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Numerische Simulation auf massiv parallelen Rechnern

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Preconditioned iterative methods for monotone nonlinear eigenvalue problems

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Abstract This paper proposes new iterative methods for the efficient computation of the smallest eigenvalue of symmetric nonlinear matrix eigenvalue problems of large order with a monotone dependence on the spectral parameter. Monotone nonlinear eigenvalue problems for differential equations have important applications in mechanics and physics. The discretization of these eigenvalue problems leads to ill-conditioned nonlinear eigenvalue problems with very large sparse matrices monotone depending on the spectral parameter. To compute the smallest eigenvalue of large-scale matrix nonlinear eigenvalue problems, we suggest preconditioned iterative methods: preconditioned simple iteration method, preconditioned steepest descent method, and preconditioned conjugate gradient method. These methods use only matrix-vector multiplications, preconditioner-vector multiplications, linear operations with vectors and inner products of vectors. We investigate the convergence and derive grid-independent error estimates of these methods for computing eigenvalues. Numerical experiments demonstrate the practical effectiveness of the proposed methods for a class of mechanical problems.

Key Words eigenvalue, eigenelement, symmetric eigenvalue problem, nonlinear eigenvalue problem, iterative method, preconditioned iterative method, gradient method, steepest descent method, conjugate gradient method

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1 Introduction

After the discretization of eigenvalue problems for symmetric elliptic differential operators we get the matrix eigenvalue problem $Au = \lambda Bu$ with large and sparse symmetric matrices A and B. Usually matrices A and B are very large, the matrix A is ill-conditioned and their are not stored explicitly, but only routines are available for computing the matrix-vector products Av and Bv. In applied eigenvalue problems describing vibrations of mechanical structures, only a few of the smallest eigenvalues defining the base frequencies are of interest.

Classical methods for solving eigenvalue problems can not be applied in our situation, since the computer storage for matrices A and B is not available. Lanczos method has slow convergence since the condition number of the matrix A increases for decreasing mesh size h. In indicated practical problems the condition number usually behaves like h^{-m}, $2 \le m \le 4$.

To find the smallest simple eigenvalue λ_1 of the matrix problem $Au = \lambda Bu$, we can use the gradient method. It is well know that λ_1 is the minimum of the Rayleigh quotient $R(v) = (Av, v)/(Bv, v)$ and its stationary point is the eigenvector u_1 corresponding to λ_1. Hence we can construct a minimizing sequence of nonzero vectors u^n, $n = 1, 2, \ldots$, $\mu^n = R(u^n) \to \lambda_1$, $u^n \to u_1$, $n \to \infty$, using the relations

$$\begin{aligned}
\tilde{u}^{n+1} &= u^n - \tau^n (A - \mu^n B) u^n, \\
u^{n+1} &= \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^{n+1} = R(u^{n+1}), \quad n = 0, 1, \ldots,
\end{aligned}$$

for a suitable choice of the scalar parameter τ^n. This iteration method is called a *gradient method* for computing the smallest eigenvalue of the matrix problem since

$$\operatorname{grad} R(v) = \frac{2}{(Bv, v)} (A - R(v)B) v$$

and

$$\tilde{u}^{n+1} = u^n - c_0 \operatorname{grad} R(u^n),$$

where $c_0 = \tau^n (Bu^n, u^n)/2$. Thus, in the gradient method we move from a given iteration vector u^n in the direction $-\operatorname{grad} R(u^n)$.

The described gradient method has a maximal simplicity and a low storage requirement. Therefore, this method is called also a *simple iteration method*. But, unfortunately, this method has poor convergence properties for an ill-conditioned matrix A.

To improve the convergence of the simple iteration method, we introduce the preconditioner C^{-1}, where C is the matrix approximating the matrix A, and calculate sequences μ^n, u^n, $n = 1, 2, \ldots$, by the relations

$$\begin{aligned}
\tilde{u}^{n+1} &= u^n - \tau^n C^{-1} (A - \mu^n B) u^n, \\
u^{n+1} &= \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^{n+1} = R(u^{n+1}), \quad n = 0, 1, \ldots,
\end{aligned}$$

The matrix C is assumed a symmetric positive definite matrix, which can be easily inverted. The last method uses the gradient of the Rayleigh quotient in the vector space with scalar product $(C.,.)$:

$$\operatorname{grad}_C R(v) = \frac{2}{(Bv,v)} C^{-1}(A - R(v)B)v$$

and we obtain

$$\tilde{u}^{n+1} = u^n - c_0 \operatorname{grad}_C R(u^n),$$

where $c_0 = \tau^n(Bu^n, u^n)/2$. Therefore, this method is called a *preconditioned gradient method* or a *preconditioned simple iteration method* (PSIM).

The convergence of PSIM can be improved if we shall minimize the Rayleigh quotient in the subspace $V_{n+1} = \operatorname{span}\{u^n, w^n\}$ or $W_{n+1} = \operatorname{span}\{u^{n-1}, u^n, w^n\}$, $w^n = C^{-1}(A - \mu^n B)u^n$. Obtained iterative methods are called a *preconditioned steepest descent method* (PSDM) and a *preconditioned conjugate gradient method* (PCGM), respectively.

Preconditioned gradient iterative methods for the symmetric eigenvalue problem $Au = \lambda Bu$ have been first studied in the paper [28]. Grid-independent convergence estimates were first obtained in [6]. Papers [8], [12], [13], [23], [24], [43], continue the investigations of these methods. In the recent papers [19], [20], [21], [16], sharp convergence estimates have been derived. A survey of results on preconditioned iterative methods is presented in the papers [14], [16].

Iterations of several vectors allow to compute several leading eigenvalues and eigenvectors [5], [7], [9], [16], [22]. These methods are called preconditioned block iterative methods or preconditioned subspace iterative methods.

In the present paper, we propose the methodology for constructing and investigating preconditioned iterative methods for large-scale monotone nonlinear eigenvalue problems of the form: $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, $A(\lambda)u = \lambda B(\lambda)u$, where H is a real Euclidean space, Λ is an interval on the real axis, $A(\mu)$ and $B(\mu)$ are large sparse symmetric matrices, $A(\mu)$ is ill-conditioned for fixed $\mu \in \Lambda$. We assume that these matrices can not be stored, and only routines for computing the matrix-vector products $A(\mu)v$ and $B(\mu)v$ are available. Here we assume that the Rayleigh quotient $R(\mu, v) = (A(\mu)v, v)/(B(\mu), v, v)$, $\mu \in \Lambda$, is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e., $R(\mu, v) \geq R(\eta, v)$, $\mu < \eta$, $\mu, \eta \in \Lambda$, $v \in H \setminus \{0\}$. For solving nonlinear eigenvalue problems we suggest PSIM of the following kind:

$$\begin{aligned}
\tilde{u}^{n+1} &= u^n - \tau^n C^{-1}(\mu^n)(A(\mu^n) - \mu^n B(\mu^n))u^n, \\
u^{n+1} &= \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_{B(\mu^{n+1})}}, \quad \mu^{n+1} = R(\mu^{n+1}, \tilde{u}^{n+1}), \quad n = 0, 1, \ldots,
\end{aligned}$$

where the symmetric positive definite matrix $C(\mu)$ is an easily inverted matrix and the following condition is valid: $\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v)$, $v \in H \setminus \{0\}$, $\mu \in \Lambda$, the iteration parameter τ^n is defined by the formula $\tau^n = \delta_1^{-1}(\mu^n)$. In this method for each $n \geq 1$ we minimize the Rayleigh quotient $R(\mu^n, v)$, $v \in H \setminus \{0\}$, and find the root of a scalar equation. In PSIM we move from a given iteration vector u^n in the direction $-\operatorname{grad}_{C(\mu^n)} R(\mu^n, u^n)$.

The convergence of PSIM can be improved if we shall minimize the Rayleigh quotient $R(\mu^n, v)$, $v \in H \setminus \{0\}$, in the subspace $V_{n+1} = \mathrm{span}\{u^n, w^n\}$ or $W_{n+1} = \mathrm{span}\{u^{n-1}, u^n, w^n\}$, $w^n = C^{-1}(\mu^n)(A(\mu^n) - \mu^n B(\mu^n))u^n$. Obtained iterative methods for solving nonlinear eigenvalue problems are called PSDM and PCGM, respectively.

Our approach allows to construct block variants of iterative methods for solving nonlinear eigenvalue problems [34], [36], [37], [38].

Monotone nonlinear matrix eigenvalue problems arise after the discretization of eigenvalue problems for differential and integral equations with nonlinear appearance of the spectral parameter. Note that monotone nonlinear eigenvalue problems have important applications in optical telecommunications and in integrated optics [37], [39], and in structural mechanics [1], [41], [42], [40].

A survey on iterative methods for relatively small nonlinear matrix eigenvalue problems is presented in [25], [11]. Recent papers [2], [3], [4], [17], [18], propose efficient structured methods for solving large polynomial matrix eigenvalue problems with matrices of special structures.

The present paper is organized as follows. In Section 2, we give the statement of a symmetric matrix eigenvalue problem with nonlinear occurrence of the spectral parameter. In Section 3, results about existence and properties of the eigenvalues of the nonlinear eigenvalue problem are proved. Similar results were obtained earlier in the papers [29], [30], [31], [32], [33]. In Section 4, we describe auxiliary results obtained in the papers [6], [10]. These results are used further for constructing and investigating the iterative methods. In Sections 5, 6, and 7, we formulate the preconditioned iterative methods for the nonlinear eigenvalue problem, and we investigate the convergence and the error of these methods for computing the smallest eigenvalue. In Section 8, we discuss numerical experiments for a model problem.

2 Formulation of the problem

Let H be an N-dimensional real Euclidean space with the scalar product $(.,.)$ and the norm $\|.\|$, and let Λ be an interval on the real axis $\mathbb{R}$, $\Lambda = (\alpha, \beta)$, $0 \leq \alpha < \beta \leq \infty$. Introduce the real symmetric N-by-N matrices $A(\mu)$ and $B(\mu)$ for fixed $\mu \in \Lambda$ satisfying the following conditions:

(a) positive definiteness, i.e., there exist positive continuous functions $\alpha_1(\mu)$ and $\beta_1(\mu)$, $\mu \in \Lambda$, such that

$$(A(\mu)v, v) \geq \alpha_1(\mu)\|v\|^2, \quad (B(\mu)v, v) \geq \beta_1(\mu)\|v\|^2 \quad \forall v \in H, \mu \in \Lambda;$$

(b) continuity with respect to the numerical argument, i.e.,

$$\|A(\mu) - A(\eta)\| \to 0, \quad \|B(\mu) - B(\eta)\| \to 0,$$

as $\mu \to \eta$, $\mu, \eta \in \Lambda$. By $\|.\|$ we also denote the matrix norm corresponding to the defined vector norm.

Define the Rayleigh quotient by the formula:

$$R(\mu, v) = \frac{(A(\mu)v, v)}{(B(\mu)v, v)}, \quad v \in H \setminus \{0\}, \mu \in \Lambda.$$

Assume that the following additional conditions are fulfilled:

(c) the Rayleigh quotient $R(\mu, v)$, $\mu \in \Lambda$, is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e.,

$$R(\mu, v) \geq R(\eta, v), \quad \mu < \eta, \mu, \eta \in \Lambda, v \in H \setminus \{0\};$$

(d) there exists $\eta = \eta_{min} \in \Lambda$ such that

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0;$$

(e) there exists $\eta = \eta_{max} \in \Lambda$ such that

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0.$$

Consider the following nonlinear eigenvalue problem: find $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, such that

$$A(\lambda)u = \lambda B(\lambda)u. \tag{1}$$

The number λ that satisfies (1) is called an eigenvalue, and the element u is called an eigenelement of problem (1) corresponding to λ. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue λ and the zero element is a closed subspace in H, which is called the eigensubspace corresponding to the eigenvalue λ. The dimension of this subspace is called a multiplicity of the eigenvalue λ.

3 Existence of the eigenvalues

For fixed $\mu \in \Lambda$, we introduce the auxiliary linear eigenvalue problem: find $\gamma(\mu) \in \mathbb{R}$, $u \in H \setminus \{0\}$, such that

$$A(\mu)u = \gamma(\mu)B(\mu)u. \tag{2}$$

For a symmetric positive definite N-by-N matrix A, denote by H_A the Euclidean space of elements from H with the scalar product $(u, v)_A = (Au, v)$ and the norm $\|v\|_A = (v, v)_A^{1/2}$, $u, v \in H_A$.

Lemma 1 *For fixed $\mu \in \Lambda$, problem (2) has N real positive eigenvalues $0 < \gamma_1(\mu) \leq \gamma_2(\mu) \leq \ldots \leq \gamma_N(\mu)$. The eigenelements $u_i = u_i(\mu)$, $i = 1, 2, \ldots, N$, corresponding to these eigenvalues can be chosen so that:*

$$(A(\mu)u_i, u_j) = \gamma_i(\mu)\delta_{ij}, \quad (B(\mu)u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, N.$$

The elements $u_i = u_i(\mu)$, $i = 1, 2, \ldots, N$, form an orthonormal basis of the space $H_{B(\mu)}$.

Proof The assertion is proved, for example, in [26]. □

Lemma 2 *The formula of the minimax principle is valid:*

$$\gamma_i(\mu) = \min_{W_i \subset H} \max_{v \in W_i \setminus \{0\}} R(\mu, v), \quad i = 1, 2, \ldots, N,$$

where W_i is an i-dimensional subspace of the space H. In particular, the following relations hold:

$$\gamma_1(\mu) = \min_{v \in H \setminus \{0\}} R(\mu, v), \quad \gamma_N(\mu) = \max_{v \in H \setminus \{0\}} R(\mu, v).$$

Proof The assertion is proved, for example, in [26]. □

For a fixed segment $[a, b]$ on Λ, we set

$$\begin{aligned}
&\alpha_{1,min}(a, b) = \min_{\mu \in [a,b]} \alpha_1(\mu), \quad \beta_{1,min}(a, b) = \min_{\mu \in [a,b]} \beta_1(\mu), \\
&\Delta_A(\mu, \eta) = \frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)}, \quad \Delta_B(\mu, \eta) = \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a, b)}, \\
&\Delta(\mu, \eta) = \Delta_A(\mu, \eta) + \Delta_B(\mu, \eta),
\end{aligned}$$

for $\mu, \eta \in [a, b]$.

Lemma 3 *Suppose that $\Delta_A(\mu, \eta) \le 1/2$ for $\mu, \eta \in [a, b]$. Then the following inequality is valid:*

$$|R(\mu, v) - R(\eta, v)| \le 2\Delta(\mu, \eta) R(\eta, v), \quad v \in H \setminus \{0\}, \quad \mu, \eta \in [a, b].$$

Proof It is easy to verify that

$$\begin{aligned}
R(\mu, v) - R(\eta, v) \;\; = \;\; & R(\eta, v) \frac{(A(\mu)v, v) - (A(\eta)v, v)}{(A(\mu)v, v)} + \\
& + R(\eta, v) \frac{(B(\eta)v, v) - (B(\mu)v, v)}{(B(\mu)v, v)} + \\
& + (R(\mu, v) - R(\eta, v)) \frac{(A(\mu)v, v) - (A(\eta)v, v)}{(A(\mu)v, v)}, \quad \mu, \eta \in \Lambda.
\end{aligned}$$

This relation implies the inequality

$$|R(\mu, v) - R(\eta, v)| \le \Delta(\mu, \eta) R(\eta, v) + |R(\mu, v) - R(\eta, v)| \, \Delta_A(\mu, \eta)$$

for $\mu, \eta \in [a, b]$. Consequently, the following estimate holds

$$|R(\mu, v) - R(\eta, v)| \le \frac{1}{1 - \Delta_A(\mu, \eta)} \Delta(\mu, \eta) R(\eta, v) \le 2\Delta(\mu, \eta) R(\eta, v)$$

for $\mu, \eta \in [a, b]$. Thus, the lemma is proved. □

Lemma 4 *Suppose that $\Delta_A(\mu,\eta) \le 1/2$ for $\mu,\eta \in [a,b]$. Then the following inequality is valid:*

$$|\gamma_i(\mu) - \gamma_i(\eta)| \le 2\Delta(\mu,\eta)\gamma_i(a), \quad i = 1,2,\dots,N.$$

Proof Denote by $E_i(\mu)$ the subspace spanned on the eigenelements $u_j = u_j(\mu)$, $j = 1,2,\dots,i$, which correspond to the eigenvalues $\gamma_j(\mu)$, $j = 1,2,\dots,i$, of problem (2) for fixed $\mu \in \Lambda$, $1 \le i \le N$. Using the minimax principle of Lemma 2, we obtain

$$\begin{aligned}
\gamma_i(\mu) &= \min_{W_i \subset H} \max_{v \in W_i \setminus \{0\}} R(\mu,v) \le \\
&\le \max_{v \in E_i(\eta)\setminus\{0\}} R(\mu,v) \le \\
&\le \max_{v \in E_i(\eta)\setminus\{0\}} R(\eta,v) + \max_{v \in E_i(\eta)\setminus\{0\}} |R(\mu,v) - R(\eta,v)| = \\
&= \gamma_i(\eta) + \sigma_i(\mu,\eta),
\end{aligned}$$

where

$$\sigma_i(\mu,\eta) = \max_{v \in E_i(\eta)\setminus\{0\}} |R(\mu,v) - R(\eta,v)|, \quad \mu,\eta \in \Lambda.$$

Hence we get

$$|\gamma_i(\mu) - \gamma_i(\eta)| \le \max\{\sigma_i(\mu,\eta), \sigma_i(\eta,\mu)\}, \quad \mu,\eta \in \Lambda.$$

Now, by Lemma 3, we have

$$\sigma_i(\mu,\eta) \le 2\Delta(\mu,\eta)\gamma_i(\eta)$$

for $\mu,\eta \in [a,b]$. Consequently, the following estimate holds

$$\begin{aligned}
&\max\{\sigma_i(\mu,\eta), \sigma_i(\eta,\mu)\} \le 2\Delta(\mu,\eta)\max\{\gamma_i(\mu),\gamma_i(\eta)\} \le \\
&\le 2\Delta(\mu,\eta) \max_{\mu \in [a,b]} \gamma_i(\mu) = 2\Delta(\mu,\eta)\gamma_i(a)
\end{aligned}$$

for $\mu,\eta \in [a,b]$. This proves the lemma. □

Lemma 5 *The functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1,2,\dots,N$, are continuous nonincreasing functions with positive values.*

Proof The continuity of the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1,2,\dots,N$, follows from Lemma 4 and condition (b). Using the minimax principle of Lemma 2 and condition (c), we obtain that the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1,2,\dots,N$, are nonincreasing functions. Thus, the lemma is proved. □

Lemma 6 *The functions $\mu - \gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1,2,\dots,N$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

Proof The increase of the functions $\mu - \gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, follows from Lemma 5.

Taking into account condition (d), we obtain that there exists a number $\eta = \eta_{min} \in \Lambda$, for which the following relations are valid:

$$\mu - \gamma_i(\mu) < \eta - \gamma_i(\eta) \leq \eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0$$

for $\mu \in (\alpha, \eta)$, $i = 1, 2, \ldots, N$.

According to condition (e), there exists $\eta = \eta_{max} \in \Lambda$ such that the following inequalities hold:

$$\mu - \gamma_i(\mu) > \eta - \gamma_i(\eta) \geq \eta - \gamma_N(\eta) = \eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0$$

for $\mu \in (\eta, \beta)$, $i = 1, 2, \ldots, N$. Thus, the lemma is proved. $\square$

Lemma 7 *A number $\lambda \in \Lambda$ is an eigenvalue of problem* (1) *if and only if the number λ is a solution of an equation from the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

Proof If λ is a solution of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, for some i, $1 \leq i \leq N$, then it follows from (1) and (2) that λ is an eigenvalue of problem (1). If λ is an eigenvalue of problem (1), then (1) and (2) imply $\lambda - \gamma_i(\lambda) = 0$ for some i, $1 \leq i \leq N$. This proves the lemma. $\square$

Theorem 8 *Problem* (1) *has N eigenvalues λ_i, $i = 1, 2, \ldots, N$, which are repeated according to their multiplicity: $\alpha < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N < \beta$. Each eigenvalue λ_i is a unique root of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

Proof By Lemma 6, each equation of the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, has a unique solution. Denote these solutions by λ_i, $i = 1, 2, \ldots, N$, i.e., $\lambda_i - \gamma_i(\lambda_i) = 0$, $i = 1, 2, \ldots, N$. To check that the numbers λ_i, $i = 1, 2, \ldots, N$, are put in a nondecreasing order, let us assume the opposite, i.e., $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 5, we obtain a contradiction, namely

$$\lambda_i = \gamma_i(\lambda_i) \leq \gamma_i(\lambda_{i+1}) \leq \gamma_{i+1}(\lambda_{i+1}) = \lambda_{i+1}.$$

By Lemma 7, the numbers λ_i, $i = 1, 2, \ldots, N$, are eigenvalues of problem (1). Thus, the theorem is proved. $\square$

Remark 9 If $\alpha = 0$, then condition (d) follows from condition (c).

Proof Let us fix $\nu \in \Lambda$ and put $\eta = \min\{\gamma_1(\nu), \nu\}/2$. Taking into account condition (c), Lemma 2, and the relations $\eta \leq \gamma_1(\nu)/2$, $\eta \leq \nu/2 < \nu$, we have

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_1(\eta) \leq \gamma_1(\nu)/2 - \gamma_1(\nu) = -\gamma_1(\nu)/2 < 0.$$

Thus, condition (d) is satisfied for chosen $\eta \in \Lambda$. $\square$

Remark 10 If $\beta = \infty$, then condition (e) follows from condition (c).

Proof For fixed $\nu \in \Lambda$, put $\eta = 2\max\{\gamma_N(\nu), \nu\}$. Since $\eta \geq 2\gamma_N(\nu)$ and $\eta \geq 2\nu > \nu$, according to condition (c) and Lemma 2, we obtain the relations:

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_N(\eta) \geq 2\gamma_N(\nu) - \gamma_N(\nu) = \gamma_N(\nu) > 0,$$

which implies that condition (e) is satisfied. □

Remark 11 We may write conditions (d) and (e) as the following conditions:
(d.1) there exists $\eta = \eta_{min} \in \Lambda$ such that $\eta - \gamma_1(\eta) \leq 0$;
(e.1) there exists $\eta = \eta_{max} \in \Lambda$ such that $\eta - \gamma_N(\eta) \geq 0$.

Conditions (d.1) and (e.1) imply the existence N roots λ_i, $i = 1, 2, \ldots, N$, of the set of equations $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$ (see Theorem 8).

We may change conditions (d.1) and (e.1) to the following conditions:
(d.2) there exists $\eta = \eta_{min} \in \Lambda$ such that $\eta - \gamma_m(\eta) \leq 0$;
(e.2) there exists $\eta = \eta_{max} \in \Lambda$ such that $\eta - \gamma_n(\eta) \geq 0$;
where $1 \leq m \leq n \leq N$.

Conditions (d.2) and (e.2) imply the existence $n - m + 1$ roots λ_i, $i = m, m+1, \ldots, n$, of the set of equations $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$. In this case, we obtain new existence theorem instead of Theorem 8.

4 Auxiliary results

In this section we shall introduce one iteration step of the preconditioned simple iteration method for linear eigenvalue problem (2) for fixed parameter $\mu \in \Lambda$ and state well known convergence results. In the following sections we shall use this results for defining and investigating preconditioned iterative methods for solving nonlinear eigenvalue problem (1).

Assume that the symmetric positive definite N-by-N matrix $C(\mu)$ is given for fixed $\mu \in \Lambda$, and that there exist continuous functions $\delta_0(\mu)$, $\delta_1(\mu)$, $\mu \in \Lambda$, $0 < \delta_0(\mu) \leq \delta_1(\mu)$, $\mu \in \Lambda$, such that

$$\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v), \quad v \in H, \quad \mu \in \Lambda.$$

For a given element $v^0 \in H$, $\|v^0\|_{B(\mu)} = 1$, we define an element $v^1 \in H$ and numbers ν^0 and ν^1 by the formulae:

$$\begin{aligned} &\tilde{v}^1 = v^0 - \tau^0 w^0, \quad \tau^0 = \delta_1^{-1}(\mu), \\ &w^0 = C(\mu)^{-1}(A(\mu) - \nu^0 B(\mu))v^0, \\ &v^1 = \frac{\tilde{v}^1}{\|\tilde{v}^1\|_{B(\mu)}}, \\ &\nu^0 = R(\mu, v^0), \quad \nu^1 = R(\mu, v^1), \end{aligned}$$

for fixed $\mu \in \Lambda$.

Lemma 12 *Let $\gamma_1(\mu)$ and $\gamma_2(\mu)$ be eigenvalues of problem* (2) *with $\mu \in \Lambda$ such that $\gamma_1(\mu) < \gamma_2(\mu)$. Assume that $\nu^0 < \gamma_2(\mu)$. Then $\gamma_1(\mu) \le \nu^1 \le \nu^0$, and the following estimate is valid:*

$$\nu^1 - \gamma_1(\mu) \le \rho(\mu, \nu^0)(\nu^0 - \gamma_1(\mu)),$$

where $0 < \rho(\mu, \nu) < 1$,

$$\rho(\mu, \nu) = \frac{1 - \delta(\mu)(1 - \nu/\gamma_2(\mu))}{1 + \delta(\mu)(1 - \nu/\gamma_2(\mu))(\nu/\gamma_1(\mu) - 1)},$$
$$\delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \nu \in [\gamma_1(\mu), \gamma_2(\mu)), \quad \mu \in \Lambda.$$

Proof The assertion of the lemma is proved in [6], [10]. □

5 Preconditioned iterative methods

Let us consider the following iterative methods for solving nonlinear eigenvalue problem (1).

Method 1. PSIM: Preconditioned Simple Iteration Method.

(1) *Select $\tilde{u}^0 \in H \setminus \{0\}$. Compute μ^0 as the solution of the equation $\mu - \varphi_0(\mu) = 0$, $\mu \in \Lambda$, $\varphi_0(\mu) = R(\mu, \tilde{u}^0)$. Define $u^0 = \tilde{u}^0/\|\tilde{u}^0\|_{B(\mu^0)}$.*

(2) *For $n = 0, 1, \ldots,$ do:*

(2a) *Compute the vector $\tilde{u}^{n+1}$ by the following formulae:*

$$\tilde{u}^{n+1} = u^n - \tau^n w^n, \quad \tau^n = \delta_1^{-1}(\mu^n),$$
$$w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n.$$

(2b) *Compute the value μ^{n+1} as the solution of the equation $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, $\varphi_{n+1}(\mu) = R(\mu, \tilde{u}^{n+1})$. Define $u^{n+1} = \tilde{u}^{n+1}/\|\tilde{u}^{n+1}\|_{B(\mu^{n+1})}$.*

Method 2. PSDM: Preconditioned Steepest Descent Method.

(1) *Select $\tilde{u}^0 \in H \setminus \{0\}$. Compute μ^0 as the solution of the equation $\mu - \varphi_0(\mu) = 0$, $\mu \in \Lambda$, $\varphi_0(\mu) = R(\mu, \tilde{u}^0)$. Define $u^0 = \tilde{u}^0/\|\tilde{u}^0\|_{B(\mu^0)}$.*

(2) *For $n = 0, 1, \ldots,$ do:*

(2a) *Compute the vector $\tilde{u}^{n+1}$ to minimize the Rayleigh quotient $R(\mu^n, v)$, $v \in H \setminus \{0\}$, on the two-dimensional subspace*

$$V_{n+1} = \mathrm{span}\{u^n, w^n\}, \quad w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n,$$

by using the Rayleigh–Ritz method, i.e.,

$$R(\mu^n, \tilde{u}^{n+1}) = \min_{v \in V_{n+1} \setminus \{0\}} R(\mu^n, v).$$

(2b) *Compute the value* μ^{n+1} *as the solution of the equation* $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, $\varphi_{n+1}(\mu) = R(\mu, \tilde{u}^{n+1})$. *Define* $u^{n+1} = \tilde{u}^{n+1}/\|\tilde{u}^{n+1}\|_{B(\mu^{n+1})}$.

Method 3. PCGM: Preconditioned Conjugate Gradient Method.

(1) *Select* $\tilde{u}^0 \in H \setminus \{0\}$. *Compute* μ^0 *as the solution of the equation* $\mu - \varphi_0(\mu) = 0$, $\mu \in \Lambda$, $\varphi_0(\mu) = R(\mu, \tilde{u}^0)$. *Define* $u^0 = \tilde{u}^0/\|\tilde{u}^0\|_{B(\mu^0)}$.

(2) *Compute* $\tilde{u}^1$ *to minimize the Rayleigh quotient* $R(\mu^n, v)$, $v \in H \setminus \{0\}$, *on the two-dimensional subspace*

$$V_1 = \operatorname{span}\{u^0, w^0\}, \quad w^0 = C(\mu^0)^{-1}(A(\mu^0) - \mu^0 B(\mu^0))u^0,$$

by using the Rayleigh–Ritz method, i.e.,

$$R(\mu^0, \tilde{u}^1) = \min_{v \in V_1 \setminus \{0\}} R(\mu^0, v).$$

Compute the value μ^1 *as the solution of the equation* $\mu - \varphi_1(\mu) = 0$, $\mu \in \Lambda$, $\varphi_1(\mu) = R(\mu, \tilde{u}^1)$. *Define* $u^1 = \tilde{u}^1/\|\tilde{u}^1\|_{B(\mu^1)}$.

(3) *For* $n = 1, 2, \ldots$, *do:*

(3a) *Compute the vector* $\tilde{u}^{n+1}$ *to minimize the Rayleigh quotient* $R(\mu^n, v)$, $v \in H \setminus \{0\}$, *on the trial subspace*

$$W_{n+1} = \operatorname{span}\{u^{n-1}, u^n, w^n\}, \quad w^n = C(\mu^n)^{-1}(A(\mu^n) - \mu^n B(\mu^n))u^n,$$

by using the Rayleigh–Ritz method, i.e.,

$$R(\mu^n, \tilde{u}^{n+1}) = \min_{v \in W_{n+1} \setminus \{0\}} R(\mu^n, v).$$

(3b) *Compute the value* μ^{n+1} *as the solution of the equation* $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, $\varphi_{n+1}(\mu) = R(\mu, \tilde{u}^{n+1})$. *Define* $u^{n+1} = \tilde{u}^{n+1}/\|\tilde{u}^{n+1}\|_{B(\mu^{n+1})}$.

Remark 13 To compute the vector $\tilde{u}^{n+1}$ in PSDM, the following formulae can be used:

$$\begin{aligned}
&\tilde{u}^{n+1} = u^n - \tau^n w^n, \\
&\tau^n = \frac{2}{\theta + [\theta^2 - 4\theta(B(\mu^n)u^n, w^n) + 4(B(\mu^n)w^n, w^n)]^{1/2}}, \\
&\theta = \frac{((A(\mu^n) - \mu^n B(\mu^n))w^n, w^n)}{(C(\mu^n)w^n, w^n)}, \\
&w^n = C^{-1}(\mu^n)(A(\mu^n) - \mu^n B(\mu^n))u^n.
\end{aligned}$$

6 Convergence of iterative methods

In this section we study the convergence of the methods PSIM, PSDM, and PCGM, introduced in Section 5.

Assume that the sequences μ^n, u^n, $n = 0, 1, \ldots$, are computed by one of these methods. We start with investigating the properties of the functions $\varphi_n(\mu) = R(\mu, \tilde{u}^n)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, and the function $\rho(\mu, \nu)$, $\mu, \nu \in \Lambda$.

Lemma 14 *The functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous nonincreasing functions with positive values. In addition, the following inequalities are valid:*

$$\gamma_1(\mu) \leq \varphi_n(\mu) \leq \gamma_N(\mu),$$

$\mu \in \Lambda$, $n = 0, 1, \ldots$

Proof The proof follows from Lemmata 2 and 3. □

Lemma 15 *The functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

Proof The increase of the functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, follows from the condition (c).

Taking into account condition (d), we obtain that there exists a number $\eta = \eta_{min} \in \Lambda$, for which the following relations are valid:

$$\mu - \varphi_n(\mu) < \eta - \varphi_n(\eta) \leq \eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0$$

for $\mu \in (\alpha, \eta)$, $n = 0, 1, \ldots$

According to condition (e), there exists $\eta = \eta_{max} \in \Lambda$ such that the following inequalities hold:

$$\mu - \varphi_n(\mu) > \eta - \varphi_n(\eta) \geq \eta - \gamma_N(\eta) = \eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0$$

for $\mu \in (\eta, \beta)$, $n = 0, 1, \ldots$ Thus, the lemma is proved. □

Let λ_1 and λ_2 be eigenvalues of problem (1) such that $\lambda_1 < \lambda_2$. Put

$$\rho(\nu) = \frac{1 - d(1 - \nu/\lambda_2)}{1 + d(1 - \nu/\lambda_2)(\nu/\lambda_1 - 1)}, \quad \nu \in [\lambda_1, \lambda_2),$$
$$d = \min_{\mu \in [\lambda_1, \lambda_2]} \delta(\mu), \quad \delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \mu \in \Lambda.$$

Note that $0 < d \leq 1$, $0 < \rho(\nu) < 1$ for $\nu \in [\lambda_1, \lambda_2)$.

Lemma 16 *The half-open interval $[\lambda_1, \lambda_2)$ is contained in the half-open interval $[\gamma_1(\mu), \gamma_2(\mu))$ for any $\mu \in [\lambda_1, \lambda_2)$.*

Proof Taking into account Lemma 5, we get $\gamma_1(\mu) \leq \lambda_1$ and $\gamma_2(\mu) \geq \lambda_2$ for $\mu \in [\lambda_1, \lambda_2)$. These inequalities prove the lemma. □

Lemma 17 *The following inequality holds: $\rho(\mu, \nu) \leq \rho(\nu)$ for $\mu, \nu \in [\lambda_1, \lambda_2)$.*

Proof By Lemma 16, if $\nu \in [\lambda_1, \lambda_2)$ and $\lambda_1 < \lambda_2$, then $\nu \in [\gamma_1(\mu), \gamma_2(\mu))$ and $\gamma_1(\mu) < \gamma_2(\mu)$ for $\mu \in [\lambda_1, \lambda_2)$. Now relations $\gamma_1(\mu) \leq \lambda_1$, $\gamma_2(\mu) \geq \lambda_2$, $\mu \in [\lambda_1, \lambda_2)$, imply the desired inequality:

$$\begin{aligned} \rho(\mu,\nu) &= \frac{1-\delta(\mu)(1-\nu/\gamma_2(\mu))}{1+\delta(\mu)(1-\nu/\gamma_2(\mu))(\nu/\gamma_1(\mu)-1)} \leq \\ &\leq \frac{1-d(1-\nu/\lambda_2)}{1+d(1-\nu/\lambda_2)(\nu/\lambda_1-1)} = \rho(\nu) \end{aligned}$$

for $\mu, \nu \in [\lambda_1, \lambda_2) \subset [\gamma_1(\mu), \gamma_2(\mu))$. Thus, the lemma is proved. □

Now we formulate the main result of the paper.

Theorem 18 *Let λ_1 and λ_2 be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Suppose that the sequence μ^n, $n = 0, 1, \ldots$, is calculated by one of the iterative methods* PSIM, PSDM, *or* PCGM, *introduced in Section* 5, *$\mu^0 < \lambda_2$. Then $\mu^n \to \lambda_1$ as $n \to \infty$ and the following inequalities are valid*

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1.$$

Moreover, the following estimate holds:

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_1, \lambda_2)$, $n = 0, 1, \ldots$

Proof Let us show that the solutions μ^n, $n = 0, 1, \ldots$, of the equations $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, satisfy the following inequalities:

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1.$$

Assume that the equation $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, has the solution μ^n such that

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \lambda_1, \quad n \geq 0.$$

Hence we obtain

$$\nu^0 = \varphi_n(\mu^n) = \mu^n < \lambda_2 = \gamma_2(\lambda_2) \leq \gamma_2(\mu^n).$$

Consequently, by Lemma 12, we have

$$\nu^1 = \varphi_{n+1}(\mu^n) \leq \nu^0 = \varphi_n(\mu^n) = \mu^n.$$

It follows from Lemmata 14 and 15 that the equation $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, has the unique solution μ^{n+1} and

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \mu^{n+1} \geq \lambda_1.$$

Let us prove that $\mu^n \to \lambda_1$ as $n \to \infty$. Taking into account Lemmata 12, 16, 17, we obtain the following relations:

$$\begin{aligned}
\mu^{n+1} - \gamma_1(\mu^{n+1}) &= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_1(\mu^{n+1})) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_1(\mu^n)) = \\
&= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\nu^1 - \gamma_1(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n, \nu^0)(\nu^0 - \gamma_1(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),
\end{aligned}$$

where $\nu^0 = \varphi_n(\mu^n) = \mu^n$, $\nu^1 = \varphi_{n+1}(\mu^n)$.

Since $\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1$, there exists $\xi \in [\lambda_1, \lambda_2)$ such that $\mu^n \to \xi$ as $n \to \infty$.

By condition (a) and the relations $\|u^n\|_{B(\mu^n)} = 1, n = 0, 1, \ldots$, we obtain that there exists a constant $c > 0$ such that

$$\begin{aligned}
&\|u^n\| \leq \frac{\|u^n\|_{B(\mu^n)}}{\sqrt{\beta_1(\mu^n)}} = \frac{1}{\sqrt{\beta_1(\mu^n)}} \leq c, \quad n = 0, 1, \ldots, \\
&c = \max_{\mu \in [\lambda_1, \lambda_2]} \frac{1}{\sqrt{\beta_1(\mu)}}.
\end{aligned}$$

Hence there exists an element $w \in H$ and a subsequence u^{n_i+1}, $i = 1, 2, \ldots$, such that $u^{n_i+1} \to w$ as $i \to \infty$.

Let us prove that $\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) \to 0$ as $i \to \infty$. We have

$$0 \leq \mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) = R(\mu^{n_i+1}, u^{n_i+1}) - R(\mu^{n_i}, u^{n_i+1}) \to 0$$

as $i \to \infty$. Here, we have taken into account that

$$R(\mu^{n_i+1}, u^{n_i+1}) \to R(\xi, w), \quad R(\mu^{n_i}, u^{n_i+1}) \to R(\xi, w),$$

as $i \to \infty$.

Using the relations

$$0 \leq \mu^{n_i+1} - \gamma_1(\mu^{n_i+1}) \leq (\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i})) + \rho(\mu^{n_i})(\mu^{n_i} - \gamma_1(\mu^{n_i}))$$

as $i \to \infty$, we get

$$0 \leq \xi - \gamma_1(\xi) \leq \rho(\xi)(\xi - \gamma_1(\xi)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_1, \lambda_2)$. Hence the number $\xi \in [\lambda_1, \lambda_2)$ satisfies the equation $\xi - \gamma_1(\xi) = 0$, i.e., $\xi = \lambda_1$ is an eigenvalue of problem (1) and $\mu^n \to \lambda_1$ as $n \to \infty$. This completes the proof of the theorem. □

7 Error estimates of iterative methods

Assume that there exist positive continuous functions $\alpha_0(\mu,\eta)$ and $\beta_0(\mu,\eta)$, $\mu,\eta\in\Lambda$, such that

$$\|A(\mu)-A(\eta)\|\le\alpha_0(\mu,\eta)|\mu-\eta|,\quad \|B(\mu)-B(\eta)\|\le\beta_0(\mu,\eta)|\mu-\eta|,$$

for $\mu,\eta\in\Lambda$.

For a fixed segment $[a,b]$ on Λ, we set

$$\alpha_{0,max}(a,b)=\max_{\mu,\eta\in[a,b]}\alpha_0(\mu,\eta),\quad \beta_{0,max}(a,b)=\max_{\mu,\eta\in[a,b]}\beta_0(\mu,\eta).$$

Lemma 19 *Assume that the following inequality holds:*

$$\frac{\alpha_{0,max}(a,b)}{\alpha_{1,min}(a,b)}(b-a)\le\frac{1}{2},$$

for a fixed segment $[a,b]$ on Λ. Then the following estimate is valid:

$$|R(\mu,v)-R(\eta,v)|\le r(a,b,v)\,|\mu-\eta|,\quad \mu,\eta\in[a,b],\quad v\in H\setminus\{0\},$$

where

$$r(a,b,v)=2\left(\frac{\alpha_{0,max}(a,b)}{\alpha_{1,min}(a,b)}+\frac{\beta_{0,max}(a,b)}{\beta_{1,min}(a,b)}\right)R(a,v).$$

Proof The proof follows from Lemma 3. □

Put

$$q(\mu)=\max\{\rho(\lambda_1),\rho(\mu)\},\quad \mu\in[\lambda_1,\lambda_2),$$
$$\omega=\frac{\lambda_2\sqrt{1-d}}{1+\sqrt{1-d}}.$$

Note that $0<q(\mu)<1$ for $\mu\in[\lambda_1,\lambda_2)$.

Lemma 20 *The following equality is valid:*

$$\max_{\mu\in[\lambda_1,\mu^0]}\rho(\mu)=q(\mu^0)$$

for $\mu^0\in[\lambda_1,\lambda_2)$. If $0\le\omega\le\lambda_1$, then $q(\mu^0)=\rho(\mu^0)$. If $\lambda_1\le\omega<\lambda_2$ and $\lambda_1\le\mu^0\le\omega$, then $q(\mu^0)=\rho(\lambda_1)$.

Proof It is not difficult to make sure (see also [10]) that $\rho'(\omega)=0$, $\rho'(\mu)<0$ for $\mu\in(0,\omega)$, $\rho'(\mu)>0$ for $\mu\in(\omega,\lambda_2)$. These relations imply desired results. Thus, the lemma is proved. □

Theorem 21 *Let λ_1 and λ_2 be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Assume that the sequence μ^n, $n = 0, 1, \ldots$, is calculated by one of the iterative methods* PSIM, PSDM, *or* PCGM, *introduced in Section* 5, *$\mu^0 < \lambda_2$, and that numbers $n_0 \geq 0$ and $\varepsilon > 0$ such that $\lambda_1 \leq \mu^{n+1} \leq \mu^n \leq \lambda_1 + \varepsilon < \lambda_2$ and*

$$\frac{\alpha_{0,max}(\lambda_1, \lambda_1 + \varepsilon)}{\alpha_{1,min}(\lambda_1, \lambda_1 + \varepsilon)} \varepsilon \leq \frac{1}{2}$$

for $n \geq n_0$. Then the following estimate is valid:

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \leq q_n(\mu^n - \gamma_1(\mu^n)),$$

where $q_n = r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1}) + \rho(\mu^n)$, $n \geq n_0$.

Suppose $r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1}) \leq \sigma$, $n \geq n_0$. Then

$$\begin{array}{rcl} \mu^{n+1} - \gamma_1(\mu^{n+1}) & \leq & q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \\ \mu^{n+1} - \lambda_1 & \leq & q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \end{array}$$

for $q_0 = \sigma + q(\mu^0)$, $n \geq n_0$.

Proof According to Lemma 19, for $n \geq n_0$, we obtain the following relation:

$$\begin{array}{rcl} \mu^{n+1} - \varphi_{n+1}(\mu^n) & = & \varphi_{n+1}(\mu^{n+1}) - \varphi_{n+1}(\mu^n) = \\ & = & R(\mu^{n+1}, u^{n+1}) - R(\mu^n, u^{n+1}) \leq \\ & \leq & r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1})(\mu^n - \mu^{n+1}) \leq \\ & \leq & r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1})(\mu^n - \gamma_1(\mu^n)), \end{array}$$

in which we have taken into account that

$$\gamma_1(\mu^n) \leq \varphi_{n+1}(\mu^n) \leq \varphi_{n+1}(\mu^{n+1}) = \mu^{n+1}.$$

Now, by Theorem 18 and Lemma 20, we obtain desired estimates. Thus, the theorem is proved. □

8 Numerical experiments

Consider the following model differential eigenvalue problem: find numbers $\lambda \in \Lambda$ and nontrivial functions $u(x)$, $x \in [0, 1]$, such that

$$\begin{array}{l} -u''(x) = \lambda u(x), \quad x \in (0, 1), \\ u(0) = 0, \quad -u'(1) = \varphi(\lambda)\, u(1), \end{array} \tag{3}$$

where $\Lambda = (\varkappa, \infty)$, $\varphi(\mu) = \mu\varkappa M/(\mu - \varkappa)$, $\mu \in \Lambda$, $\varkappa = K/M$, K and M are given positive numbers. The differential equations (3) describe eigenvibrations of a string with a load of mass M attached by an elastic spring of stiffness K.

Investigations of this section can be easily generalized for the cases of more complicated and important problems on eigenvibrations of mechanical structures (beams, plates, shells) with elastically attached loads [1], [41], [29], [42], [40].

We denote by $\mathcal{H} = L_2(0,1)$ and $\mathcal{V} = \{v : v \in W_2^1(0,1), v(0) = 0\}$ the Lebesgue and Sobolev spaces equipped with the norms

$$|u|_0 = \left(\int_0^1 u^2\, dx\right)^{1/2}, \quad |u|_1 = \left(\int_0^1 (u')^2\, dx\right)^{1/2}.$$

Note that the space $\mathcal{V}$ is compactly embedded into the space $\mathcal{H}$, any function from $\mathcal{V}$ is continuous on $[0,1]$. The semi-norm $|.|_1$ is a norm over the space $\mathcal{V}$, which is equivalent to the usual norm $\|.\|_1$, $\|.\|_1^2 = |.|_0^2 + |.|_1^2$.

Define the bilinear forms

$$a(u,v) = \int_0^1 u'v'dx, \quad u, v \in \mathcal{V}, \quad b(u,v) = \int_0^1 uv\, dx, \quad u, v \in \mathcal{H},$$
$$c(u,v) = u(1)v(1), \quad u, v \in \mathcal{V}.$$

The variational formulation of the differential problem (3) has the following form: find $\lambda \in \Lambda$, $u \in \mathcal{V} \setminus \{0\}$, such, that

$$a(u,v) + \varphi(\lambda)\, c(u,v) = \lambda b(u,v) \quad \forall v \in \mathcal{V}. \tag{4}$$

To approximate problem (4), we define the partition of the interval $[0,1]$ by the nodes x_i, $i = 0, 1, \ldots, N$, $h = 1/N$. The finite-element space $\mathcal{V}_h$ is the space of continuous functions on $[0,1]$ that are linear on each interval (x_{k-1}, x_k), $k = 1, 2, \ldots, N$, and $\mathcal{V}_h$ is subspace of the space $\mathcal{V}$. Problem (4) is approximated by the following discrete problem: find $\lambda^h \in \Lambda$, $u^h \in \mathcal{V}_h \setminus \{0\}$, such that

$$a(u^h, v^h) + \varphi(\lambda^h)\, c(u^h, v^h) = \lambda^h b(u^h, v^h) \quad \forall v^h \in \mathcal{V}_h. \tag{5}$$

Note that the following error estimate is valid $0 \le \lambda^h - \lambda \le \tilde{c}(\lambda) h^2 \lambda^2$, where λ^h is a sequence of eigenvalues of problem (5) converging to an eigenvalue λ of problem (3) as $h \to 0$ [33].

Let H be the real Euclidean space of vectors $y = (y_1, y_2, \ldots, y_N)^\top$ with the scalar product $(y,z) = \sum_{i=1}^N y_i z_i$, $y, z \in H$. The discrete problem (5) is equivalent to the following matrix eigenvalue problem: find $\lambda \in \Lambda$, $y \in H \setminus \{0\}$, such that

$$A(\lambda) y = \lambda B y, \tag{6}$$

k	1	2	3	4	5
λ_k	4.482176546	24.223573113	63.723821142	123.031221068	202.200899143

Table 1: Five minimal eigenvalues

where $A(\mu) = A_0 + \varphi(\mu)C_0$, $\mu \in \Lambda$, the square matrix C_0 of order N has zero coefficients c_{ij}^0 except the coefficient $c_{NN}^0 = 1$, $A_0 = \mathcal{M}(a_1, a_2)$, $B = \mathcal{M}(b_1, b_2)$, $a_1 = 2/h$, $a_2 = -1/h$, $b_1 = 4h/6$, $b_2 = h/6$, $\mathcal{M}(c_1, c_2)$ is the square matrix of order N defined by the formula

$$\mathcal{M}(c_1, c_2) = \begin{pmatrix} c_1 & c_2 & & & \\ c_2 & c_1 & c_2 & & \\ & & \cdots & & \\ & & c_2 & c_1 & c_2 \\ & & & c_2 & c_1/2 \end{pmatrix}.$$

We can define exact eigenvalues of problem (6) as the numbers $\lambda \in \Lambda$, $\lambda = \psi(\sigma)$,

$$\psi(\sigma) = \frac{2a_2 \cos \sigma h + a_1}{2b_2 \cos \sigma h + b_1},$$

where the numbers σ are solutions of the following equations (see, for example, [27]):

$$\frac{\tan \sigma}{\sin \sigma h} = \frac{a_2 - \psi(\sigma) b_2}{\varphi(\psi(\sigma))}. \tag{7}$$

Let $M = 1$, $K = 1$, $\varkappa = 1$. Five smallest eigenvalues λ_i, $i = 1, 2, 3, 4, 5$, of problem (6) for $N = 100$, $h = 0.01$, are given in Table 1. These eigenvalues were calculated by using equation (7).

Note that condition (e) is satisfied according to Remark 10. Condition (d) follows from the relations

$$\eta - \gamma_1(\eta) = \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0$$

for $\eta \in (\varkappa, \lambda_1)$.

Using the inequality $|v(1)| \leq |v|_1$, $v \in \mathcal{V}$, we obtain

$$\alpha_1 |v|_1^2 = a(v, v) \leq a(v, v) + \varphi(\mu) c(v, v) \leq \alpha_2(\mu) |v|_1^2, \quad v \in \mathcal{V},$$

where $\alpha_1 = 1$, $\alpha_2(\mu) = 1 + \varphi(\mu)$, $\mu \in \Lambda$. Hence we have the inequalities

$$\delta_0 (A_0 v, v) \leq (A(\mu) v, v) \leq \delta_1(\mu)(A_0 v, v), \quad v \in H,$$

for $\delta_0 = 1$, $\delta_1(\mu) = 1 + \varphi(\mu)$, $\mu \in \Lambda$.

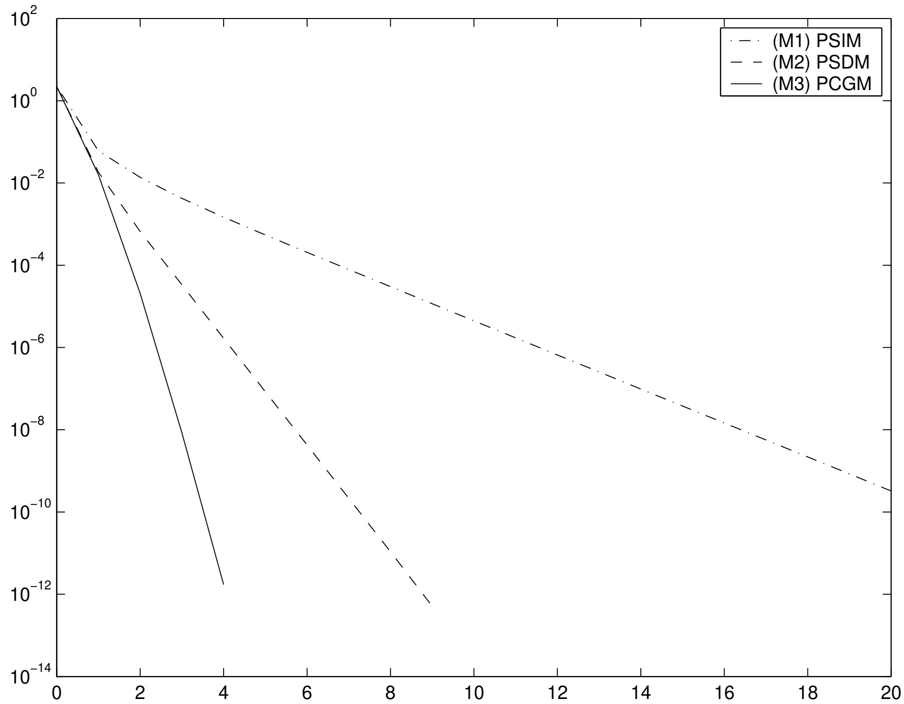

Figure 1: Error of methods PSIM, PSDM, and PCGM

To solve problem (6), we apply the iterative methods PSIM, PSDM, and PCGM, introduced in Section 5. We set $C = A_0$.

The equation $\mu^n = R(\mu^n, \tilde{u}^n)$ arising in these methods can be solved by the explicit formula

$$\mu^n = \frac{1}{2}\left(b_n + \sqrt{b_n^2 - 4a_n\varkappa}\right),$$

where $b_n = \varkappa + a_n + \varkappa M c_n$, $a_n = (A_0\tilde{u}^n, \tilde{u}^n)$, $c_n = (C_0\tilde{u}^n, \tilde{u}^n)$.

Figure 1 illustrates the convergence of methods PSIM, PSDM, and PCGM, for the initial vector $\tilde{u}^0 = (\tilde{u}_1^0, \tilde{u}_2^0, \ldots, \tilde{u}_N^0)^\top$, $\tilde{u}_i^0 = \sin(\alpha\pi x_i)$, $i = 1, 2, \ldots, N$, $\alpha = 0.9$.

Numerical experiments show that the convergence also holds, if the condition $\lambda_1 < \mu^0 < \lambda_2$ is not valid.

Figure 1 is not changed if we take $N = 1000, 10000$. This means that the convergence rates of these methods do not depend on mesh size. Figure 1 is not changed if we take $\varphi(\mu) = 0$. Hence the proposed iterative methods for the nonlinear string eigenvalue problem have the same convergence properties as analogous iterative methods for the linear string eigenvalue problem. This result can be easily generalized for nonlinear eigenvalue problems on eigenvibrations of beams, plates, shells, [1], [41], [29], [42], [40].

9 Conclusion

This paper presents a new methodology for constructing and investigating efficient preconditioned iterative methods for numerical solving large-scale monotone nonlinear eigenvalue problems. Theoretical analysis and numerical experiments show that proposed methods for the class of *nonlinear* eigenvalue problems describing the natural oscillations of mechanical structures with elastically attached loads are approximately as efficient as the analogous methods for solving *linear* eigenvalue problems describing the natural oscillations of these mechanical structures without loads.

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