**Technische Universität Chemnitz**

**Sonderforschungsbereich 393**

*Numerische Simulation auf massiv parallelen Rechnern*

S. I. Solov′ëv

# Preconditioned iterative methods for a class of nonlinear eigenvalue problems

**Preprint SFB393/03-19**

**Abstract** In this paper we develop new preconditioned iterative methods for solving monotone nonlinear eigenvalue problems. We investigate the convergence and derive grid-independent error estimates. Numerical experiments demonstrate the practical effectiveness of the proposed methods for a model problem.

**Key Words** symmetric eigenvalue problem, nonlinear eigenvalue problem, preconditioned iterative method, gradient method, steepest descent method, conjugate gradient method

**AMS(MOS) subject classification** 65F15, 65F50, 65N25

**Preprint-Reihe des Chemnitzer SFB 393**

**ISSN 1619-7178** (Print) **ISSN 1619-7186** (Internet)

**SFB393/03-19** **November 2003**

# Contents

**1 Introduction** **1**

**2 Formulation of the problem** **3**

**3 Existence of the eigenvalues** **4**

**4 Auxiliary results** **6**

**5 Preconditioned iterative methods** **7**

**6 Convergence of iterative methods** **8**

**7 Error estimates of iterative methods** **11**

**8 Numerical experiments** **13**

**9 Conclusion** **15**

**References** **16**

Current address of the author:

Sergey I. Solov′ëv
Fakultät für Mathematik
TU Chemnitz
09107 Chemnitz, Germany

solovev@mathematik.tu-chemnitz.de

Address of the author:

Sergey I. Solov′ëv
Faculty of computer science and cybernetics
Kazan State University
Kremlevskaya 18
420008 Kazan, Russia

sergei.solovyev@ksu.ru

# 1 Introduction

After the discretization of eigenvalue problems for symmetric elliptic differential operators we get the matrix eigenvalue problem $Au = \lambda Bu$ with large and sparse symmetric positive definite matrices $A$ and $B$. Usually matrices $A$ and $B$ are very large and the matrix $A$ is ill-conditioned. We assume that matrices $A$ and $B$ are not stored explicitly and routines for computing the matrix-vector products $Av$ and $Bv$ are only available. In applied eigenvalue problems describing vibrations of mechanical structures, only a few of the smallest eigenvalues defining the base frequencies are of interest.

Classical methods for solving eigenvalue problems can not be applied in our situation since the computer storage for matrices $A$ and $B$ is not available. Lanczos method has slow convergence since the condition number of the matrix $A$ increases for decreasing mesh size $h$. In indicated practical problems the condition number usually behaves like $h^{-m}$, $2 \le m \le 4$.

In order to find the smallest simple eigenvalue $\lambda_1$ of the matrix problem $Au = \lambda Bu$, we can use a gradient method. It is well known that $\lambda_1$ is the minimum of the Rayleigh quotient $R(v) = (Av, v)/(Bv, v)$ and its stationary point is the eigenvector $u_1$ corresponding to $\lambda_1$. Hence we can construct a minimizing sequence of nonzero vectors $u^n$, $n = 1, 2, \ldots$, $\mu^n = R(u^n) \to \lambda_1$, $u^n \to u_1$, $n \to \infty$, using the formulae

$$\tilde{u}^{n+1} = u^n - \tau^n (A - \mu^n B) u^n,$$
$$u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^{n+1} = R(u^{n+1}), \quad n = 0, 1, \ldots$$

for a suitable choice of the scalar parameter $\tau^n$, $\|u\|_B^2 = (Bu, u)$. This iteration method is called the *gradient method* for computing the smallest eigenvalue of the matrix problem since

$$\operatorname{grad} R(v) = \frac{2}{(Bv, v)} (A - R(v)B) v$$

and

$$\tilde{u}^{n+1} = u^n - c_0 \operatorname{grad} R(u^n),$$

where $c_0 = \tau^n (Bu^n, u^n)/2$. Thus, in the gradient method we move from a given iteration vector $u^n$ in the direction $-\operatorname{grad} R(u^n)$.

The described gradient method has a maximal simplicity and low storage requirement. Therefore this method is called also a *simple iteration method*. But, unfortunately, this method has poor convergence properties for an ill-conditioned matrix $A$.

To improve the convergence of the simple iteration method, we introduce the preconditioner $C^{-1}$, where $C$ is a matrix approximating $A$, and calculate sequences $\mu^n$, $u^n$, $n = 1, 2, \ldots$ by the relationships

$$\tilde{u}^{n+1} = u^n - \tau^n C^{-1} (A - \mu^n B) u^n,$$
$$u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^{n+1} = R(u^{n+1}), \quad n = 0, 1, \ldots$$

The matrix $C$ is assumed to be a symmetric positive definite matrix that can be easily inverted. The last method uses the gradient of the Rayleigh quotient in the vector space with scalar product $(C.,.)$:

$$\operatorname{grad}_C R(v) = \frac{2}{(Bv,v)} C^{-1}(A - R(v)B)v$$

and we obtain

$$\tilde{u}^{n+1} = u^n - c_0 \operatorname{grad}_C R(u^n),$$

where $c_0 = \tau^n (Bu^n, u^n)/2$. Therefore this method is called the *preconditioned gradient method* or *preconditioned simple iteration method* (PSIM).

The convergence of PSIM can be improved if we minimize the Rayleigh quotient in the subspace $V_{n+1} = \operatorname{span}\{u^n, w^n\}$ or $W_{n+1} = \operatorname{span}\{u^{n-1}, u^n, w^n\}$, $w^n = C^{-1}(A - \mu^n B)u^n$. The corresponding iterative methods are called *preconditioned steepest descent method* (PSDM) and *preconditioned conjugate gradient method* (PCGM), respectively.

PSDM for the symmetric eigenvalue problem $Au = \lambda Bu$ has been first studied by Samokish in the paper [14]. Grid-independent convergence estimates for PSIM were first obtained in D'yakonov and Orekhov [2]. Knyazev has suggested LOPCGM in [5] and analyzed this method and its new variants in the papers [6], [7], [8], [9]. Some other versions of PCGM can be found in Freng and Owen [3].

In the recent papers [10], [11], [9], sharp convergence estimates have been derived. A survey of results on preconditioned iterative methods is presented in the papers [6], [7], [9].

In the present paper, we propose a methodology for constructing and investigating preconditioned iterative methods for large-scale monotone nonlinear eigenvalue problems of the form: find $\lambda \in \Lambda$ and $u \in H \setminus \{0\}$ such that $A(\lambda)u = \lambda B(\lambda)u$, where $H$ is a real Euclidean space, $\Lambda$ is an interval on the real axis, $A(\mu)$ and $B(\mu)$ are large sparse symmetric positive definite matrices, $A(\mu)$ is ill-conditioned for fixed $\mu \in \Lambda$. Here we assume that the Rayleigh quotient $R(\mu, v) = (A(\mu)v, v)/(B(\mu), v, v)$, $\mu \in \Lambda$ is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e., $R(\mu, v) \geq R(\eta, v)$, $\mu < \eta$, $\mu, \eta \in \Lambda$, $v \in H \setminus \{0\}$. We consider the situation when matrices $A(\mu)$ and $B(\mu)$ can not be stored and routines for computing the matrix-vector products $A(\mu)v$ and $B(\mu)v$ are only available.

Monotone nonlinear matrix eigenvalue problems arise after the discretization of eigenvalue problems for differential and integral equations with nonlinear appearance of the spectral parameter. Note that monotone nonlinear eigenvalue problems with nonincreasing Rayleigh quotient have important applications in optical telecommunications and in integrated optics [23], [25] and in structural mechanics [1], [28], [29], [26].

For solving monotone nonlinear eigenvalue problems we suggest PSIM of the following kind:

$$\tilde{u}^{n+1} = u^n - \tau^n C^{-1}(\mu^n)(A(\mu^n) - \mu^n B(\mu^n))u^n,$$
$$u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_{B(\mu^{n+1})}}, \quad \mu^{n+1} = R(\mu^{n+1}, \tilde{u}^{n+1}), \quad n = 0, 1, \ldots,$$

where the symmetric positive definite matrix $C(\mu)$ is assumed to be easily inverted matrix satisfying the following condition: $\delta_0(\mu)(C(\mu)v,v) \le (A(\mu)v,v) \le \delta_1(\mu)(C(\mu)v,v)$, $v \in H \setminus \{0\}$, $\mu \in \Lambda$, the iteration parameter $\tau^n$ is defined by the formula $\tau^n = 2/(\delta_0(\mu^n) + \delta_1(\mu^n))$, $\|u\|^2_{B(\mu)} = (B(\mu)u,u)$. In this method for each $n \ge 1$ we minimize the Rayleigh quotient $R(\mu^n, v)$, $v \in H \setminus \{0\}$ and find the unique root of a scalar equation. In PSIM we move from a given iteration vector $u^n$ in the direction $-\mathrm{grad}_{C(\mu^n)} R(\mu^n, u^n)$.

The convergence of PSIM can be improved if we minimize the Rayleigh quotient in the subspace $V_{n+1} = \mathrm{span}\{u^n, w^n\}$ or $W_{n+1} = \mathrm{span}\{u^{n-1}, u^n, w^n\}$, $w^n = C^{-1}(\mu^n)(A(\mu^n) - \mu^n B(\mu^n))u^n$. The corresponding iterative methods for solving nonlinear eigenvalue problems are called PSDM and PCGM, respectively.

Simpler and slower variants of PSIM, PSDM, and PCGM for solving nonlinear eigenvalue problems have been studied in [27].

Our approach allows us to construct block variants of iterative methods for solving nonlinear eigenvalue problems [20], [22], [23], [24] and to treat a monotone dependence on the parameter of other forms (see, for example, [23]).

A survey on iterative methods for relatively small nonlinear matrix eigenvalue problems is given in [12], [4].

The present paper is organized as follows. In Section 2, we give the statement of a symmetric matrix eigenvalue problem with nonlinear occurrence of the spectral parameter. In Section 3, results about existence and properties of the eigenvalues of the nonlinear eigenvalue problem are proved. Similar results were obtained earlier in the papers [15], [16], [17], [18], [19]. In Section 4, we describe auxiliary results obtained in the paper [9]. These results are used further for constructing and investigating the iterative methods. In Sections 5, 6, and 7, we formulate the preconditioned iterative methods for the nonlinear eigenvalue problem, and we investigate the convergence and error of these methods for computing the smallest eigenvalue. In Section 8, we discuss numerical experiments for a model problem.

# 2 Formulation of the problem

Let $H$ be an $N$-dimensional real Euclidean space with the scalar product $(.,.)$ and the norm $\|.\|$, and let $\Lambda$ be an interval on the real axis $\mathbb{R}$, $\Lambda = (\alpha, \beta)$, $0 \le \alpha < \beta \le \infty$. Introduce continue matrix functions $A(\mu)$ and $B(\mu)$, $\mu \in \Lambda$. We assume that $A(\mu)$ and $B(\mu)$ are real symmetric positive definite $N$-by-$N$ matrices for fixed $\mu \in \Lambda$.

Define the Rayleigh quotient by the formula

$$R(\mu, v) = \frac{(A(\mu)v, v)}{(B(\mu)v, v)}, \quad v \in H \setminus \{0\}, \mu \in \Lambda.$$

Assume that the following conditions are fulfilled:

(a) the Rayleigh quotient $R(\mu, v)$, $\mu \in \Lambda$ is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e.,

$$R(\mu, v) \ge R(\eta, v), \quad \mu < \eta, \mu, \eta \in \Lambda, v \in H \setminus \{0\};$$

(b) there exists $\eta = \eta_{\min} \in \Lambda$ such that

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0;$$

(c) there exists $\eta = \eta_{\max} \in \Lambda$ such that

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0.$$

Consider the following nonlinear eigenvalue problem: find $\lambda \in \Lambda$ and $u \in H \setminus \{0\}$ such that

$$A(\lambda)u = \lambda B(\lambda)u. \tag{1}$$

The number $\lambda$ that satisfies (1) is called an eigenvalue, and the element $u$ is called an eigenelement of problem (1) corresponding to $\lambda$. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue $\lambda$ and the zero element is a closed subspace in $H$, which is called the eigensubspace corresponding to the eigenvalue $\lambda$. The dimension of this subspace is called the multiplicity of the eigenvalue $\lambda$.

# 3 Existence of the eigenvalues

For fixed $\mu \in \Lambda$, we introduce the auxiliary linear eigenvalue problem: find $\gamma(\mu) \in \mathbb{R}$ and $u \in H \setminus \{0\}$ such that

$$A(\mu)u = \gamma(\mu)B(\mu)u. \tag{2}$$

Problem (2) has $N$ real positive eigenvalues $0 < \gamma_1(\mu) \leq \gamma_2(\mu) \leq \ldots \leq \gamma_N(\mu)$ for fixed $\mu \in \Lambda$.

**Lemma 1** *The functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ are continuous nonincreasing functions with positive values.*

**Proof** The assertion follows from the continuity of the matrix functions $A(\mu)$ and $B(\mu)$, $\mu \in \Lambda$ and condition (a). □

**Lemma 2** *The functions $\mu - \gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points $\alpha$ and $\beta$, respectively.*

**Proof** The increase of the functions $\mu - \gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ follows from Lemma 1.

Taking into account condition (b), we obtain that there exists a number $\eta = \eta_{\min} \in \Lambda$, for which the following relationships are valid:

$$\mu - \gamma_i(\mu) < \eta - \gamma_i(\eta) \leq \eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0$$

for $\mu \in (\alpha, \eta)$, $i = 1, 2, \ldots, N$.

According to condition (c), there exists $\eta = \eta_{\max} \in \Lambda$ such that the following inequalities hold:

$$\mu - \gamma_i(\mu) > \eta - \gamma_i(\eta) \geq \eta - \gamma_N(\eta) = \eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0$$

for $\mu \in (\eta, \beta)$, $i = 1, 2, \ldots, N$. Thus, the lemma is proved. □

**Lemma 3** *A number $\lambda \in \Lambda$ is an eigenvalue of problem (1) if and only if the number $\lambda$ is a solution of an equation from the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

**Proof** If $\lambda$ is a solution of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$ for some $i$, $1 \leq i \leq N$, then it follows from (1) and (2) that $\lambda$ is an eigenvalue of problem (1). If $\lambda$ is an eigenvalue of problem (1), then (1) and (2) imply $\lambda - \gamma_i(\lambda) = 0$ for some $i$, $1 \leq i \leq N$. This proves the lemma. □

**Theorem 4** *Problem (1) has exactly $N$ eigenvalues $\lambda_i$, $i = 1, 2, \ldots, N$, which are repeated according to their multiplicity: $\alpha < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N < \beta$. Each eigenvalue $\lambda_i$ is a unique root of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

**Proof** By Lemma 2, each equation of the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ has a unique solution. Denote these solutions by $\lambda_i$, $i = 1, 2, \ldots, N$, i.e., $\lambda_i - \gamma_i(\lambda_i) = 0$, $i = 1, 2, \ldots, N$. To check that the numbers $\lambda_i$, $i = 1, 2, \ldots, N$ are put in a nondecreasing order, let us assume the opposite, i.e., $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 1, we obtain a contradiction, namely,

$$\lambda_i = \gamma_i(\lambda_i) \leq \gamma_i(\lambda_{i+1}) \leq \gamma_{i+1}(\lambda_{i+1}) = \lambda_{i+1}.$$

By Lemma 3, the numbers $\lambda_i$, $i = 1, 2, \ldots, N$ are eigenvalues of problem (1). Thus, the theorem is proved. □

**Remark 5** If $\alpha = 0$, then condition (b) follows from condition (a).

**Proof** Let us fix $\nu \in \Lambda$ and put $\eta = \min\{\gamma_1(\nu), \nu\}/2$. Taking into account condition (a), Lemma 1, and the relationships $\eta \leq \gamma_1(\nu)/2$, $\eta \leq \nu/2 < \nu$, we have

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_1(\eta) \leq \gamma_1(\nu)/2 - \gamma_1(\nu) = -\gamma_1(\nu)/2 < 0.$$

Thus, condition (b) is satisfied for chosen $\eta \in \Lambda$. □

**Remark 6** If $\beta = \infty$, then condition (c) follows from condition (a).

**Proof** For fixed $\nu \in \Lambda$, put $\eta = 2\max\{\gamma_N(\nu), \nu\}$. Since $\eta \geq 2\gamma_N(\nu)$ and $\eta \geq 2\nu > \nu$, according to condition (a) and Lemma 1, we obtain the relationships:

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_N(\eta) \geq 2\gamma_N(\nu) - \gamma_N(\nu) = \gamma_N(\nu) > 0,$$

which implies that condition (c) is satisfied. □

**Remark 7** We may write conditions (b) and (c) as the following conditions:

(b.1) there exists $\eta = \eta_{\min} \in \Lambda$ such that $\eta - \gamma_1(\eta) \leq 0$;

(c.1) there exists $\eta = \eta_{\max} \in \Lambda$ such that $\eta - \gamma_N(\eta) \geq 0$.

Conditions (b.1) and (c.1) imply the existence $N$ roots $\lambda_i$, $i = 1, 2, \ldots, N$ of the set of equations $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ (see Theorem 4).

We may change conditions (b.1) and (c.1) to the following conditions:

(b.2) there exists $\eta = \eta_{\min} \in \Lambda$ such that $\eta - \gamma_m(\eta) \leq 0$;

(c.2) there exists $\eta = \eta_{\max} \in \Lambda$ such that $\eta - \gamma_n(\eta) \geq 0$;

where $1 \leq m \leq n \leq N$.

Conditions (b.2) and (c.2) imply the existence of $n-m+1$ roots $\lambda_i$, $i = m, m+1, \ldots, n$ of the set of equations $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$. In this case, we obtain a new existence theorem instead of Theorem 4.

# 4 Auxiliary results

In this section we introduce one iteration step of the preconditioned simple iteration method for linear eigenvalue problem (2) with fixed parameter $\mu \in \Lambda$ and state recent convergence results. In the following sections we shall use these results for defining and investigating preconditioned iterative methods for solving nonlinear eigenvalue problem (1).

Assume that the symmetric positive definite $N$-by-$N$ matrix $C(\mu)$ is given for fixed $\mu \in \Lambda$, and there exist continuous functions $\delta_0(\mu)$, $\delta_1(\mu)$, $\mu \in \Lambda$, $0 < \delta_0(\mu) \leq \delta_1(\mu)$, $\mu \in \Lambda$ such that

$$\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v), \quad v \in H, \quad \mu \in \Lambda.$$

For a given element $v^0 \in H$, $\|v^0\|_{B(\mu)} = 1$, we define an element $v^1 \in H$ and numbers $\nu^0$ and $\nu^1$ by the formulae:

$$\begin{aligned}
&\tilde{v}^1 = v^0 - \tau^0 w^0, \quad \tau^0 = 2/(\delta_0(\mu) + \delta_1(\mu)),\\
&w^0 = C(\mu)^{-1}(A(\mu) - \nu^0 B(\mu))v^0,\\
&v^1 = \frac{\tilde{v}^1}{\|\tilde{v}^1\|_{B(\mu)}},\\
&\nu^0 = R(\mu, v^0), \quad \nu^1 = R(\mu, v^1),
\end{aligned}$$

for fixed $\mu \in \Lambda$, $\|u\|^2_{B(\mu)} = (B(\mu)u, u)$.

**Lemma 8** *Let $\gamma_1(\mu)$ and $\gamma_2(\mu)$ be eigenvalues of problem (2) with $\mu \in \Lambda$ such that $\gamma_1(\mu) < \gamma_2(\mu)$. Suppose that $\nu^0 < \gamma_2(\mu)$. Then $\gamma_1(\mu) \le \nu^1 \le \nu^0$ and the following estimate is valid*

$$\frac{\nu^1 - \gamma_1(\mu)}{\gamma_2(\mu) - \nu^1} \le \rho^2(\mu) \frac{\nu^0 - \gamma_1(\mu)}{\gamma_2(\mu) - \nu^0},$$

*where* $0 < \rho(\mu) < 1$,

$$\begin{aligned} &\rho(\mu) = 1 - (1 - \xi(\mu))(1 - \gamma_1(\mu)/\gamma_2(\mu)), \\ &\xi(\mu) = (1 - \delta(\mu))/(1 + \delta(\mu)), \\ &\delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \mu \in \Lambda. \end{aligned}$$

**Proof** The assertion of the lemma is proved in [9]. □

**Remark 9** Suppose that $\nu^1$ and $v^1$ are calculated by using one step of PSDM or PCGM. Then results of Lemma 8 are valid.

# 5 Preconditioned iterative methods

Let us consider the following iterative methods for solving nonlinear eigenvalue problem (1).

**Method 1. PSIM: Preconditioned Simple Iteration Method.**
(1) *Select $\mu^0$ and $u^0$ such that $\mu^0 = R(\mu^0, u^0)$, $\|u^0\|_{B(\mu^0)} = 1$.*
(2) *For $n = 0, 1, \ldots$ do:*
*Compute $\mu^{n+1}$ and $u^{n+1}$ such that*

$$\begin{aligned} &\mu^{n+1} = R(\mu^{n+1}, u^{n+1}) = R(\mu^{n+1}, v^{n+1}), \quad \|u^{n+1}\|_{B(\mu^{n+1})} = 1, \\ &v^{n+1} = u^n - \tau^n w^n, \quad \tau^n = 2/(\delta_0(\mu^n) + \delta_1(\mu^n)), \\ &w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n. \end{aligned}$$

**Method 2. PSDM: Preconditioned Steepest Descent Method.**
(1) *Select $\mu^0$ and $u^0$ such that $\mu^0 = R(\mu^0, u^0)$, $\|u^0\|_{B(\mu^0)} = 1$.*
(2) *For $n = 0, 1, \ldots$ do:*
*Compute $\mu^{n+1}$ and $u^{n+1}$ such that*

$$\begin{aligned} &\mu^{n+1} = R(\mu^{n+1}, u^{n+1}) = \min_{v \in V_{n+1} \setminus \{0\}} R(\mu^{n+1}, v), \quad \|u^{n+1}\|_{B(\mu^{n+1})} = 1, \\ &V_{n+1} = \mathrm{span}\{u^n, w^n\}, \quad w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n. \end{aligned}$$

**Method 3. PCGM: Preconditioned Conjugate Gradient Method.**
(1) *Select* $\mu^0$ *and* $u^0$ *such that* $\mu^0 = R(\mu^0, u^0)$, $\|u^0\|_{B(\mu^0)} = 1$.
(2) *Compute* $\mu^1$ *and* $u^1$ *such that*

$$\mu^1 = R(\mu^1, u^1) = \min_{v \in V_1 \setminus \{0\}} R(\mu^1, v), \quad \|u^1\|_{B(\mu^1)} = 1,$$
$$V_1 = \mathrm{span}\{u^0, w^0\}, \quad w^0 = C(\mu^0)^{-1}(A(\mu^0) - \mu^0 B(\mu^0))u^0.$$

(3) *For* $n = 1, 2, \ldots$ *do:*
*Compute* $\mu^{n+1}$ *and* $u^{n+1}$ *such that*

$$\mu^{n+1} = R(\mu^{n+1}, u^{n+1}) = \min_{v \in W_{n+1} \setminus \{0\}} R(\mu^{n+1}, v), \quad \|u^{n+1}\|_{B(\mu^{n+1})} = 1,$$
$$W_{n+1} = \mathrm{span}\{u^{n-1}, u^n, w^n\}, \quad w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n.$$

**Remark 10** By Lemma 12 we obtain that the equation $\mu^n = R(\mu^n, u^n)$ has a unique solution $\mu^n$ for fixed $u^n$. Hence, according to [20], we conclude that the sequence $\mu^n$, $n = 0, 1, \ldots$ is uniquely defined in PSIM, PSDM, and PCGM.

# 6 Convergence of iterative methods

In this section we study the convergence of the methods PSIM, PSDM, and PCGM introduced in Section 5. Assume that the sequences $\mu^n$, $u^n$, $n = 0, 1, \ldots$ are computed by one of these methods. We start with investigating properties of the functions $\varphi_n(\mu) = R(\mu, u^n)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$ and the function $\rho(\mu)$, $\mu \in \Lambda$.

**Lemma 11** *The functions* $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$ *are continuous nonincreasing functions with positive values. In addition, the following inequalities are valid:*

$$\gamma_1(\mu) \le \varphi_n(\mu) \le \gamma_N(\mu),$$

$\mu \in \Lambda$, $n = 0, 1, \ldots$

**Proof** The proof follows from the definition of the Rayleigh quotient and the minimax principle for eigenvalues $\gamma_1(\mu)$ and $\gamma_N(\mu)$. □

**Lemma 12** *The functions* $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$ *are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points* $\alpha$ *and* $\beta$, *respectively.*

**Proof** The increase of the functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ follows from the condition (a).

Taking into account condition (b), we obtain that there exists a number $\eta = \eta_{\min} \in \Lambda$, for which the following relationships are valid

$$\mu - \varphi_n(\mu) < \eta - \varphi_n(\eta) \le \eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \le 0$$

for $\mu \in (\alpha, \eta)$, $n = 0, 1, \ldots$

According to condition (c), there exists $\eta = \eta_{\max} \in \Lambda$ such that the following inequalities hold

$$\mu - \varphi_n(\mu) > \eta - \varphi_n(\eta) \ge \eta - \gamma_N(\eta) = \eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \ge 0$$

for $\mu \in (\eta, \beta)$, $n = 0, 1, \ldots$ Thus, the lemma is proved. □

Let $\lambda_1$ and $\lambda_2$ be eigenvalues of problem (1) such that $\lambda_1 < \lambda_2$. Put

$$\begin{aligned} &\rho_0 = 1 - (1 - \xi_0)(1 - \lambda_1/\lambda_2), \\ &\xi_0 = (1 - d)/(1 + d), \\ &d = \min_{\mu \in [\lambda_1, \lambda_2]} \delta(\mu), \quad \delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \mu \in \Lambda. \end{aligned} \tag{3}$$

Note that $0 < d \le 1$, $0 < \rho_0 < 1$.

**Lemma 13** *The half-open interval* $[\lambda_1, \lambda_2)$ *is contained in the half-open interval* $[\gamma_1(\mu), \gamma_2(\mu))$ *for any* $\mu \in [\lambda_1, \lambda_2)$.

**Proof** Taking into account Lemma 1, we get $\gamma_1(\mu) \le \lambda_1$ and $\gamma_2(\mu) > \lambda_2$ for $\mu \in [\lambda_1, \lambda_2)$. These inequalities prove the lemma. □

**Lemma 14** *The following inequality holds:* $\rho(\mu) \le \rho_0$ *for* $\mu \in [\lambda_1, \lambda_2)$.

**Proof** The relationships $\gamma_1(\mu) \le \lambda_1$, $\gamma_2(\mu) > \lambda_2$, $\mu \in [\lambda_1, \lambda_2)$ imply the desired inequality

$$\begin{aligned} \rho(\mu) &= 1 - (1 - \xi(\mu))(1 - \gamma_1(\mu)/\gamma_2(\mu)) \le \\ &\le 1 - (1 - \xi_0)(1 - \lambda_1/\lambda_2) = \rho_0 \end{aligned}$$

for $\mu \in [\lambda_1, \lambda_2)$. Thus, the lemma is proved. □

**Lemma 15** *Let* $\lambda_1 < \mu^{n+1} \le \mu^n < \lambda_2$. *Then the following inequality holds*

$$\frac{\mu^{n+1} - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} \le s_n \frac{\mu^{n+1} - \varphi_{n+1}(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} + \rho_0^2 \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n},$$

*where*

$$s_n = \frac{\gamma_2(\mu^n) - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}}.$$

**Proof** We first write the equality

$$\frac{\mu^{n+1}-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\mu^{n+1}}=\left[\frac{\mu^{n+1}-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\mu^{n+1}}-\frac{\varphi_{n+1}(\mu^n)-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\varphi_{n+1}(\mu^n)}\right]+\frac{\varphi_{n+1}(\mu^n)-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\varphi_{n+1}(\mu^n)}. \quad (4)$$

Note that the conditions $\lambda_1<\mu^{n+1}\le\mu^n<\lambda_2$ imply that $\gamma_1(\mu^n)\le\mu^n<\gamma_2(\mu^n)$, $\gamma_2(\mu^n)-\mu^n>0$, $\gamma_2(\mu^n)-\mu^{n+1}>0$, $\gamma_2(\mu^n)-\varphi_{n+1}(\mu^n)>0$. Therefore, by Lemmata 8, 13, and 14, for second term in the right hand side of (4) we have

$$\frac{\varphi_{n+1}(\mu^n)-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\varphi_{n+1}(\mu^n)}\le\rho_0^2\,\frac{\mu^n-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\mu^n}.$$

To estimate first term in the right hand side of (4) we consider the function $\psi(x)=(x-a_1)/(a_2-x)$, $x\in(a_1,a_2)$ for fixed $a_1$ and $a_2$, $a_1<a_2$. We obtain $\psi'(x)=(a_2-a_1)/(a_2-x)^2$, $x\in(a_1,a_2)$ and hence there exists $c\in(a,b)$ for $a_1\le a<b\le a_2$ such that $\psi(b)-\psi(a)=\psi'(c)(b-a)\le\psi'(b)(b-a)$.

Now setting $a_1=\gamma_1(\mu^n)$, $a_2=\gamma_2(\mu^n)$, $a=\varphi_{n+1}(\mu^n)$, $b=\mu^{n+1}$, where $b=\mu^{n+1}=\varphi_{n+1}(\mu^{n+1})\ge\varphi_{n+1}(\mu^n)=a$, and using the property of $\psi(x)$, we derive the inequality

$$\frac{\mu^{n+1}-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\mu^{n+1}}-\frac{\varphi_{n+1}(\mu^n)-\gamma_1(\mu^n)}{\gamma_2(\mu^n)-\varphi_{n+1}(\mu^n)}\le s_n\frac{\mu^{n+1}-\varphi_{n+1}(\mu^n)}{\gamma_2(\mu^n)-\mu^{n+1}},$$

which completes the proof of the lemma. □

We are now in a position to formulate the convergence result.

**Theorem 16** *Let $\lambda_1$ and $\lambda_2$ be eigenvalues of problem* (1) *such that $\lambda_1<\lambda_2$. Suppose that the sequence $\mu^n$, $n=0,1,\ldots$ is calculated by one of the iterative methods* PSIM, PSDM, *or* PCGM *introduced in Section* 5, *and $\mu^0<\lambda_2$. Then $\mu^n\to\lambda_1$ as $n\to\infty$ and the following inequalities are valid*

$$\lambda_2>\mu^0\ge\mu^1\ge\ldots\ge\mu^n\ge\ldots\ge\lambda_1. \quad (5)$$

**Proof** Let us show that solutions $\mu^n$, $n=0,1,\ldots$ of the equations $\mu-\varphi_n(\mu)=0$, $\mu\in\Lambda$, $n=0,1,\ldots$ satisfy inequalities (5). Assume that the equation $\mu-\varphi_n(\mu)=0$, $\mu\in\Lambda$ has the solution $\mu^n$ such that

$$\lambda_2>\mu^0\ge\mu^1\ge\ldots\ge\mu^n\ge\lambda_1,\quad n\ge0.$$

Therefore we obtain

$$\gamma_1(\mu^n)\le\nu^0=\varphi_n(\mu^n)=\mu^n<\lambda_2=\gamma_2(\lambda_2)\le\gamma_2(\mu^n).$$

Consequently, by Lemmata 8 and 13, we have

$$\nu^1=\varphi_{n+1}(\mu^n)\le\nu^0=\varphi_n(\mu^n)=\mu^n.$$

It follows from Lemmata 11 and 12 that the equation $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$ has a unique solution $\mu^{n+1}$ and

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \mu^{n+1} \geq \lambda_1.$$

Let us prove that $\mu^n \to \lambda_1$ as $n \to \infty$. By (5) there exists $\xi \in [\lambda_1, \lambda_2)$ such that $\mu^n \to \xi$ as $n \to \infty$. Now from the condition $(B(\mu)v, v) \geq \beta_1(\mu)\|v\|^2, v \in H, \mu \in \Lambda$ with a continuous function $\beta_1(\mu)$, $\mu \in \Lambda$ and from the relationships $\|u^n\|_{B(\mu^n)} = 1, n = 0, 1, \ldots$, we obtain that there exists a constant $c > 0$ such that

$$\|u^n\| \leq \frac{\|u^n\|_{B(\mu^n)}}{\sqrt{\beta_1(\mu^n)}} = \frac{1}{\sqrt{\beta_1(\mu^n)}} \leq c, \quad n = 0, 1, \ldots,$$

$$c = \max_{\mu \in [\lambda_1, \lambda_2]} \frac{1}{\sqrt{\beta_1(\mu)}}.$$

Hence there exists an element $w \in H$ and a subsequence $u^{n_i+1}$, $i = 1, 2, \ldots$ such that $u^{n_i+1} \to w$ as $i \to \infty$.

In order to prove that $\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) \to 0$ as $i \to \infty$, we write

$$0 \leq \mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) = R(\mu^{n_i+1}, u^{n_i+1}) - R(\mu^{n_i}, u^{n_i+1}) \to 0$$

as $i \to \infty$, where we have taken into account that

$$R(\mu^{n_i+1}, u^{n_i+1}) \to R(\xi, w), \quad R(\mu^{n_i}, u^{n_i+1}) \to R(\xi, w)$$

as $i \to \infty$.

Using Lemma 15 we write the relationships

$$\frac{\mu^{n_i+1} - \gamma_1(\mu^{n_i})}{\gamma_2(\mu^{n_i}) - \mu^{n_i+1}} \leq s_n \frac{\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i})}{\gamma_2(\mu^{n_i}) - \mu^{n_i+1}} + \rho_0^2 \frac{\mu^{n_i} - \gamma_1(\mu^{n_i})}{\gamma_2(\mu^{n_i}) - \mu^{n_i}},$$

as $i \to \infty$, from which we get

$$0 \leq \xi - \gamma_1(\xi) \leq \rho_0^2(\xi - \gamma_1(\xi)),$$

where $0 < \rho_0 < 1$. Hence the number $\xi \in [\lambda_1, \lambda_2)$ satisfies the equation $\xi - \gamma_1(\xi) = 0$, i.e., $\xi = \lambda_1$ is an eigenvalue of problem (1) and $\mu^n \to \lambda_1$ as $n \to \infty$. This completes the proof of the theorem. □

# 7 Error estimates of iterative methods

Assume that there exists a positive number $r_0$ such that

$$|R(\mu, v) - R(\eta, v)| \leq r_0 |\mu - \eta| R(\nu, v) \tag{6}$$

for $\mu, \eta \in (\lambda_1, \beta)$, $\nu = \min\{\mu, \eta\}$, $v \in H \setminus \{0\}$.

The next theorems state the main results of the paper.

**Theorem 17** *Let $\lambda_1$ and $\lambda_2$ be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Assume that the sequence $\mu^n$, $n = 0, 1, \ldots$ is calculated by one of the iterative methods* PSIM, PSDM, *or* PCGM *defined in Section* 5, *$\mu^0 < \lambda_2$. Then the following estimate is valid*

$$\frac{\mu^{n+1} - \gamma_1(\mu^{n+1})}{\gamma_2(\mu^{n+1}) - \mu^{n+1}} \le q_n \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n},$$

*where $n = 0, 1, \ldots$, $q_n \le q^* < 1$,*

$$q_n = \frac{\rho_0^2 + r_0\, s_n\, \mu^n}{1 + r_0\, s_n\, \mu^n}, \quad q^* = \frac{\rho_0^2 + r_0\, s^*\, \mu^0}{1 + r_0\, s^*\, \mu^0}, \quad s^* = \frac{\gamma_2(\lambda_1) - \gamma_1(\lambda_2)}{\gamma_2(\mu^0) - \mu^0}.$$

**Proof** According to (6), we write the following relationships

$$\begin{aligned}
\mu^{n+1} - \varphi_{n+1}(\mu^n) &= \varphi_{n+1}(\mu^{n+1}) - \varphi_{n+1}(\mu^n) = \\
&= R(\mu^{n+1}, u^{n+1}) - R(\mu^n, u^{n+1}) \le \\
&\le r_0\,(\mu^n - \mu^{n+1}) R(\mu^{n+1}, u^{n+1}) = \\
&= r_0\,(\mu^n - \mu^{n+1})\,\mu^{n+1} \le \\
&\le r_0\,(\mu^n - \mu^{n+1})\,\mu^n.
\end{aligned}$$

Therefore by Lemma 15 we obtain

$$\frac{\mu^{n+1} - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} \le \frac{r_0\, s_n\,(\mu^n - \mu^{n+1})\,\mu^n}{\gamma_2(\mu^n) - \mu^{n+1}} + \rho_0^2 \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n}$$

and

$$\begin{aligned}
(1 + r_0\, s_n\, \mu^n) \frac{\mu^{n+1} - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} &\le \frac{r_0\, s_n\, (\mu^n)^2}{\gamma_2(\mu^n) - \mu^{n+1}} - \frac{r_0\, s_n\, \mu^n \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} + \rho_0^2 \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n} \le \\
&\le (\rho_0^2 + r_0\, s_n\, \mu^n) \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n}.
\end{aligned}$$

Hence

$$\frac{\mu^{n+1} - \gamma_1(\mu^{n+1})}{\gamma_2(\mu^{n+1}) - \mu^{n+1}} \le \frac{\mu^{n+1} - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^{n+1}} \le q_n \frac{\mu^n - \gamma_1(\mu^n)}{\gamma_2(\mu^n) - \mu^n}. \tag{7}$$

Introducing the function $f(x) = (t + cx)/(1 + cx)$, $x \in (0, \infty)$ for fixed $c > 0$, $0 < t < 1$, we have $f(x) < 1$, $f'(x) = c(1 - t)/(1 + cx)^2 > 0$, $x \in (0, \infty)$. Using properties of $f(x)$, we conclude $q_n \le q^* < 1$. This proves the theorem. □

**Theorem 18** *Let $\lambda_1$ and $\lambda_2$ be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Assume that the sequence $\mu^n$, $n = 0, 1, \ldots$ is calculated by one of the iterative methods* PSIM, PSDM, *or* PCGM *described in Section* 5, *$\mu^0 < \lambda_2$. Then the following estimate is valid*

$$\frac{\mu^{n+1} - \lambda_1}{\lambda_2 - \mu^{n+1}} \le q_n \frac{\mu^n - \lambda_1}{\lambda_2 - \mu^n},$$

*where $n = 0, 1, \ldots$, $q_n \le q^* < 1$, $q_n$ and $q^*$ are defined in* Theorem 17.

**Proof** Introducing the function $g(x) = (a-x)/(b-x)$ for fixed $a$ and $b$, we have $g'(x) = (a-b)/(b-x)^2$. Using properties of $g(x)$ and (7), we get the inequalities

$$\frac{\mu^{n+1}-\lambda_1}{\mu^n-\lambda_1} \le \frac{\mu^{n+1}-\gamma_1(\mu^n)}{\mu^n-\gamma_1(\mu^n)}, \quad \frac{\mu^n-\lambda_2}{\mu^{n+1}-\lambda_2} \le \frac{\mu^n-\gamma_2(\mu^n)}{\mu^{n+1}-\gamma_2(\mu^n)},$$

which imply the desired estimate. Thus, the theorem is proved. □

**Remark 19** Theorem 18 implies the estimate

$$\mu^n - \lambda_1 \le c_0 (q^*)^n,$$

where $q^* < 1$ are defined in Theorem 17, $c_0 = (\lambda_2 - \lambda_1)\,(\mu_0 - \lambda_1)/(\lambda_2 - \mu_0)$.

**Remark 20** Suppose that there exist $r_1$ and $r_2$ such that

$$|((A(\mu) - A(\eta))v, v)| \le r_1\,|\mu - \eta|\,(A(\nu)v, v),$$
$$|((B(\mu) - B(\eta))v, v)| \le r_2\,|\mu - \eta|\,(B(\nu)v, v),$$

for $\mu, \eta \in (\lambda_1, \beta)$, $\nu = \min\{\mu, \eta\}$, $v \in H \setminus \{0\}$. Then (6) is valid with $r_0 = r_1 + r_2$.

# 8 Numerical experiments

Consider the following model differential eigenvalue problem: find numbers $\lambda \in \Lambda$ and nontrivial functions $u(x)$, $x \in [0,1]$ such that

$$\begin{aligned} &-u''(x) = \lambda u(x), \quad x \in (0,1), \\ &u(0) = 0, \quad -u'(1) = \varphi(\lambda)\,u(1), \end{aligned} \tag{8}$$

where $\Lambda = (\varkappa, \infty)$, $\varphi(\mu) = \mu\varkappa M/(\mu - \varkappa)$, $\mu \in \Lambda$, $\varkappa = K/M$, $K$ and $M$ are given positive numbers. Differential equations (8) describe eigenvibrations of a string with a load of mass $M$ attached by an elastic spring of stiffness $K$.

Investigations of this section can be easily generalized for cases of more complicated and important problems on eigenvibrations of mechanical structures (beams, plates, shells) with elastically attached loads [1], [28], [15], [29], [26].

We denote by $\mathcal{H} = L_2(0,1)$ and $\mathcal{V} = \{v : v \in W_2^1(0,1), v(0) = 0\}$ the Lebesgue and Sobolev spaces equipped with the norms

$$|u|_0 = \left(\int_0^1 u^2\,dx\right)^{1/2}, \quad |u|_1 = \left(\int_0^1 (u')^2\,dx\right)^{1/2}.$$

Note that the space $\mathcal{V}$ is compactly embedded into the space $\mathcal{H}$, any function from $\mathcal{V}$ is continuous on $[0,1]$. The semi-norm $|.|_1$ is a norm over the space $\mathcal{V}$, which is equivalent to the usual norm $\|.\|_1$, $\|.\|_1^2 = |.|_0^2 + |.|_1^2$.

Define the bilinear forms

$$a(u,v) = \int_0^1 u'v'dx, \quad u,v \in \mathcal{V}, \quad b(u,v) = \int_0^1 uv\,dx, \quad u,v \in \mathcal{H},$$
$$c(u,v) = u(1)v(1), \quad u,v \in \mathcal{V}.$$

The variational formulation of differential problem (8) has the following form: find $\lambda \in \Lambda$ and $u \in \mathcal{V} \setminus \{0\}$ such that

$$a(u,v) + \varphi(\lambda)\, c(u,v) = \lambda b(u,v) \quad \forall v \in \mathcal{V}. \tag{9}$$

To approximate problem (9), we define the partition of the interval $[0,1]$ by the nodes $x_i = ih$, $i = 0,1,\ldots,N$, $h = 1/N$. The finite-element space $\mathcal{V}_h$ is the space of continuous functions on $[0,1]$ that are linear on each interval $(x_{k-1}, x_k)$, $k = 1,2,\ldots,N$, and $\mathcal{V}_h$ is subspace of the space $\mathcal{V}$. Problem (9) is approximated by the following discrete problem: find $\lambda^h \in \Lambda$ and $u^h \in \mathcal{V}_h \setminus \{0\}$ such that

$$a(u^h, v^h) + \varphi(\lambda^h)\, c(u^h, v^h) = \lambda^h b(u^h, v^h) \quad \forall v^h \in \mathcal{V}_h. \tag{10}$$

Note that the following error estimate is valid $0 \le \lambda^h - \lambda \le \tilde{c}(\lambda)h^2\lambda^2$, where $\lambda^h$ is a sequence of eigenvalues of problem (10) converging to an eigenvalue $\lambda$ of problem (8) as $h \to 0$ [19].

Let $H$ be the real Euclidean space of vectors $y = (y_1, y_2, \ldots, y_N)^\top$ with the scalar product $(y,z) = \sum_{i=1}^N y_i z_i$, $y, z \in H$. Discrete problem (10) is equivalent to the following matrix eigenvalue problem: find $\lambda \in \Lambda$ and $y \in H \setminus \{0\}$ such that

$$A(\lambda)y = \lambda B y, \tag{11}$$

where $A(\mu) = A_0 + \varphi(\mu)C_0$, $\mu \in \Lambda$, a square matrix $C_0$ of order $N$ has zero coefficients $c^0_{ij}$ except the coefficient $c^0_{NN} = 1$, $A_0 = \mathcal{M}(a_1, a_2)$, $B = \mathcal{M}(b_1, b_2)$, $a_1 = 2/h$, $a_2 = -1/h$, $b_1 = 4h/6$, $b_2 = h/6$, $\mathcal{M}(c_1, c_2)$ is the square matrix of order $N$ defined by the formula

$$\mathcal{M}(c_1,c_2) = \begin{pmatrix} c_1 & c_2 & & & \\ c_2 & c_1 & c_2 & & \\ & & \ldots & & \\ & & c_2 & c_1 & c_2 \\ & & & c_2 & c_1/2 \end{pmatrix}.$$

We can define the exact eigenvalues of problem (11) as numbers $\lambda \in \Lambda$, $\lambda = \psi(\sigma)$,

$$\psi(\sigma) = \frac{2a_2 \cos\,\sigma h + a_1}{2b_2 \cos\,\sigma h + b_1},$$

where numbers $\sigma$ are solutions of the following equation (see, for example, [13]):

$$\frac{\tan\sigma}{\sin\,\sigma h} = \frac{a_2 - \psi(\sigma)b_2}{\varphi(\psi(\sigma))}. \tag{12}$$

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\lambda_k$ | 4.482176546 | 24.223573113 | 63.723821142 | 123.031221068 | 202.200899143 |

Table 1: The five smallest eigenvalues

Let $M = 1$, $K = 1$, $\varkappa = 1$. The five smallest eigenvalues $\lambda_i$, $i = 1, 2, 3, 4, 5$ of problem (11) for $N = 100$ and $h = 0.01$ are given in Table 1. These eigenvalues were calculated by using equation (12).

Note that condition (c) is satisfied according to Remark 6. Condition (b) follows from the relationships

$$\eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0$$

for $\eta \in (\varkappa, \lambda_1)$.

Using the inequality $|v(1)| \leq |v|_1$, $v \in \mathcal{V}$, we obtain

$$\alpha_1 |v|_1^2 = a(v, v) \leq a(v, v) + \varphi(\mu) c(v, v) \leq \alpha_2(\mu) |v|_1^2, \quad v \in \mathcal{V},$$

where $\alpha_1 = 1$, $\alpha_2(\mu) = 1 + \varphi(\mu)$, $\mu \in \Lambda$. Hence we have the inequalities

$$\delta_0 (A_0 v, v) \leq (A(\mu) v, v) \leq \delta_1(\mu) (A_0 v, v), \quad v \in H$$

for $\delta_0 = 1$, $\delta_1(\mu) = 1 + \varphi(\mu)$, $\mu \in \Lambda$.

To solve problem (11), we set $C = A_0$ and apply the iterative methods PSIM, PSDM, and PCGM described in Section 5. Figure 1 illustrates the convergence of methods PSIM, PSDM, and PCGM for the initial vector $\tilde{u}^0 = (\tilde{u}_1^0, \tilde{u}_2^0, \ldots, \tilde{u}_N^0)^\top$, $\tilde{u}_i^0 = \sin(\alpha \pi x_i)$, $x_i = ih$, $i = 1, 2, \ldots, N$, $h = 1/N$, $\alpha = 0.9$. We show graphically in Figure 1 the error $\mu^n - \lambda_1$ as a function of the iteration number $n$ for each method.

It is not difficult to verify that condition (6) holds for $r_0 = 1/(\lambda_1 - \varkappa)$. Hence convergence results of Section 7 are valid with above $r_0$. These results can be easily obtained for nonlinear eigenvalue problems on eigenvibrations of beams, plates, shells, [1], [28], [15], [29], [26] since the discretization of these problems leads to the matrix eigenvalue problem similar to (11).

# 9 Conclusion

This paper presents a new methodology for constructing and investigating efficient preconditioned iterative methods for numerical solution of large-scale monotone nonlinear eigenvalue problems. Theoretical analysis and numerical experiments show that proposed methods for *nonlinear* eigenvalue problems describing natural oscillations of mechanical structures with elastically attached loads are approximately as efficient as the analogous methods for solving *linear* eigenvalue problems describing natural oscillations of these mechanical structures without loads.

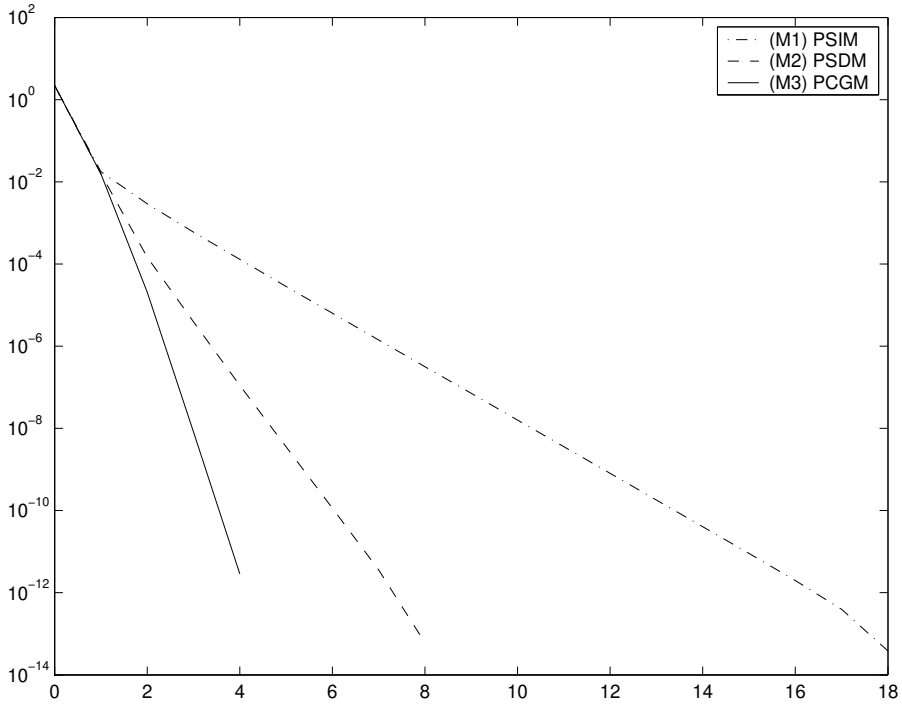


Figure 1: Error of methods PSIM, PSDM, and PCGM

**Acknowledgement.** The work of the author was supported by Alexander von Humboldt – Stiftung (Alexander von Humboldt Foundation) and Deutsche Forschungsgemeinschaft (German Research Foundation), Sonderforschungsbereich (Collaborative Research Center) 393.

# References

[1] L. V. Andreev, A. L. Dyshko, and I. D. Pavlenko. *Dynamics of plates and shells with concentrated masses.* Mashinostroenie, Moscow, 1988. In Russian.

[2] E. G. D'yakonov and M. Yu. Orekhov. Minimization of the computational labor in determining the first eigenvalues of differential operators. *Math. Notes*, 27:382–391, 1980.

[3] Y. T. Feng and D. R. J. Owen. Conjugate gradient methods for solving the smallest eigenpair of large symmetric eigenvalue problems. *Internat. J. Numer. Methods Engrg.*, 39:2209–2229, 1996.

[4] A. V. Goolin and S. V. Kartyshov. Numerical study of stability and nonlinear eigenvalue problems. *Surv. Math. Ind.*, 3:29–48, 1993.

[5] A. V. Knyazev. A preconditioned conjugate gradient method for eigenvalue problems and its implementation in a subspace. In *International Ser. Numerical Mathematics, v.* 96, *Eigenwertaufgaben in Natur- und Ingenieurwissenschaften und ihre numerische Behandlung, Oberwolfach*, 1990, pages 143–154, Birkhauser, Basel, 1991.

[6] A. V. Knyazev. Preconditioned eigensolvers—an oxymoron? *Electron. Trans. Numer. Anal.*, 7:104–123, 1998.

[7] A. V. Knyazev. Preconditioned eigensolvers (Section 11.3). In Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst, editors, *Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide*, pages 352–368, SIAM, Philadelphia, 2000.

[8] A. V. Knyazev. Toward the optimal preconditioned eigensolver: locally optimal block preconditioned conjugate gradient method. *SIAM J. Sci. Comput.*, 23:517–541, 2001.

[9] A. V. Knyazev and K. Neymeyr. A geometric theory for preconditioned inverse iteration. III: A short and sharp convergence estimate for generalized eigenvalue problems. *Linear Algebra Appl.*, 358:95–114, 2003.

[10] K. Neymeyr. A geometric theory for preconditioned inverse iteration. I: Extrema of the Rayleigh quotient. *Linear Algebra Appl.*, 322:61–85, 2001.

[11] K. Neymeyr. A geometric theory for preconditioned inverse iteration. II: Convergence estimates. *Linear Algebra Appl.*, 322:87–104, 2001.

[12] A. Ruhe. Algorithms for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 10:674–689, 1973.

[13] A. A. Samarskiĭ. *Theory of difference schemes.* Nauka, Moscow, 1983. In Russian.

[14] B. A. Samokish. The method of steepest descent in the problem of eigenelements of semi-bounded operators. *Izv. Vyssh. Uchebn. Zaved. Matematika*, 5:105–114, 1958. In Russian.

[15] S. I. Solov′ëv. The finite element method for symmetric eigenvalue problems with nonlinear occurrence of the spectral parameter. *PhD thesis*, Kazan State University, Kazan, 1990. In Russian.

[16] S. I. Solov′ëv. Error of the Bubnov-Galerkin method with perturbations for symmetric spectral problems with nonlinear entrance of the parameter. *Comput. Math. Math. Phys.*, 32:579–593, 1992.

[17] S. I. Solov′ëv. Approximation of the symmetric spectral problems with nonlinear dependence on a parameter. *Russ. Math.*, 37:59–67, 1993.

[18] S. I. Solov′ëv. Error bounds of finite element method for symmetric spectral problems with nonlinear dependence on parameter. *Russ. Math.*, 38:69–76, 1994.

[19] S. I. Solov′ëv. The finite element method for symmetric nonlinear eigenvalue problems. *Comput. Math. Math. Phys.*, 37:1269–1276, 1997.

[20] S. I. Solov′ëv. Convergence of modified subspace iteration method for nonlinear eigenvalue problems. Preprint SFB393/99-35, TU Chemnitz, 1999.

[21] S. I. Solov′ëv. Preconditioned gradient iterative methods for nonlinear eigenvalue problems. Preprint SFB393/00-28, TU Chemnitz, 2000.

[22] S. I. Solov′ëv. Block iterative methods for nonlinear eigenvalue problems. In E. I. Levanov, A. D. Lyashko, M. Ph. Pavlova, and B. N. Chetverushkin, editors, *Theory of mesh methods for nonlinear boundary value problems*, Proceeding of Russia workshop, pages 106–108, Kazan Mathematical Society, Kazan, 2000. In Russian.

[23] S. I. Solov′ëv. Finite element approximation of a nonlinear eigenvalue problem for an integral equation. In E. I. Levanov, A. D. Lyashko, M. Ph. Pavlova, and B. N. Chetverushkin, editors, *Theory of mesh methods for nonlinear boundary value problems*, Proceeding of Russia workshop, pages 108–111, Kazan Mathematical Society, Kazan, 2000. In Russian.

[24] S. I. Solov′ëv. Iterative methods for solving nonlinear eigenvalue problems. In A. A. Arzamastsev, editor, *Computer and mathematical modelling in natural and technical sciences*, Russia Internet conference, pages 36–37, Tambov State University, Tambov, 2001. In Russian.

[25] S. I. Solov′ëv. Existence of the guided modes of an optical fiber. Preprint SFB393/03-02, TU Chemnitz, 2003.

[26] S. I. Solov′ëv. Eigenvibrations of a plate with elastically attached load. Preprint SFB393/03-06, TU Chemnitz, 2003.

[27] S. I. Solov′ëv. Preconditioned iterative methods for monotone nonlinear eigenvalue problems. Preprint SFB393/03-08, TU Chemnitz, 2003.

[28] Yu. P. Zhigalko, A. D. Lyashko, and S. I. Solov′ëv. Vibrations of a cylindrical shell with joined rigid annular elements. *Model. Mekh.*, 2:68–85, 1988. In Russian.

[29] Yu. P. Zhigalko and S. I. Solov′ëv. Natural oscillations of a beam with a harmonic oscillator. *Russ. Math.*, 45:33–35, 2001.