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Existence of the guided modes of an optical fiber

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Abstract The present paper is devoted to the investigation of the guided wave problem. This problem is formulated as the eigenvalue problem with a compact self-adjoint operator pencil. Applying the minimax principle for the compact operators in the Hilbert space we obtain a necessary and sufficient condition for the existence of a preassigned number of linearly independent guided modes. As a consequence of this result we also derive simple sufficient conditions, which can be easily applied in practice. We give a statement of the problem in a bounded domain and propose an efficient method for solving the problem.

Key Words eigenvalue problem, guided modes, optical fiber

AMS(MOS) subject classification 78A50, 47A75, 49R50, 65N25

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1 Introduction

In this paper we investigate an eigenvalue problem, which has important applications in optical telecommunications and in integrated optics. To present obtained results we first introduce the mathematical statement of this problem. We shall use the physical model and the variational formulation, which are described in detail in the paper [1]. The investigated eigenvalue problem has the following mathematical formulation: find $\lambda = \lambda(p) \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$c(\mathbb{R}^2; u, v) = \lambda\, d(\mathbb{R}^2; u, v) \quad \forall v \in V, \tag{1}$$

where $V = (H^1(\mathbb{R}^2))^3$ is the Sobolev space equipped with the norm

$$\|v\|_V = \left(\int_{\mathbb{R}^2} \{|\mathrm{grad}\, v|^2 + |v|^2\}\, dx \right)^{1/2},$$

$\mathbb{R}^2$ is the coordinate plane Ox_1x_2, $x = (x_1, x_2)^\top \in \mathbb{R}^2$, Λ is an interval of the real axis, the sesquilinear forms c and d are defined by the formulae:

$$\begin{aligned} c(G; u, v) &= \int_G \left\{ p\, \mathrm{Rot}_\beta\, u \cdot \overline{\mathrm{Rot}_\beta\, v} + p_2\, \mathrm{Div}_\beta\, u\, \overline{\mathrm{Div}_\beta\, v} \right\} dx, \\ d(G; u, v) &= \int_G u \cdot \overline{v}\, dx. \end{aligned}$$

Here β is a given positive number, $u = (u_1, u_2, u_3)^\top$, $v = (v_1, v_2, v_3)^\top$,

$$\begin{aligned} \mathrm{Rot}_\beta\, u &= (\partial_2 u_3 + i\beta u_2, -i\beta u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^\top, \\ \mathrm{Div}_\beta\, u &= \partial_1 u_1 + \partial_2 u_2 - i\beta u_3, \end{aligned}$$

$\partial_i = \partial/\partial x_i$, $i = 1, 2$, $\mathrm{grad}\, v = (\mathrm{grad}\, v_1, \mathrm{grad}\, v_2, \mathrm{grad}\, v_3)^\top$, $\mathrm{grad}\, v_i = (\partial_1 v_i, \partial_2 v_i)^\top$, $i = 1, 2, 3$, $u \cdot \overline{v} = u_1\overline{v}_1 + u_2\overline{v}_2 + u_3\overline{v}_3$, $|v|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2$. Define the function $p(x) = p(\Omega, x)$ by the formula:

$$p(\Omega, x) = \begin{cases} p_1, & x \in \Omega, \\ p_2, & x \in \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{2}$$

where Ω is a given bounded plane domain with a Lipschitz-continuous boundary, Ω may be disconnected, p_1 is a positive function from $L^\infty(\Omega)$, p_2 is a given positive number. Set

$$p_{11} = \operatorname*{ess.\,inf}_{x \in \Omega} p_1(x), \quad p_{12} = \operatorname*{ess.\,sup}_{x \in \Omega} p_1(x), \quad p_{13}(G) = \operatorname*{ess.\,sup}_{x \in G} p_1(x).$$

Assuming that $p_{11} < p_2$, $p_1 \leq p_2$ for almost all $x \in \Omega$, we put

$$\Lambda = (\sigma_1, \sigma_2), \quad \sigma_1 = \beta^2 p_{11}, \quad \sigma_2 = \beta^2 p_2.$$

The guided wave problem (1) has been derived from Maxwell's equations by eliminating the electric field [1]. In this case, $u = H$ is the magnetic field, Ω is the core region, the exterior domain $\mathbb{R}^2 \setminus \Omega$ is the cladding, β is the propagation constant, $\lambda = k^2$, k is the wavenumber, $p = 1/n^2$, n is the index profile, $p_1 = 1/n^2$, $x \in \Omega$, $p_2 = 1/n_\infty^2$, n_∞ is the refractive index of the cladding, $p_{11} = 1/n_+^2$, $n_+ = \operatorname{ess.sup}_{x\in\Omega} n(x)$.

Denote by $D_R = \{x : x \in \mathbb{R}^2, |x| < R\}$ the open disk with center O and radius R. Assume that q_1 and q_2 are positive numbers such that $q_1 < q_2$, $q(x) = q(D_R, x)$ is the function of the form (2). The guided modes for this case are well known. We can also compute the eigenvalues $\lambda_i(q)$, $i = 1, 2, \dots, m$, as roots of transcendental equations [17], [18], [22]. In the general case, to compute eigenvalues of problem (1) we need to apply numerical methods.

There exists a vast amount of literature on computational methods for solving waveguide problems. Surveys of obtained experimental and theoretical results are contained in [9], [10], [21], [34]. But the existence of guided modes in waveguides of arbitrary cross-section has been proved only recently in [1], [32].

Bamberger and Bonnet [1] have investigated eigenvalue problem (1) using the operator formulation $C_\beta u = \lambda u$ with an unbounded self-adjoint operator C_β. Urbach [32] has analyzed the domain integral formulation with a symmetric bounded noncompact integral operator in the case of guided electromagnetic waves in anisotropic inhomogeneous guides. Authors of the papers [1], [32], have proved the existence of at least two linearly independent guided modes. Bamberger and Bonnet [1] give also a complete description of the dispersion curves. These questions are studied in Dautov and Karchevskiĭ [5] by applying the spectral theory of bounded operators.

In the present paper problem (1) is written in the form: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$c(\mathbb{R}^2; u, v) - \lambda\, d(\mathbb{R}^2 \setminus D; u, v) + \alpha_0 d(D; u, v) = (\lambda + \alpha_0)\, d(D; u, v) \quad \forall v \in V, \tag{3}$$

where D is a given bounded plane domain with a Lipschitz-continuous boundary, $\Omega \subseteq D$,

$$\alpha_0 = \begin{cases} (\sigma_2 - \sigma_1)^2/\varepsilon\sigma_1, & \xi \le 0, \\ 0, & \xi > 0, \end{cases} \tag{4}$$

$\xi = 2p_{11} - p_2$, $\varepsilon \in (0, 1)$. In Section 2, we prove the positive definiteness of the sesquilinear form in the left hand side of equation (3). Then problem (3) is formulated as the eigenvalue problem: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that $u = (\lambda + \alpha_0)A(\lambda)u$, with the compact self-adjoint positive definite operator pencil $A(\mu) : V \to V$, $\mu \in \Lambda$. In Section 3, applying the minimax principle for the compact operators in the Hilbert space we obtain a necessary and sufficient condition for the existence of a preassigned number of linearly independent guided modes. As a consequence of this result we derive results for comparing eigenvalues $\lambda(p)$ for various functions p. For example, the following result is proved. Suppose that $D_R \subseteq \Omega$, $q_1 = p_{13}(D_R)$ and $q_2 = p_2$, $q(x) = q(D_R, x)$ is the function of the form (2). If there exist $\lambda_i(q)$, $i = 1, 2, \dots, m$, then there exist at least m, $m \ge 2$, eigenvalues $\lambda_i(p)$, $i = 1, 2, \dots, m$, of problem (1) and $\lambda_i(p) \le \lambda_i(q)$, $i = 1, 2, \dots, m$. In Section 4 we obtain simple sufficient conditions, which can be easily applied in practice. In particular, we prove the following result. Suppose that $S_B \subseteq \Omega$, where S_B is a square with the side length B,

$\mu_1^1 \le \mu_2^1 \le \ldots \le \mu_k^1 \le \ldots$ are numbers of the form $\pi\sqrt{i^2+j^2}$, $i, j = 1, 2, \ldots$, enumerated in ascending order,

$$\lambda_{2i-1}^0 = \left(\frac{\mu_i^1}{B}\right)^2, \quad \lambda_{2i}^0 = \left(\frac{\mu_i^1}{B}\right)^2, \quad i = 1, 2, \ldots$$

Define $m = \max\{i : p_2\lambda_i^0 < \sigma_2 - \sigma_0, i \ge 1\}$, where $\sigma_0 = \beta^2 p_0$, $p_0 = p_{13}(S_B)$. Then problem (1) has at least m, $m \ge 2$, real eigenvalues of finite multiplicity $\lambda_i = \lambda_i(p)$, $i = 1, 2, \ldots, m$, which are repeated according to their multiplicity: $\sigma_1 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_m < \sigma_2$. Similar results have been established in [30] for the scalar equation of the weak-guidance approximation. In this paper the analysis analogous to [26] is applied. In Section 5 we propose the equivalent formulation in a bounded domain for problem (1) and suggest an efficient method for solving the problem. Let us indicate that the proposed approach uses an eigenvalue problem in a bounded domain, which is equivalent to the initial eigenvalue problem (1). In contrast to our approach, previous papers (see review [10]) have applied approaches when the initial eigenvalue problem (1) is reduced approximately to an eigenvalue problem in a bounded domain. More precisely, we define a variational eigenvalue problem in a bounded domain, which leads to the differential eigenvalue problem with the exact boundary condition, and not with approximate one as in previous investigations of problem (1).

2 Eigenvalue problem with compact operator pencil

By $\mathbb{R}$ and $\mathbb{C}$ denote the real axis and the complex plane, respectively. Let D be a given bounded plane domain with a Lipschitz-continuous boundary, $\Omega \subseteq D$. Assume that $H = (L_2(D))^3$ is the Lebesgue space equipped with the norm

$$\|v\|_H = \left(\int_D |v|^2\,dx\right)^{1/2}.$$

Let us define the mappings $a : \Lambda \times V \times V \to \mathbb{C}$, $b : H \times H \to \mathbb{C}$, by the formulae:

$$\begin{aligned} a(\mu, u, v) &= c(\mathbb{R}^2; u, v) - \mu\, d(\mathbb{R}^2 \setminus D; u, v), \\ b(u, v) &= d(D; u, v). \end{aligned}$$

Then problem (3) can be written in the form: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$a(\lambda, u, v) + \alpha_0\, b(u, v) = (\lambda + \alpha_0)\, b(u, v) \quad \forall v \in V. \tag{5}$$

Introduce the following auxiliary linear eigenvalue problem: find $\gamma(\mu) \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$a(\mu, u, v) + \alpha_0\, b(u, v) = (\gamma(\mu) + \alpha_0)\, b(u, v) \quad \forall v \in V \tag{6}$$

for fixed parameter $\mu \in \Lambda$.

It can be directly verified that

$$
\begin{aligned}
a(\mu,u,v) &= \int_{\mathbb{R}^2} \{p \operatorname{rot}\mathbf{u} \operatorname{rot}\overline{\mathbf{v}} + p_2 \operatorname{div}\mathbf{u} \operatorname{div}\overline{\mathbf{v}} + p_2 \operatorname{grad} u_3 \cdot \operatorname{grad}\overline{v}_3\}\, dx + \\
&+ \int_\Omega (p-p_2)\,(\operatorname{grad} u_3 + i\beta\mathbf{u})\overline{(\operatorname{grad} v_3 + i\beta\mathbf{v})}\, dx + \\
&+ \sigma_2 \int_D u\cdot\overline{v}\, dx + (\sigma_2-\mu)\int_{\mathbb{R}^2\setminus D} u\cdot\overline{v}\, dx,
\end{aligned} \tag{7}
$$

where $\mathbf{u} = (u_1,u_2)^\top$, $\mathbf{v} = (v_1,v_2)^\top$, $\operatorname{div}\mathbf{u} = \partial_1 u_1 + \partial_2 u_2$, $\operatorname{rot}\mathbf{u} = \partial_1 u_2 - \partial_2 u_1$, $\operatorname{grad}\varphi = (\partial_1\varphi, \partial_2\varphi)^\top$.

Lemma 1 *For $v \in V$, $\mu \in \Lambda$, the following inequality holds:*

$$
a(\mu,v,v) \geq \alpha_1(\mu)\|v\|_V^2 - \alpha_0\|v\|_H^2,
$$

where α_0 is defined by (4)*,*

$$
\alpha_1(\mu) = \begin{cases} \min\{(1-\varepsilon)p_{11}, \sigma_1, \sigma_2-\mu\}, & \xi \leq 0, \\ \min\{p_{11}, \xi, \beta^2\xi, \sigma_2, \sigma_2-\mu\}, & \xi > 0, \end{cases}
$$

$\xi = 2p_{11} - p_2$, $\varepsilon \in (0,1)$.

Proof By (7) we have

$$
\begin{aligned}
a(\mu,v,v) &= \int_{\mathbb{R}^2} \left\{p\,|\operatorname{rot}\mathbf{v}|^2 + p_2\,|\operatorname{div}\mathbf{v}|^2 + p_2\,|\operatorname{grad} v_3|^2\right\}\, dx + \\
&\quad + c_0(v,v) + \sigma_2\int_D |v|^2\, dx + (\sigma_2-\mu)\int_{\mathbb{R}^2\setminus D} |v|^2\, dx,
\end{aligned} \tag{8}
$$

where

$$
c_0(v,v) = \int_\Omega (p-p_2)\,|\operatorname{grad} v_3 + i\beta\mathbf{v}|^2\, dx.
$$

For this term we obtain the following estimates:

$$
\begin{aligned}
c_0(v,v) &\geq -2(p_2-p_{11})\int_\Omega \left\{|\operatorname{grad} v_3|^2 + \beta^2|\mathbf{v}|^2\right\}\, dx, \\
c_0(v,v) &\geq -(p_2-p_{11})\int_\Omega \left\{|\operatorname{grad} v_3|^2 + \beta^2|\mathbf{v}|^2 + 2\beta|\operatorname{grad} v_3|\cdot|\mathbf{v}|\right\}\, dx \geq \\
&\geq -(p_2-p_{11})\int_\Omega \left\{|\operatorname{grad} v_3|^2 + \beta^2|\mathbf{v}|^2\right\}\, dx - \\
&\quad -\varepsilon p_{11}\int_\Omega |\operatorname{grad} v_3|^2\, dx - \frac{(p_2-p_{11})^2\beta^2}{\varepsilon p_{11}}\int_\Omega |\mathbf{v}|^2\, dx.
\end{aligned}
$$

Substituting the previous estimates in (8), we derive the desired properties:

$$
\begin{aligned}
a(\mu, v, v) &\geq p_{11} \int_{\mathbb{R}^2} |\operatorname{grad} \mathbf{v}|^2 \, dx + p_2 \int_{\mathbb{R}^2 \setminus \Omega} |\operatorname{grad} v_3|^2 \, dx + \\
&+\xi \int_{\Omega} |\operatorname{grad} v_3|^2 \, dx + \xi \beta^2 \int_{\Omega} |\mathbf{v}|^2 \, dx + \\
&+\sigma_2 \int_{D} |v_3|^2 \, dx + \sigma_2 \int_{D \setminus \Omega} |\mathbf{v}|^2 \, dx + (\sigma_2 - \mu) \int_{\mathbb{R}^2 \setminus D} |v|^2 \, dx \geq \qquad (9) \\
&\geq \min\{p_{11}, \xi, \beta^2 \xi, \sigma_2, \sigma_2 - \mu\} \int_{\mathbb{R}^2} \left\{ |\operatorname{grad} v|^2 + |v|^2 \right\} \, dx = \\
&= \alpha_1(\mu) \|v\|_V^2, \quad \xi > 0, \\
a(\mu, v, v) &\geq p_{11} \int_{\mathbb{R}^2} |\operatorname{grad} \mathbf{v}|^2 \, dx + p_2 \int_{\mathbb{R}^2 \setminus \Omega} |\operatorname{grad} v_3|^2 \, dx + \\
&+(1 - \varepsilon) p_{11} \int_{\Omega} |\operatorname{grad} v_3|^2 \, dx + (\sigma_1 - \alpha_0) \int_{\Omega} |\mathbf{v}|^2 \, dx + \\
&+\sigma_2 \int_{D} |v_3|^2 \, dx + \sigma_2 \int_{D \setminus \Omega} |\mathbf{v}|^2 \, dx + (\sigma_2 - \mu) \int_{\mathbb{R}^2 \setminus D} |v|^2 \, dx \geq \qquad (10) \\
&\geq \min\{(1 - \varepsilon) p_{11}, \sigma_1, \sigma_2 - \mu\} \int_{\mathbb{R}^2} \left\{ |\operatorname{grad} v|^2 + |v|^2 \right\} \, dx - \alpha_0 \int_{D} |v|^2 \, dx = \\
&= \alpha_1(\mu) \|v\|_V^2 - \alpha_0 \|v\|_H^2, \quad \xi \leq 0.
\end{aligned}
$$

□

According to Lemma 1 we obtain the positive definiteness of the sesquilinear form in the left hand side of equation (6), i.e.,

$$a(\mu, v, v) + \alpha_0 \, b(v, v) \geq \alpha_1(\mu) \|v\|_V^2 \quad \forall v \in V$$

for fixed $\mu \in \Lambda$. Moreover, it is easy to get the boundedness property:

$$a(\mu, v, v) + \alpha_0 \, b(v, v) \leq \alpha_2 \|v\|_V^2 \quad \forall v \in V$$

for fixed $\mu \in \Lambda$, $\alpha_2 = \max\{p_2, \sigma_2 + \alpha_0\}$. Therefore, one can define the self-adjoint positive definite operator $A(\mu) : V \to V$ by the following equality:

$$a(\mu, A(\mu)u, v) + \alpha_0 b(A(\mu)u, v) = b(u, v) \quad \forall u, v \in V$$

for fixed parameter $\mu \in \Lambda$. Because $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, the operator $A(\mu) : V \to V$ is compact self-adjoint positive definite operator in the Hilbert space V.

Now we rewrite problems (5) and (6) in the following operator forms:

find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$u = (\lambda + \alpha_0) \, A(\lambda) u, \qquad (11)$$

find $\gamma(\mu) \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$u = (\gamma(\mu) + \alpha_0)\, A(\mu) u \tag{12}$$

for fixed parameter $\mu \in \Lambda$.

Using spectral theory of self-adjoint compact operators in the Hilbert space [2], we obtain that problem (12) has a denumerable set of real eigenvalues of finite multiplicity $\gamma_k(\mu)$, $k = 1, 2, \ldots$, which are repeated according to their multiplicity, such that

$$-\alpha_0 < \gamma_1(\mu) \le \gamma_2(\mu) \le \ldots \le \gamma_k(\mu) \le \ldots, \qquad \lim_{k\to\infty} \gamma_k(\mu) = \infty.$$

The following minimax principle holds:

$$\gamma_k(\mu) = \min_{W_k \subset V} \max_{v \in W_k \setminus \{0\}} \frac{a(\mu, v, v)}{b(v, v)}, \quad k = 1, 2, \ldots, \tag{13}$$

where $\mu \in \Lambda$, W_k is k-dimensional subspace of the space V, $k = 1, 2, \ldots$

Now we can define eigenvalues of problem (11) or (5) as roots of the equations:

$$\mu - \gamma_k(\mu) = 0, \quad k = 1, 2, \ldots \tag{14}$$

Thus, the question on the existence of eigenvalues of problem (5) or (1) is the existence question of roots of equations (14).

3 Existence of eigenvalues

First we shall study properties of the functions $\gamma_k(\mu)$, $k = 1, 2, \ldots$

Lemma 2 *The functions $\gamma_k(\mu)$, $\mu \in \Lambda$, $k = 1, 2, \ldots$, are continuous nonincreasing functions.*

Proof The assertion of this lemma follows from minimax principle (13). □

We set

$$\gamma_i(\sigma_1) = \lim_{\mu\to\sigma_1,\ \mu\in\Lambda} \gamma_i(\mu), \quad \gamma_i(\sigma_2) = \lim_{\mu\to\sigma_2,\ \mu\in\Lambda} \gamma_i(\mu), \quad i = 1, 2, \ldots$$

Lemma 3 *The following inequalities hold: $\gamma_k(\sigma_1) > \sigma_1$, $k = 1, 2, \ldots$*

Proof The desired inequalities follow from minimax principle (13) and the relations:

$$\begin{aligned}
&a(\sigma_1, v, v) = \\
&= c(\mathbb{R}^2; v, v) - \sigma_1\, d(\mathbb{R}^2 \setminus D; v, v) \ge \\
&\ge p_{11} \int_{\mathbb{R}^2} \left\{ |\mathrm{Rot}_\beta\, v|^2 + \beta^2 |\mathrm{Div}_\beta\, v|^2 \right\} dx - \sigma_1 \int_{\mathbb{R}^2\setminus D} |v|^2\, dx = \\
&= p_{11} \int_{\mathbb{R}^2} \left\{ |\mathrm{grad}\, v|^2 + \beta^2 |v|^2 \right\} dx - \sigma_1 \int_{\mathbb{R}^2\setminus D} |v|^2\, dx = \\
&= p_{11} \int_{\mathbb{R}^2} |\mathrm{grad}\, v|^2\, dx + \sigma_1 b(v, v) > \\
&> \sigma_1 b(v, v)
\end{aligned}$$

for all $v \in V \setminus \{0\}$. $\square$

Lemma 4 *The following inequalities are valid:* $\gamma_1(\sigma_2) \le \gamma_2(\sigma_2) < \sigma_2$.

Proof For $D \subseteq D_r$, $0 < r < R$, we introduce the function

$$\varphi_R(x) = \begin{cases} 1, & x \in D_r, \\ \dfrac{\log|x| - \log R}{\log r - \log R}, & x \in D_R \setminus D_r, \\ 0, & x \in \mathbb{R}^2 \setminus D_R, \end{cases}$$

and define the subspace

$$W_2(\varphi_R) = \{w : w = (\delta_1\varphi_R, \delta_2\varphi_R, 0)^\top, (\delta_1, \delta_2)^\top \in \mathbb{C}^2\}.$$

According to minimax principle (13), we obtain the relations

$$\begin{aligned} \gamma_2(\mu) &= \min_{W_2 \subset V} \max_{v \in W_2 \setminus \{0\}} \frac{a(\mu, v, v)}{b(v, v)} \le \\ &\le \max_{v \in W_2(\varphi_R) \setminus \{0\}} \frac{a(\mu, v, v)}{b(v, v)} \le \\ &\le \sigma_2 + \varkappa_R(\mu) \end{aligned}$$

with

$$\varkappa_R(\mu) = \frac{1}{|D|}\left(2\pi p_2 \left(\log\frac{R}{r}\right)^{-1} - \beta^2 \int_\Omega (p_2 - p)\,dx + (\sigma_2 - \mu)\,\pi R^2\right),$$

where $\mu \in \Lambda$, $|D|$ denotes the area of the domain D. Consequently, by taking R large enough and $\sigma_2 - \mu$ small enough, we obtain $\varkappa_R(\mu) < 0$ and, therefore, $\gamma_1(\sigma_2) \le \gamma_2(\sigma_2) \le \gamma_2(\mu) \le \sigma_2 + \varkappa_R(\mu) < \sigma_2$. $\square$

Lemma 5 *The following relations hold:* $\gamma_k(\sigma_2) \to \infty$ *as* $k \to \infty$.

Proof The numbers $\gamma_k(\sigma_2)$, $k = 1, 2, \ldots$, are eigenvalues of the following problem: find $\eta \in \mathbb{R}$, $u \in W \setminus \{0\}$, such that

$$a(\sigma_2, u, v) + \alpha_0\, b(u, v) = (\eta + \alpha_0)\, b(u, v) \quad \forall v \in W,$$

where W denotes the Sobolev space equipped with the norm

$$\|v\|_W = \left(\int_{\mathbb{R}^2} |\operatorname{grad} v|^2\,dx + \int_D |v|^2\,dx\right)^{1/2}.$$

Since $\eta_k \to \infty$ as $k \to \infty$, we obtain the assertion of the lemma. $\square$

Now we formulate the main result of the paper.

Theorem 6 *Let* $m = \max\{i : \gamma_i(\sigma_2) < \sigma_2, i \geq 1\}$. *Then*

(a) *Problem* (1) *has exactly* m, $2 \leq m < \infty$, *real eigenvalues of finite multiplicity* $\lambda_i = \lambda_i(p)$, $i = 1, 2, \ldots, m$, *which are repeated according to their multiplicity:*

$$\sigma_1 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < \sigma_2,$$

i.e., if for $1 \leq i \leq m$ *we have*

$$\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+k} < \lambda_{i+k+1},$$

with $\lambda_0 = \sigma_1$, $\lambda_{m+1} = \sigma_2$, *then* $\dim U(\lambda_i) = k + 1$, *where* $U(\lambda_i)$ *is the eigensubspace corresponding to the eigenvalue* λ_i.

(b) *Each eigenvalue* λ_i, $1 \leq i \leq m$, *is a unique root of the equation* $\mu - \gamma_i(\mu) = 0$, $1 \leq i \leq m$. *The following relations hold:*

$$\lambda_k = \min_{W_k \subset V} \max_{v \in W_k \setminus \{0\}} \frac{a(\lambda_k, v, v)}{b(v, v)}, \quad k = 1, 2, \ldots, m,$$

where W_k *is* k*-dimensional subspace of the space* V, $k = 1, 2, \ldots, m$.

(c) *The corresponding eigenelements* $u^{(i)}$, $i = 1, 2, \ldots, m$, *form a orthogonal system in* $(L_2(\mathbb{R}^2))^3$ *such that*

$$\int_{\mathbb{R}^2} u^{(i)} \cdot u^{(j)}\, dx = \delta_{ij}, \quad i, j = 1, 2, \ldots, m.$$

Proof By Lemmata 2 and 3 each equation of the set

$$\mu - \gamma_k(\mu) = 0, \quad \mu \in \Lambda, \quad k = 1, 2, \ldots, m,$$

has a unique solution. Denote these solutions by λ_i, $i = 1, 2, \ldots, m$, i.e., $\lambda_i - \gamma_i(\lambda_i) = 0$, $i = 1, 2, \ldots, m$. Applying Lemmata 4 and 5 we get $2 \leq m < \infty$. To check that the numbers λ_i, $i = 1, 2, \ldots, m$, are put in a nondecreasing order, let us assume the opposite, i.e., $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 2, we obtain a contradiction, namely

$$\lambda_i = \gamma_i(\lambda_i) \leq \gamma_i(\lambda_{i+1}) \leq \gamma_{i+1}(\lambda_{i+1}) = \lambda_{i+1},$$

As was noted in Section 2 the numbers λ_i, $i = 1, 2, \ldots, m$, are eigenvalues of problem (1).

Let us prove if $\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+k} < \lambda_{i+k+1}$, then $\dim U(\lambda_i) = k + 1$. Since $\lambda_j = \gamma_j(\lambda_j)$, $j = i, i + 1, \ldots, i + k$, we have

$$\gamma_{i-1}(\lambda_{i-1}) < \gamma_i(\lambda_i) = \ldots = \gamma_{i+k}(\lambda_{i+k}) < \gamma_{i+k+1}(\lambda_{i+k+1}).$$

Now by Lemma 2 we get

$$\gamma_{i-1}(\lambda_i) < \gamma_i(\lambda_i) = \ldots = \gamma_{i+k}(\lambda_i) < \gamma_{i+k+1}(\lambda_i).$$

Hence $\gamma_i(\lambda_i)$ is the eigenvalue of problem (6) for $\mu = \lambda_i$ and $\dim \tilde{U}(\gamma_i(\lambda_i)) = k + 1$, where $\tilde{U}(\gamma_i(\lambda_i))$ is the eigensubspace corresponding to the eigenvalue $\gamma_i(\lambda_i)$. Therefore, we conclude that $\dim U(\lambda_i) = \dim \tilde{U}(\gamma_i(\lambda_i)) = k + 1$.

It is well known that the eigenelements $u^{(i)}$ and $u^{(j)}$ of problem (1) corresponding to the eigenvalues λ_i and λ_j for $\lambda_i \neq \lambda_j$ are orthogonal in $(L_2(\mathbb{R}^2))^3$. For each eigensubspace $U(\lambda_i)$, $\dim U(\lambda_i) = k+1$, $\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+k} < \lambda_{i+k+1}$, one can define orthogonal in $(L_2(\mathbb{R}^2))^3$ system of eigenelements $u^{(j)}$, $j = i, i+1, \ldots, i+k$. Thus we have constructed the orthogonal in $(L_2(\mathbb{R}^2))^3$ system of eigenelements $u^{(i)}$, $i = 1, 2, \ldots, m$, corresponding to the eigenvalues λ_i, $i = 1, 2, \ldots, m$. $\square$

Theorem 6 states the necessary and sufficient conditions for the existence of a preassigned number of eigenvalues of problem (1). These conditions contain the values $\eta_i^0 = \gamma_i(\sigma_2)$, $i = 1, 2, \ldots$, which can be calculated by numerical methods (Remark 21). Using minimax principle (13) we can construct more simple sufficient conditions formulated in the following corollaries.

Corollary 7 *Suppose that* $\Omega_2 \subseteq \Omega_1$, $p^{(1)}(x) = p^{(1)}(\Omega_1, x)$ *and* $p^{(2)}(x) = p^{(2)}(\Omega_2, x)$ *are the functions of the form* (2), $p^{(2)} \geq p^{(1)}$, *there exist* $\lambda_i(p^{(2)})$, $i = 1, 2, \ldots, m$. *Then there exist at least* m *eigenvalues* $\lambda_i(p^{(1)})$, $i = 1, 2, \ldots, m$, *and the following inequalities hold:*

$$\lambda_i(p^{(1)}) \leq \lambda_i(p^{(2)}), \quad i = 1, 2, \ldots, m. \tag{15}$$

Proof Assume that $\Omega_1 \subseteq D$ and consider problems (5) and (6). Denote by $\gamma_i(p^{(j)}, \mu)$, $i = 1, 2, \ldots$, the eigenvalues of problem (6) corresponding to the function $p^{(j)}$, $j = 1, 2$. If $\lambda_i(p^{(2)})$, $i = 1, 2, \ldots, m$, exist, then the numbers $\lambda_i(p^{(2)})$, $i = 1, 2, \ldots, m$, are the roots of the equations $\mu - \gamma_i(p^{(2)}, \mu) = 0$, $i = 1, 2, \ldots, m$. Therefore, $\gamma_i(p^{(2)}, \sigma_2) < \sigma_2$, $i = 1, 2, \ldots, m$. By minimax principle (13) we obtain

$$\gamma_i(p^{(1)}, \sigma_2) \leq \gamma_i(p^{(2)}, \sigma_2) < \sigma_2 \quad i = 1, 2, \ldots, m.$$

Hence the equations $\mu - \gamma_i(p^{(1)}, \mu) = 0$, $i = 1, 2, \ldots, m$, have the roots $\lambda_i(p^{(1)})$, $i = 1, 2, \ldots, m$. $\square$

Corollary 8 *Suppose that* $D_R \subseteq \Omega$, $q_1 = p_{13}(D_R)$, $q_2 = p_2$, $q(x) = q(D_R, x)$ *is the function of the form* (2), *there exist* $\lambda_i(q)$, $i = 1, 2, \ldots, m$. *Then there exist at least* m *eigenvalues* $\lambda_i(p)$, $i = 1, 2, \ldots, m$, *and the following inequalities hold:*

$$\lambda_i(p) \leq \lambda_i(q), \quad i = 1, 2, \ldots, m. \tag{16}$$

Proof The assertion of the corollary is a consequence of Corollary 7. $\square$

Note that Corollary 8 is very important for practice because we may calculate eigenvalues $\lambda_i(q)$, $i = 1, 2, \ldots, m$, by the analytical method [17], [18], [22].

Corollary 9 *Suppose that* Ω_0 *is a given bounded plane domain with a Lipschitz-continuous boundary* Γ_0, $\Omega_0 \subseteq \Omega$, $p^{(0)}(x) = p^{(0)}(\Omega_0, x)$ *is the function of the form* (2), $p^{(0)} \geq p$, $\eta_i = \eta_i(p^{(0)})$, $i = 1, 2, \ldots$, *are the eigenvalues of the following problem: find* $\eta = \eta(p^{(0)}) \in \Lambda$, $u \in V_0 \setminus \{0\}$, *such that*

$$c(\Omega_0; u, v) = \eta\, d(\Omega_0; u, v) \quad \forall v \in V_0,$$

where $V_0 = (H_0^1(\Omega_0))^3 \equiv \{v : v \in (H^1(\Omega_0))^3, v|_{\Gamma_0} = 0\}$*, the sesquilinear forms* c *and* d *are defined in Section* 1 *by using the function* $p^{(0)}$*. If* $\eta_m < \sigma_2$*, then there exist at least* m *real eigenvalues of finite multiplicity* $\lambda_i(p)$*,* $i = 1, 2, \ldots, m$*, which are repeated according to their multiplicity, and the following inequalities hold:*

$$\lambda_i(p) \leq \eta_i, \quad i = 1, 2, \ldots, m.$$

Proof By minimax principle (13) we get

$$\gamma_i(p, \sigma_2) \leq \eta_i < \sigma_2 \quad i = 1, 2, \ldots, m,$$

where $\gamma_i(p, \mu)$ is defined in the proof of Corollary 7. Hence the equations $\mu - \gamma_i(p, \mu) = 0$, $i = 1, 2, \ldots, m$, have the roots $\lambda_i(p)$, $i = 1, 2, \ldots, m$. □

Corollary 10 *Let* $m = \max\{i : \gamma_i(\delta) < \sigma_2, i \geq 1\}$*,* $\delta \in (\sigma_1, \sigma_2)$*. Then problem* (1) *has at least* m*,* $2 \leq m < \infty$*, real eigenvalues of finite multiplicity* $\lambda_i = \lambda_i(p)$*,* $i = 1, 2, \ldots, m$*, which are repeated according to their multiplicity:*

$$\sigma_1 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < \delta.$$

Proof By the definition of the number m and Lemma 2, we get the relations $\gamma_i(\sigma_2) \leq \gamma_i(\delta) < \sigma_2$, $i = 1, 2, \ldots, m$, which with Theorem 6 imply the desired result. □

4 Simple sufficient conditions

Let Ω_0 be a given bounded plane domain with a Lipschitz-continuous boundary Γ_0, $\Omega_0 \subseteq \Omega$. Assume that $\mathbf{V}_0 = (H_0^1(\Omega_0))^2 \equiv \{\mathbf{v} : \mathbf{v} \in (H^1(\Omega_0))^2, \mathbf{v}|_{\Gamma_0} = 0\}$ and $\mathbf{H}_0 = (L_2(\Omega_0))^2$ are the Sobolev and Lebesgue spaces equipped with the norms

$$\|\mathbf{v}\|_{\mathbf{V}_0} = \left(\int_{\Omega_0} |\operatorname{grad} \mathbf{v}|^2 \, dx\right)^{1/2}, \quad \|\mathbf{v}\|_{\mathbf{H}_0} = \left(\int_{\Omega_0} |\mathbf{v}|^2 \, dx\right)^{1/2}.$$

Let us define the mappings $a_0 : \mathbf{V}_0 \times \mathbf{V}_0 \to \mathbb{C}$, $b_0 : \mathbf{H}_0 \times \mathbf{H}_0 \to \mathbb{C}$, by the formulae:

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_0} \operatorname{grad} \mathbf{u} \cdot \operatorname{grad} \overline{\mathbf{v}} \, dx, \\ b_0(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_0} \mathbf{u} \cdot \overline{\mathbf{v}} \, dx. \end{aligned}$$

Here we denote $\mathbf{v} = (v_1, v_2)^\top$, $\operatorname{grad} \mathbf{v} = (\operatorname{grad} v_1, \operatorname{grad} v_2)^\top$, $\mathbf{u} \cdot \overline{\mathbf{v}} = u_1 \overline{v}_1 + u_2 \overline{v}_2$, $|\mathbf{v}|^2 = |v_1|^2 + |v_2|^2$.

Let us introduce the following linear eigenvalue problem: find $\lambda^0 \in \mathbb{R}$, $\mathbf{u} \in \mathbf{V}_0 \setminus \{0\}$, such that

$$a_0(\mathbf{u}, \mathbf{v}) = \lambda^0 b_0(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0. \tag{17}$$

Problem (17) has a denumerable set of real positive eigenvalues of finite multiplicity λ_k^0, $k = 1, 2, \ldots$, [2], which are repeated according to their multiplicity, such that

$$0 < \lambda_1^0 \leq \lambda_2^0 \leq \ldots \leq \lambda_k^0 \leq \ldots, \quad \lim_{k\to\infty} \lambda_k^0 = \infty.$$

The following relations hold:

$$\lambda_k^0 = \min_{\mathbf{W}_k \subset \mathbf{V}_0} \max_{\mathbf{v} \in \mathbf{W}_k \setminus \{0\}} \frac{a_0(\mathbf{v}, \mathbf{v})}{b_0(\mathbf{v}, \mathbf{v})}, \quad k = 1, 2, \ldots,$$

where $\mathbf{W}_k$ is k-dimensional subspace of the space $\mathbf{V}_0$, $k = 1, 2, \ldots$

Theorem 11 *Let $m = \max\{i : p_2\lambda_i^0 < \sigma_2 - \sigma_0, i \geq 1\}$, where $\sigma_0 = \beta^2 p_0$, $p_0 = p_{13}(\Omega_0)$. Then problem* (1) *has at least m, $2 \leq m < \infty$, real eigenvalues of finite multiplicity $\lambda_i = \lambda_i(p)$, $i = 1, 2, \ldots, m$, which are repeated according to their multiplicity:*

$$\sigma_1 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < \sigma_2.$$

Moreover, assertions (b) *and* (c) *of Theorem* 6 *are valid.*

Proof Taking into account minimax principle (13) and representation (7), we obtain

$$\begin{aligned} \gamma_k(\mu) &= \min_{W_k \subset V} \max_{v \in W_k \setminus \{0\}} \frac{a(\mu, v, v)}{b(v, v)} \leq \\ &\leq \min_{W_k \subset V_0} \max_{v \in W_k \setminus \{0\}} \frac{a(\mu, v, v)}{b(v, v)} \leq \\ &\leq \sigma_0 + p_2 \min_{\mathbf{W}_k \subset \mathbf{V}_0} \max_{\mathbf{v} \in \mathbf{W}_k \setminus \{0\}} \frac{a_0(\mathbf{v}, \mathbf{v})}{b_0(\mathbf{v}, \mathbf{v})} = \\ &= \sigma_0 + p_2 \lambda_k^0, \end{aligned}$$

where

$$V_0 = \{v : v = (v_1, v_2, 0)^\top, v \in V, v(x) = 0, x \in \mathbb{R}^2 \setminus \Omega_0\}.$$

Hence the relations $\gamma_k(\sigma_2) \leq \sigma_0 + p_2\lambda_i^0 < \sigma_2$ imply the condition $p_2\lambda_i^0 < \sigma_2 - \sigma_0$. Thus by Theorem 6 we get desired assertions. □

For $z = (z_1, z_2)^\top \in \mathbb{R}^2$ we set

$$\begin{aligned} D_R(z) &= \{x : x \in \mathbb{R}^2, |x - z| < R\}, \\ S_B(z) &= \{x : x = (x_1, x_2)^\top \in \mathbb{R}^2, z_i < x_i < z_i + B, i = 1, 2\}. \end{aligned}$$

The formulae for eigenvalues of the Laplace operator in a disk or square [33] are well known. Using these formulae in Theorem 11 we can derive simple sufficient conditions for the existence of eigenvalues of problem (1). Below we indicate examples of simple formulae for eigenvalues of problem (17).

Remark 12 Let $\Omega_0 = D_R(z)$, $\Omega_0 \subseteq \Omega$, R is a fixed positive number, $z \in \mathbb{R}^2$, μ_k^0, $k = 1,2,\ldots$, $\mu_1^0 \le \mu_2^0 \le \ldots \le \mu_k^0 \le \ldots$, are the positive roots t_{ni}, $n = 0,1,\ldots$, $i = 1,2,\ldots$, enumerated in ascending order, of the equations $J_n(t) = 0$, $t \in (0,\infty)$, $n = 0,1,\ldots$, where $J_n(t)$ is the Bessel functions of first kind of order n, $n = 0,1,\ldots$ Then

$$\lambda_{2i-1}^0 = \left(\frac{\mu_i^0}{R}\right)^2, \quad \lambda_{2i}^0 = \left(\frac{\mu_i^0}{R}\right)^2, \quad i = 1,2,\ldots$$

Remark 13 Let $\Omega_0 = S_B(z)$, $\Omega_0 \subseteq \Omega$, B is a fixed positive number, $z \in \mathbb{R}^2$, μ_k^1, $k = 1,2,\ldots$, $\mu_1^1 \le \mu_2^1 \le \ldots \le \mu_k^1 \le \ldots$, are numbers of the form $\pi\sqrt{i^2+j^2}$, $i,j = 1,2,\ldots$, enumerated in ascending order. Then

$$\lambda_{2i-1}^0 = \left(\frac{\mu_i^1}{B}\right)^2, \quad \lambda_{2i}^0 = \left(\frac{\mu_i^1}{B}\right)^2, \quad i = 1,2,\ldots$$

Remark 14 Let

$$\Omega_0 = \left(\bigcup_{i=1}^{n_0} D_{R_i}(a^{(i)})\right) \bigcup \left(\bigcup_{j=1}^{n_1} S_{B_j}(b^{(j)})\right),$$

where

$$\begin{aligned}
\overline{D_{R_i}(a^{(i)})} \cap \overline{D_{R_j}(a^{(j)})} &= \emptyset, \quad i \ne j, \quad i,j = 1,2,\ldots,n_0,\\
\overline{S_{B_i}(b^{(i)})} \cap \overline{S_{B_j}(b^{(j)})} &= \emptyset, \quad i \ne j, \quad i,j = 1,2,\ldots,n_1,\\
\overline{D_{R_i}(a^{(i)})} \cap \overline{S_{B_j}(b^{(j)})} &= \emptyset, \quad i = 1,2,\ldots,n_0, \quad j = 1,2,\ldots,n_1,
\end{aligned}$$

$\Omega_0 \subseteq \Omega$, R_i, $i = 1,2,\ldots,n_0$, B_j, $j = 1,2,\ldots,n_1$, are fixed positive numbers, $a^{(i)} \in \mathbb{R}^2$, $i = 1,2,\ldots,n_0$, $b^{(j)} \in \mathbb{R}^2$, $j = 1,2,\ldots,n_1$. Here for a domain G with the boundary Γ we define $\overline{G} = G \cup \Gamma$. Denote

$$\eta_{2i-1}^{0,j} = \left(\frac{\mu_i^0}{R_j}\right)^2, \quad \eta_{2i}^{0,j} = \left(\frac{\mu_i^0}{R_j}\right)^2, \quad i = 1,2,\ldots, \quad j = 1,2,\ldots,n_0,$$

$$\eta_{2i-1}^{1,j} = \left(\frac{\mu_i^1}{B_j}\right)^2, \quad \eta_{2i}^{1,j} = \left(\frac{\mu_i^1}{B_j}\right)^2, \quad i = 1,2,\ldots, \quad j = 1,2,\ldots,n_1,$$

where μ_k^0, μ_k^1, $k = 1,2,\ldots$, are defined in Remarks 12 and 13. Enumerating the numbers

$$\eta_i^{k,j_k}, \quad i = 1,2,\ldots, \quad j_k = 1,2,\ldots,n_k, \quad k = 0,1,$$

in ascending order, we obtain the sequence λ_i^0, $i = 1,2,\ldots$

5 Eigenvalue problem in a bounded domain

Assume that $\Omega \subseteq D_R$ and put $V_R = (H^1(D_R))^3$, $H_R = (L_2(D_R))^3$. Let us denote by Γ the boundary of D_R and by ∂_ν the outward normal derivative on Γ, i.e., $\partial_\nu \varphi = \operatorname{grad} \varphi \cdot \nu$, where $\nu = (\nu_1, \nu_2)^\top$ is the unit outward normal to Γ.

Applying the following representation for the sesquilinear form c:

$$\begin{aligned}
c(\mathbb{R}^2; u, v) &= c_1(D_R; u, v) + c_2(\mathbb{R}^2 \setminus D_R; u, v), \\
c_1(G; u, v) &= \int_G (p - p_2) \operatorname{Rot}_\beta u \cdot \overline{\operatorname{Rot}_\beta v} \, dx + c_2(G; u, v), \\
c_2(G; u, v) &= p_2 \int_G \left\{ \operatorname{grad} u \cdot \operatorname{grad} \overline{v} + \beta^2 \, u \cdot \overline{v} \right\} \, dx,
\end{aligned}$$

we write problem (3) for $D = D_R$ in the equivalent form: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, such that

$$\begin{aligned}
&c_1(D_R; u, v) + c_2(\mathbb{R}^2 \setminus D_R; u, v) - \lambda \, d(\mathbb{R}^2 \setminus D_R; u, v) + \alpha_0 \, d(D_R; u, v) = \\
&= (\lambda + \alpha_0) \, d(D_R; u, v) \quad \forall v \in V.
\end{aligned} \tag{18}$$

To transform equation (18) in the unbounded domain $\mathbb{R}^2$ to an equation in the bounded domain D_R, let us note that the eigenelement u of problem (1) satisfies the following equation [1]:

$$-\Delta u + \sigma^2 u = 0, \quad x \in \mathbb{R}^2 \setminus D_R, \tag{19}$$

where $\sigma = \sigma(\lambda) = \sqrt{(\sigma_2 - \lambda)/p_2}$, $\lambda \in \Lambda$. This equation is the consequence of equation (18).

The solution of equation (19) is defined by the well known formulae [19]:

$$u(r \cos \varphi, r \sin \varphi) = \sum_{n=-\infty}^{\infty} \frac{K_n(\sigma r)}{K_n(\sigma R)} \, a_n(u) \, e^{in\varphi},$$

$$a_n(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(R\cos\varphi, R\sin\varphi) e^{-in\varphi} \, d\varphi, \quad n = 0, \pm 1, \pm 2, \ldots,$$

where $K_n(t)$ is the modified Bessel function of order n, $n = 0, \pm 1, \pm 2, \ldots$ Hence one can obtain the explicit formula for the normal derivative $\partial_\nu u$ on Γ of the eigenelement u:

$$\partial_\nu u\big|_\Gamma = \frac{\partial}{\partial r} u(r \cos \varphi, r \sin \varphi) \bigg|_\Gamma = -\frac{1}{R} \sum_{n=-\infty}^{\infty} H_n(\sigma R) \, a_n(u) \, e^{in\varphi},$$

where

$$H_n(t) = |n| + t \, \frac{K_{|n|-1}(t)}{K_{|n|}(t)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Then for the term from (18) containing the integral over $\mathbb{R}^2 \setminus D_R$, we get the following representation:

$$
\begin{aligned}
&c_2(\mathbb{R}^2 \setminus D_R; u, v) - \lambda\, d(\mathbb{R}^2 \setminus D_R; u, v) = \\
&= p_2 \int_{\mathbb{R}^2\setminus D_R} \left\{\operatorname{grad} u \cdot \operatorname{grad} \overline{v} + \beta^2\, u \cdot \overline{v}\right\}\, dx - \lambda \int_{\mathbb{R}^2\setminus D_R} u \cdot \overline{v}\, dx = \\
&= p_2 \int_{\mathbb{R}^2\setminus D_R} \left\{\operatorname{grad} u \cdot \operatorname{grad} \overline{v} + \sigma^2\, u \cdot \overline{v}\right\}\, dx = \\
&= p_2 \int_{\mathbb{R}^2\setminus D_R} (-\Delta u + \sigma^2\, u) \cdot \overline{v}\, dx - p_2 \int_\Gamma \partial_\nu u \cdot \overline{v}\, d\gamma = \\
&= -p_2 \int_\Gamma \partial_\nu u \cdot \overline{v}\, d\gamma = \\
&= 2\pi p_2 \sum_{n=-\infty}^{\infty} H_n(\sigma R)\, a_n(u) \cdot \overline{a_n(v)}.
\end{aligned}
$$

Define the mapping $s : \Lambda \times V_R \times V_R \to \mathbb{C}$, by the formula:

$$
s(\mu, u, v) = 2\pi p_2 \sum_{n=-\infty}^{\infty} H_n(\sigma(\mu) R)\, a_n(u) \cdot \overline{a_n(v)},
$$

where $\sigma(\mu) = \sqrt{(\sigma_2 - \mu)/p_2}$, $\mu \in \Lambda$.

Now one can write the following equivalent formulation of problem (1) in the bounded domain D_R: find $\lambda \in \Lambda$, $u \in V_R \setminus \{0\}$, such that

$$
c_1(D_R; u, v) + s(\lambda, u, v) + \alpha_0\, d(D_R; u, v) = (\lambda + \alpha_0)\, d(D_R; u, v) \quad \forall v \in V_R. \tag{20}
$$

Note that similar approaches of obtaining problems in a bounded domain have been applied in the papers [8], [7], [3], [16], [4], for the cases of scalar equations, and in the paper [5] for the case of the vector equation. In the present paper, for problem (1) we propose variational formulation (20), which leads to the differential eigenvalue problem with the more simple boundary condition than in [5].

Introduce the mappings $a_R : \Lambda \times V_R \times V_R \to \mathbb{C}$, $b_R : H_R \times H_R \to \mathbb{C}$, by the formulae:

$$
\begin{aligned}
a_R(\mu, u, v) &= c_1(D_R; u, v) + s(\mu, u, v), \\
b_R(u, v) &= d(D_R; u, v).
\end{aligned}
$$

Lemma 15 *For $v \in V_R$, $\mu \in \Lambda$, the following inequality is valid:*

$$
a_R(\mu, v, v) \geq \tilde{\alpha}_1 \|v\|^2_{V_R} - \alpha_0 \|v\|^2_{H_R},
$$

where α_0 is defined by (4),

$$
\tilde{\alpha}_1 = \begin{cases} \min\{(1-\varepsilon)p_{11}, \sigma_1\}, & \xi \leq 0, \\ \min\{p_{11}, \xi, \beta^2\xi, \sigma_2\}, & \xi > 0, \end{cases}
$$

$\xi = 2p_{11} - p_2$, $\varepsilon \in (0, 1)$.

Proof For fixed number $\mu \in \Lambda$ and for fixed function $v \in V_R$, using $\partial_n v|_\Gamma$ we can construct the function $v \in V$, if we define $v(x)$, $x \in \mathbb{R}^2 \setminus D_R$, as the solution of problem (19) for $\mu = \lambda$. Then applying (9) and (10) we get

$$\begin{aligned}
&a_R(\mu, v, v) = c_1(D_R; v, v) + s(\mu, v, v) = \\
&= \ c_1(D_R; v, v) + c_2(\mathbb{R}^2 \setminus D_R; v, v) - \mu\, d(\mathbb{R}^2 \setminus D_R; v, v) = \\
&= \ c(\mathbb{R}^2; v, v) - \mu\, d(\mathbb{R}^2 \setminus D_R; v, v) = \\
&= \ a(\mu, v, v) \geq \\
&\geq \ p_{11} \int_{\mathbb{R}^2} |\mathrm{grad}\, \mathbf{v}|^2\, dx + p_2 \int_{\mathbb{R}^2\setminus\Omega} |\mathrm{grad}\, v_3|^2\, dx + \\
&\quad +\xi \int_\Omega |\mathrm{grad}\, v_3|^2\, dx + \xi\beta^2 \int_\Omega |\mathbf{v}|^2\, dx + \\
&\quad +\sigma_2 \int_{D_R} |v_3|^2\, dx + \sigma_2 \int_{D_R\setminus\Omega} |\mathbf{v}|^2\, dx + (\sigma_2 - \mu) \int_{\mathbb{R}^2\setminus D_R} |v|^2\, dx \geq \\
&\geq \ \min\{p_{11}, \xi, \beta^2\xi, \sigma_2\} \int_{D_R} \left\{|\mathrm{grad}\, v|^2 + |v|^2\right\}\, dx = \\
&= \ \tilde{\alpha}_1 \|v\|_V^2, \quad \xi > 0, \\
&a_R(\mu, v, v) = a(\mu, v, v) \geq \\
&\geq \ p_{11} \int_{\mathbb{R}^2} |\mathrm{grad}\, \mathbf{v}|^2\, dx + p_2 \int_{\mathbb{R}^2\setminus\Omega} |\mathrm{grad}\, v_3|^2\, dx + \\
&\quad +(1-\varepsilon)p_{11} \int_\Omega |\mathrm{grad}\, v_3|^2\, dx + (\sigma_1 - \alpha_0) \int_\Omega |\mathbf{v}|^2\, dx + \\
&\quad +\sigma_2 \int_{D_R} |v_3|^2\, dx + \sigma_2 \int_{D_R\setminus\Omega} |\mathbf{v}|^2\, dx + (\sigma_2 - \mu) \int_{\mathbb{R}^2\setminus D_R} |v|^2\, dx \geq \\
&\geq \ \min\{(1-\varepsilon)p_{11}, \sigma_1\} \int_{D_R} \left\{|\mathrm{grad}\, v|^2 + |v|^2\right\}\, dx - \alpha_0 \int_{D_R} |v|^2\, dx = \\
&= \ \tilde{\alpha}_1 \|v\|_{V_R}^2 - \alpha_0 \|v\|_{H_R}^2, \quad \xi \leq 0.
\end{aligned}$$

□

It is well known [14], [15], that for $v \in V_R$ we have $v|_\Gamma \in W_R \equiv (H^{1/2}(\Gamma))^3$ and there exists $\delta > 0$ such that

$$\|v\|_{W_R} \leq \delta \|v\|_{V_R} \quad \forall v \in V_R,$$

where

$$\|v\|_{W_R}^2 = \sum_{n=-\infty}^{\infty} (|n| + 1)\, |a_n(v)|^2.$$

Lemma 16 *For $v \in V_R$, $\mu \in \Lambda$, the following inequality holds:*

$$a_R(\mu, v, v) \leq \tilde{\alpha}_2(\mu) \|v\|_{V_R}^2,$$

where

$$\begin{aligned}
&\tilde\alpha_2(\mu) = p_2 \max\{1, \beta^2\} + 2\pi p_2 \delta^2 \max\{1, H_0(\sigma(\mu)R)\},\\
&\sigma(\mu) = \sqrt{(\sigma_2 - \mu)/p_2},\\
&H_0(t) = t\,\frac{K_1(t)}{K_0(t)}.
\end{aligned}$$

Proof Using definitions of sesquilinear forms we get the desired relations:

$$\begin{aligned}
& a_R(\mu, v, v) = c_1(D_R; v, v) + s(\mu, v, v) = \\
= \;& \int_{D_R} (p - p_2)\, |\mathrm{Rot}_\beta\, v|^2\, dx + p_2 \int_{D_R} \left\{ |\mathrm{grad}\, v|^2 + \beta^2\, |v|^2 \right\} dx + \\
+ \;& 2\pi p_2 \sum_{n=-\infty}^{\infty} H_n(\sigma(\mu)R)\, |a_n(v)|^2 \le \\
\le \;& p_2 \max\{1, \beta^2\} \|v\|^2_{V_R} + 2\pi p_2 \max\{1, H_0(\sigma(\mu)R)\} \|v\|^2_{W_R} \le \\
\le \;& \left(p_2 \max\{1, \beta^2\} + 2\pi p_2 \delta^2 \max\{1, H_0(\sigma(\mu)R)\}\right) \|v\|^2_{V_R} = \\
= \;& \tilde\alpha_2(\mu) \|v\|^2_{V_R}.
\end{aligned}$$

□

Now problem (20) can be written in the form: find $\lambda \in \Lambda$, $u \in V_R \setminus \{0\}$, such that

$$a_R(\lambda, u, v) + \alpha_0\, b_R(u, v) = (\lambda + \alpha_0)\, b_R(u, v) \quad \forall v \in V_R. \tag{21}$$

According to Lemmata 15 and 16 we obtain the positive definiteness and boundedness of the sesquilinear form in the left hand side of equation (21), i.e.,

$$\tilde\alpha_1 \|v\|^2_{V_R} \le a_R(\mu, v, v) + \alpha_0\, b_R(v, v) \le (\tilde\alpha_2(\mu) + \alpha_0) \|v\|^2_{V_R} \quad \forall v \in V_R \tag{22}$$

for fixed $\mu \in \Lambda$.

Define the sesquilinear forms $a_R^0 : V_R \times V_R \to \mathbb{C}$, $s_1 : V_R \times V_R \to \mathbb{C}$, by the formulae:

$$\begin{aligned}
a_R^0(u, v) &= c_1(D_R; u, v) + s_1(u, v),\\
s_1(u, v) &= 2\pi p_2 \sum_{n=-\infty}^{\infty} |n|\, a_n(u) \cdot \overline{a_n(v)}.
\end{aligned}$$

Lemma 17 *The following relation holds:* $s(\mu, u, v) \to s_1(u, v)$ *as* $\mu \to \sigma_2$, $\mu \in \Lambda$, $u, v \in V_R$.

Proof For $\mu \in \Lambda$ we have

$$s(\mu, u, v) = s_1(u, v) + s_2(\mu, u, v),$$

where

$$\begin{aligned} |s_2(\mu,u,v)| &= 2\pi p_2 \left| \sum_{n=-\infty}^{\infty} \sigma(\mu) R \,\frac{K_{|n|-1}(\sigma(\mu)R)}{K_{|n|}(\sigma(\mu)R)}\, a_n(u)\cdot \overline{a_n(v)} \right| \leq \\ &\leq 2\pi p_2\, H_0(\sigma(\mu)R)\, \|u\|_{W_R} \|v\|_{W_R}. \end{aligned}$$

Since $H_0(\sigma(\mu)R) \to 0$ as $\mu \to \sigma_2$, $s_2(\mu,u,v) \to 0$ as $\mu \to \sigma_2$. □

Introduce the linear eigenvalue problem: find $\eta^0 \in \mathbb{R}$, $u \in V_R \setminus \{0\}$, such that

$$a_R^0(u,v) + \alpha_0\, b_R(u,v) = (\eta^0 + \alpha_0)\, b_R(u,v) \quad \forall v \in V_R. \tag{23}$$

It can be easily show that

$$\alpha_1^0 \|v\|_{V_R}^2 \leq a_R^0(v,v) + \alpha_0\, b_R(v,v) \leq \alpha_2^0 \|v\|_{V_R}^2 \quad \forall v \in V_R, \tag{24}$$

where $\alpha_1^0 = \tilde{\alpha}_1$, $\alpha_2^0 = p_2 \max\{1,\beta^2\} + 2\pi p_2 \delta^2$. Therefore, problem (23) has a denumerable set of real eigenvalues of finite multiplicity η_k^0, $k = 1,2,\ldots$, [2], which are repeated according to their multiplicity, such that

$$-\alpha_0 < \eta_1^0 \leq \eta_2^0 \leq \ldots \leq \eta_k^0 \leq \ldots, \quad \lim_{k\to\infty} \eta_k^0 = \infty.$$

Lemma 18 *The following relations hold:* $\gamma_k(\mu) \to \eta_k^0$ *as* $\mu \to \sigma_2$, $\mu \in \Lambda$, $k = 1,2,\ldots$

Proof The assertion of the lemma follows from minimax principles for eigenvalues $\gamma_k(\mu)$ and η_k^0, $k = 1,2,\ldots$ □

The following theorem states the necessary and sufficient condition for the existence of eigensolutions of problem (1) in terms of eigenvalues of the linear eigenvalue problem (23) in the bounded domain D_R.

Theorem 19 *Let* $m = \max\{i : \eta_i^0 < \sigma_2, i \geq 1\}$. *Then the results of Theorem* 6 *are valid.*

Proof By Lemma 18 we obtain that $\gamma_k(\sigma_2) = \eta_k^0$, $k = 1,2,\ldots$ Therefore, the assertion of the theorem follows from Theorem 6. □

Remark 20 To solve problem (21) one can use the finite element method in a bounded domain. The abstract convergence results of the finite element method for problem (21) are presented in the papers [23], [24], [25], [26]. The matrix nonlinear eigenvalue problem of the finite element method [26] has the form: find $\lambda \in \Lambda$, $y \in \mathbb{R}^N \setminus \{0\}$, such that

$$A(\lambda) y = (\lambda + \alpha_0) B y \tag{25}$$

with large sparse symmetric positive definite matrices $A(\lambda)$ and B of order N. Since the functions $f_n(\mu) = H_n(R\sqrt{(\sigma_2 - \mu)/p_2})$, $\mu \in \Lambda$, $n = 0, \pm 1, \pm 2, \ldots$, are decreasing functions, the matrix function $A(\mu)$, $\mu \in \Lambda$, has the monotonicity property $(A(\mu)y, y) \geq (A(\eta)y, y)$ for $\mu < \eta$, $\mu, \eta \in \Lambda$, $y \in \mathbb{R}^N$. By (22) we get the inequalities:

$$\tilde{\alpha}_1(Cy, y) \leq (A(\mu)y, y) \leq (\tilde{\alpha}_2(\mu) + \alpha_0)(Cy, y) \quad \forall y \in \mathbb{R}^N, \tag{26}$$

where $\mu \in \Lambda$, the matrix C is the finite element matrix for the differential operator L defined by $Lu = -\Delta u + u$ in the bounded domain D_R, $u = (u_1, u_2, u_3)^\top$, $\Delta u = \partial_1^2 u + \partial_2^2 u$. It follows from (26) that the matrices $A(\mu)$ and C are spectrally equivalent [6]. Therefore, the matrix C can be chosen as the preconditioner for $A(\mu)$. Efficient preconditioned iterative methods for solving large monotone nonlinear eigenvalue problems of the form (25) have been suggested in the papers [27], [28], [29], [30], [31].

Remark 21 The finite element method for linear eigenvalue problem (23) leads to the matrix linear eigenvalue problem: find $\eta \in \mathbb{R}$, $y \in \mathbb{R}^N \setminus \{0\}$, such that

$$A_0 y = (\eta + \alpha_0) B y$$

with large sparse symmetric positive definite matrices A_0 and B of order N. This problem can be solved by efficient preconditioned eigensolvers suggested and investigated in the recent papers [20], [11], [12], [13]. Relations (24) imply the following properties:

$$\alpha_1^0(Cy, y) \leq (A_0 y, y) \leq \alpha_2^0(Cy, y) \quad \forall y \in \mathbb{R}^N,$$

where the matrix C is defined in Remark 20. Hence the matrix C can be chosen as the preconditioner.

6 Conclusion

This paper presents a theoretical investigation of an eigenvalue problem describing the guided modes of an optical fiber. We consider the questions on the existence of eigenvalues and eigenfunctions and study their properties. We propose the statement of the problem in a bounded domain with the exact boundary condition and show that this problem belongs to the class of monotone positive definite nonlinear eigenvalue problems. This allows to apply the finite element method for solving the problem and to use the theoretical results on the convergence and error estimates obtained by the author in the previous papers. The finite element method leads to a monotone nonlinear matrix eigenvalue problem with large sparse matrices. We show that for solving this matrix eigenvalue problem one can apply efficient preconditioned iterative methods, which have been suggested in the previous papers of the author.

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