A Matrix Model for $\nu_{k_1 k_2} = \frac{k_1+k_2}{k_1 k_2}$ Fractional Quantum Hall States

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1. Introduction

Recently, Susskind[1)] showed that an Abelian non-commutative Chern-Simons theory at level k is actually equivalent to Laughlin theory:[2)]

$$S = \frac{k}{4\pi}\int d^3y \epsilon^{\mu\nu\lambda}\left[A_\mu \star \partial_\nu A_\lambda + \frac{2}{3}A_\mu \star A_\nu \star A_\lambda\right] \quad (1)$$

where the star–product is the usual Moyal product with parameter θ. Therefore, he obtained the filling factor

$$\nu_{\rm S} = \frac{1}{k}. \quad (2)$$

He also pointed out that the above theory can be formulated in terms of a matrix model involving classical Hermitian matrix variables A_0, X^i, $i = 1, 2$. The Lagrangian for the matrix theory is

$$L = B \ {\rm Tr}\left\{\epsilon_{ij}(\dot{X}^i + i[A_0, X^i])X^j + 2\theta A_0\right\}, \quad (3)$$

B is the magnetic field. The equation of motion for the coordinate A_0 (Gauss law constraint) is

$$[X^1, X^2] = i\theta \quad (4)$$

which can only be solved if the matrices are infinite dimensional. This corresponds to an infinite number of electrons on an infinite plane.

For a finite system, Polychronakos[3)] has introduced an additional set of bosonic degrees of freedom ψ_m, $m = 1, 2, ..., M$, such that $\psi = (\psi_1, \cdots, \psi_M)$,

$$L_\psi = \psi^\dagger(i\dot{\psi} - A_0\psi). \quad (5)$$

Considering $L + L_\psi$, Polychronakos[3)] found a quantum correction to Susskinds filling factor such that

$$\nu_{\rm P} = \frac{1}{k+1}. \quad (6)$$

In this case, the Gauss law constraint becomes

$$[X^1, X^2] = i\theta\left(\mathbf{1} - \frac{1}{k+1}\psi\psi^\dagger\right). \quad (7)$$

Later Hellerman and Van Raamsdonk[4)] built the corresponding wavefunctions for $L + L_\psi$,

$$|k\rangle = \left\{\epsilon^{i_1\cdots i_M}(\psi^\dagger)_{i_1}(\psi^\dagger A^\dagger)_{i_2}...(\psi^\dagger {A^\dagger}^{M-1})_{i_M}\right\}^k |0\rangle \quad (8)$$

where $\epsilon^{i_1\cdots i_M}$ is the fully antisymmetric tensor. These are similar to Laughlins wavefunction.[2)] Subsequently, three of us generalised[5)] the above results to any filling factor given by

$$\nu_{k_1 k_2} = \frac{1}{k_1} + \frac{1}{k_2}, \qquad k_2 > k_1. \quad (9)$$

In what follows, we propose a matrix model to describe such FQH states that are not of Laughlin type.

2. $\nu_{k_1 k_2}$ fractional quantum Hall states

Although the $\nu = \frac{2}{5}$ FQH state is not of the Laughlin type, it shares some basic features of Laughlin fluids. The point is that from the standard definition of the filling factor $\nu = \frac{N}{N_\phi}$, the state $\nu = \frac{2}{5}$ can naively be thought of as corresponding to $\nu = \frac{N}{N_\phi}$ where the number N_ϕ of flux quanta is given by a fractional amount of the electron number; that is

$$N_\phi = (3 - \frac{1}{2})N. \quad (10)$$

In fact this way of viewing things reflects the original idea of a hierarchical construction of FQH states for general filling factor $\frac{p}{q}$. In Haldane's hierarchy,[6)] the elements of the series

$$\nu_{p_1 p_2} = \frac{p_2}{p_1 p_2 - 1} \quad (11)$$

correspond to taking N_ϕ as given by a specific rational factor of the electron number, i.e.,

$$N_\phi = (p_1 - \frac{1}{p_2})N. \quad (12)$$

Upon setting

$$k_1 = p_1, \quad k_2 = k_1(k_1 p_2 - 1) \equiv r k_1 \quad (13)$$

we have $\nu_{p_1 p_2} \equiv \nu_{k_1 k_2}$. For $\nu = \frac{2}{5}$, e.g.,

$$\nu = \frac{2}{5} \equiv \frac{1}{3} + \frac{1}{15}. \quad (14)$$

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3. Matrix model analysis

To describe FQH fluids at $\nu_{k_1k_2}$, we consider the following action for a system of $N = N_1 + N_2$ particles[5)]

$$\begin{aligned}\mathcal{S} = \int dt\ \textstyle\sum_{i=1}^{2}\left[\frac{k_i}{4\theta}\mathrm{Tr}\left(i\bar{Z}_iDZ_i - \omega\bar{Z}_iZ_i + 2\theta A_{0,i}\right)\right] \\ + h.c. + \int dt\left[\frac{i}{2}\bar{\Psi}^{\alpha a}\left[\partial_t + A_{0,1}{}_{\alpha}^{\beta}\delta_a^b + A_{0,2}{}_a^b\delta_\alpha^\beta\right]\Psi_{\beta b} \\ + \lambda\bar{\Psi}^{\alpha a}Z_1{}_{\alpha}^{\beta}Z_{2a}^{b}\Psi_{\beta b}\right] + h.c.\end{aligned} \tag{15}$$

where $1 \leq \alpha, \beta \leq N_1$, $1 \leq a, b \leq N_2$, $Z_l = X_l^1 + iX_l^2$ and $A_{0,i}$ the gauge for the ith particle. The $J_{\alpha\alpha}^{(1)}$ and $J_{aa}^{(2)}$ currents (Gauss law constraints) read as

$$\begin{aligned}J_{\alpha\alpha}^{(1)} &= [Z_1, \bar{Z}_1]_{\alpha\alpha} + \frac{\theta}{2k_1}\left(\textstyle\sum_{a=1}^{N_2}\Psi_{\alpha a}\bar{\Psi}_{\alpha a} - J_0^{(1)}\right),\\ J_{aa}^{(2)} &= [Z_2, \bar{Z}_2]_{aa} + \frac{\theta}{2k_2}\left(\textstyle\sum_{\alpha=1}^{N_1}\Psi_{\alpha a}\bar{\Psi}_{\alpha a} - J_0^{(2)}\right),\end{aligned} \tag{16}$$

where the two $U(1)$ charge operators $J_0^{(1)}$ and $J_0^{(2)}$ are

$$J_0^{(1)} = J_0^{(2)} = J_0 = \textstyle\sum_{\alpha=1}^{N_1}\sum_{a=1}^{N_2}\ \bar{\Psi}_{\alpha a}\Psi_{\alpha a}\ . \tag{17}$$

The wavefunctions $|\Phi\rangle$ describing the (N_1+N_2) system of electrons on the non-commutative plane $\mathbb{R}_\theta^2$ with filling factor $\nu_{k_1k_2}$ should obey the constraint[5)]

$$J_0|\Phi\rangle = \frac{N}{\nu_{k_1k_2}}|\Phi\rangle. \tag{18}$$

Once we know the fundamental state $|\Phi_{\nu_{k_1k_2}}^{(0)}\rangle$, excitations are immediately determined by applying the usual rules. Upon recalling the coordinate operators as

$$Z_{1\alpha\alpha} = \sqrt{\frac{\theta}{2}}r_{\alpha\alpha}^{+}, \quad Z_{2aa} = \sqrt{\frac{\theta}{2}}s_{aa}^{+}, \tag{19}$$

the total Hamiltonian $\mathcal{H}$ may be treated as the sum of a free part given by

$$\mathcal{H}_0 = \frac{\omega}{2}\left(2\mathcal{N}_1 + 2\mathcal{N}_2 + N_1^2 + N_2^2\right), \tag{20}$$

where $\mathcal{N}_1 = \sum_{\alpha,\beta=1}^{N_1} r_{\alpha\beta}^{\dagger}r_{\beta\alpha}^{-}$ and $\mathcal{N}_2 = \sum_{a,b=1}^{N_2} s_{ab}^{\dagger}s_{ba}^{-}$ are the operator numbers counting the N_1 and N_2 particles respectively, and an interacting part

$$\mathcal{H}_{\mathrm{int}} \sim \left(\psi_{a\alpha}^{+}r_{\alpha\beta}^{+}\ s_{ab}^{-}\psi_{\beta b}^{-} + h.c.\right) \tag{21}$$

describing couplings between the *two sectors*.[5)] The creation and annihilation operators $r_{\alpha\alpha}^{\pm}$, $s_{aa}^{\pm}$, and $\psi_{\alpha a}^{\pm}$ satisfy the Heisenberg algebra

$$\begin{aligned}&\left[(r^-)_{\alpha}^{\alpha}, (r^+)_{\beta}^{\beta}\right] = \delta_{\alpha\beta}, \quad \left[(s^-)_a^a, (s^+)_b^b\right] = \delta_{ab},\\ &\left[(\psi^-)^{\alpha a}, (\psi^+)_{\alpha a}\right] = 1,\end{aligned} \tag{22}$$

all others are given by commuting relations. A way to build the spectrum of the Hamiltonian $\mathcal{H}_0$ is given by help of the special condensate operators

$$\left(A^+\right)_{a\alpha}^{(n,m)} = \left[\left(s^+\right)^{n-1}\psi^+\left(r^+\right)^{m-1}\right]_{a\alpha}. \tag{23}$$

The wavefunctions for the vacuum $|0>$ of $\mathcal{H}_0$ read as

$$\left[\varepsilon^{\alpha_1\dots\alpha_{N_1}}\prod_{j=1}^{p}O_{\alpha_{(jN_2+1)}\dots\alpha_{(j+1)N_2}}^{(j)}\right]^{k_1}|0> \tag{24}$$

where the $O^{(j)}$'s are building blocks and given by

$$\begin{aligned}O_{\alpha_{(jN_2+1)}\dots\alpha_{(j+1)N_2}}^{(j)} = \varepsilon^{a_{(jN_2+1)}\dots a_{(j+1)N_2}}\\ \times\left(A^+\right)_{a_{(jN_2+1)}\alpha_{(jN_2+1)}}^{(1,j)}\dots\left(A^+\right)_{a_{(j+1)N_2}\alpha_{(j+1)N_2}}^{(N_2,j)}.\end{aligned} \tag{25}$$

The corresponding energy spectrum $E_c\left(\nu_{k_1k_2}\right)$ is

$$E_c = k_1\left[p\frac{(N_2-1)(N_2-2)}{2} + \frac{(p-1)(p-2)}{2}N_2\right] + \frac{N_1+N_2}{2}. \tag{26}$$

Note that for large value of N_1 and N_2 ($N_1 = rN_2$), $E_c(\nu_{k_1k_2})$ behaves quadratically in N_2,

$$E_c(\nu_{k_1k_2} \sim \frac{k_2}{2}N_2^2. \tag{27}$$

This energy relation is less than the total energy $E_d(\nu_{k_i})$ of the decoupled configuration $(|\Phi_1, v_{k_1}\rangle \otimes |\Phi_2, v_{k_2}\rangle)$:

$$E_d(\nu_{k_i}) \equiv E\left(\frac{1}{k_1}\right) + E\left(\frac{1}{k_2}\right) \sim \frac{k_2(r+1)}{2}\ N_2^2. \tag{28}$$

Therefore, we have the following relation

$$E_d \sim (r+1)\,E_c. \tag{29}$$

For the example of the FQH state at $\nu = \frac{2}{5}$, the energy of the decoupled representation reads as

$$E\left(\frac{1}{3}\right) + E\left(\frac{1}{15}\right) \sim 45N_2^2 \tag{30}$$

while that of the interacting one is

$$E_c\left(\frac{2}{5}\right) \sim \frac{15}{2}N_2^2 \tag{31}$$

leading to

$$E\left(\frac{1}{3}\right) + E\left(\frac{1}{15}\right) \sim 6E_c\left(\frac{2}{5}\right). \tag{32}$$

4. Conclusion

We have developed a matrix model for FQH states at filling factor $\nu_{k_1k_2}$ going beyond the Laughlin theory. To illustrate our idea, we have considered an FQH system of a finite number $N = (N_1 + N_2)$ of electrons with filling factor $\nu_{k_1k_2} \equiv \nu_{p_1p_2} = \frac{p_2}{p_1p_2-1}$; p_1 is an odd integer and p_2 is an even integer. The $\nu_{p_1p_2}$ series corresponds just to the level two of the Haldane hierarchy; it recovers the Laughlin series $\nu_{p_1} = \frac{1}{p_1}$ by going to the limit p_2 large and contains several observable FQH states such as $\nu = \frac{2}{3}, \frac{2}{5}, \cdots$.

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