Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

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Fast solvers for degenerated problems

Preprint SFB393/03-04

Abstract

In this paper, £nite element discretizations of the degenerated operator $-\omega^2(y)u_{xx} - \omega^2(x)u_{yy} = g$ in the unit square are investigated, where the weight function satis£es $\omega(\xi) = \xi^\alpha$ with $\alpha \geq 0$. We propose two multi-level methods in order to solve the resulting system of linear algebraic equations. The £rst method is a multi-grid algorithm with line smoother. A proof of the smoothing property is given. The second method is a BPX-like preconditioner which we call MTS-BPX preconditioner. We show that the upper eigenvalue bound of the MTS-BPX preconditioned system matrix grows proportionally to the level number.

Preprint-Reihe des Chemnitzer SFB 393

SFB393/03-04 **February 2003**

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1 Introduction

In this paper, we consider the following problem: Find $u \in H^1_{0,\omega}(\Omega)$ such that

$$a(u,v) := \int_\Omega (\omega(y))^2 u_x v_x + (\omega(x))^2 u_y v_y = \int_\Omega gv =: \langle g, v\rangle \quad \forall v \in H^1_{0,\omega}(\Omega), \tag{1}$$

where

$$H^1_{0,\omega}(\Omega) = \{u \in L^2(\Omega), \omega(x)u_y, \omega(y)u_x \in L^2(\Omega), u\mid_{\partial\Omega}= 0\}.$$

The domain $\Omega = (0,1)^2$ is the unit square.

ASSUMPTION 1.1. *The weight function $\omega(\xi)$ is assumed to be of the form $\omega(\xi) = \xi^\alpha$ with $\alpha \geq 0$.*

REMARK 1.2. *If $\alpha \neq 0$, the differential operator in* (1) *is not uniformly elliptic in the Sobolev space $H^1_0(\Omega)$, an estimate of the type*

$$a(u,u) \geq \gamma \parallel u \parallel^2_{H^1(\Omega)} \quad \forall u \in H^1_0(\Omega) \tag{2}$$

with a constant $\gamma > 0$ is not satis£ed.

The integrand on the left hand side in (1) is of the type $(\nabla u)^T \mathcal{D}(x,y)\nabla v$ with the diffusion tensor

$$\mathcal{D}(x,y) = \left[\begin{array}{cc} \omega^2(y) & 0 \\ 0 & \omega^2(x) \end{array}\right]. \tag{3}$$

Therefore, the matrix $\mathcal{D}$ is symmetric and positive de£nite for all $(x,y) \in \Omega$, but not uniformly positive de£nite for $\alpha > 0$. Moreover, the matrix $\mathcal{D}$ is bounded for each $(x,y) \in \Omega$. Such problems are called degenerated problems. In the past, degenerated problems have been considered relatively rarely. One reason is the unphysical behaviour of the partial differential equation which is quite unusual in technical applications. One work focusing on this type of partial differential equation is the book of Kufner and Sändig [14]. Nowadays, problems of this type become more and more popular because there are stochastic pde's which have a similar structure. An example of a degenerated stochastic partial differential equation is the Black-Scholes partial differential equation, [17]. Moreover, the solver related to the problem of the subdomains embedded in a domain decomposition preconditioner for the p-version of the £nite element method can be interpreted as h-version fem-discretization matrix of (1) in the case of the weight function $\omega(\xi) = \xi$. We refer to [5], [4] for more details.

We discretize problem (1) by £nite elements. For this purpose, some notation is introduced. Let k be the level of approximation and $n = 2^k$. Let $x^k_{ij} = (\frac{i}{n}, \frac{j}{n})$, where $i,j = 0,\ldots,n$. The domain Ω is divided into congruent, isosceles, right-angled triangles $\tau^{s,k}_{ij}$, where $0 \leq i,j < n$ and $s = 1,2$, see Figure 1. The triangle $\tau^{1,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i,j+1}$, $\tau^{2,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i+1,j}$, see Figure 1. Furthermore, let $\mathcal{E}^k_{ij} = \overline{\tau^{1,k}_{ij} \cup \tau^{2,k}_{ij}}$ be the macro-element

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

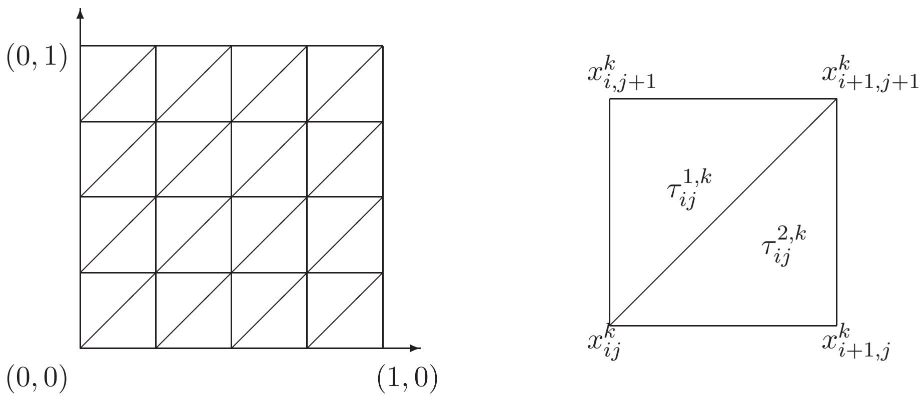

Figure 1: Mesh for the £nite element method (left), Notation within a macro-element $\mathcal{E}^k_{ij}$ (right).

Piecewise linear £nite elements are used on the mesh

$$T_k = \{\tau^{s,k}_{ij}\}^{n-1,n-1,2}_{i=0,j=0,s=1}.$$

The subspace of piecewise linear functions ϕ^k_{ij} with

$$\phi^k_{ij} \in H^1_0(\Omega),\ \phi^k_{ij}\,|_{\tau^{s,k}_{lm}} \in \mathbb{P}^1(\tau^{s,k}_{lm})$$

is denoted by $\mathbb{V}_k$, where $\mathbb{P}^1$ is the space of polynomials of degree ≤ 1. A basis of $\mathbb{V}_k$ is the system of the usual hat-functions $\{\phi^k_{ij}\}^{n-1}_{i,j=1}$ uniquely de£ned by

$$\phi^k_{ij}(x^k_{lm}) = \delta_{il}\delta_{jm} \tag{4}$$

and $\phi^k_{ij} \in \mathbb{V}_k$, where δ_{il} is the Kronecker delta. Now, we can formulate the discretized problem. Find $u^k \in \mathbb{V}_k$ such that

$$a(u^k, v^k) = \langle g, v^k \rangle \quad \forall v^k \in \mathbb{V}_k \tag{5}$$

holds. Problem (5) is equivalent to solving the system of linear algebraic equations

$$K_{\alpha,k}\underline{u}_k = \underline{g}_k, \tag{6}$$

where

$$\begin{aligned} K_{\alpha,k} &= \left[a(\phi^k_{ij}, \phi^k_{lm})\right]^{n-1}_{i,j,l,m=1}, \\ \underline{u}_k &= \left[u_{ij}\right]^{n-1}_{i,j=1}, \\ \underline{g}_k &= \left[\langle g, \phi^k_{lm}\rangle\right]^{n-1}_{l,m=1}. \end{aligned}$$

The index α denotes the parameter of the weight function $\omega(\xi) = \xi^\alpha$. Then, $u^k = \sum^{n-1}_{i,j=1} u_{ij}\phi^k_{ij}$ is the solution of (5). In this paper, we will derive fast solution methods for (6). Because of

the right-angled triangles $\tau_{ij}^{r,k}$, and the diagonal matrix $\mathcal{D}(x,y)$ (3), the matrix $K_{\alpha,k}$ is a sparse matrix with 5-point stencil structure and $\mathcal{O}(n^2)$ nonzero matrix entries.
Therefore, it is important to £nd a method which solves (6) in $\mathcal{O}(n^2)$ arithmetical operations. Using the usual Cholesky decomposition with lexicographic ordering of the unknowns, the arithmetical cost is proportional to n^4, and the memory requirement is of order n^3. Using the method of nested dissection developed by George, [10], the arithmetical cost can be reduced to $\mathcal{O}(n^3)$ and the memory requirement to $\mathcal{O}(n^2 \log(1+n))$, if only the nonzero elements of the matrix are stored. However, this method is not arithmetically optimal, too. Moreover, the order of the arithmetical cost and memory requirement cannot be improved by taking another reordering for the Cholesky decomposition, [11].
Using iterative methods, no additional memory requirement in order to save the matrix $K_{\alpha,k}$ is necessary.
However, ef£cient preconditioners are needed. For systems of £nite element equations arising from the discretization of boundary value problems as e.g. $-u_{xx} - u_{yy} = f$, ef£cient solution techniques are developed in the last two decades. Examples for such solvers are the preconditioned conjugate gradient (pcg) method with BPX preconditioners, [8], or hierarchical basis preconditioners, [21], and multi-grid methods, [12], [13].
However, the differential operator in (1) is not spectrally equivalent to the Laplacian. It is an elliptic, but not uniformly elliptic differential operator, cf. (2). In a certain way, this differential operator can be interpreted as an operator with local anisotropies, where the range of anisotropy ε goes to zero, if the discretization parameter h tends to zero.
A typical anisotropic model problem considered in the literature, see [12], is

$$-\frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^2 u}{\partial y^2} = f, \quad \varepsilon \text{ small.}$$

One iterative method with a rate of convergence independent of the choice of ε is the multi-grid algorithm with a line Gauß-Seidel (GS) smoother, cf. [13]. Bramble and Zhang, [9], considered multi-grid methods in a more general case as for the Laplace equation. They proved multi-grid convergence for differential operators of the type $-(f(x,y)u_x)_x - (g(x,y)u_y)_y$, where $0 < g(x,y) \leq g_{max}$ and $0 < f_{min} < f(x,y) < f_{max}$, i.e. one of the coef£cients can be arbitrarily small. However, both coef£cients can be arbitrarily small in (1). Thus, we have to £nd a modi£ed solution technique.
In [5], the special case of the singular weight function $\omega(\xi) = \xi$ in (1) is considered. Using the techniques of Braess, [7], Schieweck, [18], and P¤aum, [16], a meshsize independent multi-grid convergence rate $\rho < 1$ has been shown. Moreover numerical experiments, see [6], for discretizations of differential operators as (1) indicate a mesh-size independent convergence rate $\rho < 1$ for multi-grid algorithms with semi-coarsening and line-smoother. In [3], a BPX-like preconditioner which we call MTS-BPX preconditioner $\hat{C}_{1,k}$ for $K_{1,k}$ is proposed (i.e. $\alpha = 1$). Numerical experiments indicate a small increasing condition number of $\hat{C}_{1,k}^{-1} K_{1,k}$.
The aim of this paper is to extend the MTS-BPX preconditioner of [3] and the multi-grid algorithm of [5] to the more general problem of $K_{\alpha,k}\underline{w} = \underline{r}$, where $\alpha \geq 0$. This paper is organized as follows. In section 2, the multi-grid algorithm is considered. We state the main assumptions

required for the algebraic convergence theory, the constant in the strengthened Cauchy inequality and the smoothing property. Then, the de£nition of the smoother in [5] for $K_{1,k}$ is generalized to a smoother for $K_{\alpha,k}$. Moreover, a proof of the smoothing property is given. In section 3, the MTS-BPX preconditioner $\hat{C}_{\alpha,k}$ for $K_{\alpha,k}$ is de£ned. Finally, the upper eigenvalue estimate of $\hat{C}_{\alpha,k}^{-1}K_{\alpha,k}$ is proved and some numerical experiments are given.

2 Multi-grid for degenerated problems

In the typical multi-grid proofs, cf. [12], one splits the multi-grid operator in a product of two operators $\mathcal{A}$ and $\mathcal{B}$. One proves a smoothing property for the operator $\mathcal{A}$, whereas an approximation property has to be shown for $\mathcal{B}$. Helpful tools for this aim are the approximation theorems for £nite elements as the Aubin-Nitsche-trick. In order to prove such a result, the boundedness and the ellipticity of the bilinear form are required in the Sobolev space $H^1(\Omega)$. However, the ellipticity of the bilinear form (1) cannot be guaranteed, cf. relation (2).
Another technique in order to prove a mesh-size independent convergence rate has been introduced by Braess, [7]. In this method, the approximation space $\mathbb{V}_k$ is split into a direct sum of the space $\mathbb{V}_{k-1}$ and a complementary space $\mathbb{W}_k$. One obtains a multiplicative solver for the problem on $\mathbb{V}_k$ by solving the problems on $\mathbb{V}_{k-1}$ and $\mathbb{W}_k$. Schieweck, [18], and P¤aum, [16], have extended this technique. This method does not require regularity assumptions to the bilinear form. Moreover, for triangulations of simple geometry as for (5), the required assumptions are quite simple to handle.

REMARK 2.1. *Note that the bilinear form $a(\cdot,\cdot)$ is positive de£nite on the space $\mathbb{V}_k$.*

2.1 Multi-grid algorithm

In this subsection, the multi-grid algorithm in order to solve (6) is introduced. The space $\mathbb{V}_k$ is represented as the direct sum

$$\mathbb{V}_k = \mathbb{V}_{k-1} \oplus \mathbb{W}_k,$$

where

$$\mathbb{W}_k = \operatorname{span}\{\phi_{ij}^k\}_{(i,j)\in N_k}, \tag{7}$$

see e.g. [15], [7], [18], [19], [20]. The index subset $N_k \subset \mathbb{N}^2$ contains the indices of the new nodes on level k and is given by

$$N_k := \{(i,j) \in \mathbb{N}^2, 1 \le i,j \le n-1, i = 2m-1 \text{ or } j = 2m-1, m \in \mathbb{N}\}. \tag{8}$$

Let $u_0 \in \mathbb{V}_k$ be the initial guess. One step $u_1 = MULT(k, u_0, g)$ of the multi-grid algorithm $MULT$ is de£ned recursively as follows.

ALGORITHM 2.2 (**$MULT$**)**.** *Set $l = k$.*

- *If $l > 1$, then do*

1. *Pre-smoothing on $\mathbb{W}_l$: Solve*

$$a(w, v) = \langle g, v\rangle - a(u_0, v) \quad \forall v \in \mathbb{W}_l$$

approximately by using ν steps of a simple iterative method S, the approximate solution is $\tilde{w}$. Set $u_0^1 = u_0 + \tilde{w}$.

2. *Coarse grid correction on $\mathbb{V}_{l-1}$: Find $w \in \mathbb{V}_{l-1}$ such that*

$$a(w, v) = \langle g, v\rangle - a(u_0^1, v) = \langle r, v\rangle \quad \forall v \in \mathbb{V}_{l-1}.$$

Compute an approximate solution $\tilde{w}$ by using μ_{l-1} steps of the algorithm $MULT(l-1, 0, r)$. Set $u_0^2 = u_0^1 + \tilde{w}$.

3. *Post-smoothing on $\mathbb{W}_l$: Solve*

$$a(w, v) = \langle g, v\rangle - a(u_0^2, v) \quad \forall v \in \mathbb{W}_l$$

approximately by using ν steps of a simple iterative method S, the approximate solution is $\tilde{w}$. Set $u_1 = u_0^2 + \tilde{w}$.

- *else*

  - *Solve $a(w, v) = \langle g, v\rangle - a(u_0, v) \quad \forall v \in \mathbb{V}_1$ exactly.*

- *end-if.*

2.2 Algebraic convergence theory for multi-grid

Our aim is to prove the convergence of the multi-grid Algorithm 2.2 $MULT$ in order to solve (6) using $\mu = \mu_l = 3$ and a special line smoother $S = \ _{0,k}$ on level k which will be de£ned in (30). From [16], [18], the following convergence theorem is known for multi-grid algorithms of the type of the algorithm $MULT$.

THEOREM 2.3. *Let us assume that the following assumptions are ful£lled.*

- *Let $a(\cdot, \cdot)$ be a symmetric and positive de£nite bilinear form on $\mathbb{V}_k$. Let*

$$\| \cdot \|_a^2 := a(\cdot, \cdot)$$

be the energy norm.

- *Let S be a smoother satisfying*

$$\| S^\nu w \|_a \leq c\rho^\nu \| w \|_a \quad \forall w \in \mathbb{W}_k, \tag{9}$$

where $0 \leq \rho < 1$ independent of k and $c > 0$.

- *There is a constant* $0 \leq \gamma < 1$ *independent of* k *such that*

$$(a(v,w))^2 \leq \gamma^2 a(v,v)a(w,w) \quad \forall w \in \mathbb{W}_k, \forall v \in \mathbb{V}_{k-1} \tag{10}$$

holds.

- *Let* $u_{j+1,k} = MULT(k, u_{j,k}, g)$*, let* u^* *be the exact solution of* (6) *and let*

$$\sigma_k = \sup_{u_{j,k}-u^* \in \mathbb{V}_k} \frac{\| u_{j+1,k} - u^* \|_a}{\| u_{j,k} - u^* \|_a} \tag{11}$$

be the convergence rate of $MULT$ *in the energy norm with* ν *smoothing operations.*

Then, the recursion formula

$$\sigma_k \leq \sigma_{k-1}^{\mu_{k-1}} + (1 - \sigma_{k-1}^{\mu_{k-1}})(c\rho^\nu + (1 - c\rho^\nu)\gamma)^2 \tag{12}$$

is valid.

Proof: This theorem has been proved by Schieweck, Theorem 2.2 of [18] with $\rho = \rho_1 = \rho_3$, and P¤aum, see Theorem 4 of [16]. □
This theorem is the key in order to prove a mesh-size independent convergence rate $\sigma_k < \sigma < 1$. If $\gamma^2 < \frac{\mu-1}{\mu}$, the estimate $\sigma_k < \sigma < 1$ follows for $\nu \geq \nu_0$, cf. [18], [5]. By Remark 2.1, the £rst assumption of Theorem 2.3 is satis£ed for the bilinear form $a(\cdot,\cdot)$ (1). The constant γ^2 in the strengthened Cauchy inequality (10) can be determined by the techniques described in [5].
In [5], Theorem 2.2, we have proved

$$(a(v,w))^2 \leq \frac{95}{176} a(v,v)a(w,w) \quad \forall v \in \mathbb{V}_k, \forall w \in \mathbb{W}_{k+1}$$

for the bilinear form $a(\cdot,\cdot)$ (1) with the weight function $\omega(\xi) = \xi$, i.e. $\alpha = 1$. In subsection 2.4, we want to prove (9) for more general weight functions as in [5]. Therefore, the stiffness matrices restricted to the elements $\tau_{ij}^{1,k}$ and $\tau_{ij}^{2,k}$ are needed. This is done in subsection 2.3.

2.3 Calculation of the macroelement stiffness matrices

In this subsection, we determine the stiffness matrix on the macro-elements $\mathcal{E}_{ij}^k$ with respect to the basis built by the basis functions of $\mathbb{W}_{k+1} \mid_{\mathcal{E}_{ij}^k}$. We start with the introduction of the basis functions on $\mathcal{E}_{ij}^k$. Note that the triangle $\tau_{ij}^{2,k}$ is the union of the triangles $\tau_{2i,2j}^{2,k+1}$, $\tau_{2i+1,2j}^{1,k+1}$, $\tau_{2i+1,2j}^{2,k+1}$, and $\tau_{2i+1,2j+1}^{2,k+1}$, the triangle $\tau_{ij}^{1,k}$ is the union of the triangles $\tau_{2i,2j}^{1,k+1}$, $\tau_{2i,2j+1}^{1,k+1}$, $\tau_{2i,2j+1}^{2,k+1}$, and $\tau_{2i+1,2j+1}^{1,k+1}$. The nodes x_{ij}^k, $x_{i,j+1}^k$, $x_{i+1,j}^k$, and $x_{i+1,j+1}^k$ are the coarse grid nodes, the nodes $x_{2i+1,2j}^{k+1}$, $x_{2i,2j+1}^{k+1}$, $x_{2i+2,2j+1}^{k+1}$, $x_{2i+1,2j+2}^{k+1}$, and $x_{2i+1,2j+1}^{k+1}$ are new in level $k+1$, compare Figure 2. Using this notation, we have

$$\mathbb{W}_{k+1} \mid_{\mathcal{E}_{ij}^k} = \operatorname{span}\{\phi_{lm}^{k+1}\}_{(l,m) \in N_{ij}^{\mathbb{W}_{k+1}}}. \tag{13}$$

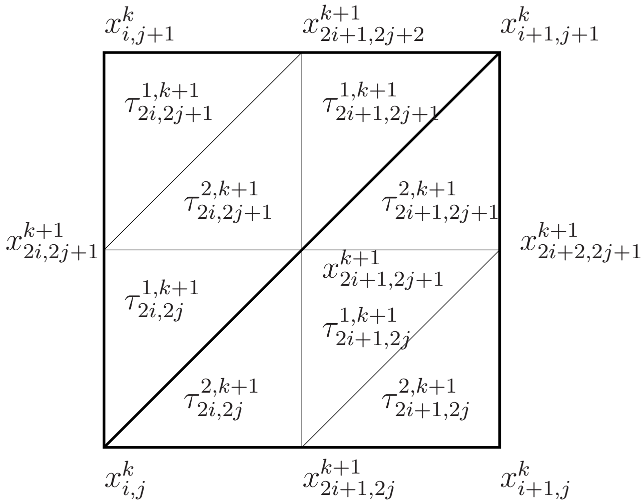

Figure 2: Local numbering of the nodes and sub-triangles of $\mathcal{E}^k_{ij}$.

For reasons of simplicity, we write only ϕ^{k+1}_{lm} instead of $\phi^{k+1}_{lm} \mid_{\mathcal{E}^k_{ij}}$ for the restriction of ϕ^{k+1}_{lm} on $\mathcal{E}^k_{ij}$. The index set in (13) is given by

$$N^{\mathbb{W}_{k+1}}_{ij} = N_{k+1} \cap \{(l,m) \in \mathbb{N}^2_0, 2i \le l \le 2i+2, 2j \le m \le 2j+2\},$$

where N_{k+1} was de£ned in (8). Since $\mathbb{V}_k \subset H^1_0(\Omega)$, some modi£cations are necessary for boundary macro-elements $\mathcal{E}^k_{ij}$, i.e. with $i=0$, $j=0$, $i=n-1$, or $j=n-1$.
On the elements $\tau^{2,k}_{ij}$, we introduce the matrices

$$J_{q,ij} := \left[a^{\tau^{q,k}_{ij}}(\phi^{k+1}_{lm}, \phi^{k+1}_{rs})\right]_{(r,s),(l,m)\in N^{q,\mathbb{W}_{k+1}}_{ij}}$$

with $N^{q,\mathbb{W}_{k+1}}_{ij} := T^q_{ij} \cap N^{\mathbb{W}_{k+1}}_{ij}$, where $T^1_{ij} := \{(l,m) \in \mathbb{N}^2_0, l-m \le i-j\}$ and $T^2_{ij} := \{(l,m) \in \mathbb{N}^2_0, l-m \ge i-j\}$. Namely, we obtain

$$\begin{aligned} N^{2,\mathbb{W}_{k+1}}_{ij} &= \{(2i+1,2j),(2i+2,2j+1),(2i+1,2j+1)\} \quad \text{and} \\ N^{1,\mathbb{W}_{k+1}}_{ij} &= \{(2i,2j+1),(2i+1,2j+2),(2i+1,2j+1)\}. \end{aligned}$$

The entries of the matrices $J_{q,ij}$ can be determined by a straightforward calculation. We compute those for the case of a general weight function $\omega(\xi)$. The following parameters depending on the integer j are introduced:

$$\begin{aligned} d_j &= \frac{1}{4}\int_{\tau^{1,k+1}_{2i,2j}\cup\tau^{2,k+1}_{2i,2j+1}} (\omega(y))^2 \,\mathrm{d}(x,y), \\ e_j &= \frac{1}{4}\int_{\tau^{2,k+1}_{2i,2j}\cup\tau^{2,k+1}_{2i+1,2j}} (\omega(y))^2 \,\mathrm{d}(x,y), \\ f_j &= \frac{1}{4}\int_{\tau^{1,k+1}_{2i,2j+1}\cup\tau^{1,k+1}_{2i+1,2j+1}} (\omega(y))^2 \,\mathrm{d}(x,y). \end{aligned} \tag{14}$$

Note that d_j, e_j and f_j are independent of the integer i. The values d_i, e_i and f_i are de£ned by a permutation of i and j, x and y, and $\tau_{ij}^{2,k}$ and $\tau_{ji}^{1,k}$ in (14). One obtains the following proposition.

PROPOSITION 2.4. *Let* $0 < i, j < n-1$. *Then, one has*

$$J_{2,ij} \quad = \quad 4\begin{bmatrix} d_i + e_j & 0 & -d_i \\ 0 & f_i + d_j & -d_j \\ -d_i & -d_j & d_i + d_j \end{bmatrix}. \tag{15}$$

By exchanging the indices i and j in (15), one derives the matrices $J_{1,ij} = J_{2,ji}$.

2.4 Construction of the smoother

In order to apply multi-grid to the linear system (6), we need an ef£cient smoother. This smoother will be contructed by the local behaviour of the differential operator. An idea of Axelsson and Padiy, [1], for anisotropic problems is extended to bilinear forms as in problem (1). This smoother operates on the space $\mathbb{W}_{k+1}$ only.

Consider the triangle $\tau_{ij}^{2,k}$. For our discussion, only the sub-matrices $J_{s,ij}$, where $0 \leq i, j \leq n-1$ and $s = 1, 2$, are needed which correspond to the nodal basis functions on $\mathbb{W}_{k+1}$. The two cases $i < j$ and $i \geq j$ are discussed. We start with $i < j$. By Proposition 2.4,

$$J_{2,ij} = 4\begin{bmatrix} d_i + e_j & 0 & -d_i \\ 0 & f_i + d_j & -d_j \\ -d_i & -d_j & d_i + d_j \end{bmatrix}.$$

The index k is omitted. For $i < j$, the matrix

$$M_{2,ij} = 4\begin{bmatrix} d_i + e_j & 0 & 0 \\ 0 & f_i + d_j & -d_j \\ 0 & -d_j & d_i + d_j \end{bmatrix} \tag{16}$$

is introduced. In the matrix $M_{2,ij}$, we set all off diagonal entries of $J_{2,ij}$ to 0 which have relatively small absolute values in comparison to the corresponding main diagonal entries. Since ω is monotonic increasing, the relation $d_i < d_j$ is valid for $i < j$. Thus, we set the $-d_i$ entries of $J_{2,ij}$ in $M_{2,ij}$ to 0. We prove now the following lemma.

LEMMA 2.5. *For* $0 \leq i < j < n$, *the eigenvalue estimates*

$$\begin{aligned} \lambda_{min}\left({M_{2,ij}}^{-1} J_{2,ij}\right) \quad &\geq \quad 1 - \frac{1}{3}\sqrt{3} \quad \textit{and} \\ \lambda_{max}\left({M_{2,ij}}^{-1} J_{2,ij}\right) \quad &\leq \quad 1 + \frac{1}{3}\sqrt{3} \end{aligned}$$

hold.

Proof: Let

$$\beta = d_i f_i + d_i d_j + f_i d_j.$$

Then, we have

$$M_{2,ij}^{-1} J_{2,ij} = \begin{bmatrix} 1 & 0 & \frac{-d_i}{d_i+e_j} \\ \frac{-d_i d_j}{\beta} & 1 & 0 \\ \frac{-d_i f_i - d_i d_j}{\beta} & 0 & 1 \end{bmatrix}.$$

This matrix has the characteristical polynomial

$$\det(\lambda I - M_{2,ij}^{-1} J_{2,ij}) = (\lambda - 1)\left((1-\lambda)^2 - \frac{d_i}{d_i+e_j}\frac{d_i f_i + d_i d_j}{d_i f_i + d_i d_j + f_i d_j}\right).$$

The roots λ_i, $i = 1, 2, 3$, of this polynomial are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_{2,3} &= 1 \pm \sqrt{\rho}, \end{aligned}$$

where

$$\rho = \frac{d_i}{d_i+e_j}\frac{d_i f_i + d_i d_j}{d_i f_i + d_i d_j + f_i d_j}. \tag{17}$$

Note that for all i and j, the values d_j, e_j and f_j are mean values of the positive function $(\omega(y))^2$ over the union of two triangles having a volume of $\frac{1}{8n^2}$. By the monotony of the weight function, the inequality $d_i \le f_i$ holds for all $i \in \mathbb{N}$, cf. (14) and Figure 2. Therefore,

$$\frac{d_i f_i + d_i d_j}{d_i f_i + d_i d_j + f_i d_j} \le \frac{d_i f_i + d_i d_j}{d_i f_i + 2 d_i d_j} = \frac{f_i + d_j}{f_i + 2d_j} = \frac{1}{1 + \frac{1}{\frac{f_i}{d_j}+1}}.$$

Moreover, by $i \le j-1$ and the monotony of the weight function, one has $\omega(x) \le \omega(y)$ for all $x, y \in \tau_{ij}^{2,k}$. Thus, by integration over sub-triangles of $\tau_{ij}^{2,k}$ with volume $\frac{1}{8n^2}$, cf. Figure 2,

$$\begin{aligned} f_i = \frac{1}{4}\int_{\tau_{2i+1,2j}^{2,k+1} \cup \tau_{2i+1,2j+1}^{2,k+1}} (\omega(x))^2 \, \mathrm{d}(x,y) &\le \frac{1}{4}\int_{\tau_{2i+1,2j}^{1,k+1} \cup \tau_{2i+1,2j+1}^{2,k+1}} (\omega(y))^2 \, \mathrm{d}(x,y) = d_j, \\ d_i = \frac{1}{4}\int_{\tau_{2i,2j}^{2,k+1} \cup \tau_{2i+1,2j}^{1,k+1}} (\omega(x))^2 \, \mathrm{d}(x,y) &\le \frac{1}{4}\int_{\tau_{2i,2j}^{2,k+1} \cup \tau_{2i+1,2j}^{2,k+1}} (\omega(y))^2 \, \mathrm{d}(x,y) = e_j. \end{aligned}$$

Therefore, we obtain the estimates

$$\frac{d_i f_i + d_i d_j}{d_i f_i + d_i d_j + f_i d_j} \le \frac{2}{3} \tag{18}$$

and

$$\frac{d_i}{d_i + e_j} \le \frac{1}{2}. \tag{19}$$

Inserting the estimates (18) and (19) into (17), one has

$$1-\sqrt{\frac{1}{3}} \le \lambda_3 \le \lambda_1 \le \lambda_2 \le 1+\sqrt{\frac{1}{3}}.$$

Hence, the assertion follows immediately. $\square$
Now, consider the case $i \geq j$. Introducing the matrix

$$M_{2,ij} = 4\begin{bmatrix} d_i+e_j & 0 & -d_i \\ 0 & f_i+d_j & 0 \\ -d_i & 0 & d_i+d_j \end{bmatrix}, \tag{20}$$

we will show that $\kappa\left({M_{2,ij}}^{-1}J_{2,ij}\right) \le c$ independent of the parameters j, i, and n. In order to prove this result, the following estimate concerning the weight function is necessary.

LEMMA 2.6. *Let $\omega(\cdot)$ satisfy Assumption 1.1. Then, one has the inequality*

$$0 \le \left(\omega\left(y+\frac{1}{2n}\right)\right)^2 \le c(\omega(y))^2 \quad \forall y \ge \frac{1}{n}, \tag{21}$$

where the constant c is independent of n and y.

The inequality

$$\left(\frac{\xi+1.5}{\xi+1}\right)^{2\alpha} = \left(1+\frac{1}{2\xi+2}\right)^{2\alpha} \le \left(\frac{3}{2}\right)^{2\alpha} = c$$

holds for all $\xi \geq 0$ and $\alpha \geq 0$ with $c = \left(\frac{3}{2}\right)^{2\alpha}$. Thus,

$$\begin{aligned} \left(\xi+\frac{3}{2}\right)^{2\alpha} &\le c\left(\xi+1\right)^{2\alpha}, \quad \text{or} \\ \left(\frac{\xi+\frac{3}{2}}{n}\right)^{2\alpha} &\le c\left(\frac{\xi+1}{n}\right)^{2\alpha} \end{aligned}$$

with some $n > 0$. Using $(\omega(\xi))^2 = \xi^{2\alpha}$, we have $\left(\omega\left(\frac{\xi+\frac{3}{2}}{n}\right)\right)^2 \le c\left(\omega\left(\frac{\xi+1}{n}\right)\right)^2$, or, substituting $y = \frac{\xi+1}{n}$,

$$0 \le \left(\omega\left(y+\frac{1}{2n}\right)\right)^2 \le c(\omega(y))^2 \quad \forall y \ge \frac{1}{n}$$

which is the desired result. $\square$

LEMMA 2.7. *For $0 \le j \le i < n$, one has*

$$\begin{aligned} \lambda_{min}\left({M_{2,ij}}^{-1}J_{2,ij}\right) &\asymp 1 \quad \textit{and} \\ \lambda_{max}\left({M_{2,ij}}^{-1}J_{2,ij}\right) &\asymp 1. \end{aligned}$$

The constants are independent of i, j *and* n. *For* $\omega(\xi) = \xi$, *the eigenvalue estimates*

$$\begin{aligned}
\lambda_{min}\left({M_{2,ij}}^{-1} J_{2,ij}\right) &\geq 1 - \frac{2}{11}\sqrt{11} \quad \textit{and} \\
\lambda_{max}\left({M_{2,ij}}^{-1} J_{2,ij}\right) &\leq 1 + \frac{2}{11}\sqrt{11}
\end{aligned}$$

are valid.

Proof: We start with the case $i < n-1$ and $j > 0$. The proof is similar to the proof of Lemma 2.5. A short calculation yields

$$\det(\lambda I - M_{2,ij}^{-1} J_{2,ij}) = (\lambda - 1)\left((\lambda-1)^2 - \frac{d_j}{d_j + f_i}\frac{d_i d_j + e_j d_j}{d_i e_j + d_i d_j + e_j d_j}\right).$$

By $i \geq j$ and the monotony of the weight function ω, we have

$$\begin{aligned}
\int_{\tau_{2i+1,2j}^{2,k+1}} (\omega(x))^2 \,\mathrm{d}(x,y) &= \int_{\tau_{2i+1,2j+1}^{2,k+1}} (\omega(x))^2 \,\mathrm{d}(x,y) \\
&\geq \int_{\tau_{2j+1,2i}^{2,k+1}} (\omega(x))^2 \,\mathrm{d}(x,y) \\
&= \int_{\tau_{2i,2j+1}^{1,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y) \\
&\geq \int_{\tau_{2i+1,2j}^{1,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y).
\end{aligned} \tag{22}$$

For the same reason,

$$\int_{\tau_{2i,2j}^{2,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y) \leq \int_{\tau_{2i+1,2j}^{1,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y). \tag{23}$$

Using (22) and (23),

$$f_i = \int_{\tau_{2i+1,2j}^{2,k+1} \cup \tau_{2i+1,2j+1}^{2,k+1}} (\omega(x))^2 \,\mathrm{d}(x,y) \geq \int_{\tau_{2i,2j}^{2,k+1} \cup \tau_{2i+1,2j}^{1,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y) = d_j. \tag{24}$$

By Lemma 2.6, we have

$$0 \leq \left(\omega\left(y + \frac{1}{2n}\right)\right)^2 \leq c(\omega(y))^2 \quad \forall y \geq \frac{1}{n}.$$

Integration over $\tau_{2i+1,2j}^{2,k+1}$ gives

$$\int_{\tau_{2i+1,2j}^{2,k+1}} \left(\omega\left(y + \frac{1}{2n}\right)\right)^2 \,\mathrm{d}(x,y) \leq c \int_{\tau_{2i+1,2j}^{2,k+1}} (\omega(y))^2 \,\mathrm{d}(x,y)$$

with $j \geq 1$. With a change of variables $\tilde{y} = y + \frac{1}{2n}$ in the left integral, the integration will be done now over $\tau^{2,k+1}_{2i+1,2j+1}$,

$$\int_{\tau^{2,k+1}_{2i+1,2j+1}} (\omega(y))^2 \, \mathrm{d}(x,y) \leq c \int_{\tau^{2,k+1}_{2i+1,2j}} (\omega(y))^2 \, \mathrm{d}(x,y). \tag{25}$$

Using (25),

$$\int_{\tau^{2,k+1}_{2i,2j}} (\omega(y))^2 \, \mathrm{d}(x,y) = \int_{\tau^{2,k+1}_{2i+1,2j}} (\omega(y))^2 \, \mathrm{d}(x,y),$$

and

$$\int_{\tau^{1,k+1}_{2i+1,2j}} (\omega(y))^2 \, \mathrm{d}(x,y) \leq \int_{\tau^{2,k+1}_{2i+1,2j+1}} (\omega(y))^2 \, \mathrm{d}(x,y),$$

we have

$$d_j = \frac{1}{4} \int_{\tau^{1,k+1}_{2i+1,2j} \cup \tau^{2,k+1}_{2i+1,2j+1}} (\omega(y))^2 \, \mathrm{d}(x,y) \leq \frac{c}{4} \int_{\tau^{2,k+1}_{2i,2j} \cup \tau^{2,k+1}_{2i+1,2j}} (\omega(y))^2 \, \mathrm{d}(x,y) = e_j. \tag{26}$$

Using (26) and $d_j \leq d_i$ for $j \leq i$, one can estimate

$$e_j d_j + d_j d_i \leq (c+1) e_j d_i.$$

Equivalently, one obtains

$$(c+2)(e_j d_j + d_j d_i) \leq (c+1)(e_j d_i + e_j d_j + d_j d_i).$$

Together with (24), the assertion follows as in the proof of Lemma 2.5.
Consider now $i = n-1$. Then, the second row and column of $M_{2,ij}$ and $J_{2,ij}$ has to be canceled. Thus, the matrices $M_{2,n-1,j}$ and $J_{2,n-1,j}$ are identical and

$$\lambda_1(M^{-1}_{2,n-1,j} J_{2,n-1,j}) = \lambda_2(M^{-1}_{2,n-1,j} J_{2,n-1,j}) = 1.$$

The last case is $j = 0$. We have to omit the £rst row and column in $M_{2,i,0}$ and $J_{2,i,0}$. A short calculation shows

$$\det(\lambda I - M^{-1}_{2,i,0} J_{2,i,0}) = (\lambda - 1)^2 - \frac{d_0}{f_i + d_0} \frac{d_0}{d_0 + d_i}.$$

Since $d_0 \leq d_i$ and $d_0 \leq f_i$ for $i \geq 0$, cf. relation (24), $\frac{d_0}{d_0 + d_i} \leq \frac{1}{2}$ and $\frac{d_0}{d_0 + f_i} \leq \frac{1}{2}$ follows. Hence, the estimates

$$\frac{1}{2} \leq \lambda_2 < \lambda_1 \leq \frac{3}{2}$$

are obtained for the roots of the characteristical polynomial of the matrix $M^{-1}_{2,i,0} J_{2,i,0}$. $\square$

In (16), (20), we have de£ned a local preconditioner $M_{2,ij}$ for the macro-element stiffness matrices $J_{2,ij}$ corresponding to the triangle $\tau_{ij}^{2,k}$. On the triangles $\tau_{ij}^{1,k}$, we de£ne matrices $M_{1,ij}$ in the same way as $M_{2,ij}$ for $\tau_{ij}^{2,k}$:

$$M_{1,ij} = \begin{cases} 4\begin{bmatrix} e_i + d_j & 0 & -d_j \\ 0 & d_i + f_j & 0 \\ -d_j & 0 & d_i + d_j \end{bmatrix} & \text{for} \quad i \leq j, \\ \\ 4\begin{bmatrix} e_i + d_j & 0 & 0 \\ 0 & d_i + f_j & -d_i \\ 0 & -d_i & d_i + d_j \end{bmatrix} & \text{for} \quad i > j. \end{cases} \tag{27}$$

REMARK 2.8. *By the symmetry of the differential operator with respect to the variables x and y, we obtain the same results for the triangles $\tau_{ij}^{1,k}$ as in Lemmata 2.5 and 2.7.*

Following [5], a global preconditioner $M_{\mathbb{W}_{k+1}}$ for $K_{\mathbb{W}_{k+1}}$ is de£ned using the local matrices $M_{s,ij}$, where $0 \leq i, j \leq n-1$, $s = 1, 2$. The matrix $K_{\mathbb{W}_{k+1}}$ is de£ned as stiffness matrix K_{k+1} (6) restricted to the space $\mathbb{W}_{k+1}$, i.e.

$$K_{\mathbb{W}_{k+1}} = \left[a(\phi_{lm}^{k+1}, \phi_{ij}^{k+1})\right]_{(i,j),(l,m)\in N_{k+1}}$$

(compare (8), (9)). The matrix $K_{\mathbb{W}_{k+1}}$ is the result of assembling the local stiffness matrices $J_{s,ij}$, $s = 1, 2$ and $i, j = 0, \ldots, n-1$, i.e.

$$K_{\mathbb{W}_{k+1}} = \sum_{s=1}^{2} \sum_{i,j=0}^{n-1} L_{s,ij}^T J_{s,ij} L_{s,ij}. \tag{28}$$

The matrices $L_{s,ij} \in \mathbb{R}^{3\times 3\cdot 4^{k-1}-2^k}$ are the usual £nite element connectivity matrices. Since

$$(2^k - 1)^2 - (2^{k-1} - 1)^2 = 3 \cdot 4^{k-1} - 2^k,$$

the dimension of $L_{s,ij}$ is $3 \times 3(4^{k-1} - 2^k)$.

DEFINITION 2.9. *We de£ne the matrix $M_{\mathbb{W}_{k+1}}$ by*

$$M_{\mathbb{W}_{k+1}} = \sum_{s=1}^{2} \sum_{i,j=0}^{n-1} L_{s,ij}^T M_{s,ij} L_{s,ij}. \tag{29}$$

Because of the properties of the local preconditioners $M_{s,ij}$, the matrix $M_{\mathbb{W}_{k+1}}$ is a good preconditioner for $K_{\mathbb{W}_{k+1}}$. This result is stated as main theorem of this subsection.

THEOREM 2.10. *Let $\omega(\xi)$ satisfy Assumption 1.1, let $M_{\mathbb{W}_{k+1}}$ and $K_{\mathbb{W}_{k+1}}$ be de£ned in* (29) *and* (28)*, respectively. Then, one obtains*

$$\begin{aligned} \lambda_{min}\left((M_{\mathbb{W}_{k+1}})^{-1} K_{\mathbb{W}_{k+1}}\right) &\asymp 1, \\ \lambda_{max}\left((M_{\mathbb{W}_{k+1}})^{-1} K_{\mathbb{W}_{k+1}}\right) &\asymp 1. \end{aligned}$$

Proof: Take Lemma 2.5 of [5]. By Lemma 2.5 and Lemma 2.7, and Remark 2.8, the assertions follow. □

REMARK 2.11. *This result can be extended to more general weight functions ω. The weight function should ful£ll an estimate of the type* (26) *which means that the weight function does not change rapidly. Another possible assumption is that the weight function $\omega(\xi) \geq 0$ satis£es the following properties:*

- *ω is monotonic increasing,*
- *ω is Lipschitz-continuous with a Lipschitz constant L,*
- *$\omega(\xi) \geq \frac{c}{\xi}$ for $\xi \in (0, \delta)$, $\delta > 0$ with some $c > 0$.*

Proof: Using the last assumption and the monotony of ω,

$$\omega\,(y) \geq \frac{c}{2n} \quad \forall y \geq \frac{1}{n}.$$

Therefore, $\frac{L}{2n} + \omega(y) \leq \left(1 + \frac{L}{c}\right)\omega(y)$. By the monotony of w and the Lipschitz continuity, one derives

$$\omega\left(y + \frac{1}{2n}\right) \leq \frac{L}{2n} + \omega(y) \leq \left(1 + \frac{L}{c}\right)\omega(y)$$

which gives (26).□

Applying Theorem 2.10, a preconditioned Richardson iteration can be built as a preconditioned simple iteration method. The error transion operator $_{\alpha,k+1}$ of this method is de£ned by

$$S_{\alpha,k+1} = I - \zeta(M_{\mathbb{W}_{k+1}})^{-1}K_{\mathbb{W}_{k+1}}, \tag{30}$$

where $S_{\alpha,k+1}$ denotes the matrix representation of $_{\alpha,k+1}$ by the usual fem-isomorphism. This smoother $S = \ _{\alpha,k+1}$ can be used for the Algorithm $MULT$.

COROLLARY 2.12. *Let*

$$\| \, w \, \|_a^2 = a(w,w)$$

be the energy norm of the bilinear form a. Then, for all $w \in \mathbb{W}_{k+1}$

$$\| \ _{\alpha,k+1}^{\nu} w \, \|_a \leq \rho^\nu \, \| \, w \, \|_a$$

holds, where $\zeta = 1$ and $\rho < \rho_0 < 1$.

Proof: The proof is similar to the proof of Corollary 2.3 in [5]. □

2.5 Interpretation of the smoother

In order to apply the smoother $S_{\alpha,l}$ (30), a linear system of the type

$$M_{\mathbb{W}_l}\underline{u} = \underline{g}, \quad l = 1, \ldots, k \tag{31}$$

has to be solved. Therefore, it is important to £nd an ef£cient solution technique for the system (31). In this subsection, it will be shown that $M_{\mathbb{W}_k}$ is a block diagonal matrix consisting of tridiagonal blocks. Then, using Cholesky/Crout-decomposition, the system (31) can be solved in $\mathcal{O}(m_k)$ arithmetical operations, where m_k is the number of unknowns on level k. Furthermore, we show that the smoother $S_{\alpha,k}$ (30) is a line smoother operating on lines ℓ_{2m-1} which will be de£ned below. According to (16), (20), and (27), the matrix $M_{\mathbb{W}_k}$ has the structure

$$M_{\mathbb{W}_k} = \mathrm{diag}(K_{\mathbb{W}_k}) + R,$$

where $\mathrm{diag}(K_{\mathbb{W}_k})$ is the diagonal part of the matrix $K_{\mathbb{W}_k}$, de£ned in (28). The matrix R will be de£ned below. Let $b : \mathbb{W}_k \times \mathbb{W}_k \to \mathbb{R}$ be the following non-symmetric bilinear form uniquely determined by the values of the basis functions $\{\phi_{ij}^k\}_{(i,j)\in N_k} \in \mathbb{W}_k$

$$b(\phi_{ij}^k, \phi_{lm}^k) = \begin{cases} a(\phi_{ij}^k, \phi_{lm}^k) & \text{if} \quad \begin{array}{lll} i = l = 2r-1, & j = 2, \ldots, i, & m = j-1 \\ j = m = 2r-1, & i = 2, \ldots, j, & l = i-1 \end{array} \\ 0 & \qquad\qquad\qquad \text{otherwise} \end{cases}$$

for $r = 1, \ldots, \frac{n}{2}$. Note that by this de£nition, $a(\phi_{ij}^k, \phi_{lm}^k)$ is equal to the element $(i,j), (l,m)$ of the matrix $K_{\alpha,k}$, if $i = l = 2r-1$, $j = 2, \ldots, i$, $m = j-1$, or $j = m = 2r-1$, $i = 2, \ldots, j$, $l = i-1$. The matrix R is de£ned as the symmetric part of the bilinear form b. More precisely, let

$$R = \left[b(\phi_{ij}^k, \phi_{lm}^k) + b(\phi_{lm}^k, \phi_{ij}^k)\right]_{(i,j),(l,m)\in N_k}.$$

After a proper permutation P, we have

$$M_{\mathbb{W}_k} = P^T \mathrm{blockdiag}\left[M_{\mathbb{W}_k,r}\right]_{r=0}^{\frac{n}{2}} P$$

with

$$M_{\mathbb{W}_k,r} = \begin{cases} \left[a(\phi_{ij}^k, \phi_{lm}^k)\right]_{(i,j),(l,m)\in\tilde{N}_{2r-1}} & \text{for} \quad r > 0 \\ \left[a(\phi_{ij}^k, \phi_{lm}^k)\right]_{(i,j),(l,m)\in\cup_{r=1}^{n/2-1}(\tilde{N}_{2r}\cap N_{k+1})} & \text{for} \quad r = 0 \end{cases}.$$

The index set $\tilde{N}_r$ is de£ned as

$$\tilde{N}_r = \left\{(i,j),(l,m) \in \{1,\ldots,r\}^4 : \ i = l = r \quad \text{or} \quad j = m = r\right\} \tag{32}$$

and N_k has been de£ned in (8). Thus, the matrices $M_{\mathbb{W}_k,r}$, $r \geq 1$, are tridiagonal matrices and the matrix $M_{\mathbb{W}_k,0}$ is a diagonal matrix. The shape functions of one block $M_{\mathbb{W}_k,r}$ correspond to one edge of the left picture of Figure 3 which has been marked by a bold line. Therefore, the system (31) can be solved using Cholesky decomposition in $\mathcal{O}(n^2)$ ¤ops. Hence, the operation $S_{\alpha,k}\underline{w}$ is arithmetically optimal. Additionally, we build a smoother $\tilde{S}_{\alpha,k} = I - \omega L_{\alpha,k}^{-1} K_{\alpha,k}$ which

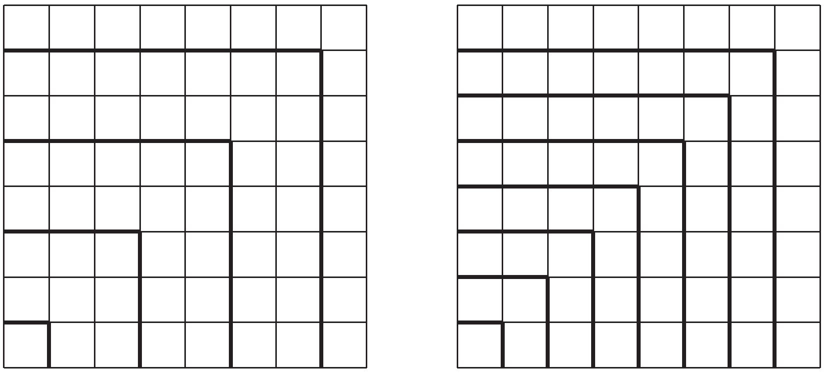

Figure 3: Nonzero entries of the matrices R (left) and $\tilde{R}$ (right).

uses the ideas of (30). This smoother operates on the space $\mathbb{V}_k$. Let

$$L_{\alpha,k} = \operatorname{diag}(K_{\alpha,k}) + \tilde{R}, \tag{33}$$

where

$$\tilde{R} = \left[\tilde{b}(\phi_{ij}^k, \phi_{lm}^k) + \tilde{b}(\phi_{lm}^k, \phi_{ij}^k)\right]_{(i,j),(l,m)=(1,1)}^{(n-1,n-1)},$$

with the bilinear form $\tilde{b} : \mathbb{V}_k \times \mathbb{V}_k \to \mathbb{R}$

$$\tilde{b}(\phi_{ij}^k, \phi_{lm}^k) = \begin{cases} a(\phi_{ij}^k, \phi_{lm}^k) & \text{if} \quad \begin{array}{ll} & i = l = r, \quad j = 2, \ldots, i, \quad m = j-1 \\ \text{or} & j = m = r, \quad i = 2, \ldots, j, \quad l = i-1 \end{array} \\ 0 & \qquad\qquad \text{otherwise} \end{cases} \tag{34}$$

for $r = 1, \ldots, n-1$. As well as $S_{\alpha,k}$ (30), $\tilde{S}_{\alpha,k}$ is a line smoother. However, it operates on each bold line in the right picture of Figure 3. The matrix $L_{\alpha,k}$ is a block diagonal matrix of tridiagonal blocks. After a proper permutation P,

$$L_{\alpha,k} = P^T \text{blockdiag}\,[L_{k,r}]_{r=1}^{n-1}\, P$$

where

$$L_{k,r} = \left[a(\phi_{ij}^k, \phi_{lm}^k)\right]_{(i,j),(l,m)\in\tilde{N}_r}$$

with the index set $\tilde{N}_r$ (32). The matrices $L_{k,r}$ are tridiagonal matrices, where the shape functions correspond to nodes marked by one bold line in the right picture of Figure 3. Analogously to $S_{\alpha,k}$, the operation

$$\tilde{S}_{\alpha,k}\underline{w} = \underline{r}$$

can be done arithmetically optimal in $\mathcal{O}(n^2)$ ¤ops using Cholesky- or Crout-decompostion.

3 BPX preconditioner

3.1 De£nition of the preconditioner

Recall the £nite element discretization of problem (1):
Find $u \in \mathbb{V}_k$ such that

$$\int_\Omega \left(\omega^2(y)u_x v_x + \omega^2(x)u_y v_y\right)\, \mathrm{d}(x,y) = \int_\Omega fv\, \mathrm{d}(x,y) \tag{35}$$

holds for all $v \in \mathbb{V}_k$ with a weight function $\omega(\xi)$ satisfying Assumption 1.1.
For the ef£cient solution of systems of linear equations arising from discretizations of uniformly elliptic problems by £nite elements, Bramble, Pasciak, and Xu have developed a preconditioner, [8], which has been called the BPX preconditioner. For this preconditioner, the spectral equivalence to the original stiffness matrix can be shown. Later, this preconditioner has been improved by the multiple diagonal scaling version, [22]. As mentioned in [2], a BPX preconditioner with multiple diagonal scaling does not show good numerical results in order to solve $K_{\alpha,k}\underline{u} = \underline{g}_k$, the system of linear algebraic equations resulting from the £nite element discretization of (35). One reason is that this preconditioner cannot handle the anisotropies resulting from the degenerated elliptic operator. However, with a modi£cation, the so called multiple tridiagonal scaling BPX (MTS-BPX), this behaviour of the BPX preconditioner can be improved, [3].
In subsection 2.5, the smoother

$$\tilde{S}_{\alpha,k} = I - \omega L_{\alpha,k}^{-1} K_{\alpha,k}$$

has been considered as smoother for $K_{\alpha,k}$. In this smoother, the matrix $L_{\alpha,k}$ is a preconditioner for $K_{\alpha,k}$ which can handle anisotropies. The idea is now to apply the matrix $L_{\alpha,k}$ as "scaling" on each level instead of a diagonal scaling. We expect a stabilization of the BPX preconditioner. The following MTS-BPX preconditioner is now de£ned. Let Q_l^k, $l = 0,\ldots,k$ be the basis transformation matrix from the basis $\{\phi_{ij}^l\}_{i,j}^{n_l-1} \in \mathbb{V}_l$ to the basis $\{\phi_{ij}^k\}_{i,j=1}^{n_k-1} \in \mathbb{V}_k$, where $n_j = 2^j$. Let Q_k^l be the transposed operator. Furthermore, let $L_{\alpha,k}$ be the matrix in (33). Then, we de£ne the preconditioner

$$\hat{C}_{\alpha,k}^{-1} = \sum_{l=1}^{k} Q_l^k L_{\alpha,l}^{-1} Q_k^l. \tag{36}$$

This preconditioner is called the MTS-BPX preconditioner for $K_{\alpha,k}$. In the case of $\alpha = 1$, this de£nition corresponds to the de£nition of the MTS-BPX preconditioner given in [3].
For the correct analytical de£nition of the MTS-BPX preconditioner $\hat{C}_{\alpha,k}$ for $K_{\alpha,k}$ with $\alpha \geq 0$, we recall the notation of this paper and introduce some new notation. Let $\mathbb{V}_k = \operatorname{span}\left\{\phi_{ij}^k\right\}_{i,j=1}^{n_k-1}$, where k denotes the level number and $n_k = 2^k$. Moreover, let $k' \leq k$. The domain Ω is decomposed into overlapping stripes $\hat{\Omega}_j^k$, i.e.

$$\overline{\Omega} = \bigcup_{j=1}^{n_k-1} \hat{\Omega}_j^k,$$

where $\hat{\Omega}_j^k = \hat{\Omega}_{j,x}^k \cup \hat{\Omega}_{j,y}^k$ with

$$\begin{aligned} \hat{\Omega}_{j,x}^k &= \left\{ (x,y) \in \mathbb{R}^2, 0 \le y \le x, \frac{j-1}{n_k} \le x \le \frac{j+1}{n_k} \right\}, \\ \hat{\Omega}_{j,y}^k &= \left\{ (x,y) \in \mathbb{R}^2, 0 \le x \le y, \frac{j-1}{n_k} \le y \le \frac{j+1}{n_k} \right\}, \end{aligned}$$

see Figure 4. According to this decomposition, let

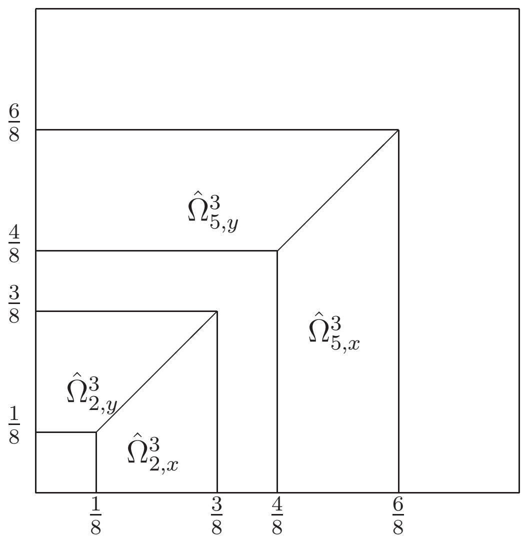

Figure 4: Stripes $\hat{\Omega}_j^k$ for $k = 3$ and $j = 2, 5$.

$$\mathbb{V}_j^k = \text{span} \left\{ \phi_{ij}^k \right\}_{i=1}^{j-1} \oplus \text{span} \left\{ \phi_{ji}^k \right\}_{i=1}^{j} \tag{37}$$

be the corresponding £nite element subspaces to the sub-domains $\hat{\Omega}_j^k$. Note that all shape functions $\phi^k \in \mathbb{V}_j^k$ vanish on the boundary of $\hat{\Omega}_j^k$. The additive Schwarz splitting of the £nite element space $\mathbb{V}_k$, i.e.

$$\mathbb{V}_k = \sum_{k'=1}^{k} \sum_{j=1}^{n_{k'}-1} \mathbb{V}_j^{k'}$$

is considered. Following Zhang, [22], let ${}_{\alpha,k} : \mathbb{V}_k \mapsto \mathbb{V}_k$ and ${}_{\alpha,i,k} : \mathbb{V}_i^k \mapsto \mathbb{V}_i^k$ be the operators

$$\begin{aligned} \langle {}_{\alpha,k} u, v \rangle &= a(u,v) \quad \forall u, v \in \mathbb{V}_k, \\ \langle {}_{\alpha,i,k} u, v \rangle &= a(u,v) \quad \forall u, v \in \mathbb{V}_i^k. \end{aligned}$$

Moreover, let ${}_{i,k'} : \mathbb{V}_k \mapsto \mathbb{V}_i^{k'}$ be the energetic projection and ${}_{i,k'} : \mathbb{V}_k \mapsto \mathbb{V}_i^{k'}$ be the L^2-projection, i.e.

$$\begin{aligned} a({}_{i,k'} u, v) &= a(u,v) \quad \forall v \in \mathbb{V}_i^{k'}, \\ \langle {}_{i,k'} u, v \rangle &= \langle u, v \rangle \quad \forall v \in \mathbb{V}_i^{k'}, \end{aligned}$$

where $u \in \mathbb{V}_k$. Then, the preconditioner $\hat{\ }_{\alpha,k}$ and the k-th level additive Schwarz operator $\ _k$ are de£ned by

$$\begin{aligned} \hat{\ }_{\alpha,k}^{-1} &= \sum_{k'=1}^{k} \sum_{i=1}^{n_{k'}-1} \ _{\alpha,i,k'}^{-1} \ _{i,k'}, & (38) \\ \ _k &= \hat{\ }_{\alpha,k}^{-1} \ _{\alpha,k} = \sum_{k'=1}^{k} \sum_{i=1}^{n_{k'}-1} \ _{i,k'} . & (39) \end{aligned}$$

Note that the matrices $K_{\alpha,k}$ (6) and $\hat{C}_{\alpha,k}$ (36) denote the matrix representations of $\ _{\alpha,k}$ and $\hat{\ }_{\alpha,k}$ by the usual fem-isomorphism. For technical reasons, we investigate the additive Schwarz splitting

$$\mathbb{V}_k = \sum_{k'=1}^{k} \mathbb{U}_1^{k'} \oplus \mathbb{U}_2^{k'}, \qquad (40)$$

where

$$\begin{aligned} \mathbb{U}_1^{k'} &= \mathbb{V}_1^{k'} \oplus \mathbb{V}_3^{k'} \oplus \cdots \oplus \mathbb{V}_{n_{k'}-1}^{k'} \quad \text{and} & (41) \\ \mathbb{U}_2^{k'} &= \mathbb{V}_2^{k'} \oplus \mathbb{V}_4^{k'} \oplus \cdots \oplus \mathbb{V}_{n_{k'}-2}^{k'} \end{aligned}$$

as well (cf. (37)). Let $\tilde{\ }_{\alpha,s,k}$: $\mathbb{U}_s^k \mapsto \mathbb{U}_s^k$, $\tilde{\ }_{s,k'}$: $\mathbb{V}_k \mapsto \mathbb{U}_s^{k'}$ and $\tilde{\ }_{s,k'}$: $\mathbb{V}_k \mapsto \mathbb{U}_s^{k'}$ be the operators

$$\begin{aligned} \langle \tilde{\ }_{\alpha,s,k} u, v \rangle &= a(u,v) \quad \forall u, v \in \mathbb{U}_s^k, \\ a\left(\tilde{\ }_{s,k'} u, v \right) &= a(u,v) \quad \forall u \in \mathbb{V}_k, v \in \mathbb{U}_s^{k'}, \\ \langle \tilde{\ }_{s,k'} u, v \rangle &= \langle u, v \rangle \quad \forall u \in \mathbb{V}_k, v \in \mathbb{U}_s^{k'}, \end{aligned}$$

where $s = 1, 2$. Thus, the preconditioner $\hat{\ }_{\alpha,k}$ (38) and the k-th level additive Schwarz operator $\hat{\ }_k$ can be obtained as multi-level additive Schwarz preconditioner and projection operator corresponding to (40).

LEMMA 3.1. *The relations*

$$\begin{aligned} \hat{\ }_{\alpha,k}^{-1} &= \sum_{k'=1}^{k} \sum_{s=1}^{2} \tilde{\ }_{\alpha,s,k'}^{-1} \tilde{\ }_{s,k'} \quad \textit{and} & (42) \\ \ _k &= \sum_{k'=1}^{k} \sum_{s=1}^{2} \tilde{\ }_{s,k'} & (43) \end{aligned}$$

are valid.

Proof: Note that $a(u,v) = 0$ and $\langle u, v \rangle = 0$ for all $u \in \mathbb{V}_i^{k'}$ and $v \in \mathbb{V}_j^{k'}$ with $|i - j| \geq 2$. Thus, the sums in (41) are orthogonal sums with respect to $a(\cdot,\cdot)$ and $\langle \cdot, \cdot \rangle$. Hence, the L^2

and the energetic projection from $\mathbb{V}_k$ onto $\mathbb{U}_{s,k'}$ is the sum of the projections onto $\mathbb{V}^k_{2i-2+s}$, $i = 1, \ldots, \frac{n'_k}{2} + 1 - s$, i.e.

$$\tilde{\ }_{s,k'} u \;=\; \sum_{i=1}^{\frac{n'_k}{2}+1-s} \ _{2i-2+s} u, \tag{44}$$

$$\tilde{\ }_{s,k'} u \;=\; \sum_{i=1}^{\frac{n'_k}{2}+1-s} \ _{2i-2+s} u$$

hold for all $u \in \mathbb{V}_{k'} \subset \mathbb{V}_k$. Therefore, relation (43) has been proved. Moreover, let

$$u = \sum_{i=1}^{\frac{n'_k}{2}+1-s} u_{2i-2+s}, \quad u_j \in \mathbb{V}^{k'}_j, u \in \mathbb{U}^{k'}_s, s = 1, 2.$$

Since $a(u_i, u_j) = 0$ for all $u_j \in \mathbb{V}^{k'}_j$ and $u_i \in \mathbb{V}^{k'}_i$ with $|i - j| \geq 2$,

$$\tilde{\ }_{\alpha,s,k'} u = \sum_{i=1}^{\frac{n'_k}{2}+1-s} \tilde{\ }_{\alpha,2i-2+s,k'} u_{2i-2+s} \quad \text{or} \quad \left(\tilde{\ }_{\alpha,s,k'}\right)^{-1} u = \sum_{i=1}^{\frac{n'_k}{2}+1-s} \left(\tilde{\ }_{\alpha,2i-2+s,k'}\right)^{-1} u_{2i-2+s}$$

follows. Together with (44) and (41), the assertion (42) has been proved. □

3.2 Proof of the upper eigenvalue estimate

We prove now the estimate $\lambda_{max}(\ _k) \leq c\,k$ with a constant c independent of the mesh-size h. Two proofs are given.
The £rst proof is similar to the proof of Zhang for the upper eigenvalue bound of the MDS-BPX preconditioned system matrix given in [22]. Zhang has proved that the condition number of the preconditioned system is bounded by a constant independent of the level number, if the bilinear form $a(\cdot, \cdot)$ is uniformly elliptic and bounded. Using the techniques of Zhang, we can only prove the result $\lambda_{max}\left(\hat{C}_k^{-1} K_k\right) = \lambda_{max}(\ _k) \leq c\,k$ for the MTS-BPX preconditioner. The second proof uses the multi-level additive Schwarz splitting $\mathbb{V}_k = \sum_{k'=1}^{k} \mathbb{U}^{k'}_1 \oplus \mathbb{U}^{k'}_2$ (40). Using this space splitting, the result $\lambda_{max}(\ _k) \leq c\,k$ can be established by a short proof. This proof requires the positive de£niteness of the bilinear form $a(\cdot, \cdot)$ only. The Zhang-like proof is given in order to show that $\lambda_{max}(\ _k) \leq c$ cannot be concluded by a more rigorous estimate. Numerical experiments indicate that the maximal eigenvalue of $\ _k$ grows proportionally to the level number..
Now, we start with the £rst proof. For this aim, the following lemma is useful. (Recall Figure 2 for the de£nition of the triangles $\tau^{1,k}_{ij}$ and $\tau^{2,k}_{ij}$.)

LEMMA 3.2. *For weight functions satisfying Assumption 1.1, the estimate*

$$\int_{\tau_{rs}^{1,k}} \omega^2(y)\,\mathrm{d}(x,y) \asymp \int_{\tau_{rs}^{2,k}} \omega^2(y)\,\mathrm{d}(x,y) \tag{45}$$

is valid for all $r, s \in \mathbb{N}_0$.

Proof: By the monotony of the weight function, one easily checks

$$\int_{\tau_{rs}^{2,k}} \omega^2(y)\,\mathrm{d}(x,y) \leq \int_{\tau_{rs}^{1,k}} \omega^2(y)\,\mathrm{d}(x,y). \tag{46}$$

By Lemma 2.6, we have

$$0 \leq \left(\omega\left(y + \frac{1}{4n_k}\right)\right)^2 \leq c(\omega(y))^2 \quad \forall y \geq \frac{1}{2n_k}.$$

Integration with respect to the variable y gives

$$\begin{aligned} \int_{\frac{4j+2}{4n_k}}^{\frac{4j+3}{4n_k}} \left(\omega\left(y + \frac{1}{4n_k}\right)\right)^2 \mathrm{d}y &\leq c\int_{\frac{4j+2}{4n_k}}^{\frac{4j+3}{4n_k}} \omega^2(y)\,\mathrm{d}y \quad \forall j \in \mathbb{N}_0, \quad \text{or,} \\ \int_{\frac{4j+3}{4n_k}}^{\frac{4j+4}{4n_k}} \omega^2(y)\,\mathrm{d}y &\leq c\int_{\frac{4j+2}{4n_k}}^{\frac{4j+3}{4n_k}} \omega^2(y)\,\mathrm{d}y \quad \forall j \in \mathbb{N}_0. \end{aligned}$$

By integration with respect to the variable x from $\frac{4i}{4n_k}$ to $\frac{4i+1}{4n_k}$, one concludes $(4n_k = 2^{k+2})$

$$\begin{aligned} \int_{\mathcal{E}_{4i,4j+3}^{k+2}} \omega^2(y)\,\mathrm{d}(x,y) &\leq c\int_{\mathcal{E}_{4i,4j+2}^{k+2}} \omega^2(y)\,\mathrm{d}(x,y) \quad \forall i,j \in \mathbb{N}_0 \\ &= c\int_{\mathcal{E}_{4i+3,4j+2}^{k+2}} \omega^2(y)\,\mathrm{d}(x,y) \quad \forall i,j \in \mathbb{N}_0. \end{aligned} \tag{47}$$

For the last estimate, it is used that the integrand does not depend on the variable x. Note that $\mathcal{E}_{4i+3,4j+2}^{k+2} \subset \tau_{ij}^{2,k}$, cf. Figure 5. Thus, the inequality

$$\int_{\mathcal{E}_{4i+3,4j+2}^{k+2}} \omega^2(y)\,\mathrm{d}(x,y) \leq \int_{\tau_{ij}^{2,k}} \omega^2(y)\,\mathrm{d}(x,y) \tag{48}$$

holds for all $i, j \in \mathbb{N}_0$. Moreover, by $\mathcal{E}_{4i,4j+2}^{k+2} \subset \tau_{ij}^{1,k}$ and the monotony of the weight function, one easily deduces

$$8\int_{\mathcal{E}_{4i,4j+3}^{k+2}} \omega^2(y)\,\mathrm{d}(x,y) \geq \int_{\tau_{ij}^{1,k}} \omega^2(y)\,\mathrm{d}(x,y). \tag{49}$$

Combining the estimates (47), (48), and (49), one checks

$$\int_{\tau_{ij}^{1,k}} \omega^2(y)\,\mathrm{d}(x,y) \leq 8c\int_{\tau_{ij}^{2,k}} \omega^2(y)\,\mathrm{d}(x,y). \tag{50}$$

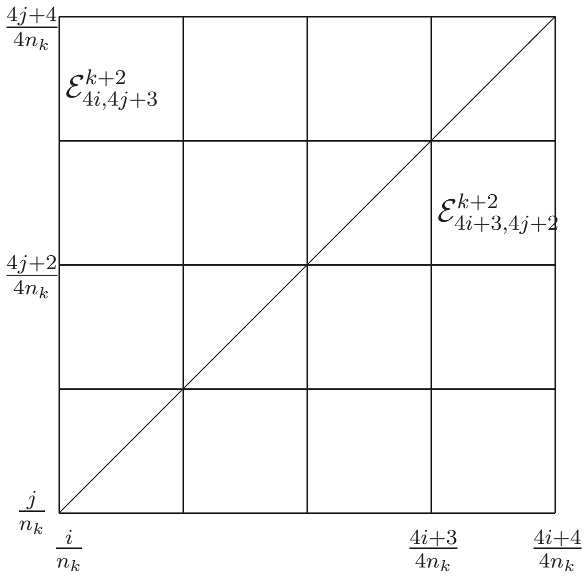

Figure 5: Notation for $\mathcal{E}^k_{ij} = \overline{\tau^{1,k}_{ij} \cup \tau^{2,k}_{ij}}$, $n_k = 2^k$.

By (46) and (50), the assertion follows immediately. □
Equivalent to the estimate (45) is that

$$\int_{\tau^{u,k}_{rs}} \omega^2(x)\ \mathrm{d}(x,y) \geq c \int_{\mathcal{E}^k_{rs}} \omega^2(x)\ \mathrm{d}(x,y)$$

is valid for $u = 1, 2$, and $r, s \in \mathbb{N}_0$ with a constant c independent of r, s, and k. The main tool in order to estimate the upper eigenvalue of the BPX preconditioner is Lemma 3.5 which is a strengthened Cauchy-inequality of the type

$$\Big(a(u^{k'}_i, u^k_j)\Big)^2 \leq c2^{|k'-k|} a(u^{k'}_i, u^{k'}_i) a(u^k_j, u^k_j) \tag{51}$$

for all $u^k_j \in \mathbb{V}^k_j$ and $u^{k'}_i \in \mathbb{V}^{k'}_i$. Our aim is to prove (51). We split this proof into several lemmata. The £rst lemma says that the mean value of the weight function $\omega(x)$ over $\tau^{u,k'}_{rs} \cap \hat{\Omega}^k_{j,x}$ can be bounded by the mean value over $\tau^{u,k'}_{rs}$.

LEMMA 3.3. *For $u = 1, 2$, $r, s \in \mathbb{N}_0$, $k' \leq k$, $j \in \mathbb{N}$, the inequalities*

$$\frac{n_{k'}}{n_k} \int_{\tau^{u,k'}_{rs}} \omega^2(y)\ \mathrm{d}(x,y) \geq c \int_{\tau^{u,k'}_{rs} \cap \hat{\Omega}^k_{j,x}} \omega^2(y)\ \mathrm{d}(x,y) \tag{52}$$

and

$$\frac{n_{k'}}{n_k} \int_{\tau^{u,k'}_{rs}} \omega^2(x)\ \mathrm{d}(x,y) \geq c \int_{\tau^{u,k'}_{rs} \cap \hat{\Omega}^k_{j,x}} \omega^2(x)\ \mathrm{d}(x,y) \tag{53}$$

are valid.

Proof: For $\overline{\tau_{rs}^{u,k'}} \cap \hat{\Omega}_{j,x}^k = \emptyset$, the assertion is trivial ($c = 0$). We assume that $\overline{\tau_{rs}^{u,k'}} \cap \hat{\Omega}_{j,x}^k \neq \emptyset$. Then, we have

$$\underline{c}\frac{r}{n_{k'}} \leq \frac{j}{n_k} \leq \overline{c}\frac{r+1}{n_{k'}}. \tag{54}$$

Now, with (45) and Assumption 1.1, we estimate

$$\begin{aligned}
\int_{\tau_{rs}^{u,k'}} \omega^2(x)\,\mathrm{d}(x,y) &\overset{(45)}{\geq} c\int_{\mathcal{E}_{rs}^{k'}} \omega^2(x)\,\mathrm{d}(x,y) \\
&= c\int_{\frac{r}{n_{k'}}}^{\frac{r+1}{n_{k'}}} \int_{\frac{s}{n_{k'}}}^{\frac{s+1}{n_{k'}}} \omega^2(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\overset{\omega(\xi)=\xi^\alpha}{=} c\frac{1}{n_{k'}}\int_{\frac{r}{n_{k'}}}^{\frac{r+1}{n_{k'}}} x^{2\alpha}\,\mathrm{d}x \\
&\geq \frac{c}{n_{k'}}\frac{(r+1)^{2\alpha}}{{n_{k'}}^{2\alpha+1}} = \frac{1}{{n_{k'}}^2}\left(\frac{r+1}{n_{k'}}\right)^{2\alpha}.
\end{aligned} \tag{55}$$

Moreover, one concludes

$$\begin{aligned}
\int_{\tau_{rs}^{u,k'}\cap\hat{\Omega}_{j,x}^k} \omega^2(x)\,\mathrm{d}(x,y) &\leq \int_{\mathcal{E}_{rs}^{k'}\cap\hat{\Omega}_{j,x}^k} \omega^2(x)\,\mathrm{d}(x,y) \\
&\leq \int_{\frac{s}{n_{k'}}}^{\frac{s+1}{n_{k'}}} \int_{\frac{j-1}{n_k}}^{\frac{j+1}{n_k}} \omega^2(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{n_{k'}}\int_{\frac{j-1}{n_k}}^{\frac{j+1}{n_k}} x^{2\alpha}\,\mathrm{d}x \\
&\leq \frac{c}{n_{k'}}\frac{j^{2\alpha}}{n_k^{2\alpha+1}} \leq \frac{c}{n_k n_{k'}}\left(\frac{j}{n_k}\right)^{2\alpha}.
\end{aligned} \tag{56}$$

Using (54) and (56), we have

$$\int_{\tau_{rs}^{u,k'}\cap\hat{\Omega}_{j,x}^k} \omega^2(x)\,\mathrm{d}(x,y) \leq \frac{c}{n_k n_{k'}}\left(\frac{r+1}{n_{k'}}\right)^{2\alpha}. \tag{57}$$

Combining (57) and (55), the inequality (53) follows immediately. The estimate (52) can be proved with similar arguments. □

Let $a_{\hat{\Omega}_j^k}$ be the restriction of the bilinear form a to $\hat{\Omega}_j^k$, i.e.

$$a_{\hat{\Omega}_j^k}(u,v) = \int_{\hat{\Omega}_j^k} \left(\omega^2(y)u_x v_x + \omega^2(x)u_y v_y\right)\,\mathrm{d}(x,y).$$

Using Lemma 3.3, the following result can be shown.

LEMMA 3.4. *Let $u_i^{k'} \in \mathbb{V}_i^{k'}$. Then for $k' \leq k$, the estimate*

$$2^{k'-k} a(u_i^{k'}, u_i^{k'}) \geq c\, a_{\hat{\Omega}_j^k}(u_i^{k'}, u_i^{k'})$$

is valid.

Proof: For each triangle $\tau_{rs}^{u,k'} \subset \hat{\Omega}_{i,x}^{k'}$, $(\nabla u_i^{k'})^T$ is constant on $\tau_{rs}^{u,k'}$. Therefore, using the estimates (52) and (53) of Lemma 3.3,

$$\begin{aligned} \int_{\tau_{rs}^{u,k'}} \left(\omega^2(y)(u_i^{k'})_x(u_i^{k'})_x + \omega^2(x)(u_i^{k'})_y(u_i^{k'})_y\right) &= \int_{\tau_{rs}^{u,k'}} \omega^2(y)(u_i^{k'})_x^2 + \int_{\tau_{rs}^{u,k'}} \omega^2(x)(u_i^{k'})_y^2 \\ &\geq \frac{cn_k}{n_{k'}} \int_{\tau_{rs}^{u,k'} \cap \hat{\Omega}_{j,x}^k} (u_i^{k'})_x^2\omega^2(y) + (u_i^{k'})_y^2\omega^2(x). \end{aligned}$$

By symmetry of the differential operator (1), the same result is valid for each triangle $\tau_{rs}^{u,k'} \subset \hat{\Omega}_{i,y}^{k'}$. Summation over all triangles $\tau_{rs}^{u,k'} \subset \hat{\Omega}_i^{k'}$ gives

$$\int_{\hat{\Omega}_i^{k'}} \left((u_i^{k'})_x^2\omega^2(y) + (u_i^{k'})_y^2\omega^2(x)\right)\, \mathrm{d}(x,y) \geq c\frac{n_k}{n_{k'}} \int_{\hat{\Omega}_j^k} \left((u_i^{k'})_x^2\omega^2(y) + (u_i^{k'})_y^2\omega^2(x)\right)\, \mathrm{d}(x,y),$$

or equivalently,

$$a(u_i^{k'}, u_i^{k'}) \geq c\frac{n_k}{n_{k'}} a_{\hat{\Omega}_j^k}(u_i^{k'}, u_i^{k'}) = c2^{k-k'} a_{\hat{\Omega}_j^k}(u_i^{k'}, u_i^{k'})$$

which proves the lemma. □

The next lemma gives a relation for the cosine of the angle between the spaces $\mathbb{V}_i^{k'}$ and $\mathbb{V}_j^k$ with respect to $a(\cdot,\cdot)$ which in general is de£ned as

$$\gamma_{\mathbb{U},\mathbb{V}} = \sup_{\substack{u \in \mathbb{U} \\ v \in \mathbb{V} \\ u,v \neq 0}} \frac{a(u,v)}{\sqrt{a(u,u)a(v,v)}}. \tag{58}$$

LEMMA 3.5. *Let $k' \leq k$ and $i \in \{1, \ldots, n_{k'} - 1\}$, $j \in \{1, \ldots, n_k - 1\}$. Then,*

$$\gamma^2_{\mathbb{V}_i^{k'}, \mathbb{V}_j^k} \leq \max\left\{c2^{-\frac{k-k'}{2}}, 1\right\}.$$

Proof: The proof is similar to the proof of Lemma 3.2. in [22]. Let $u_i^{k'} \in \mathbb{V}_i^{k'}$ and $u_j^k \in \mathbb{V}_j^k$. Then, by the usual Cauchy-inequality on $a_{\hat{\Omega}_j^k}(\cdot,\cdot)$ and Lemma 3.4,

$$\begin{aligned} \left(a(u_i^{k'}, u_j^k)\right)^2 &= \left(a_{\hat{\Omega}_j^k}(u_i^{k'}, u_j^k)\right)^2 \\ &\leq a_{\hat{\Omega}_j^k}(u_i^{k'}, u_i^{k'})\, a(u_j^k, u_j^k) \\ &\leq c2^{k'-k} a(u_i^{k'}, u_i^{k'})\, a(u_j^k, u_j^k) \end{aligned}$$

which shows the assertion.□
Following Zhang, [22], let

$$\Theta = \left[\theta_{ij}^{k',k''}\right]_{(i,k'),(j,k'')},$$

where

$$\theta_{ij}^{k',k''} = \gamma^2_{\mathbb{V}_i^{k'},\mathbb{V}_j^{k''}}, \quad 1 \leq k', k'' \leq k.$$

Our aim is to prove an estimate of the type

$$\| \Theta \|_2 \leq ck.$$

For this purpose, the following propositions and lemmata are helpful.

PROPOSITION 3.6. *Let k', k be £xed with $k' \leq k$. If $\theta_{ij}^{k',k} \neq 0$, then*

$$(i-1)2^{k-k'} \leq j \leq (i+1)2^{k-k'}.$$

Proof: By de£nition, $\phi \in \mathbb{V}_j^k$ satis£es supp $\phi \subset \hat{\Omega}_j^k$. If $\mathrm{int}(\hat{\Omega}_j^k) \cap \mathrm{int}(\hat{\Omega}_i^{k'}) = \emptyset$, then $\theta_{ij}^{k',k} = 0$. By de£nition of the stripes $\hat{\Omega}_j^k$, the assertion follows. □
Now, we consider one block of the matrix Θ, i.e.

$$\Theta^{k',k''} = \left[\theta_{ij}^{k',k''}\right]_{i=1,j=1}^{n_{k'},n_{k''}}.$$

Then, the following proposition is valid.

PROPOSITION 3.7. *The Frobenius norm of $\Theta^{k',k''}$ can be estimated by a constant, independent of the mesh-size h, i.e.*

$$\| \Theta^{k',k''} \|_F \leq c \quad \textit{for } 1 \leq k', k'' \leq k.$$

Proof: Without loss of generality, let $k' \leq k''$. By Proposition 3.6, each row of $\Theta^{k',k''}$ has maximal $2^{k''-k'+1}+1$ nonzero matrix entries, and each column maximal 2 nonzero matrix entries. Therefore, the total number of nonzero matrix entries is less than or equal to $2^{k''-k'+2}+2$. By Lemma 3.5, $\theta_{ij}^{k',k''} \leq c2^{\frac{k'-k''}{2}}$ holds. Summing up over all $(\theta_{ij}^{k',k''})^2$ gives

$$\| \Theta^{k',k''} \|_F^2 = \sum_{i,j} (\theta_{ij}^{k',k''})^2 \leq c2^{k'-k''}(2^{k''-k'+2}+2) \leq 6c$$

which proves the lemma.□
The following lemma, [22], gives a relation between the Frobenius norm of the block matrix Θ and the Frobenius norm of $\tilde{\Theta}$, where the entries of the matrix $\tilde{\Theta}$ are the Frobenius norms of the blocks of Θ.

LEMMA 3.8. *The estimate $\| \Theta \|_F \leq ck$ is valid, where c is independent of the level number k.*

Proof: We introduce the $n \times n$ block-matrix

$$\Theta_{k',k''} = \left[\theta_{ij}^{k',k''}\right]_{i,j}, \quad 1 \le k', k'' \le k,$$

and the matrix

$$\tilde{\Theta} = [\| \Theta_{k',k''} \|_F]_{k',k''=1}^{k} .$$

By Proposition 3.7, $\| \Theta_{k',k''} \|_F \le c$. Computing the Frobenius norm of $\tilde{\Theta}$, one has

$$\| \tilde{\Theta} \|_F^2 \le ck^2.$$

By $\| \Theta \|_F = \| \tilde{\Theta} \|_F$, one easily checks

$$\| \Theta \|_F = \le ck$$

which is the desired result. □
The main result of this section is the upper eigenvalue estimate of the MTS-BPX preconditioner.

THEOREM 3.9. *For $u \in \mathbb{V}_k$, let*

$$|||u|||^2 = \min_{u=\sum_{l,i} u_i^l} \sum_{l=1}^{k} \sum_{i} a(u_i^l, u_i^l).$$

Then, one obtains

$$a(u,u) \le ck \ ||| \ u \ |||^2 .$$

Proof: We give two proofs. The £rst proof follows by Lemma 3.1 and Lemma 3.5 of Zhang, [22], the fact $\| A \|_2 \le \| A \|_F$ and Lemma 3.8.
In the second proof, we investigate the splitting $\mathbb{V}_k = \sum_{k'=1}^{k} \mathbb{U}_1^{k'} \oplus \mathbb{U}_2^{k'}$ (40). Now, let Θ be a $k \times k$ block matrix consisting of 2×2 matrices, i.e.

$$\Theta = \left[\theta^{k',k''}\right]_{k',k''=1}^{k} \quad \text{with} \quad \theta^{k',k''} = \left[\gamma_{\mathbb{U}_i^{k'},\mathbb{U}_j^{k''}}\right]_{i,j=1}^{2} .$$

By the usual Cauchy-inequality, the cosines $\gamma_{\mathbb{U}_i^{k'},\mathbb{U}_j^{k''}}$, cf. (58), of the angles between $\mathbb{U}_i^{k'}$ and $\mathbb{U}_j^{k''}$ are bounded from above by 1. Thus, $\| \theta^{k',k''} \|_F \le 2$ follows. This is the analogous result of Proposition 3.7 for the space splitting (40). Using the proof of Lemma 3.8, the assertion follows.
□

REMARK 3.10. *The eigenvalue estimate $\lambda_{max}\left(\hat{C}_{\alpha,k}^{-1} K_{\alpha,k}\right) \le ck$ of the MTS-BPX preconditioner $\hat{C}_{\alpha,k}$ for $K_{\alpha,k}$, de£ned via relation* (36)*, follows immediately.*

REMARK 3.11. *The constant in Theorem 3.9 depends linearly on the level number. The reason is the splitting into the spaces $\mathbb{V}_i^l$, not the differential operator. For the Laplacian, i.e. $\omega(x) = 1$, only this result can be proved using this space splitting.*

3.3 Numerical results

For the MTS-BPX preconditioner $\hat{C}_{\alpha,k}$ (36), Table 1 gives the lower and upper constants in the norm equivalence

$$\underline{c} \, ||| \, u|||^2 \leq a(u,u) \leq \overline{c} \, ||| \, u \, |||^2 \quad \forall u \in \mathbb{V}_l$$

computed by a vector iteration and inverse vector iteration for the corresponding matrices and the weight functions $\omega(\xi) = \xi^\alpha$, $(\alpha = 0, \frac{1}{2}, 1, 2, 10)$. One can see that the constant $\overline{c}$ is proportional to the level number for all weight functions which indicates that the estimate of Theorem 3.9 is sharp. The lower constant $\underline{c}$ seems to be bounded from below by a constant of about 0.488 uniformly with respect to α. However, we cannot prove the boundedness of $\underline{c}$ from below.

Level	$\overline{c}$				
	$\omega(\xi) = 1$	$\omega(\xi) = \sqrt{\xi}$	$\omega(\xi) = \xi$	$\omega(\xi) = \xi^2$	$\omega(\xi) = \xi^{10}$
2	1.86	1.80	1.77	1.82	2.00
3	2.73	2.65	2.59	2.51	2.93
4	3.44	3.41	3.39	3.34	3.75
5	4.00	4.01	4.03	4.06	4.59
6	4.45	4.47	4.52	4.70	5.50
7	4.81	4.85	4.91	5.34	6.44
8	5.11	5.14	5.23	6.03	7.40
9	5.35	5.39	5.59	6.70	8.37
10	5.55	5.59	6.11	7.42	9.35
Level	$\underline{c}$				
	$\omega(\xi) = 1$	$\omega(\xi) = \sqrt{\xi}$	$\omega(\xi) = \xi$	$\omega(\xi) = \xi^2$	$\omega(\xi) = \xi^{10}$
2	0.607	0.687	0.747	0.822	0.977
3	0.522	0.607	0.647	0.690	0.844
4	0.495	0.554	0.583	0.619	0.716
5	0.489	0.527	0.543	0.569	0.664
6	0.488	0.513	0.524	0.538	0.611
7	0.488	0.504	0.512	0.522	0.569
8	0.488	0.498	0.504	0.511	0.541
9	0.488	0.495	0.498	0.503	0.524
10	0.488	0.493	0.495	0.498	0.513

Table 1: Lower (below) and upper (above) eigenvalue bounds of the MTS-BPX preconditioned system matrix.

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