for $k = 0$ and $u_h, v_h \in V_{ah}$, resp., $\mathcal{B}_{h,k}(\cdot,\cdot)$ and $\mathcal{F}_{h,k}(\cdot)$ are defined as follows:

$$\begin{aligned}\mathcal{B}_{h,k}(u_h, v_h) := &\sum_{i=1}^{2}\Big\{(\nabla u_h^i, \nabla v_h^i)_{1/2,\Omega_a^i} + k^2 (u_h^i, v_h^i)_{-1/2,\Omega_a^i}\Big\} - \Big\langle \alpha_1 \frac{\partial u_h^1}{\partial n_1} - \alpha_2 \frac{\partial u_h^2}{\partial n_2}, v_h^1 - v_h^2 \Big\rangle_{1/2,\Gamma} \\ &- \Big\langle \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}, u_h^1 - u_h^2 \Big\rangle_{1/2,\Gamma} + \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} (u_h^1 - u_h^2, v_h^1 - v_h^2)_{1/2,E}\end{aligned} \tag{20}$$

$$\mathcal{F}_{h,k}(v_h) \;:=\; \sum_{i=1}^{2} (f_k^i, v_h^i)_{1/2,\Omega_a^i}.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the $[H^{1/2}_{1/2,*}(\Gamma)]' \times H^{1/2}_{1/2,*}(\Gamma)$-duality pairing and $(\cdot,\cdot)_{1/2,E}$ the weighted $L_{2,1/2}(E)$ scalar product which is defined by analogy to (4). Moreover, γ is a sufficiently large positive constant (the restriction of γ will be given subsequently) and α_1 as well as α_2 are taken from (17). For $v_h = (v_h^1, v_h^2) \in V_{ah}$, we have $\frac{\partial v_h^i}{\partial n_i}|_\Gamma \in L_{2,1/2}(\Gamma)$. This will be used subsequently for evaluating $\langle\cdot,\cdot\rangle$ by the $L_{2,1/2}(\Gamma)$-scalar product.
The Nitsche-finite-element approximations $u_{0h} = (u_{0h}^1, u_{0h}^2) \in V_{ah}$ and $u_{kh} = (u_{kh}^1, u_{kh}^2) \in W_{ah}$, $k \in \mathbb{Z}\backslash\{0\}$, of the functions $u_k = (u_k^1, u_k^2)$ are defined to be the solutions of the equations

$$\begin{aligned} \mathcal{B}_{h,0}(u_{0h}, v_h) \;&=\; \mathcal{F}_{h,0}(v_h) \quad \forall v_h \in V_{ah} \\ \mathcal{B}_{h,k}(u_{kh}, v_h) \;&=\; \mathcal{F}_{h,k}(v_h) \quad \forall v_h \in W_{ah}, \quad k \in \mathbb{Z}\backslash\{0\}. \end{aligned} \tag{21}$$

In order to define the combined Fourier-Nitsche-finite-element approximation u_{hN} of u, we choose some $N > 0$ and carry out the Fourier synthesis of the functions $u_{kh} = (u_{kh}^1, u_{kh}^2)$ for $|k| \le N$:

$$u_{hN} = (u_{hN}^1, u_{hN}^2) \quad \text{with} \;\; u_{hN}^j = \sum_{|k|\le N} u_{kh}^j(r,z)\, e^{ik\varphi} \quad \text{for} \;\; j = 1, 2. \tag{22}$$

Clearly, the approximation u_{hN} of u depends on the two discretization parameters h and N.

4 Properties of the bilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$

The following lemma states the consistency of the solutions u_k ($k \in \mathbb{Z}$) from (10) with the variational equations (21).

Lemma 1 *Let u_k ($k \in \mathbb{Z}$) be the solution of the BVPs (10). Then $u_k = (u_k^1, u_k^2)$ satisfies:*

$$\begin{aligned} \mathcal{B}_{h,0}(u_0, v_h) \;&=\; \mathcal{F}_{h,0}(v_h) \quad \forall v_h \in V_{ah} \\ \mathcal{B}_{h,k}(u_k, v_h) \;&=\; \mathcal{F}_{h,k}(v_h) \quad \forall v_h \in W_{ah}, \quad k \in \mathbb{Z}\backslash\{0\}. \end{aligned} \tag{23}$$

Proof: Obviously, for any v_h indicated at (23), $\mathcal{B}_{h,k}(u_k, v_h)$ are well defined for $k \in \mathbb{Z}$. Since $\Delta_{r,z} u_k^i := \left\{ \frac{\partial^2 u_k^i}{\partial r^2} + \frac{\partial^2 u_k^i}{\partial z^2} + \frac{1}{r}\frac{\partial u_k^i}{\partial r} \right\} \in L_{2,1/2}(\Omega_a^i)$ as well as $u_k^i, v_h^i \in H^1_{1/2}(\Omega_a^i)$ hold ($i = 1, 2$, $k \in \mathbb{Z}$), the generalized version of Green's formula with the weight r

$$-\int_{\Omega_a^i} \Delta_{r,z} u_k^i \, v_h^i \, r dr dz = \int_{\Omega_a^i} \nabla u_k^i \, \nabla v_h^i \, r dr dz - \int_{\partial\Omega_a^i \backslash \Gamma_0} \frac{\partial u_k^i}{\partial n_i} v_h^i \, r ds$$

(see e.g. [9, Theorem 3.1]) may be applied with $i = 1, 2$. Using the properties $u_k^1|_\Gamma = u_k^2|_\Gamma$ and $\frac{\partial u_k^1}{\partial n_1}\Big|_\Gamma = -\frac{\partial u_k^2}{\partial n_2}\Big|_\Gamma$ (see (12)), we get (23). ■

For the proof of the boundedness and ellipticity of the bilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$ we shall need an estimate of the term $\sum_{E\in\mathcal{E}_h} h_E \, \|\frac{\partial v_h^i}{\partial n_i}\|^2_{L_{2,1/2}(E)}$ for $i \in \{1, 2\}$ where $\alpha_i \ne 0$.

Lemma 2 *Let Assumptions 1 and 2 be satisfied. Furthermore, let $F \in \mathcal{E}_h^i$ denote the side of a triangle $T_F \in \mathcal{T}_h^i$ touching Γ by F ($T_F \cap \Gamma = F$). Then the inequalities*

$$\sum_{E\in\mathcal{E}_h} h_E \left\|\frac{\partial v_h^i}{\partial n_i}\right\|^2_{L_{2,1/2}(E)} \le C_I^{(i)} \sum_{F\in\mathcal{E}_h^i} \|\nabla v_h^i\|^2_{L_{2,1/2}(T_F)} \tag{24}$$

hold for $i \in \{1,2\}$: $0 < \alpha_i \le 1$ and $v_h \in V_{ah}$, where the constants $C_I^{(i)}$ are independent of h, h_T, and h_E.

Proof: By means of the Cauchy-Schwarz inequality we get

$$\left\|\frac{\partial v_h^i}{\partial n_i}\right\|^2_{L_{2,1/2}(E)} = \int_E |(n_i, \nabla v_h^i)|^2 r ds \le \int_E (n_i, n_i)(\nabla v_h^i, \nabla v_h^i)\, r ds = \|\nabla v_h^i\|^2_{L_{2,1/2}(E)}. \tag{25}$$

Moreover, using Assumption 2 we can state the inequality

$$\sum_{E\in\mathcal{E}_h} h_E \|\nabla v_h^i\|^2_{L_{2,1/2}(E)} \le C \sum_{F\in\mathcal{E}_h^i} h_F \|\nabla v_h^i\|^2_{L_{2,1/2}(F)} \tag{26}$$

where $F \in \mathcal{E}_h^i$ is the side of a triangle $T_F \in \mathcal{T}_h^i$ touching Γ by F and h_F denotes the length of F. In the following, we shall estimate the norm $\|\nabla v_h^i\|^2_{L_{2,1/2}(F)}$.
The vertices of T_F and their coordinates are denoted by $P_j = (r_j, z_j)$, $j = 1, 2, 3$, and P_1, P_2 are the end points of $F \in \mathcal{E}_h^i$ ($i = 1$ or $i = 2$). Since the functions from V_{ah}^i are linear on each triangle, we can use the representation $v_h^i|_{T_F} = a_0 + a_1 r + a_2 z$ for any $v_h^i \in V_{ah}^i$ ($i = 1$ or $i = 2$) where the coefficients a_j ($j = 0, 1, 2$) depend on the triangle $T = T_F$. Some obvious calculations yield

$$\|\nabla v_h^i\|^2_{L_{2,1/2}(F)} = \frac{h_F}{2}(a_1^2 + a_2^2)(r_1 + r_2). \tag{27}$$

Now, the norm square from (27) has to be bounded by $\|\nabla v_h^i\|^2_{L_{2,1/2}(T_F)}$. We get by means of some cubature formula being exact for linear functions

$$\|\nabla v_h^i\|^2_{L_{2,1/2}(T_F)} = (a_1^2 + a_2^2) \int_{T_F} r dr dz = \frac{h_F\, h_F^{\perp}}{6}(a_1^2 + a_2^2)(r_1 + r_2 + r_3), \tag{28}$$

where $h_F^{\perp}$ is the height of the triangle T_F over the side F. The estimates $r_3 \ge 0$ and $h_F \le C h_F^{\perp}$ yield together with (27) and (28):

$$h_F \|\nabla v_h^i\|^2_{L_{2,1/2}(F)} \le C \|\nabla v_h^i\|^2_{L_{2,1/2}(T_F)} \quad \forall F \in \mathcal{E}_h^i. \tag{29}$$

Summing up (29) for all $F \in \mathcal{E}_h^i$ and using (25) as well as (26) we obtain the estimate (24). ∎

Taking (24) we can easily derive the inequalities

$$\begin{aligned}\sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}\right\|^2_{L_{2,1/2}(E)} &\le C_I \sum_{i=1}^{2} \sum_{F\in\mathcal{E}_h^i} \alpha_i^2 \|\nabla v_h^i\|^2_{L_{2,1/2}(T_F)} \\ &\le C_I \sum_{i=1}^{2} \alpha_i^2 \|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)},\end{aligned} \tag{30}$$

with $C_I := 2\max\{C_I^{(1)}, C_I^{(2)}\}$, or $C_I := C^{(i)}$ for $\alpha_i = 1$, $i \in \{1,2\}$.
In special cases, we can easily give an estimate for the constant C_I. For instance, choosing $\mathcal{E}_h = \mathcal{E}_h^1$ and $\alpha_1 = 1$, we get by means of the relations (27) and (28)

$$C_I = C^{(1)} = \sup_{h \le h_0} \max_{F \in \mathcal{E}_h^1} \Big(3 \frac{h_F}{h_F^{\perp}} \Big).$$

For deriving the boundedness and ellipticity of the bilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$ we introduce the weighted discrete norms $\|\cdot\|_{1,h,k}$ ($k \in \mathbb{Z}$) as follows:

$$\|v_h\|_{1,h,k}^2 := \sum_{i=1}^{2} \Big\{ \|\nabla v_h^i\|_{L_{2,1/2}(\Omega_a^i)}^2 + k^2 \|v_h^i\|_{L_{2,-1/2}(\Omega_a^i)}^2 \Big\} + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|v_h^1 - v_h^2\|_{L_{2,1/2}(E)}^2. \tag{31}$$

For mortar methods, norms $\|\cdot\|_{1,h}$ are employed (see e.g. [1, 5, 11, 15, 16, 19]) which are similar to (31). But the norm given by (31) involves a new term depending on k and norm terms provided with the weight r^α, $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$.
Now we are ready to prove the following theorem.

Theorem 1 *Let Assumptions 1 and 2 for $\mathcal{T}_h^i$ $(i = 1,2)$ and for $\mathcal{E}_h$ be satisfied. Then there exists a constant $\mu_1 > 0$ such that the following estimate holds,*

$$|\mathcal{B}_{h,k}(w_h, v_h)| \le \mu_1 \|w_h\|_{1,h,k} \|v_h\|_{1,h,k} \quad \forall w_h, v_h \in W_{ah}, k \in \mathbb{Z}\backslash\{0\} \; (w_h, v_h \in V_{ah}, k = 0, \textit{resp.}).$$

If the constant γ in (20) is independent of h and k and fulfills $\gamma > C_I$ (C_I from (30)), then the inequality

$$\mathcal{B}_{h,k}(v_h, v_h) \ge \mu_2 \|v_h\|_{1,h,k}^2 \quad \forall v_h \in W_{ah}, k \in \mathbb{Z}\backslash\{0\} \; (v_h \in V_{ah}, k = 0, \textit{resp.})$$

holds with a positive constant μ_2. Both constants μ_1, μ_2 are independent of h and k.

Proof: In order to prove the boundedness of $\mathcal{B}_{h,k}(\cdot,\cdot)$, the integrals on Γ arising from (20) are represented as sums of integrals on $E \in \mathcal{E}_h$. Moreover, we employ the triangle inequality as well as the Cauchy-Schwarz inequality and obtain

$$\begin{aligned} |\mathcal{B}_{h,k}(w_h, v_h)| \le & \sum_{i=1}^{2} \Big\{ \|\nabla w_h^i\|_{L_{2,1/2}(\Omega_a^i)} \|\nabla v_h^i\|_{L_{2,1/2}(\Omega_a^i)} + k^2 \|w_h^i\|_{L_{2,-1/2}(\Omega_a^i)} \|v_h^i\|_{L_{2,-1/2}(\Omega_a^i)} \Big\} \\ + & \sum_{E \in \mathcal{E}_h} \Big\| \alpha_1 \frac{\partial w_h^1}{\partial n_1} - \alpha_2 \frac{\partial w_h^2}{\partial n_2} \Big\|_{L_{2,1/2}(E)} \|v_h^1 - v_h^2\|_{L_{2,1/2}(E)} \\ + & \sum_{E \in \mathcal{E}_h} \Big\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \Big\|_{L_{2,1/2}(E)} \|w_h^1 - w_h^2\|_{L_{2,1/2}(E)} \\ + & \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \|w_h^1 - w_h^2\|_{L_{2,1/2}(E)} \, \|v_h^1 - v_h^2\|_{L_{2,1/2}(E)}. \end{aligned} \tag{32}$$

Using Hölder's inequality and introducing the term $h_E^{-1}h_E$ in the third and fourth terms on the right-hand side of (32) we get

$$\begin{aligned}
|\mathcal{B}_{h,k}(w_h,v_h)| \;\le\; & \left(\sum_{i=1}^{2}\|\nabla w_h^i\|^2_{L_{2,1/2}(\Omega_a^i)}\right)^{1/2}\left(\sum_{i=1}^{2}\|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)}\right)^{1/2}\\
&+\; k^2\left(\sum_{i=1}^{2}\|w_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\right)^{1/2}\left(\sum_{i=1}^{2}\|v_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\right)^{1/2}\\
&+\; \left(\sum_{E\in\mathcal{E}_h} h_E\left\|\alpha_1\frac{\partial w_h^1}{\partial n_1}-\alpha_2\frac{\partial w_h^2}{\partial n_2}\right\|^2_{L_{2,1/2}(E)}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1-v_h^2\|^2_{L_{2,1/2}(E)}\right)^{1/2}\\
&+\; \left(\sum_{E\in\mathcal{E}_h} h_E\left\|\alpha_1\frac{\partial v_h^1}{\partial n_1}-\alpha_2\frac{\partial v_h^2}{\partial n_2}\right\|^2_{L_{2,1/2}(E)}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h} h_E^{-1}\|w_h^1-w_h^2\|^2_{L_{2,1/2}(E)}\right)^{1/2}\\
&+\; \left(\gamma\sum_{E\in\mathcal{E}_h} h_E^{-1}\|w_h^1-w_h^2\|^2_{L_{2,1/2}(E)}\right)^{1/2}\left(\gamma\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1-v_h^2\|^2_{L_{2,1/2}(E)}\right)^{1/2}.
\end{aligned}$$

For bounding the terms containing normal derivatives we apply inequality (30) and again Hölder's inequality. This leads to

$$\begin{aligned}
|\mathcal{B}_{h,k}(w_h,v_h)| \le \mu_1 & \Big\{\sum_{i=1}^{2}\Big(\|\nabla w_h^i\|^2_{L_{2,1/2}(\Omega_a^i)}+k^2\|w_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\Big)+\sum_{E\in\mathcal{E}_h} h_E^{-1}\|w_h^1-w_h^2\|^2_{L_{2,1/2}(E)}\Big\}^{1/2}\\
&\times\Big\{\sum_{i=1}^{2}\Big(\|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)}+k^2\|v_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\Big)+\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1-v_h^2\|^2_{L_{2,1/2}(E)}\Big\}^{1/2}\\
= \mu_1 & \|w_h\|_{1,h,k}\|v_h\|_{1,h,k}
\end{aligned}$$

where $\mu_1 := \max(1+C_I, 1+\gamma)$.
It remains to prove the ellipticity of $\mathcal{B}_{h,k}(\cdot,\cdot)$. For $v_h\in W_{ah}$, $k\in\mathbb{Z}\setminus\{0\}$, and for $v_h\in V_{ah}$, $k=0$, resp., we get

$$\begin{aligned}
\mathcal{B}_{h,k}(v_h,v_h) = & \sum_{i=1}^{2}\Big\{(\nabla v_h^i,\nabla v_h^i)_{1/2,\Omega_a^i}+k^2(v_h^i,v_h^i)_{-1/2,\Omega_a^i}\Big\} - 2\Big\langle \alpha_1\frac{\partial v_h^1}{\partial n_1}-\alpha_2\frac{\partial v_h^2}{\partial n_2}, v_h^1-v_h^2\Big\rangle_{1/2,\Gamma}\\
&+\gamma\sum_{E\in\mathcal{E}_h} h_E^{-1}(v_h^1-v_h^2,v_h^1-v_h^2)_{1/2,E}\,. \qquad (33)
\end{aligned}$$

By means of Young's inequality, the second term on the right-hand side of (33) can be estimated as follows:

$$\begin{aligned}
-2&\Big\langle \alpha_1\frac{\partial v_h^1}{\partial n_1}-\alpha_2\frac{\partial v_h^2}{\partial n_2}, v_h^1-v_h^2\Big\rangle_{1/2,\Gamma}\\
&\ge -\frac{1}{\varepsilon}\sum_{E\in\mathcal{E}_h} h_E\left\|\alpha_1\frac{\partial v_h^1}{\partial n_1}-\alpha_2\frac{\partial v_h^2}{\partial n_2}\right\|^2_{L_{2,1/2}(E)} - \varepsilon\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1-v_h^2\|^2_{L_{2,1/2}(E)}.
\end{aligned}$$

This, together with inequality (30) and relation (33) leads to

$$
\begin{aligned}
\mathcal{B}_{h,k}(v_h, v_h) \;&\geq\; \sum_{i=1}^{2}\Big\{\|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)} + k^2\|v_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\Big\} - \frac{1}{\varepsilon}\,C_I\sum_{i=1}^{2}\|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)} \\
&+\; (\gamma-\varepsilon)\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1 - v_h^2\|^2_{L_{2,1/2}(E)} \\
&\geq\; \mu_2\Big\{\sum_{i=1}^{2}\Big(\|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)} + k^2\|v_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)}\Big) + \sum_{E\in\mathcal{E}_h} h_E^{-1}\|v_h^1 - v_h^2\|^2_{L_{2,1/2}(E)}\Big\} \\
&=\; \mu_2\|v_h\|^2_{1,h,k}\,,
\end{aligned}
$$

where ε is chosen such that $\gamma > \varepsilon > C_I$ is satisfied. Then we have $\mu_2 := \min\{1 - \frac{C_I}{\varepsilon}, \gamma - \varepsilon\}$.
■

5 Error estimates

For the error analysis of the Fourier-FEM, we have to study the approximation properties of the interpolation operator and the projection-interpolation operator, see e.g. [17, 14, 21]. Since we now consider the FEM with mortaring, these operators have to be slightly adapted. The interpolation operator Π_h which will be employed for estimating the approximation error for the Fourier coefficients u_k with $|k| \leq 1$ is now defined as follows:

$$\Pi_h u_k := (\Pi_h u_k^1, \Pi_h u_k^2), \tag{34}$$

where $\Pi_h u_k^i$ $(i = 1, 2)$ denotes the usual Lagrange interpolant of u_k^i in the space V_{ah}^i.
The use of $\Pi_h u_k$ for $|k| \geq 2$ does not lead to optimal error estimates with respect to the discretization parameters h and N^{-1}, cf. [17]. Therefore, for $|k| \geq 2$ we shall apply some projection-interpolation operator P_h. In order to define P_h, some notations are introduced. For any node $Q \in \mathcal{T}_h^i$ $(i = 1, 2)$, let S_Q^i be the polygon consisting of all triangles $T \in \mathcal{T}_h^i$ having Q as vertex. B_h^0 denotes the interior of the union of all triangles $T \in \mathcal{T}_h$ with $T \cap \overline{\Gamma}_0 \neq \emptyset$. For $i \in \{1, 2\}$ define $\overline{B}_{h,i}^0 := \overline{B}_h^0 \cap \overline{\Omega}_a^i$ (see Figure 5) and let $B_{h,i}$ be the interior of the union of all triangles $T \in \mathcal{T}_h$ with $T \cap \overline{B}_{h,i}^0 \neq \emptyset$. The set of all nodes of $\mathcal{T}_h^i$ $(i = 1, 2)$ is called Σ_h^i, and the set $\Sigma_h^{i,*}$ consists of all nodes $Q \in \Sigma_h^i$ with $Q \notin \overline{B}_h^0$ and $Q \notin (\partial\Omega_a^i \cap \Gamma_a)$.

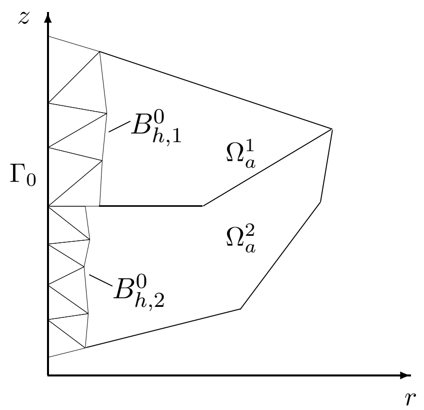

Figure 5

Introduce Courant's basis function $\Phi_Q^i \in C(\overline{\Omega}_a^i)$ associated with the node $Q \in \Sigma_h^i$:

$$\Phi_Q^i \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h, \quad \Phi_Q^i(Q') = \begin{cases} 1 & \text{for} \quad Q' = Q \\ 0 & \text{for} \quad Q' \neq Q,\ Q' \in \Sigma_h^i. \end{cases} \tag{35}$$

Define the orthogonal projection operator $P_Q^i : L_2(S_Q^i) \longrightarrow \mathbb{P}_1(S_Q^i)$ by $v \longrightarrow P_Q^i v$ ($Q \in \Sigma_h^i$, $i = 1, 2$) via the relation

$$(v - P_Q^i v, p)_{L_2(S_Q^i)} = 0 \quad \forall p \in \mathbf{P}_1(S_Q^i).$$

Taking Φ_Q^i from (35) and $v_Q^i := (P_Q^i v)(Q)$ we define the projection-interpolation operator P_h as follows,

$$P_h u_k := (P_h^1 u_k^1, P_h^2 u_k^2) \quad \text{with} \ \ P_h^i v := \sum_{Q \in \Sigma_h^{i,\star}} v_Q^i \Phi_Q^i, \quad i = 1, 2. \tag{36}$$

For P_h we can easily verify that $P_h u_k = 0$ on $\overline{B}_h^0$ and $P_h u_k \in W_{ah}$ hold for $|k| \geq 2$.

In addition to $\|\cdot\|_{1,h,k}^2$ at (31), we introduce the weighted mesh-dependent norm $\|\cdot\|_{h,k,\Omega_a}$:

$$\begin{aligned} \|v\|_{h,k,\Omega_a}^2 := \ & \sum_{i=1}^2 \left\{ \|\nabla v^i\|_{L_{2,1/2}(\Omega_a^i)}^2 + k^2 \|v^i\|_{L_{2,-1/2}(\Omega_a^i)}^2 + \sum_{E \in \mathcal{E}_h} h_E \left\| \alpha_i \frac{\partial v^i}{\partial n_i} \right\|_{1/2,E}^2 \right\} \\ & + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|v^1 - v^2\|_{L_{2,1/2}(E)}^2 \end{aligned} \tag{37}$$

for functions v satisfying $v \in V_a$ for $k = 0$, $v \in W_a$ for $k \in \mathbb{Z} \setminus \{0\}$, and $\frac{\partial v^i}{\partial n_i}|_\Gamma \in L_{2,1/2}(\Gamma)$ ($i = 1, 2$). Because of the regularity $u_k \in H_{1/2}^2(\Omega_a)$ (see Section 2), we have $\frac{\partial u_k^i}{\partial r}, \frac{\partial u_k^i}{\partial z} \in H_{1/2}^1(\Omega_a)$ and, consequently, $\frac{\partial u_k^i}{\partial n_i}|_\Gamma \in L_{2,1/2}(\Gamma)$ ($i = 1, 2$) is satisfied for all u_k, $k \in \mathbb{Z}$ (cf. [4, Section II.1.]).

The norms of the approximation errors $u_k - u_{kh}$, $k \in \mathbb{Z}$, can be bounded by means of the norms of $u_k - \Pi_h u_k$ and $u_k - P_h u_k$, resp. This is stated in the following lemma.

Lemma 3 *Let Assumptions 1 and 2 for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$ be satisfied, moreover, $\gamma > C_I$. Then for the error $u_k - u_{kh}$ (u_k, u_{kh} from (11), (21)), the following estimates hold,*

$$\begin{aligned} \|u_k - u_{kh}\|_{1,h,k} \ &\leq \ C\, \|u_k - \Pi_h u_k\|_{h,k,\Omega_a} \quad \text{for} \ \ |k| \leq 1 \\ \|u_k - u_{kh}\|_{1,h,k} \ &\leq \ C\, \|u_k - P_h u_k\|_{h,k,\Omega_a} \quad \text{for} \ \ |k| \geq 2. \end{aligned} \tag{38}$$

Proof: Using Lemma 1 and Theorem 1, the proof can be carried out by analogy to the proof of Lemma 3 in [15]. ■

In order to derive bounds of the norms on the right-hand sides of (38), we need a refined trace theorem in weighted norms. In [20], a refined trace theorem involving the L_2-norm of some function v and its gradient is given. Now, the occurrence of norms with the weight r requires new techniques for the proof if the triangle T is situated near the z-axis.

Theorem 2 *Let $T \in \mathcal{T}_h^i$ ($i \in \{1,2\}$) be a triangle with $T \cap \Gamma_0 = \emptyset$ or $T \cap \Gamma_0 = \{P\}$, where P denotes a vertex of T. Let F be a side of this triangle, then the following inequality is valid,*

$$\|v\|^2_{L_{2,1/2}(F)} \le C \left(h_T^{-1} \|v\|^2_{L_{2,1/2}(T)} + \|v\|_{L_{2,1/2}(T)} \, \|\nabla v\|_{L_{2,1/2}(T)} \right) \quad \forall v \in H^1_{1/2}(T). \tag{39}$$

Proof: We distinguish two cases concerning the position of the triangle T.

Case 1: We suppose that $T \cap \Gamma_0 = \emptyset$ holds, then we employ the refined trace theorem from [20, p. 645] and obtain

$$\begin{aligned} \|v\|^2_{L_{2,1/2}(F)} &\le \sup_F r \, \|v\|^2_{L_2(F)} \le C \sup_F r \left\{ h_T^{-1} \|v\|^2_{L_2(T)} + \|v\|_{L_2(T)} \, \|\nabla v\|_{L_2(T)} \right\} \\ &\le C \sup_T r \, (\inf_T r)^{-1} \left\{ h_T^{-1} \|v\|^2_{L_{2,1/2}(T)} + \|v\|_{L_{2,1/2}(T)} \, \|\nabla v\|_{L_{2,1/2}(T)} \right\}. \end{aligned} \tag{40}$$

Setting $r_0 := \inf_T r$ and using the inequality $r_0 \ge C h_T$ which is a consequence of Assumption 1, the factor $\sup_T r \, (\inf_T r)^{-1}$ in (40) can be bounded as follows:

$$\sup_T r \, (\inf_T r)^{-1} \le (r_0 + h_T) \, r_0^{-1} \le 1 + \frac{h_T}{C h_T} \le C. \tag{41}$$

This, together with (40) proves inequality (39).

Case 2: Assuming $T \cap \Gamma_0 = \{P\}$ we perform the proof by means of an affin-linear transformation of the reference triangle $\hat{T}$ to the triangle T.

Let $P_j = (r_j, z_j)$, $j = 1,2,3$, denote the vertices of T. Without loss of generality we suppose that P_1 is the vertex lying on Γ_0, i.e., $r_1 = 0$. Then the mapping $\hat{T} \to T$ can be described as follows:

$$\begin{pmatrix} r \\ z \end{pmatrix} = B \begin{pmatrix} \hat{r} \\ \hat{z} \end{pmatrix} + b = \begin{pmatrix} r_2 & r_3 \\ z_2 - z_1 & z_3 - z_1 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{z} \end{pmatrix} + \begin{pmatrix} 0 \\ z_1 \end{pmatrix}. \tag{42}$$

Taking into account Assumption 1 and the fact that the triangle T has only one common point with Γ_0, we easily show that the inequality

$$c h_T (\hat{r} + \hat{z}) \le r \le h_T (\hat{r} + \hat{z}) \tag{43}$$

is valid for all r with $(r,z) \in T$ and $(\hat{r}, \hat{z})$ related to (r,z) by (42).

The next step is the proof of the following inequality on the reference triangle:

$$\|(\hat{r}+\hat{z})^{1/2} \hat{v}\|^2_{L_2(\hat{F})} \le C \Big\{ \|(\hat{r}+\hat{z})^{1/2} \hat{v}\|^2_{L_2(\hat{T})} + \|(\hat{r}+\hat{z})^{1/2} \hat{v}\|_{L_2(\hat{T})} \, \|(\hat{r}+\hat{z})^{1/2} \, \nabla \hat{v}\|_{L_2(\hat{T})} \Big\}. \tag{44}$$

It should be noted that in [16, Proof of Theorem 5.4], a similar inequality is given. But there the variable $\hat{r}$ has another meaning: it denotes the distance of a point of the reference triangle to the origin. Therefore, for proving (44) we follow the ideas of [16], but with some essential modifications.

We use the following decomposition of the function $\hat{v}$ on $\hat{F}$: $\hat{v} = v_1 + v_2$, where $v_i = \lambda_i \hat{v}$, $i = 1,2$ and λ_1 (resp. λ_2) is the barycentric coordinate associated with $(0,0)$ (resp. $(1,0)$).

Without loss of generality we assume that the end points of $\hat{F}$ are $(0,0)$ and $(1,0)$. By using the relation $v_1(\hat{r}, 1-\hat{r}) = 0 \ \forall \hat{r} \in [0,1]$ we obtain for $\hat{r} \in (0,1)$:

$$\begin{aligned}
\hat{r}|v_1(\hat{r},0)|^2 &= \hat{r}(|v_1(\hat{r},0)|^2 - |v_1(\hat{r},1-\hat{r})|^2) = -\hat{r}\int_0^{1-\hat{r}} \partial_{\hat{z}}|v_1(\hat{r},\hat{z})|^2 d\hat{z} \\
&= -\hat{r}\int_0^{1-\hat{r}} 2v_1(\hat{r},\hat{z})\partial_{\hat{z}}v_1(\hat{r},\hat{z})d\hat{z}.
\end{aligned}$$

Integration of this equality on $\hat{r} \in (0,1)$ yields

$$\int_0^1 \hat{r}|v_1(\hat{r},0)|^2 d\hat{r} = -2\int_{\hat{T}} \hat{r}v_1(\hat{r},\hat{z})\partial_{\hat{z}}v_1(\hat{r},\hat{z})d\hat{r}d\hat{z},$$

and by means the Cauchy-Schwarz inequality as well as relations $\hat{z} = 0$ on $\hat{F}$, $\hat{z} \geq 0$ on $\hat{T}$ we get

$$\begin{aligned}
\|\hat{r}^{1/2}v_1\|^2_{L_2(\hat{F})} = \|(\hat{r}+\hat{z})^{1/2}v_1\|^2_{L_2(\hat{F})} &\leq 2\,\|\hat{r}^{1/2}v_1\|_{L_2(\hat{T})}\,\|\hat{r}^{1/2}\nabla v_1\|_{L_2(\hat{T})} \\
&\leq 2\,\|(\hat{r}+\hat{z})^{1/2}v_1\|_{L_2(\hat{T})}\,\|(\hat{r}+\hat{z})^{1/2}\nabla v_1\|_{L_2(\hat{T})}.
\end{aligned} \tag{45}$$

Concerning the function v_2 we utilize $v_2(0,x) = 0 \ \forall x \in [0,1]$ (here, we temporarily use the variables x, y because a transformation of the coordinates will be performed later) and obtain

$$\begin{aligned}
x|v_2(x,0)|^2 &= x(|v_2(x,0)|^2 - |v_2(0,x)|^2) = x\int_0^1 \partial_t|v_2(tx,(1-t)x|^2 dt \\
&= 2x\int_0^1 v_2(tx,(1-t)x)\{\partial_x v_2(tx,(1-t)x) - \partial_y v_2(tx,(1-t)x)\}xdt.
\end{aligned}$$

Integration on $x \in (0,1)$ leads to

$$\int_0^1 x|v_2(x,0)|^2 dx = 2\int_0^1\int_0^1 xv_2(tx,(1-t)x)\{\partial_x v_2(tx,(1-t)x) - \partial_y v_2(tx,(1-t)x)\}xdxdt.$$

For the integral on the right-hand side of this equality we perform the transformation $\hat{r} = tx$, $\hat{z} = (1-t)x$. This allows us to use the relation $x = \hat{r}+\hat{z}$, such that we get

$$\int_0^1 \hat{r}|v_2(\hat{r},0)|^2 d\hat{r} \leq 2\int_{\hat{T}} (\hat{r}+\hat{z})|v_2(\hat{r},\hat{z})||\partial_{\hat{r}}v_2(\hat{r},\hat{z}) - \partial_{\hat{z}}v_2(\hat{r},\hat{z})|d\hat{r}d\hat{z},$$

and by means of the Cauchy-Schwarz inequality we have

$$\|\hat{r}^{1/2}v_2\|^2_{L_2(\hat{F})} = \|(\hat{r}+\hat{z})^{1/2}v_2\|^2_{L_2(\hat{F})} \leq 4\,\|(\hat{r}+\hat{z})^{1/2}v_2\|_{L_2(\hat{T})}\,\|(\hat{r}+\hat{z})^{1/2}\nabla v_2\|_{L_2(\hat{T})}. \qquad (46)$$

By analogy to the inequalities (5.12) in [16] we get for $v_i = \lambda_i \hat{v}$ $(i = 1, 2)$:

$$\begin{aligned} \|(\hat{r}+\hat{z})^{1/2}v_i\|_{L_2(\hat{T})} &\leq \|(\hat{r}+\hat{z})^{1/2}\hat{v}\|_{L_2(\hat{T})} \\ \|(\hat{r}+\hat{z})^{1/2}\nabla v_i\|_{L_2(\hat{T})} &\leq 2\|(\hat{r}+\hat{z})^{1/2}\hat{v}\|_{L_2(\hat{T})} + \|(\hat{r}+\hat{z})^{1/2}\nabla\hat{v}\|_{L_2(\hat{T})}, \end{aligned} \qquad (47)$$

and inequality (44) can be concluded from (45)-(47).
Using estimate (43), the properties of transformation (42) as well as inequality (44) we arrive at

$$\begin{aligned} \|r^{1/2}v\|^2_{0,F} &\leq Ch_T^2\,\|(\hat{r}+\hat{z})^{1/2}\hat{v}\|^2_{L_2(\hat{F})} \\ &\leq Ch_T^2\left(\|(\hat{r}+\hat{z})^{1/2}\hat{v}\|^2_{L_2(\hat{T})} + \|(\hat{r}+\hat{z})^{1/2}\hat{v}\|_{L_2(\hat{T})}\|(\hat{r}+\hat{z})^{1/2}\nabla\hat{v}\|_{L_2(\hat{T})}\right) \\ &\leq Ch_T^2\left(h_T^{-1}h_T^{-2}\|r^{1/2}v\|^2_{L_2(T)} + h_T^{-1/2}h_T^{-1}\|r^{1/2}v\|_{L_2(T)}h_T^{-1/2}\|r^{1/2}\nabla v\|_{L_2(T)}\right) \\ &\leq C\left(h_T^{-1}\|v\|^2_{L_{2,1/2}(T)} + \|v\|_{L_{2,1/2}(T)}\,\|\nabla v\|_{L_{2,1/2}(T)}\right) \end{aligned} \qquad (48)$$

which completes the proof of Theorem 2. ■

Local interpolation error estimates in the $L_{2,-1/2}$-norm and in the $H^1_{1/2}$-seminorm are given e.g. in [14, 17, 21]. Nevertheless, for further error estimates by means of the inequalities given in Lemma 3 and Theorem 2 we also need an estimate for the local interpolation error in the $L_{2,1/2}$-norm.
For each triangle $T \in \mathcal{T}_h$ we introduce the local interpolation operator by $\Pi_T : C(T) \longrightarrow \mathbb{P}_1(T)$ and $(\Pi_T\psi)(Q_j) = \psi(Q_j)$ (Q_j, $j = 1, 2, 3$: the vertices of T). Further, let $Q_0 \in \Gamma_0$ be a node of the triangulation $\mathcal{T}_h^i$ ($i = 1$ or $i = 2$), then for $\psi \in C(S^i_{Q_0})$ the function $\Pi_{S^i_{Q_0}}\psi \in C(S^i_{Q_0})$ is defined such that $\Pi_{S^i_{Q_0}}\psi|_{T_j} = \Pi_{T_j}\psi$ holds $\forall\, T_j \in S^i_{Q_0}$, $1 \leq j \leq j_0$ (j_0: number of triangles with vertex at Q_0).
The following lemma gives estimates for the local interpolation error in the $L_{2,1/2}$-norm.

Lemma 4 *Let Assumptions 1 and 2 for $\mathcal{T}_h^i$ $(i = 1, 2)$ be satisfied. If $Q_0 \in \Gamma_0$ is a node of the triangulation $\mathcal{T}_h^i$ ($i = 1$ or $i = 2$), the following inequality is valid*

$$\|v - \Pi_{S^i_{Q_0}}v\|_{L_{2,1/2}(S^i_{Q_0})} \leq Ch^2|v|_{H^2_{1/2}(S^i_{Q_0})} \quad \forall v \in H^2_{1/2}(S^i_{Q_0}). \qquad (49)$$

If $T \in \mathcal{T}_h^i$ is a triangle with $T \cap \Gamma_0 = \emptyset$, the estimate

$$\|v - \Pi_T v\|_{L_{2,1/2}(T)} \leq Ch_T^2|v|_{H^2_{1/2}(T)} \qquad (50)$$

holds for all $v \in H^2_{1/2}(T)$.

Proof: For (49), we will only give a sketch of the proof, because it is very similar to the proof of Lemma 6.1 in [17]. The transformation of the reference element $\hat{S}$ to the polygon $S^i_{Q_0}$ as well as the operator $\hat{\Pi}$ can be chosen by analogy to [17, pp. 434-437]. Using relation (6.6) from [17] we get

$$\|v - \Pi_{S^i_{Q_0}} v\|^2_{L_{2,1/2}(S^i_{Q_0})} \le C d(S^i_{Q_0}) \sup_{1\le j\le j_0} |\det B_j| \, \|\hat{v} - \hat{\Pi}\hat{v}\|^2_{L_{2,1/2}(\hat{S})}, \tag{51}$$

where B_j, $1 \le j \le j_0$, is the matrix of the transformation $\hat{T}_j \longrightarrow T_j$ ($\hat{T}_j \in \hat{S},\ T_j \in S^i_{Q_0}$). Applying Corollary 4.2 (with $X := L_{2,1/2}(\hat{S})$) and inequality (6.11) from [17] we obtain

$$\|\hat{v} - \hat{\Pi}\hat{v}\|^2_{L_{2,1/2}(\hat{S})} \le C|\hat{v}|^2_{H^2_{1/2}(\hat{S})} \le C\,\frac{h^2}{d(S_{Q^i_0})}\,|v|^2_{H^2_{1/2}(S^i_{Q_0})}. \tag{52}$$

This, together with (51) and with the inequalities $|\det B_j| \le Ch^2$ ($1 \le j \le j_0$), yields estimate (49). Finally, in order to prove (50), we can use the classical interpolation error estimates (see e.g. [8]) and inequality (41). ■

For simplicity of further presentations, we impose an additional condition on the triangulations.

Assumption 3 *For each $F \in \mathcal{E}^i_h$, $i = 1, 2$, the triangle T with $T \cap F = F$ has at most one common point with Γ_0.*

Now we have the following error estimate for the operator Π_h.

Theorem 3 *Under the Assumptions 1-3, for the Fourier coefficients u_k of u, with $|k| \le 1$, the following error estimate holds:*

$$\|u_k - \Pi_h u_k\|_{h,k,\Omega_a} \le Ch\|u_k\|_{H^2_{1/2}(\Omega_a)}. \tag{53}$$

Proof: For the sake of brevity we set $w_k := u_k - \Pi_h u_k$ and $w^i_k := u^i_k - \Pi_h u^i_k$. Then, according to (37), we have

$$\begin{aligned}\|w_k\|^2_{h,k,\Omega_a} &= \sum_{i=1}^{2}\left\{\|\nabla w^i_k\|^2_{L_{2,1/2}(\Omega^i_a)} + k^2\|w^i_k\|^2_{L_{2,-1/2}(\Omega^i_a)} + \sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_i\frac{\partial w^i_k}{\partial n_i}\right\|^2_{L_{2,1/2}(E)}\right\}\\ &+ \sum_{E\in\mathcal{E}_h} h_E^{-1}\|w^1_k - w^2_k\|^2_{L_{2,1/2}(E)} =: \sum_{i=1}^{2}\{S^i_1 + S^i_2 + S^i_3\} + S_4, \end{aligned}\tag{54}$$

where S^i_1, S^i_2, S^i_3, and S_4 are abbreviations for the corresponding norm terms on the left-hand side, with $S^i_2 = 0$ for $k = 0$. In order to estimate S^i_1 and S^i_2, $i = 1, 2$, we employ Proposition 6.1 from [17] and obtain

$$\sum_{i=1}^{2} S^i_1 \le Ch^2 \sum_{i=1}^{2}\|u^i_k\|^2_{H^2_{1/2}(\Omega^i_a)} = Ch^2\|u_k\|^2_{H^2_{1/2}(\Omega_a)} \quad \text{for } |k| \le 1, \tag{55}$$

$$\sum_{i=1}^{2} S^i_2 \le Ch^2 \sum_{i=1}^{2}\|u^i_k\|^2_{H^2_{1/2}(\Omega^i_a)} = Ch^2\|u_k\|^2_{H^2_{1/2}(\Omega_a)} \quad \text{for } |k| = 1. \tag{56}$$

Moreover, the summation over $E \in \mathcal{E}_h$ in S_3^i and S_4 ($i = 1$ or $i = 2$) can be replaced by a summation over $F \in \mathcal{E}_h^i$ (cf. [15], Proof of Theorem 2), and $\|\frac{\partial w_k^i}{\partial n_i}\|_{L_{2,1/2}(E)}$ can be bounded by $\|\nabla w_k^i\|_{L_{2,1/2}(E)}$. This leads to

$$S_3^i \leq C \sum_{F \in \mathcal{E}_h^i} h_F \, \|\nabla w_k^i\|^2_{L_{2,1/2}(F)} \tag{57}$$

$$S_4 \leq 2 \sum_{i=1}^{2} \sum_{E \in \mathcal{E}_h} h_E^{-1} \|w_k^i\|^2_{L_{2,1/2}(E)} \leq C \sum_{i=1}^{2} \sum_{F \in \mathcal{E}_h^i} h_F^{-1} \|w_k^i\|^2_{L_{2,1/2}(F)}, \tag{58}$$

where inequality (57) is only considered for $i \in \{1, 2\} : \alpha_i > 0$.
Owing to Assumption 3, we may apply Theorem 2 for the estimation of the terms on the right-hand sides of (57) and (58). Hence we obtain for each $F \in \mathcal{E}_h^i$

$$\|\nabla w_k^i\|^2_{L_{2,1/2}(F)} \leq C \, \{h_T^{-1} |w_k^i|^2_{H^1_{1/2}(T)} + |w_k^i|_{H^1_{1/2}(T)} \, |w_k^i|_{H^2_{1/2}(T)}\} \tag{59}$$

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \leq C \, \{h_T^{-1} \|w_k^i\|^2_{L_{2,1/2}(T)} + \|w_k^i\|_{L_{2,1/2}(T)} \, \|\nabla w_k^i\|_{L_{2,1/2}(T)}\}. \tag{60}$$

We first consider the case that $F \cap \Gamma_0 = \emptyset$ holds. Then we can employ inequality (50) and [17, Lemma 6.2], which leads to the inequalities

$$\|\nabla w_k^i\|^2_{L_{2,1/2}(F)} \leq C \left\{ h_T^{-1} h_T^2 |u_k^i|^2_{H^2_{1/2}(T)} + h_T |u_k^i|^2_{H^2_{1/2}(T)} \right\} \leq C h_T |u_k^i|^2_{H^2_{1/2}(T)} \tag{61}$$

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \leq C \left\{ h_T^{-1} h_T^4 |u_k^i|^2_{H^2_{1/2}(T)} + h_T^2 |u_k^i|_{H^2_{1/2}(T)} h_T |u_k^i|_{H^2_{1/2}(T)} \right\} \leq C h_T^3 |u_k^i|^2_{H^2_{1/2}(T)}. \tag{62}$$

In the case $F \cap \Gamma_0 \neq \emptyset$ we utilize inequality (49) as well as [17, estimate (6.1)], and get

$$\|\nabla w_k^i\|^2_{L_{2,1/2}(F)} \;\leq\; C h |u_k^i|^2_{H^2_{1/2}(S^i_{Q_0})} \tag{63}$$

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \;\leq\; C h^3 |u_k^i|^2_{H^2_{1/2}(S^i_{Q_0})}. \tag{64}$$

Using $h_F^{-1} \leq C h^{-1}$ and summing up inequalities (61) and (62) for all $F : F \cap \Gamma_0 = \emptyset$ (resp. (63), (64) for all $F : F \cap \Gamma_0 \neq \emptyset$), together with (54)-(58), we are led to the desired estimate (53). ∎

Theorem 4 *Let Assumptions 1-3 be satisfied. Then, for P_h from (36) and the Fourier coefficients u_k, $|k| \geq 2$, the following inequality is fulfilled:*

$$\|u_k - P_h u_k\|_{h,k,\Omega_a} \leq C h \left\{ k^2 \|u_k\|^2_{H^1_{-1/2}(\Omega_a)} + \|u_k\|^2_{H^2_{1/2}(\Omega_a)} \right\}^{1/2}. \tag{65}$$

Proof: With the abbreviations $w_k := u_k - P_h u_k$ and $w_k^i := u_k^i - P_h^i u_k^i$, we obtain by analogy to (54), (57) and (58):

$$\|w_k\|^2_{h,k,\Omega_a} \;\le\; C\sum_{i=1}^{2}\Big\{\|\nabla w_k^i\|^2_{L_{2,1/2}(\Omega_a^i)} + k^2\|w_k^i\|^2_{L_{2,-1/2}(\Omega_a^i)} \tag{66}$$

$$+\;\alpha_i\sum_{F\in\mathcal{E}_h^i} h_F\,\|\nabla w_k^i\|^2_{1/2,F} + \sum_{F\in\mathcal{E}_h^i} h_F^{-1}\|w_k^i\|^2_{L_{2,1/2}(F)}\Big\} =: C\sum_{i=1}^{2}\{S_1^i+S_2^i+S_3^i+S_4^i\},$$

where S_1^i, S_2^i, S_3^i, and S_4^i are now abbreviations for the corresponding norm terms in (66). If $\alpha_i = 0$ for $i=1$ or $i=2$, the term S_3^i in (66) vanishes for this i.
Applying Theorems 7.1 and 7.2 of [17], we get for $i=1,2$ the inequalities

$$S_1^i \;\le\; Ch^2\Big\{\|u_k^i\|^2_{H^1_{-1/2}(\Omega_a^i)} + \|u_k^i\|^2_{H^2_{1/2}(\Omega_a^i)}\Big\} \tag{67}$$

$$S_2^i \;\le\; Ck^2h^2|u_k^i|_{H^1_{-1/2}(\Omega_a^i)}. \tag{68}$$

For the terms S_3^i and S_4^i we can use the refined trace theorem again, e.g. inequalities (60) and (59) hold, but now the functions w_k^i contain the operator P_h^i instead of Π_h. The terms occurring in this relations will now be estimated. For this purpose we distinguish three cases concerning the position of the triangle T having F as a side.
Case 1: Let $T\subset \overline{B}^0_{h,i}$ with $i=1$ or $i=2$ be satisfied, with $\overline{B}^0_{h,i}$ introduced at the beginning of this Section. Because $P_h^i u_k^i = 0$ holds on $\overline{B}^0_{h,i}$, we only need estimates for the norms of the functions u_k^i themselves. Using the relation $\sup_T r \le h_T$ which holds for all triangles considered in this case we get

$$\|u_k^i\|_{L_{2,1/2}(T)} \le \sup_T r\|u_k^i\|_{L_{2,-1/2}(T)} \le h_T\|u_k^i\|_{L_{2,-1/2}(T)}.$$

This, together with $\|\nabla u_k^i\|_{L_{2,1/2}(T)} = |u_k^i|_{H^1_{1/2}(T)}$ and estimate (39) yields

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \;\le\; Ch_T\Big\{\|u_k^i\|^2_{L_{2,-1/2}(T)} + \|u_k^i\|_{L_{2,-1/2}(T)}|u_k^i|_{H^1_{1/2}(T)}\Big\} \tag{69}$$

$$\;\le\; Ch_T\Big\{\|u_k^i\|^2_{L_{2,-1/2}(T)} + |u_k^i|^2_{H^1_{1/2}(T)}\Big\}, \tag{70}$$

where $F\in\mathcal{E}_h^i$ ($i=1$ or $i=2$) is a side of T.
Case 2: We consider the triangles T with $T\subset \overline{B}_{h,i}\setminus B^0_{h,i}$, $i=1$ or $i=2$. For these triangles, at least one vertex belongs to $\overline{B}^0_{h,i}$. Without loss of generality we suppose that $Q_1\in\overline{B}^0_{h,i}$ holds (Q_j, $j=1,2,3$: the local numbers of the vertices of T). Then, following the ideas of [17, Proof of Theorem 7.1], we obtain the inequalities

$$\|w_k^i\|^2_{L_{2,1/2}(T)} \;\le\; (\sup_T r)^2\|w_k^i\|^2_{L_{2,-1/2}(T)} \le Ch^2\|w_k^i\|^2_{L_{2,-1/2}(T)}$$

$$\;\le\; Ch^2\Big\{\|u_k^i\|^2_{L_{2,-1/2}(T)} + h^2\sum_{j=2}^{3}|u_k^i|^2_{H^1_{-1/2}(S^i_{Q_j})}\Big\}, \tag{71}$$

$$\|\nabla w_k^i\|^2_{L_{2,1/2}(T)} = |w_k^i|^2_{H^1_{1/2}(T)} \leq C\Big\{|u_k^i|^2_{H^1_{1/2}(T)} + h^2\sum_{j=2}^{3}|u_k^i|^2_{H^1_{-1/2}(S^i_{Q_j})}\Big\}. \tag{72}$$

Combining relations (71) and (72) with the refined trace theorem, we are led to the estimate

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \leq Ch\Big\{\|u_k^i\|^2_{L_{2,-1/2}(T)} + |u_k^i|^2_{H^1_{1/2}(T)} + h^2\sum_{j=2}^{3}|u_k^i|^2_{H^1_{-1/2}(S^i_{Q_j})}\Big\}. \tag{73}$$

Case 3: Let $T \not\subset \overline{B}_{h,i}$ $(i = 1, 2)$ be satisfied. The local numbering of the vertices of T is again: Q_j, $j = 1, 2, 3$. Then, using Theorem 1 and inequality (7) from [10] as well as estimate (41) we get

$$\begin{aligned}
\|w_k^i\|^2_{L_{2,1/2}(T)} &\leq \sup_T r\|w_k^i\|^2_{L_2(T)} \leq Ch^4\sup_T r\sum_{j=1}^{3}|u_k^i|^2_{H^2(S^i_{Q_j})} \\
&\leq Ch^4\sup_T r(\inf_T r)^{-1}\sum_{j=1}^{3}|u_k^i|^2_{H^2_{1/2}(S^i_{Q_j})} \leq Ch^4\sum_{j=1}^{3}|u_k^i|^2_{H^2_{1/2}(S^i_{Q_j})}, \\
\|\nabla w_k^i\|^2_{L_{2,1/2}(T)} &\leq \sup_T r|w_k^i|^2_{H^1(T)} \leq Ch^2\sup_T r\sum_{j=1}^{3}|u_k^i|^2_{H^2(S^i_{Q_j})} \leq Ch^2\sum_{j=1}^{3}|u_k^i|^2_{H^2_{1/2}(S^i_{Q_j})}.
\end{aligned}$$

Therefore, the estimate

$$\|w_k^i\|^2_{L_{2,1/2}(F)} \leq Ch^3\sum_{j=1}^{3}|u_k^i|^2_{H^2_{1/2}(S^i_{Q_j})}$$

holds, where $F \in \mathcal{E}_h^i$ ($i = 1$ or $i = 2$) is a side of $T \subset S^i_{Q_j}$.

Now, after discussing these three cases, we additionally need the following estimates (cf. [17], Lemma 7.2 and relations (7.26), (7.27)):

$$\begin{aligned}
\|u_k^i\|_{L_{2,-1/2}(B)} &\leq Ch|u_k^i|^2_{H^1_{-1/2}(B_{l_i(h)})} \\
|u_k^i|_{H^1_{1/2}(B)} &\leq Ch|u_k^i|^2_{H^1_{-1/2}(B_{l_i(h)})}, \quad B \in \{B^0_{h,i}, B_{h,i}\}, \quad i = 1, 2,
\end{aligned} \tag{74}$$

where $B_{l_i(h)} := \{(r, z) \in \Omega_a : 0 < r < l_i(h)\}$, $l_i(h) = \sup_{B_{h,i}} r$.

Now, summing up the estimates for all $F \in \mathcal{E}_h^i$ $(i = 1, 2)$ and using $h_F^{-1} \leq Ch^{-1}$ as well as inequalities (74), we arrive at

$$\sum_{i=1}^{2} S_4^i \leq Ch^2\sum_{i=1}^{2}\Big\{|u_k^i|^2_{H^1_{-1/2}(\Omega_a^i)} + |u_k^i|^2_{H^2_{1/2}(\Omega_a^i)}\Big\} = Ch^2\Big\{|u_k|^2_{H^1_{-1/2}(\Omega_a)} + |u_k|^2_{H^2_{1/2}(\Omega_a)}\Big\}. \tag{75}$$

The estimate of S_3^i $(i = 1, 2)$ can be performed by analogy to that of S_4^i given above, and we obtain

$$\sum_{i=1}^{2} S_3^i \leq Ch^2\sum_{i=1}^{2}\Big\{|u_k^i|^2_{H^1_{-1/2}(\Omega_a^i)} + |u_k^i|^2_{H^2_{1/2}(\Omega_a^i)}\Big\} = Ch^2\Big\{|u_k|^2_{H^1_{-1/2}(\Omega_a)} + |u_k|^2_{H^2_{1/2}(\Omega_a)}\Big\}. \tag{76}$$

Collecting the inequalities (66)-(68), (75), and (76) completes the proof. ■

Now it remains to state an estimate for the error $u - u_{hN}$, where u is the solution of (8) and u_{hN} the approximate solution defined by (22). For this purpose we introduce a suitable norm in 3D:

$$\|v\|^2_{1,h,\Omega} := \sum_{j=1}^{2} |v^j|^2_{X^1_{1/2}(\Omega^j)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|v^1 - v^2\|^2_{X^0_{1/2}(E\times(-\pi,\pi])}, \tag{77}$$

where $|\cdot|_{X^1_{1/2}(\Omega^j)}$, with $\Omega^j := \Omega^j_a \times (-\pi,\pi]$ $(j = 1,2)$, is defined by analogy to $|\cdot|_{X^1_{1/2}(\Omega)}$ at (2), and the norm $\|\cdot\|_{X^0_{1/2}(E\times(-\pi,\pi])}$ is determined by the completeness relation

$$\|v\|^2_{X^0_{1/2}(E\times(-\pi,\pi])} := 2\pi \sum_{k\in\mathbb{Z}} \|v_k\|^2_{L_{2,1/2}(E)}. \tag{78}$$

It should be noted that we have in general $u_{hN} \notin X^1_{1/2}(\Omega)$, but only $u_{hN} \in X^1_{1/2}(\Omega^j)$, $j = 1,2$.

The final result of this paper is given in the next theorem.

Theorem 5 *Let u be the solution of the BVP (8) and u_{hN} its approximation given by (21), (22). Then the following error estimate holds,*

$$\|u - u_{hN}\|_{1,h,\Omega} \le C(h + N^{-1}) \|f\|_{X^0_{1/2}(\Omega)}. \tag{79}$$

Clearly, relation (79) states also the convergence $u_{hN} \to u$ as $h \to 0$, $N \to \infty$. In particular, h and N can be chosen independently from each other.

Proof: By means of the auxiliary function $u_N = (u^1_N, u^2_N)$ defined by

$$u^j_N = \sum_{|k|\le N} u^j_k(r,z)\, e^{ik\varphi} \quad j = 1,2, \tag{80}$$

we easily get

$$\|u - u_{hN}\|_{1,h,\Omega} \le \|u - u_N\|_{1,h,\Omega} + \|u_N - u_{hN}\|_{1,h,\Omega} =: S_1 + S_2, \tag{81}$$

where S_1 and S_2 denote the corresponding norm terms. We shall now find estimates of S_1 and S_2 in terms of powers of h and N. According to (77) we have

$$S_1^2 = \sum_{j=1}^{2} |u^j - u^j_N|^2_{X^1_{1/2}(\Omega^j)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|u^1 - u^1_N - (u^2 - u^2_N)\|^2_{X^0_{1/2}(E\times(-\pi,\pi])}. \tag{82}$$

The first term on the right-hand side of (82) is equal to $|u - u_N|^2_{X^1_{1/2}(\Omega)}$ since $u_k \in H^1_{1/2}(\Omega_a)$ $(k \in \mathbb{Z})$ and, therefore, $u - u_N \in X^1_{1/2}(\Omega)$ holds. Further we have the estimate

$$|u - u_N|^2_{X^1_{1/2}(\Omega)} \le N^{-2} \|u\|^2_{X^2_{1/2}(\Omega)} \tag{83}$$

(for the proof we refer to [17, Proof of Theorem 8.1]). The second term on the right-hand side of (82) vanishes. This is clear by $u^1|_{E\times(-\pi,\pi]} = u^2|_{E\times(-\pi,\pi]}$; the same holds for u^1_N, u^2_N.

This, together with (82) and (83) as well as with the estimate $\|u\|_{X^2_{1/2}(\Omega)} \le C\|f\|_{X^0_{1/2}(\Omega)}$ yields

$$S_1 \le CN^{-1}\|f\|_{X^0_{1/2}(\Omega)}. \tag{84}$$

Applying completeness relations (cf. [12, Lemma 3.2] and (78)) we obtain for S_2 the relation

$$\begin{aligned} S_2^2 &= \sum_{j=1}^{2} |u_N^j - u_{hN}^j|^2_{X^1_{1/2}(\Omega^j)} + \sum_{E\in\mathcal{E}_h} h_E^{-1}\|u_N^1 - u_{hN}^1 - (u_N^2 - u_{hN}^2)\|^2_{X^0_{1/2}(E\times(-\pi,\pi])} \\ &= 2\pi \sum_{j=1}^{2}\sum_{|k|\le N}\Big\{\|\nabla(u_k^j - u_{kh}^j)\|^2_{L_{2,1/2}(\Omega_a^j)} + k^2\|u_k^j - u_{kh}^j\|^2_{L_{2,-1/2}(\Omega_a^j)}\Big\} \\ &+ 2\pi \sum_{E\in\mathcal{E}_h} h_E^{-1}\Big\{\sum_{|k|\le N}\|u_k^1 - u_{kh}^1 - (u_k^2 - u_{kh}^2)\|^2_{L_{2,1/2}(E)}\Big\}. \end{aligned} \tag{85}$$

Changing the sequence of summation and employing (31) as well as (38) we are led to

$$S_2^2 = 2\pi\sum_{|k|\le N}\|u_k - u_{kh}\|^2_{1,h,k} \le C\Big\{\sum_{|k|\le 1}\|u_k - \Pi_h u_k\|^2_{h,k,\Omega_a} + \sum_{2\le|k|\le N}\|u_k - P_h u_k\|^2_{h,k,\Omega_a}\Big\}.$$

At the right-hand side of this estimate, we may apply Theorems 3 and 4 and get

$$S_2^2 \le Ch^2\Big\{\sum_{|k|\le N}\|u_k\|^2_{H^2_{1/2}(\Omega_a)} + \sum_{2\le|k|\le N} k^2\|u_k\|^2_{H^1_{-1/2}(\Omega_a)}\Big\}.$$

By means of completeness relations (see [12, Lemma 3.2]), the terms on the right-hand side of this inequality can be bounded by $\|u\|^2_{X^2_{1/2}(\Omega)}$. This yields

$$S_2 \le Ch\|u\|_{X^2_{1/2}(\Omega)} \le Ch\|f\|_{X^0_{1/2}(\Omega)}. \tag{86}$$

The assertion of Theorem 5 is a consequence of (81), (84), (86). ■

6 Numerical example

For verifying the convergence rate of the Fourier-finite-element method with Nitsche mortaring, we consider some BVP of type (6), (7). The meridian domain Ω_a which generates $\widehat{\Omega}$, is a rectangle with two subdomains $\Omega_a^1 = (0,1)\times(1,2)$ and $\Omega_a^2 = (0,1)\times(0,1)$, cf. also Figure 6. The right-hand is chosen such that the solution of the BVP (7) is

$$u(r,\varphi,z) = \sum_{k=1}^{128} u_k(r,z)\,\sin k\varphi, \quad \text{with } u_k(r,z) = k^{-\frac{5}{2}}(r^{\frac{5}{2}} - r^{\frac{3}{2}})(z^2 - 2z). \tag{87}$$

We easily check that $u = 0|_{\Gamma_a\times(-\pi,\pi]}$ and $u \in X^2_{1/2}(\Omega)$ holds.

For the experiments, the initial mesh shown in Figure 6 is used. This mesh is refined globally by dividing each triangle into four equal triangles such that the mesh parameters form a sequence $\{h_1, h_2, \ldots\}$ given by $h_{i+1} = 0.5\, h_i$, here with six levels. The ratio of the number of mesh segments on the mortar interface is given by $2 : 3$. The mortar parameters (cf. Section 3) are chosen as follows: $\alpha_1 = 1$, $\alpha_2 = 0$, and $\gamma = 4$. The trace $\mathcal{E}_h^1$ of the triangulation $\mathcal{T}_h^1$ of Ω_a^1 on the interface Γ is taken to form the partition $\mathcal{E}_h$. Furthermore, for the discretization with respect to N (the number of Fourier modes for the approximate solution), we employ five levels N_i, where $N_1 = 4$ and $N_{i+1} = 2\, N_i$ for $i = 1, 2, 3, 4$ holds.

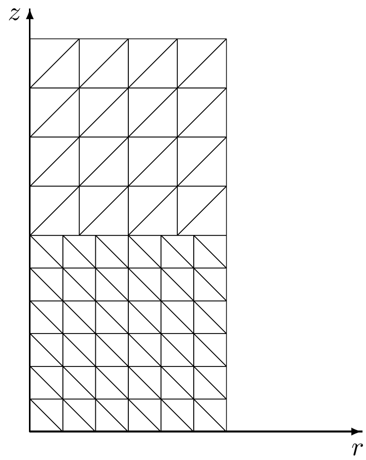

Figure 6

For the approximate measuring of the convergence rate stated in (79), the hypothesis for the tests is

$$\|u - u_{hN}\|_{1,h,\Omega} \approx C_1 h^\alpha + C_2 N^{-\beta}, \tag{88}$$

with u from (87) and u_{hN} as its approximate solution according to (22). The parameters C_1 and C_2 are assumed to be approximately constant for two consecutive levels of h and N.
First we investigate the convergence order with respect to the discretization parameter h. Table 1 shows the approximation error norms e_{total} and e_h as well as the observed values α_{obs} of the convergence order α on the levels $h_1, \ldots, h_6$, all with $N = 64$. Here, e_{total} denotes the norm of the total error, that is $\|u - u_{hN}\|_{1,h,\Omega} =: e_{\text{total}}$, and e_h is the norm of the error with respect to h (for fixed $N = 64$). The latter is used to compute the values α_{obs}. We can state that the obtained values of the convergence order are very close to the theoretically expected value $\alpha_{exp} = 1$ (cf. Theorem 5).

level	e_{total}	e_h	α_{obs}
h_1	1.97376e-1	1.97351e-1	–
h_2	9.96031e-2	9.95527e-2	0.987
h_3	4.96306e-2	4.95293e-2	1.007
h_4	2.48453e-2	2.46422e-2	1.007
h_5	1.26857e-2	1.22833e-2	1.004
h_6	6.90228e-3	6.13132e-3	1.002

Table 1: Error norms and convergence orders for $h = h_1, \ldots h_6$ and $N = 64$

In order to observe the convergence order with respect to N, some computations on the mesh with the mesh parameter h_5 and N varying from N_1 to N_5 are carried out. The norms e_{total} and e_N as well as the observed values β_{obs} of the convergence order β are represented in Table 2 (e_N: the error norm with respect to N for fixed $h = h_5$). The convergence orders β_{obs} are computed by means of e_N. According to Theorem 5, the

level	e_{total}	e_N	β_{obs}
N_1	5.62542e-2	5.48976e-2	–
N_2	3.06833e-2	2.81175e-2	0.965
N_3	1.88183e-2	1.42566e-2	0.980
N_4	1.41588e-2	7.04220e-3	1.018
N_5	1.26857e-2	3.16992e-3	1.152

Table 2: Error norms and convergence orders for $h = h_5$ and $N = N_1, \ldots, N_5$

expected convergence order is $\beta_{exp} = 1$, and the observed rates are approximately equal to β_{exp}.
This numerical example illustrates that the Fourier-finite-element method with Nitsche mortaring is a suitable approach for the numerical treatment of the Poisson equation in three-dimensional axisymmetric domains where the exact solution is supposed to be regular, i.e. $u \in X^2_{1/2}(\Omega)$.

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