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Compression Techniques for Boundary Integral Equations – Optimal Complexity Estimates

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Abstract. In this paper matrix compression techniques in the context of wavelet Galerkin schemes for boundary integral equations are developed and analyzed that exhibit optimal complexity in the following sense. The fully discrete scheme produces approximate solutions within discretization error accuracy offered by the underlying Galerkin method at a computational expense that is proven to stay proportional to the number of unknowns. Key issues are the second compression, that reduces the near field complexity significantly, and an additional a-posteriori compression. The latter one is based on a general result concerning an optimal work balance, that applies, in particular, to the quadrature used to compute the compressed stiffness matrix with sufficient accuracy in linear time. The theoretical results are illustrated by a 3D example on a nontrivial domain.

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1. Introduction. Many mathematical models concerning e.g. field calculations, flow simulation, elasticity or visualization are based on operator equations with *global operators*, especially *boundary integral* operators. Discretizing such problems will then lead in general to possibly very large linear systems with *densely populated* matrices. Moreover, the involved operator may have an order different from zero which means that it acts on different length scales in a different way. This is well known to entail the linear systems to become more and more ill-conditioned when the level of resolution increases. Both features pose serious obstructions to the efficient numerical treatment of such problems to an extent that desirable realistic simulations are still beyond current computing capacities.

This fact has stimulated enormous efforts to overcome these obstructions. The resulting significant progress made over the past ten or fifteen years manifests itself in several different approaches such as panel clustering (PC) [18], multipole expansions (MP) [17] and wavelet compression (WC) [1]. Each of these methodologies has its recognized advantages and drawbacks whose balance may depend on the problem at hand. The first two (PC), (MP) are quite similar in spirit and exploit perhaps in the best way the (typical) smoothness of the integral kernel away from the boundary manifold. As a consequence, they are fairly robust with regard to the shape and complexity of the boundary manifold. Common experience seems to indicate that the third option (WC) depends in a more sensitive way on the underlying geometry and its performance may suffer from strong domain anisotropies. On the other hand, (WC) allows one in a natural way to incorporate preconditioning techniques which very much supports the fast solution of the resulting sparsified systems. Moreover, recent developments suggest a natural combination with adaptive discretizations to keep from the start, for a given target accuracy, the size of the linear systems as small as possible. The perhaps main difference between (PC), (MP) on one hand, and (WC) on the other hand, is that the former are essentially *agglomeration* techniques, while (WC) is more apt to refining a given coarse discretization. Since these methodologies are in that sense somewhat complementary in spirit, it is in our opinion less appropriate to contrapose them, but one should rather try to get out the best from each option.

As indicated before, a preference for any of the above mentioned solution strategies will, in general, depend on the concrete application at hand. The objective of this paper is therefore to provide a complete analysis of the wavelet approach (WC) from the following perspectives. Recall that (WC) has been essentially initiated by the pioneering paper [1] where it was observed that certain operators have an almost sparse representation in wavelet coordinates. Discarding all entries below a certain threshold in a given principal section of the wavelet representation will then give rise to a sparse matrix that can be further processed by efficient linear algebra tools. This idea has since initiated many subsequent studies. The diversity as well as the partly deceiving nature of the by now existing rich body of literature is one of the main reasons for us to take up this issue here again.

When dealing with large scale problems, a sharp *asymptotic analysis* of the complexity is in our opinion ultimately essential for assessing its potential. It is important to clarify the meaning of "complexity" in this context. It is always understood as the *work/accuracy* rate of the method under consideration when the level of resolution increases, i.e., the overall accuracy of the computed approximate solution is to be tied to the computational work required to obtain it. There is no point in increasing the number of degrees of freedom, i.e., the size of the linear systems, without

improving the accuracy of the resulting solutions. On the other hand, since one is ultimately interested in the "exact solution of the infinite dimensional problem" it makes no sense to determine the solution to the discrete finite dimensional problem with much more accuracy than that offered by the discretization. Thus, a reasonable target accuracy for the solutions of the discrete systems is *discretization error accuracy* which will guide all our subsequent considerations. A method will therefore be said to exhibit *optimal complexity* if the discrete solution can be computed within discretization error accuracy at a computational expense that stays *proportional* to the number N of unknowns, i.e., when not even logarithmic factors are permitted.

In the present context, this says that the solutions of the compressed systems should exhibit the same asymptotic convergence rates as the solutions to unperturbed discretizations. In connection with (WC) this, in turn, means that any threshold parameters and truncation strategies have to be adapted to the current number N of degrees of freedom. Such an asymptotic analysis is missing in [1] and in many subsequent investigations. The program carried out in [11, 12, 26, 27, 30] aimed at determining exactly such work/accuracy rates for various types of boundary integral equations. Roughly speaking, it could be shown that discretization error accuracy can be realized for appropriately chosen wavelet bases at a computational work that stays bounded by $C\,N(\log N)^a$ for some constants C, a independent of the size N of the linear systems. Moreover, in [30] it was shown for the first time that, by incorporating second compression, an overall optimal compression strategy can be devised that even avoids additional logarithmic factors, while the complexity estimates for a corresponding adaptive quadrature scheme was confined to collocation methods.

The purpose of the present paper can now be summarized as follows.

– We present a complete analysis of wavelet compression schemes for boundary integral equations based on Galerkin discretizations, that exhibit overall asymptotically optimal complexity, i.e., discretization error accuracy is obtained at a computational expense that stays proportional to the size N of the arising linear systems, uniformly in N.

– The analysis significantly simplifies previous studies including the effect of second compression. In particular, it reveals the right work balance for the compression and the quadrature needed to compute the compressed matrices with sufficient accuracy, so as to realize optimal computational complexity of the fully discretized scheme. Specifically, the computational work for computing and assembling the compressed stiffness matrix remains proportional to the number N of degrees of freedom.

– This also lays the foundation for an additional proper *a-posteriori compression* which together with a new preconditioning technique improves the quantitative performance of the scheme significantly.

– The performance of the scheme is demonstrated by a 3D example on a non-trivial domain geometry, namely a crankshaft. While further more extensive numerical studies will be presented elsewhere, this example already indicates that, when all ingredients of the scheme are properly chosen, the dependence on the geometry is much weaker than expected. Moreover, it also confirms in a quantitative manner that the compression strategy realizing optimal complexity, does not degrade the discretization accuracy offered by the Galerkin scheme. These facts particularly encourage us in our efforts devoted to a careful and complete analysis of (WC).

Our analysis concerns what is called the *standard* wavelet representation. A preference for using the so called *nonstandard* form is frequently reported in the literature. The reason is that the entries in this latter form only involve scaling functions and wavelets on the *same* level. This indeed simplifies assembling the matrices and offers essential advantages when dealing with shift-invariant problems. However, aside from the problem of preconditioning in connection with operators of nonzero order, to our knowledge it has so far *not* been shown that, for a fixed order of vanishing moments, optimal computational complexity in the above sense can be obtained with the nonstandard form. In fact, for regular solutions approximate solutions with prescribed accuracy can be obtained at (asymptotically) lower computational cost with the aid of the standard form when compared with the nonstandard form. This is backed by theory and confirmed by numerical experience.

As mentioned before, it is important to employ the "right" wavelet bases. This question has been discussed extensively in previous work [2, 6, 13, 14]. In this paper we shall make use

of these findings and remark here only that, depending on the order of the operator, a proper relation between the approximation order of the underlying multiresolution spaces and the order of vanishing moments matters, which often rules out orthonormal wavelets.

We shall frequently write $a \lesssim b$ to express that a is bounded by a constant multiple of b, uniformly with respect to all parameters on which a and b may depend. Then $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

2. Problem Formulation and Preliminaries. We consider boundary integral equations on a closed boundary surface Γ of a $(n+1)$-dimensional domain $\Omega \subset \mathbb{R}^n$

$$Au = f \quad \text{on } \Gamma, \tag{2.1}$$

where the boundary integral operator

$$Au(\widehat{x}) = \int_\Gamma k(\widehat{x}, \widehat{y}) u(\widehat{y}) d\Gamma_{\widehat{y}}$$

is assumed to be an operator of order $2q$, that is

$$A : H^q(\Gamma) \to H^{-q}(\Gamma).$$

The kernel functions under consideration are supposed to be smooth as functions in the variables $\widehat{x}, \widehat{y}$, apart from the diagonal $\{(\widehat{x}, \widehat{y}) \in \Gamma \times \Gamma : \widehat{x} = \widehat{y}\}$ and may have a singularity on the diagonal. Such kernel functions arise, for instance, by applying a boundary integral formulation to a second order elliptic problem. In general, they decay like a negative power of the distance of the arguments which depends on the spatial dimension n and the order $2q$ of the operator.

Throughout the remainder of this paper we shall assume that the boundary manifold Γ is given as a parametric surface consisting of smooth patches. More precisely, let $\square := [0,1]^n$ denote the unit n-cube. The manifold $\Gamma \in \mathbb{R}^{n+1}$ is partitioned into a finite number of *patches*

$$\Gamma = \bigcup_{i=1}^M \Gamma_i, \qquad \Gamma_i = \gamma_i(\square), \qquad i = 1, 2, \ldots, M, \tag{2.2}$$

where each $\gamma_i : \square \to \Gamma_i$ defines a diffeomorphism of $\square$ onto Γ_i. We also assume that there exist smooth extensions $\Gamma_i \subset\subset \widetilde{\Gamma}_i$ and $\widetilde{\gamma}_i : \widetilde{\square} := [-1,2]^n \to \widetilde{\Gamma}_i$. The intersection $\Gamma_i \cap \Gamma_{i'}$, $i \neq i'$, of the patches Γ_i and $\Gamma_{i'}$ is supposed to be either $\emptyset$ or a lower dimensional face.

A mesh of level j on Γ is induced by dyadic subdivisions of depth j of the unit cube into 2^{nj} cubes $C_{j,k} \subseteq \square$, where $k = (k_1, \ldots, k_n)$ with $0 \leq k_m < 2^j$. This generates $2^{nj} M$ *elements* (or elementary domains) $\Gamma_{i,j,k} := \gamma_i(C_{j,k}) \subseteq \Gamma_i$, $i = 1, \ldots, M$.

In order to ensure that the collection of elements $\{\Gamma_{i,j,k}\}$ on the level j forms a regular mesh on Γ, the parametric representation is subjected to the following *matching condition*: For all $\widehat{x} \in \Gamma_i \cap \Gamma_{i'}$ exists a bijective, affine mapping $\Xi : \square \to \square$ such that $\gamma_i(x) = (\gamma_{i'} \circ \Xi)(x) = \widehat{x}$ for $x = (x_1, \ldots, x_n) \in \square$ with $\gamma_i(x) = \widehat{x}$.

The first fundamental tensor of differential geometry is given by the matrix $K_i(x) \in \mathbb{R}^{n \times n}$ defined by

$$K_i(x) := \Big[\Big(\frac{\partial \gamma_i(x)}{\partial x_j}, \frac{\partial \gamma_i(x)}{\partial x_{j'}} \Big)_{l^2(\mathbb{R}^{n+1})} \Big]_{j,j'=1}^n. \tag{2.3}$$

Since γ_i is supposed to be a diffeomorphism, the matrix $K_i(x)$ is symmetric and positive definite. The canonical inner product in $L^2(\Gamma)$ is then given by

$$\langle u, v \rangle = \int_\Gamma u(x) v(x) d\Gamma_x = \sum_{i=1}^M \int_\square u\big(\gamma_i(x)\big) v\big(\gamma_i(x)\big) \sqrt{\det K_i(x)} dx. \tag{2.4}$$

The corresponding Sobolev spaces are denoted by $H^s(\Gamma)$, endowed with the norms $\|\cdot\|_s$, where for $s < 0$ it is understood that $H^s(\Gamma) = (H^{-s}(\Gamma))'$. Of course, depending on the global smoothness of the surface, the range of permitted $s \in \mathbb{R}$ is limited to $s \in (-s_\Gamma, s_\Gamma)$.

We can now specify the kernel functions. To this end, we denote by $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ multi-indices of dimension n and define $|\alpha| := \alpha_1 + \ldots + \alpha_n$. Recall that $\widehat{x}$ and $\widehat{y}$ are points on the surface, i.e. $\widehat{x} := \gamma_i(x)$ and $\widehat{y} := \gamma_{i'}(y)$ for some $1 \leq i, i' \leq M$.

DEFINITION 2.1. *A kernel $k(\widehat{x}, \widehat{y})$ is called standard kernel of order $2q$, if the partial derivatives of the transported kernel function*

$$\widetilde{k}(x, y) := k(\gamma_i(x), \gamma_{i'}(y)) \sqrt{\det K_i(x)} \sqrt{\det K_{i'}(y)} \tag{2.5}$$

are bounded by

$$|\partial_x^\alpha \partial_y^\alpha \widetilde{k}(x, y)| \lesssim \frac{1}{\operatorname{dist}(\hat{x}, \hat{y})^{n+2q+|\alpha|+|\beta|}}, \tag{2.6}$$

provided that $n + 2q + |\alpha| + |\beta| > 0$. We emphasize that this definition requires patchwise smoothness but *not* global smoothness of the geometry. The surface itself needs to be only Lipschitz. Generally, under this assumption, the kernel of a boundary integral operator A of order $2q$ is a standard kernel of order $2q$. Hence, we may assume this property in the sequel. We shall encounter further specifications below in connection with discretizations.

3. Galerkin Scheme. We shall be concerned with the Galerkin method with respect to a hierarchy of conforming trial spaces $V_J \subset V_{J+1} \subset H^q(\Gamma)$: find $u_J \in V_J$ solving the variational problem

$$\langle A u_J, v_J \rangle = \langle f, v_J \rangle \quad \text{for all} \quad v_J \in V_J. \tag{3.1}$$

Here the index J reflects a meshwidth of the order 2^{-J}. Moreover, we say that the trial spaces have *(approximation) order* $d \in \mathbb{N}$ and *regularity* $\gamma > 0$ if

$$\begin{aligned} \gamma &= \sup\{s \in \mathbb{R} : V_j \subset H^s(\Gamma)\}, \\ d &= \sup\{s \in \mathbb{R} : \inf_{v_j \in V_j} \|v - v_j\|_0 \lesssim 2^{-js} \|v\|_s\}. \end{aligned} \tag{3.2}$$

Thus conformity requires, of course, that $\gamma > \max\{0, q\}$.

In order to ensure that (3.1) is well posed we shall make the following assumptions on the operator A throughout the remainder of the paper.

ASSUMPTION:

1. A is strongly elliptic, i.e., there exists a symmetric compact operator $C : H^q(\Gamma) \to H^{-q}(\Gamma)$ such that
$$\langle (A + A^\star + C)u, u \rangle \gtrsim \|u\|_q^2.$$
2. The operator $A : H^q(\Gamma) \to H^{-q}(\Gamma)$ is injective, i.e. $\operatorname{Ker} A = \{0\}$.

REMARK: Most boundary integral equations of the first kind, resulting from a direct approach, are known to satisfy these requirements, even if Γ is supposed to be the boundary of a Lipschitz domain [7]. The boundary value problem is then known to be *strongly elliptic*. In particular, this is the case for the first kind boundary integral equations for the Laplacian, the system of Navier-Lamé equations and the Stokes system. For second kind integral equations the condition is obvious if the double layer potential operator is compact, or in case of smooth boundaries, since the principal symbol satisfies a Gårding inequality.

LEMMA 3.1. *Under the above assumptions the Galerkin discretization is stable, i.e.*

$$\langle (A + A^\star) v_J, v_J \rangle \gtrsim \|v_J\|_q^2, \quad v_J \in V_J, \tag{3.3}$$

for J sufficiently large, and

$$|\langle A v_J, w_J \rangle| \lesssim \|v_J\|_q \|w_J\|_q, \quad v_J, w_J \in V_J. \tag{3.4}$$

Furthermore, let u, u_J denote the solution of the original equation $Au = f$ respectively of (3.1). *Then one has*

$$\|u - u_J\|_t \lesssim 2^{J(t-t')}\|u\|_{t'} \tag{3.5}$$

provided that $-d-2q \le t < \gamma$, $t \le t'$, $q \le t' \le d$ and Γ is sufficiently regular. Note that the best possible convergence rate is given by

$$\|u - u_J\|_{-d+2q} \lesssim 2^{-2J(d+q)}\|u\|_d \tag{3.6}$$

provided that $u \in H^d(\Gamma)$ which is only possible when Γ is sufficiently regular. Since this case gives rise to the highest convergence rate, it will be seen later to impose the most stringent demands on the matrix compression.

4. Wavelets and Multiresolution Analysis. The nested trial spaces $V_j \subset V_{j+1}$ that we shall employ in (3.1) are spanned by so called *single-scale bases* $\Phi_j = \{\phi_{j,k} : k \in \Delta_j\}$ whose elements are normalized in $L^2(\Gamma)$ and whose compact supports scale like $\operatorname{diam}\operatorname{supp}\phi_{j,k} \sim 2^{-j}$. Associated with these collections are always *dual* bases $\widetilde{\Phi}_j = \{\widetilde{\phi}_{j,k} : k \in \Delta_j\}$, i.e., one has $\langle \phi_{j,k}, \widetilde{\phi}_{j,k'}\rangle = \delta_{k,k'}$, $k, k' \in \Delta_j$. For the current type of boundary surfaces Γ the $\Phi_j, \widetilde{\Phi}_j$ are generated by constructing first dual pairs of single-scale bases on the interval $[0,1]$, using B-splines for the primal bases and the dual components from [5] adapted to the interval [10]. Tensor products yield corresponding dual pairs on $\square$. Using the parametric liftings γ_i and gluing across patch boundaries leads to globally continuous single-scale bases $\Phi_j, \widetilde{\Phi}_j$ on Γ, [2, 6, 14, 19]. For B-splines of order d and duals of order $\widetilde{d} \ge d$ such that $d + \widetilde{d}$ is even the $\Phi_j, \widetilde{\Phi}_j$ have approximation orders $d, \widetilde{d}$, respectively. It is known that the respective regularity indices $\gamma, \widetilde{\gamma}$ (inside each patch) satisfy $\gamma = d - 1/2$ while $\widetilde{\gamma} > 0$ is known to increase proportionally to $\widetilde{d}$.

In view of the biorthogonality of $\Phi_j, \widetilde{\Phi}_j$, it will be convenient to employ the canonical projectors

$$Q_j v := \sum_{k\in\Delta_j} \langle v, \widetilde{\phi}_{j,k}\rangle \phi_{j,k}, \qquad Q_j^\star v := \sum_{k\in\Delta_j} \langle v, \phi_{j,k}\rangle \widetilde{\phi}_{j,k}, \tag{4.1}$$

associated with the *multiresolution sequences* $\{V_j\}_{j\ge j_0}, \{\widetilde{V}_j\}_{j\ge j_0}$. Here and below j_0 always stands for some fixed coarsest level of resolution that may depend on Γ.

It follows from the L^2-boundedness of the Q_j that one has the following *Jackson* and *Bernstein* type estimates uniformly in j, namely

$$\|v - Q_j v_j\|_s \lesssim 2^{-j(t-s)}\|v\|_t, \quad v \in H^t(\Gamma), \tag{4.2}$$

for all $-\widetilde{d} \le s \le t \le d$, $s < \gamma$, $-\widetilde{\gamma} < t$ and

$$\|Q_j v\|_s \lesssim 2^{j(s-t)}\|Q_j v\|_t, \quad v \in H^t(\Gamma), \tag{4.3}$$

for all $t \le s \le \gamma$.

Given the single-scale bases $\Phi_j, \widetilde{\Phi}_j$, one can construct now biorthogonal *complement bases* $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$, $\widetilde{\Psi}_j = \{\widetilde{\psi}_{j,k} : k \in \nabla_j\}$, i.e., $\langle \psi_{j,k}, \widetilde{\psi}_{j',k'}\rangle = \delta_{(j,k),(j',k')}$, such that

$$\operatorname{diam}\operatorname{supp}\psi_{j,k} \sim 2^{-j}, \quad j \ge j_0, \tag{4.4}$$

see e.g. [2, 6, 13, 14] and [19] for particularly useful local representations of important construction ingredients. In fact, for this type of bases, the dual wavelets scale in the same way, but this will not be needed and does not hold for alternative constructions based on finite elements [15].

Denoting by $W_j, \widetilde{W}_j$ the span of Ψ_j, respectively $\widetilde{\Psi}$, biorthogonality implies

$$V_{j+1} = W_j \oplus V_j, \qquad \widetilde{V}_{j+1} = \widetilde{W}_j \oplus \widetilde{V}_j, \qquad \widetilde{V}_j \perp W_j, \qquad V_j \perp \widetilde{W}_j.$$

Hence V_J and $\widetilde{V}_J$ can be written as a direct sum of the complement spaces W_j, respectively $\widetilde{W}_j$, $j_0 - 1 \le j < J$ (using the convention $W_{j_0-1} := V_{j_0}$, $\widetilde{W}_{j_0-1} := \widetilde{V}_{j_0}$, $Q_{j_0-1} = Q^\star_{j_0-1} := 0$). In fact, one has for $v_J \in V_J$, $\widetilde{v}_J \in \widetilde{V}_J$

$$v_J = \sum_{j=j_0}^{J} (Q_j - Q_{j-1}) v_J, \qquad \widetilde{v}_J = \sum_{j=j_0}^{J} (Q^\star_j - Q^\star_{j-1}) \widetilde{v}_J,$$

where

$$(Q_{j+1} - Q_j) v = \sum_{k \in \nabla_j} \langle v, \widetilde{\psi}_{j,k} \rangle \psi_{j,k}, \qquad (Q^\star_{j+1} - Q^\star_j) v = \sum_{k \in \nabla_j} \langle v, \psi_{j,k} \rangle \widetilde{\psi}_{j,k}.$$

A biorthogonal or *dual* pair of wavelet bases is now obtained by taking the coarse single-scale basis and the union of the complement bases

$$\Psi = \bigcup_{j=j_0-1} \Psi_j, \qquad \widetilde{\Psi} = \bigcup_{j=j_0-1} \widetilde{\Psi}_j,$$

where we have set for convenience $\Psi_{j_0-1} := \Phi_{j_0}$, $\widetilde{\Psi}_{j_0-1} := \widetilde{\Phi}_{j_0}$. We will refer to Ψ and $\widetilde{\Psi}$ as the *primal*, respectively *dual* basis.

Throughout the paper, all basis function (scaling functions and wavelets) are normalized in $L^2(\Gamma)$.

From biorthogonality and the fact that the dual single-scale bases on $\square$ represent all polynomials of order $\widetilde{d}$ exactly one infers vanishing polynomial moments of the primal wavelets on $\square$, which, on account of the locality (4.4) entails the first key feature of the primal wavelet bases, namely *vanishing moments* or the *cancellation property*

$$|\langle v, \psi_{j,k} \rangle| \lesssim 2^{-j(\widetilde{d}+n/2)} |v|_{W^{\widetilde{d},\infty}(\operatorname{supp} \psi_{j,k})}. \tag{4.5}$$

Here $|v|_{W^{\widetilde{d},\infty}(\Omega)} := \sup_{|\alpha|=\widetilde{d},\, x \in \Omega} |\partial^\alpha v(x)|$ denotes the semi-norm in $W^{\widetilde{d},\infty}(\Omega)$. The fact, that the concept of biorthogonality allows us to choose the order $\widetilde{d}$ of vanishing moments higher than the approximation order d, will be essential for deriving optimal compression strategies that could not be realized by orthonormal bases.

Of course, in the infinite dimensional case the notion of basis has to be made more specific. The second key feature of the basis Ψ is now the fact that (properly scaled versions of) Ψ and $\widetilde{\Psi}$ are actually *Riesz bases* for a *whole range* of Sobolev spaces, which means

$$\begin{aligned} \|v\|_t^2 &\sim \sum_{j=j_0-1}^{\infty} \sum_{k \in \nabla_j} 2^{2jt} |\langle v, \widetilde{\psi}_{j,k} \rangle|^2, \qquad t \in (-\widetilde{\gamma}, \gamma), \\ \|v\|_t^2 &\sim \sum_{j=j_0-1}^{\infty} \sum_{k \in \nabla_j} 2^{2jt} |\langle v, \psi_{j,k} \rangle|^2, \qquad t \in (-\gamma, \widetilde{\gamma}). \end{aligned} \tag{4.6}$$

The validity of these norm equivalences hinges on the estimates (4.2) and (4.3) for *both* the primal and dual multiresolution sequences. The equivalences (4.6) will be essential for preconditioning.

5. Wavelet Galerkin Schemes – Preconditioning. As before let $A : H^q(\Gamma) \to H^{-q}(\Gamma)$ be a boundary integral operator of order $2q$. Since the wavelet basis Ψ is, in particular, a Riesz basis for $L^2(\Gamma)$, the associated system matrices

$$\mathbf{A}_J = [\langle A \psi_{j',k'}, \psi_{j,k} \rangle]_{j_0-1 \le j,j' < J,\, k \in \nabla_j,\, k' \in \nabla_{j'}}$$

become more and more ill conditioned when J increases. In fact, one has $\operatorname{cond}_{l^2} \mathbf{A}_J \sim 2^{2J|q|}$. However, as a consequence of the stability of the Galerkin discretization under the given circumstances and the norm equivalences (4.6), the following simple *diagonal preconditioner* gives rise to uniformly bounded spectral condition numbers [8, 9, 11].

THEOREM 5.1. *Let the diagonal matrix* $\mathbf{D}_J^r$ *be defined by*

$$\left[\mathbf{D}_J^r\right]_{(j,k),(j',k')} = 2^{rj}\delta_{(j,k),(j',k')}, \quad k \in \nabla_j, \quad k' \in \nabla_{j'}, \quad j_0 - 1 \le j, j' < J.$$

Then, if $A : H^q(\Gamma) \to H^{-q}(\Gamma)$ *is a boundary integral operator of order* $2q$*, satisfying the assumptions 1), 2) from Section 3, and if* $\widetilde{\gamma} > -q$*, the diagonal matrix* $\mathbf{D}_J^{2q}$ *defines an asymptotically optimal preconditioner for* $\mathbf{A}_J$*, i.e.,*

$$\mathrm{cond}_{l^2}(\mathbf{D}_J^{-q}\mathbf{A}_J\mathbf{D}_J^{-q}) \sim 1.$$

It should be stressed that while the above scaling is asymptotically optimal, the quantitative performance may vary significantly among different scalings with the same asymptotic behavior. In particular, since Ψ is, on account of the mapping properties of A and the norm equivalences (4.6), also a Riesz basis with respect to the energy norm, it would be natural to normalize the wavelets in this energy norm which would suggest the specific scaling $\langle A\psi_{j,k}, \psi_{j,k}\rangle \sim 2^{2qj}$. In fact, this latter diagonal scaling improves and simplifies the wavelet preconditioning. We shall mention later another alternative offering a further significant improvement.

In view of the above simple preconditioning, the iterative solution of the Galerkin systems is feasible and its overall efficiency relies now on the cost of matrix/vector multiplications which brings us to the central issue, namely *matrix compression*.

6. Basic Estimates. The basic ingredients in the analysis of the compression procedure are estimates for the matrix entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ with $k \in \nabla_j$, $k' \in \nabla_{j'}$ and $j, j' \ge j_0 - 1$. The convex hulls of the supports of the wavelets will be denoted by

$$\Omega_{j,k} := \mathrm{conv\,hull}(\mathrm{supp}\,\psi_{j,k}). \tag{6.1}$$

A complete proof of the following estimates can be found e.g. in [30, 15].

THEOREM 6.1. *Suppose* $n + 2\widetilde{d} + 2q > 0$ *and* $j, j' \ge j_0$*. Then one has*

$$|\langle A\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim \frac{2^{-(j+j')(\widetilde{d}+n/2)}}{\mathrm{dist}(\Omega_{j,k}, \Omega_{j',k'})^{n+2q+\widetilde{d}}}$$

uniformly with respect to J*.*

However, in order to arrive ultimately at solution schemes with linear complexity, the number of nonzero entries in the compressed matrices should remain proportional to their size while preserving discretization error accuracy. To achieve this, it is not sufficient to consider only coefficients where the supports of the involved wavelet do not overlap. There are still $\mathcal{O}(N_J \log N_J) = \mathcal{O}(2^{Jn}J)$ coefficients that would remain. To avoid the logarithmic term we propose an additional, so called *second compression*. For this purpose we require that our primal basis functions are piecewise polynomial, in the sense that $\psi_{j,k}|_{\Gamma_{i,j+1,l}} = p \circ \gamma_i^{-1}$, where p is a polynomial. By

$$\Omega'_{j,k} := \mathrm{sing\,supp}\,\psi_{j,k} \tag{6.2}$$

we denote the *singular support* of $\psi_{j,k}$, which is that subset of Γ, where the function $\psi_{j,k}$ is not smooth. Thus the singular support of the wavelet $\psi_{j,k}$ consists of the boundaries of some of the elements $\Gamma_{i,j+1,l}$. The goal of the subsequent investigation is to estimate those matrix coefficients for which $\mathrm{dist}(\Omega_{j,k}, \Omega'_{j',k'})$, $j \ge j'$, is sufficiently large.

To this end, we require the following extension lemma which follows e.g. immediately from the well known extension theorem of Calderón [32].

LEMMA 6.2. *The function* $f_{i,j,k,l}$*, defined by*

$$f_{i,j,k,l} := \psi_{j,k}|_{\Gamma_{i,j+1,l}} \circ \gamma_i = (\psi_{j,k} \circ \gamma_i)|_{C_{j+1,l}} \in C^\infty(C_{j+1,l}),$$

can be extended to a function $\widetilde{f}_{i,j,k,l} \in C_0^\infty(\mathbb{R}^n)$ *in such a way that* $\mathrm{diam\,supp}\,\widetilde{f}_{i,j,k,l} \lesssim 2^{-j}$*,* $\widetilde{f}_{i,j,k,l} \equiv \psi_{j,k} \circ \gamma_i$ *on* $C_{j+1,l}$*, and that for all* $s \ge 0$

$$\|\widetilde{f}_{i,j,k,l}\|_{H^s(\mathbb{R}^n)} \lesssim 2^{js},$$

independently of i,j,k,l.

Proof. Suppose that $f_\square \in C^\infty(\square)$ with $\|f_\square\|_{H^s(\square)} \lesssim 1$. By virtue of Calderón's extension theorem, there exists an extension $f \in C_0^\infty(\mathbb{R}^n)$, i.e. $f(x) \equiv f_\square(x)$ on $\square$, satisfying $\|f\|_{H^s(\mathbb{R}^n)} \lesssim \|f_\square\|_{H^s(\square)}$. Let us consider an affine map κ with $C_{j+1,l} = \kappa(\square)$ and choose $f_{i,j,k,l}(\kappa(x)) := f_\square(x)$. The claim follows now from $|\partial^\alpha \kappa(x)| = 2^{|\alpha|(j+1)}$. □

It is well known that boundary integral operators A of order $2q$, acting on smooth surfaces, are classical *pseudodifferential operators* [34]. Since the patches Γ_i are smooth and have smooth extensions $\widetilde{\Gamma}_i$, there exists for each i, a pseudodifferential operator $A^\sharp : H^q(\mathbb{R}^n) \to H^{-q}(\mathbb{R}^n)$ such that

$$A^\sharp f(x) = \int_{\mathbb{R}^n} \chi(x)\chi(y) k(\widetilde{\gamma}_i(x), \widetilde{\gamma}_i(y)) f(y) \sqrt{\det \widetilde{K}_i(y)} dy, \tag{6.3}$$

where χ is a C^∞-cut-off function with respect to $\square$, i.e., $\chi(x) = 1$ on $\square$ and $\chi(x) = 0$ outside $[-1,2]^n$. Therefore $A^\sharp f(x) = A(f \circ \gamma_i)(\gamma_i(x))$, for all $f \in C_0^\infty(\square)$, $x \in \square$ and $A^\sharp$ is compactly supported [23]. Moreover, it is well known [23] that the Schwartz kernel of pseudodifferential operators satisfy the standard estimate (2.6).

A compactly supported pseudodifferential operator $A^\sharp : H^s(\mathbb{R}^n) \to H^{s-2q}(\mathbb{R}^n)$ of order $2q$ acts continuously on Sobolev spaces [23, 33]. Therefore, for any function $\widetilde{f}_{i,j,k,l} \in C_0^\infty(\mathbb{R}^n)$, satisfying $\operatorname{diam}\operatorname{supp} \widetilde{f}_{i,j,k,l} \sim 2^{-j}$ and $\|\widetilde{f}_{i,j,k,l}\|_{H^s(\mathbb{R}^n)} \lesssim 2^{js}$, for all $s \geq 0$, one has $A^\sharp \widetilde{f}_{i,j,k,l} \in C_0^\infty(\mathbb{R}^n)$ with

$$\|A^\sharp \widetilde{f}_{i,j,k,l}\|_{H^{s-2q}(\mathbb{R}^n)} \lesssim 2^{js}. \tag{6.4}$$

With these preparations at hand, we are able to formulate the following result.

THEOREM 6.3. *Suppose that* $n + 2\widetilde{d} + 2q > 0$ *and* $j' > j \geq j_0 - 1$. *Then, the coefficients* $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ *and* $\langle A\psi_{j,k}, \psi_{j',k'}\rangle$ *satisfy*

$$|\langle A\psi_{j',k'}, \psi_{j,k}\rangle|, \ |\langle A\psi_{j,k}, \psi_{j',k'}\rangle| \lesssim \frac{2^{jn/2} 2^{-j'(\widetilde{d}+n/2)}}{\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'})^{2q+\widetilde{d}}},$$

uniformly with respect to j, *provided that*

$$\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}) \gtrsim 2^{-j'}. \tag{6.5}$$

Proof. We shall consider three cases.

(i) The first observation concerns an estimate for disjoint supports that will be applied several times.

LEMMA 6.4. *Suppose that* $\Omega_{j,k} \cap \Omega_{j',k'} = \emptyset$ *and that* f *is any function supported on* $\Omega_{j,k}$ *satisfying* $|f(x)| \lesssim 2^{jn/2}$, $x \in \Omega_{j,k}$. *Then one has*

$$|\langle A\psi_{j',k'}, f\rangle| \lesssim 2^{jn/2} 2^{-j'(\widetilde{d}+n/2)} \operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'})^{-2q-\widetilde{d}}. \tag{6.6}$$

To prove (6.6) note that our assumption implies that $\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}) = \operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'})$. On account of the cancellation property (4.5) of the wavelet bases and the decay property (2.6) of the kernel, we obtain

$$\begin{aligned} |A\psi_{j',k'}(x)| = |\langle k(x,\cdot), \psi_{j',k'}\rangle| &\lesssim 2^{-j'(\widetilde{d}+n/2)} |k(x,\cdot)|_{W^{\infty,\widetilde{d}}(\Omega_{j',k'})} \\ &\lesssim 2^{-j'(\widetilde{d}+n/2)} \operatorname{dist}(x, \Omega_{j',k'})^{-n-2q-\widetilde{d}} \end{aligned}$$

for all $x \in \psi_{j,k}$. Therefore, we conclude that

$$\begin{aligned} |\langle A\psi_{j',k'}, f\rangle| &\lesssim \|f\|_{L^\infty(\Gamma)} \int_{\Omega_{j,k}} |A\psi_{j',k'}(x)| d\Gamma_x \\ &\lesssim 2^{jn/2} 2^{-j'(\widetilde{d}+n/2)} \int_{\Omega_{j,k}} \operatorname{dist}(x, \Omega_{j',k'})^{-n-2q-\widetilde{d}} d\Gamma_x \\ &\leq 2^{jn/2} 2^{-j'(\widetilde{d}+n/2)} \operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'})^{-2q-\widetilde{d}}, \end{aligned}$$

which proves the lemma. Of course, the same reasoning applies to the adjoint boundary integral operator $A^\star$.
(ii) Next, we treat the case $\Omega_{j,k} \cap \Omega_{j',k'} \neq \emptyset$ and $\Omega_{j,k} \subset \Gamma_i$. By (6.5) we have $\Omega_{j',k'} \subset \Omega_{j,k}$ i.e., both wavelets are supported on the same patch. We infer from (6.5) that there exists an element $\Omega_{j',k'} \subset \Gamma_{i,j+1,l} \subset \Omega_{j,k}$ such that

$$f_{i,j,k,l} := \psi_{j,k}|_{\Gamma_{i,j+1,l}} \circ \gamma_i = (\psi_{j,k} \circ \gamma_i)|_{C_{j+1,l}}$$

is a $C^\infty(C_{j+1,l})$-function. On account of Lemma 6.2, we can choose an extension of $f_{i,j,k,l}$, denoted by $\widetilde{f}_{i,j,k,l}$. Decomposing

$$\psi_{j,k} \circ \gamma_i = \widetilde{f}_{i,j,k,l} + \widetilde{f}^C_{i,j,k,l},$$

we obtain

$$\begin{aligned}
|\langle A\psi_{j,k}, \psi_{j',k'}\rangle| &= \left| \int_{\mathbb{R}^n} A^\sharp(\widetilde{f}_{i,j,k,l} + \widetilde{f}^C_{i,j,k,l})(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| \\
&\leq \left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| + \left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}^C_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right|.
\end{aligned}$$

The second term on the right hand side can be treated analogously to (6.6), i.e.,

$$\left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}^C_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| \lesssim 2^{jn/2} 2^{-j'(\tilde{d}+n/2)} \operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'})^{-2q-\tilde{d}},$$

because

$$\operatorname{dist}\left(\operatorname{supp} \widetilde{f}^C_{i,j,k,l}, \operatorname{supp}(\psi_{j',k'} \circ \gamma_i)\right) \sim \operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}).$$

Invoking (6.4) and (4.5), the first term can be estimated by

$$\left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| \lesssim 2^{-j'(\tilde{d}+n/2)} \|A^\sharp \widetilde{f}_{i,j,k,l}(x)\|_{W^{\infty,\tilde{d}}(\operatorname{supp}(\psi_{j',k'} \circ \gamma_i))}.$$

By virtue of Sobolev's embedding theorem, this implies, in view of (6.4),

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| &\lesssim 2^{-j'(\tilde{d}+n/2)} \|A^\sharp \widetilde{f}_{i,j,k,l}(x)\|_{H^{\tilde{d}+n/2}(\mathbb{R}^n)} \\
&\lesssim 2^{-j'(\tilde{d}+n/2)} 2^{j(\tilde{d}+2q+n/2)}.
\end{aligned}$$

By (6.5) and and $j' \geq j$, we arrive at the desired estimate

$$\left| \int_{\mathbb{R}^n} A^\sharp \widetilde{f}_{i,j,k,l}(x)(\psi_{j',k'} \circ \gamma_i)(x)dx \right| \lesssim 2^{jn/2} 2^{-j'(\tilde{d}+n/2)} \operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'})^{-2q-\tilde{d}}.$$

(iii) It remains to consider the case $\Omega_{j,k} \cap \Omega_{j',k'} \neq \emptyset$ where, however, $\psi_{j,k}$ is not supported completely in the patch Γ_i. In this case we decompose $\psi_{j,k} = (\psi_{j,k} - \psi_{j,k}|_{\Gamma_i}) + \psi_{j,k}|_{\Gamma_i})$. Invoking (6.6), we derive

$$|\langle A(\psi_{j,k} - \psi_{j,k}|_{\Gamma_i}), \psi_{j',k'}\rangle| \lesssim \frac{2^{jn/2} 2^{-j'(\tilde{d}+n/2)}}{\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'})^{2q+\tilde{d}}}$$

because we have again

$$\operatorname{dist}(\operatorname{supp}(\psi_{j,k} - \psi_{j,k}|_{\Gamma_i}), \Omega_{j',k'}) \geq \operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}).$$

Finally, estimating $|\langle A(\psi_{j,k}|_{\Gamma_i}), \psi_{j',k'}\rangle|$ as in step *(ii)*, finishes the proof. □

REMARK: We recall from [8, 30] that there is a general estimate which says that matrix entries for wavelets with overlapping supports decay with increasing difference of scales. In fact, for each $0 \le \delta < \gamma - q$ we have

$$|\langle A\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim 2^{-\delta|j-j'|}.$$

Since $\gamma < d$ this estimate is, however, not sufficient to achieve the optimal order of convergence within the desired linear complexity. Therefore, this estimate will not be exploited in the present study.

7. Matrix Compression. The discretization of a boundary integral operator $A : H^q(\Gamma) \to H^{-q}(\Gamma)$ by wavelets with a sufficiently strong cancellation property (4.5) yields, in view of the above estimates, quasi-sparse matrices. In a first compression step all matrix entries, for which the distance of the supports of the corresponding trial and test functions are larger than a level depending cut-off parameter $\mathcal{B}_{j,j'}$, are set to zero. In the second compression step also some of those matrix entries are neglected, for which the corresponding trial and test functions have overlapping supports.

A-priori compression. Let $\Omega_{j,k}$ and $\Omega'_{j,k}$ be given as in (6.1) and (6.2). Then, the compressed system matrix $\mathbf{A}^\epsilon_J$, corresponding to the boundary integral operator A, is defined by

$$[\mathbf{A}^\epsilon_J]_{(j,k),(j',k')} := \begin{cases} 0, & \operatorname{dist}(\Omega_{j,k},\Omega_{j',k'}) > \mathcal{B}_{j,j'} \text{ and } j, j' \ge j_0, \\ 0, & \operatorname{dist}(\Omega_{j,k},\Omega_{j',k'}) \lesssim 2^{-\min\{j,j'\}} \text{ and} \\ & \operatorname{dist}(\Omega'_{j,k},\Omega_{j',k'}) > \mathcal{B}'_{j,j'} \text{ if } j' > j \ge j_0 - 1, \\ & \operatorname{dist}(\Omega_{j,k},\Omega'_{j',k'}) > \mathcal{B}'_{j,j'} \text{ if } j > j' \ge j_0 - 1, \\ \langle A\psi_{j',k'}, \psi_{j,k}\rangle, & \text{otherwise.} \end{cases} \tag{7.1}$$

Fixing

$$a, a' > 1, \qquad d < d' < \widetilde{d} + 2q, \tag{7.2}$$

the cut-off parameters $\mathcal{B}_{j,j'}$ and $\mathcal{B}'_{j,j'}$ are set as follows

$$\begin{aligned} \mathcal{B}_{j,j'} &= a \, \max\left\{2^{-\min\{j,j'\}}, 2^{\frac{2J(d'-q)-(j+j')(d'+\tilde{d})}{2(\tilde{d}+q)}}\right\}, \\ \mathcal{B}'_{j,j'} &= a' \max\left\{2^{-\max\{j,j'\}}, 2^{\frac{2J(d'-q)-(j+j')d'-\max\{j,j'\}\tilde{d}}{\tilde{d}+2q}}\right\}. \end{aligned} \tag{7.3}$$

The parameter a is a fixed constant which determines the bandwidth in the block matrices

$$\mathbf{A}^\epsilon_{j,j'} := [\mathbf{A}^\epsilon_J]_{(j,\nabla_j),(j',\nabla_{j'})}, \quad j_0 - 1 \le j, j' < J.$$

This parameter has to be chosen in such a way that the scaled compressed matrix retains stability, see [30]. We emphasize that the parameters a and a' are independent of J.

When the entries of the compressed system matrix $\mathbf{A}^\epsilon_J$ have been computed, we apply an *a-posteriori compression* by setting all entries to zero, which are smaller than a level depending threshold. In this way, a matrix $\widetilde{\mathbf{A}}^\epsilon_J$ is obtained which has even less nonzero entries than the matrix $\mathbf{A}^\epsilon_J$. Although this does not accelerate the computation of the matrix coefficients, the amount of necessary memory is reduced when the system matrix has to be stored. For instance, this offers advantages for the coupling of FEM and BEM, cf. [20, 21].

A-posteriori compression. We define the a-posteriori compression by

$$\left[\widetilde{\mathbf{A}}^\epsilon_J\right]_{(j,k),(j',k')} = \begin{cases} 0, & \text{if } \left|[\mathbf{A}^\epsilon_J]_{(j,k),(j',k')}\right| \le \varepsilon_{j,j'}, \\ [\mathbf{A}^\epsilon_J]_{(j,k),(j',k')}, & \text{if } \left|[\mathbf{A}^\epsilon_J]_{(j,k),(j',k')}\right| > \varepsilon_{j,j'}. \end{cases} \tag{7.4}$$

Here the level dependent threshold $\varepsilon_{j,j'}$ is chosen as

$$\varepsilon_{j,j'} = a'' \min\left\{2^{-\frac{|j-j'|n}{2}}, 2^{-n(J-\frac{j+j'}{2})\frac{d'-q}{\tilde{d}+q}}\right\} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})} \tag{7.5}$$

with $a'' < 1$ and $d' \in (d, \tilde{d}+r)$ from (7.2).

8. Matrix Estimates. In order to study the accuracy of the solutions to the compressed systems we investigate the perturbation introduced by discarding specific matrix elements. The perturbation matrices are of scalewise blocks of the type

$$\mathbf{R}_{j,j'} := \mathbf{A}_{j,j'} - \mathbf{A}^{\epsilon}_{j,j'}.$$

By $\|\mathbf{R}_{j,j'}\|_p$ we denote the operator norm of the matrix $\mathbf{R}_{j,j'}$ with respect to the norm l^p.

In order to analyze the error introduced by our compression strategy, we decompose the complete compression into three subsequent steps.

THEOREM 8.1 (First Compression). *We define the matrix* $\mathbf{A}^{\epsilon_1}_J$ *by*

$$[\mathbf{A}^{\epsilon_1}_J]_{(j,k),(j',k')} := \begin{cases} 0, & \operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'}) > \mathcal{B}_{j,j'} \text{ and } j, j' \geq j_0, \\ \langle A\psi_{j',k'}, \psi_{j,k}\rangle, & \text{otherwise.} \end{cases}$$

Here the parameter $\mathcal{B}_{j,j'}$ *is given by* (7.2) *and* (7.3). *Then, one has for the perturbation matrix* $\mathbf{R}_{j,j'} := \mathbf{A}_{j,j'} - \mathbf{A}^{\epsilon_1}_{j,j'}$

$$\|\mathbf{R}_{j,j'}\|_2 \lesssim a^{-2(\tilde{d}+q)} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}.$$

Proof. We proceed in two steps.

(i) We abbreviate $\mathbf{R}_{j,j'} := \left[r_{(j,k),(j',k')}\right]_{k\in\nabla_j, k'\in\nabla_{j'}}$. Invoking Theorem 6.1, we find for the column sum

$$\begin{aligned} \sum_{k\in\nabla_j} |r_{(j,k),(j',k')}| &= \sum_{\{k\in\nabla_j:\, \operatorname{dist}(\Omega_{j,k},\Omega_{j',k'})>\mathcal{B}_{j,j'}\}} |\langle A\psi_{j',k'}, \psi_{j,k}\rangle| \\ &\lesssim \sum_{\{k\in\nabla_j:\, \operatorname{dist}(\Omega_{j,k},\Omega_{j',k'})>\mathcal{B}_{j,j'}\}} 2^{-(j+j')(\tilde{d}+n/2)} \operatorname{dist}\left(\Omega_{j,k}, \Omega_{j',k'}\right)^{-n-2(\tilde{d}+q)}. \end{aligned}$$

Since $\mathcal{B}_{j,j'} \geq a \max\{2^{-j}, 2^{-j'}\}$, we can estimate this sum by an integral which yields

$$\begin{aligned} \sum_{k\in\nabla_j} |r_{(j,k),(j',k')}| &\lesssim 2^{-(j+j')(\tilde{d}+n/2)} 2^{jn} \int_{|x|>\mathcal{B}_{j,j'}} |x|^{-n-2(\tilde{d}+q)} dx \\ &\lesssim 2^{-(j+j')(\tilde{d}+n/2)} 2^{jn} \mathcal{B}_{j,j'}^{-2(\tilde{d}+q)}. \end{aligned}$$

On the other hand, inserting the estimate $\mathcal{B}_{j,j'} \geq a 2^{\frac{2J(d'-q)-(j+j')(\tilde{d}+d')}{2(\tilde{d}+q)}}$ (see (7.3)), we arrive at

$$\sum_{k\in\nabla_j} |r_{(j,k),(j',k')}| \lesssim a^{-2(\tilde{d}+q)} 2^{\frac{(j-j')n}{2}} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}.$$

In complete analogy one proves an analogous estimate for the row sums

$$\sum_{k'\in\nabla_{j'}} |r_{(j,k),(j',k')}| \lesssim a^{-2(\tilde{d}+q)} 2^{\frac{(j'-j)n}{2}} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}.$$

(ii) From the estimate for the operator norms of matrices

$$\|\mathbf{R}_{j,j'}\|_2^2 \leq \|\mathbf{R}_{j,j'}\|_1 \|\mathbf{R}_{j,j'}\|_\infty,$$

it is easy to conclude the following version of the Schur lemma (see e.g. [24, 30])

$$\begin{aligned}\|\mathbf{R}_{j,j'}\| &\le \Big[\max_{k\in\nabla_j}\sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}}|r_{(j,k),(j',k')}|\Big]^{1/2}\Big[\max_{k'\in\nabla_{j'}}\sum_{k\in\nabla_j} 2^{\frac{(j'-j)n}{2}}|r_{(j,k),(j',k')}|\Big]^{1/2}\\ &\le \frac{1}{2}\Big[\max_{k\in\nabla_j}\sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}}|r_{(j,k),(j',k')}| + \max_{k'\in\nabla_{j'}}\sum_{k\in\nabla_j} 2^{\frac{(j'-j)n}{2}}|r_{(j,k),(j',k')}|\Big]\\ &\lesssim a^{-2(\tilde{d}+q)}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}.\end{aligned}$$

which proves the assertion. □

The following so called second compression concerns entries involving basis functions with overlapping supports. It is important that here the coarse scale basis function may be a scaling function which greatly affects the near field compression.

THEOREM 8.2 (Second Compression). *In addition to the first compression we apply the following second compression*

$$[\mathbf{A}_J^{\epsilon_2}]_{(j,k),(j',k')} := \begin{cases} 0, & \operatorname{dist}(\Omega_{j,k},\Omega_{j',k'}) \lesssim 2^{-\min\{j,j'\}} \text{ and} \\ & \operatorname{dist}(\Omega'_{j,k},\Omega_{j',k'}) > \mathcal{B}'_{j,j'} \text{ if } j'>j\ge j_0-1, \\ & \operatorname{dist}(\Omega_{j,k},\Omega'_{j',k'}) > \mathcal{B}'_{j,j'} \text{ if } j>j'\ge j_0-1, \\ [\mathbf{A}_J^{\epsilon_1}]_{(j,k),(j',k')}, & \text{otherwise.}\end{cases}$$

where the parameter $\mathcal{B}'_{j,j'}$ is set in accordance with (7.2) *and* (7.3). *Then, the corresponding perturbation matrix $\mathbf{S}_{j,j'} := \mathbf{A}^{\epsilon_1}_{j,j'} - \mathbf{A}^{\epsilon_2}_{j,j'}$ satisfies*

$$\|\mathbf{S}_{j,j'}\|_2 \lesssim (a')^{-(\tilde{d}+2q)}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}.$$

Proof. Abbreviating $\mathbf{S}_{j,j'} := [s_{(j,k),(j',k')}]_{k\in\nabla_j,k'\in\nabla_{j'}}$ and assuming without loss of generality that $j'>j$, we infer from Theorem 6.3 that

$$\begin{aligned}|s_{(j,k),(j',k')}| &\lesssim 2^{jn/2}2^{-j'(\tilde{d}+n/2)}\mathcal{B}_{j,j'}^{-2q-\tilde{d}}\\ &\lesssim (a')^{-2q-\tilde{d}}2^{jn/2}2^{-j'(\tilde{d}+n/2)}2^{-2J(d'-q)+(j+j')d'+j'\tilde{d}}\\ &= (a')^{-2q-\tilde{d}}2^{(j-j')n/2}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}.\end{aligned}$$

The condition $\operatorname{dist}(\Omega_{j,k},\Omega_{j',k'}) \lesssim 2^{-\min\{j,j'\}}$ guarantees that in each row and column of $\mathbf{S}_{j,j'}$ we have set at most $\mathcal{O}(2^{(j'-j)n})$, respectively $\mathcal{O}(1)$ entries to zero. Therefore, we obtain for the weighted row sums

$$\begin{aligned}\sum_{k\in\nabla_j} 2^{\frac{(j-j')n}{2}}|s_{(j,k),(j',k')}| &\lesssim \sum_{k'\in\nabla_{j'}} (a')^{-2q-\tilde{d}}2^{j'n}2^{(j-j')n}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}\\ &\lesssim (a')^{-2q-\tilde{d}}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})},\end{aligned}$$

and likewise for the weighted column sums

$$\sum_{k'\in\nabla_{j'}} 2^{\frac{(j'-j)n}{2}}|s_{(j,k),(j',k')}| \lesssim (a')^{-2q-\tilde{d}}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}$$

for all $j_0-1\le j<j'<J$. In complete analogy to the proof of Theorem 8.1 we conclude

$$\begin{aligned}\|\mathbf{S}_{j,j'}\|_2 &\le \Big[\max_{k\in\nabla_j}\sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}}|s_{(j,k,(j,k)}| + \max_{k'\in\nabla_{j'}}\sum_{k\in\nabla_j} 2^{\frac{(j'-j)n}{2}}|s_{(j,k,(j,k)}|\Big]\\ &\lesssim (a')^{-2q-\tilde{d}}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}.\end{aligned}$$

□

THEOREM 8.3 (A-posteriori compression). *Let the matrix $\mathbf{A}_J^\epsilon$ be compressed according to Theorems 8.1 and 8.2. Then the a-posteriori compression defined by*

$$[\widetilde{\mathbf{A}}_J^\epsilon]_{(j,k),(j',k')} = \begin{cases} 0, & \text{if } \left|[\mathbf{A}_J^\epsilon]_{(j,k),(j',k')}\right| \le \varepsilon_{j,j'}, \\ [\mathbf{A}_J^\epsilon]_{(j,k),(j',k')}, & \text{if } \left|[\mathbf{A}_J^\epsilon]_{(j,k),(j',k')}\right| > \varepsilon_{j,j'} \end{cases}$$

with the level dependent threshold $\varepsilon_{j,j'}$ from (7.5), causes a block perturbation $\mathbf{T}_{j,j'} := \widetilde{\mathbf{A}}_{j,j'}^\epsilon - \mathbf{A}_{j,j'}^\epsilon$ satisfying

$$||\mathbf{T}_{j,j'}||_2 \lesssim a'' 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}.$$

Proof. We organize the proof in four steps.
(i) Abbreviating $\mathbf{T}_{j,j'} := [t_{(j,k),(j',k')}]_{k\in\nabla_j,k'\in\nabla_{j'}}$, one obviously has

$$\left|t_{(j,k),(j',k')}\right| \le a'' \min\left\{2^{-\frac{|j-j'|n}{2}}, 2^{-n(J-\frac{j+j'}{2})\frac{d'-q}{\tilde{d}+q}}\right\} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}. \tag{8.1}$$

We shall use the first compression in order to derive from this inequality the desired. To this end, we find in each row and column of $\mathbf{T}_{j,j'}$ only $\mathcal{O}([\mathcal{B}_{j,j'}2^{j'}]^n)$, respectively $\mathcal{O}([\mathcal{B}_{j,j'}2^j]^n)$ nonzero entries. Setting $M := \frac{d'+\tilde{d}}{2(\tilde{d}+q)}$, one has

$$\begin{aligned} 2^{\frac{2J(d'-q)-(j+j')(d'+\tilde{d})}{2(\tilde{d}+q)}} &= 2^{-J} 2^{\frac{2J(d'+\tilde{d})-(j+j')(d'+\tilde{d})}{2(\tilde{d}+q)}} = 2^{-J} 2^{\frac{(J-j)(d'+\tilde{d})}{2(\tilde{d}+q)}} 2^{\frac{(J-j')(d'+\tilde{d})}{2(\tilde{d}+q)}} \\ &= 2^{-J} 2^{(J-j)M} 2^{(J-j')M}. \end{aligned}$$

Hence, by (7.3), the cut-off parameter for the first compression takes the form

$$\mathcal{B}_{j,j'} \sim \max\left\{2^{-\min\{j,j,'\}}, 2^{-J} 2^{(J-j)M} 2^{(J-j')M}\right\}. \tag{8.2}$$

From (7.2) and $q < d - \frac{1}{2}$, one concludes $\frac{1}{2} < M < 1$. Moreover, we shall make use of the identity

$$2^{-n(J-\frac{j+j'}{2})\frac{d'-q}{\tilde{d}+q}} = 2^{-2n(J-\frac{j+j'}{2})(M-\frac{1}{2})}. \tag{8.3}$$

Without the loss of generality, we assume in the sequel that $j' \ge j$.
(ii) With these preparations at hand we shall first estimate the block matrices $\mathbf{T}_{j,j'}$ with $2^{-\frac{(j'-j)n}{2}} \le 2^{-2n(J-\frac{j+j'}{2})(M-\frac{1}{2})}$. One readily verifies that this relation is equivalent to

$$2^{-j} \ge 2^{-J} 2^{(J-j)M} 2^{(J-j')M},$$

which, by (8.2), implies that the cut-off parameter satisfies $\mathcal{B}_{j,j'} \sim 2^{-j}$. Thus, from (8.1) one infers the estimate

$$\begin{aligned} \sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}} \left|t_{(j,k),(j',k')}\right| &\lesssim a'' 2^{\frac{(j+j')n}{2}} 2^{-jn} 2^{-\frac{(j'-j)n}{2}} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})} \\ &= a'' 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}, \end{aligned}$$

for the weighted row sums of $\mathbf{T}_{j,j'}$. Analogously, one derives

$$\sum_{k\in\nabla_j} 2^{\frac{(j'-j)n}{2}} \left|t_{(j,k),(j',k')}\right| \lesssim a'' 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})}$$

for the weighted column sums.

(iii) We still have to estimate the errors in the remaining blocks, where $2^{-\frac{(j'-j)n}{2}} > 2^{-2n(J-\frac{j+j'}{2})(M-\frac{1}{2})}$. Then, by (8.3), the cut-off parameter is given by $B_{j,j'} \sim 2^{-J}2^{(J-j)M}2^{(J-j')M}$. Therefore, we obtain for the weighted row sums

$$\begin{aligned}
&\sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}}|t_{(j,k),(j',k')}| \\
&\quad\lesssim a''2^{\frac{(j+j')n}{2}}2^{-Jn}2^{(J-j)Mn}2^{(J-j')Mn}2^{-2n(J-\frac{j+j'}{2})(M-\frac{1}{2})}2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})} \\
&\quad= a''2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})},
\end{aligned}$$

and a similar estimate for the weighted column sums.

(iv) Combining the estimates in steps *(ii)* and *(iii)*, we conclude that

$$\begin{aligned}
\sum_{k'\in\nabla_{j'}} 2^{\frac{(j-j')n}{2}}|t_{(j,k),(j',k')}| &\lesssim a''2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}, \\
\sum_{k\in\nabla_{j}} 2^{\frac{(j'-j)n}{2}}|t_{(j,k),(j',k')}| &\lesssim a''2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})},
\end{aligned}$$

for all $j_0 - 1 \le j, j' < J$. The proof can now be completed in complete analogy to the proof of Theorem 8.1. □

9. Consistency. We shall establish next the consistency of the compressed scheme with the original operator equation in the corresponding Sobolev norms, making crucial use of the norm equivalences induced by the wavelet bases.

To this end, note that the operator $\widetilde{A}_J^\epsilon : H^s(\Gamma) \to H^{s-2q}(\Gamma)$, $-\widetilde{\gamma} < s < \widetilde{\gamma} + 2q$, defined by

$$A_J^\epsilon u = \sum_{j,j'=j_0-1}^{J-1} \sum_{k,k'} \widetilde{\psi}_{j,k}[\widetilde{\mathbf{A}}_J^\epsilon]_{(j,k),(j',k')}\langle\widetilde{\psi}_{j,k}, u\rangle.$$

is represented by the compressed system matrix, since apparently

$$\langle\widetilde{A}_J^\epsilon\psi_{j',k'}, \psi_{j,k}\rangle = [\widetilde{\mathbf{A}}_J^\epsilon]_{(j,k),(j',k')}.$$

THEOREM 9.1. *Let $d < \widetilde{d} + 2q$ and $\widetilde{\mathbf{A}}_J^\epsilon$ the compressed matrix, defined according to Section 7. Then, for $q \le t, t' \le d$ the estimate*

$$|\langle(A - \widetilde{A}_J^\epsilon)Q_J u, Q_J v\rangle| \lesssim \epsilon 2^{J(2q-t-t')}\|u\|_t\|v\|_{t'} \tag{9.1}$$

holds uniformly with respect to J, where

$$\epsilon := a^{-2\widetilde{d}-2q} + (a')^{-\widetilde{d}-2q} + a'', \tag{9.2}$$

and a, a', a'' are the constants from (7.2) *and* (7.5).

Proof. By definition of the block perturbation matrices $\mathbf{R}_{j,j'}$, $\mathbf{S}_{j,j'}$ and $\mathbf{T}_{j,j'}$, one has

$$|\langle(A - \widetilde{A}_J^\epsilon)\psi_{j',k'}, \psi_{j,k}\rangle| \le \big|[\mathbf{R}_{j,j'} + \mathbf{S}_{j,j'} + \mathbf{T}_{j,j'}]_{k,k'}\big|.$$

Hence, we can estimate

$$
\begin{aligned}
&|\langle (A-\widetilde{A}_J^\epsilon)Q_J u, Q_J v\rangle| \qquad (9.3)\\
&\quad = \Big| \sum_{j,j'=j_0}^{J} \langle (A-\widetilde{A}_J^\epsilon)(Q_{j'}-Q_{j'-1})u, (Q_j-Q_{j-1})v\rangle \Big|\\
&\quad \le \sum_{j,j'=j_0}^{J} |\langle (A-\widetilde{A}_J^\epsilon)(Q_{j'}-Q_{j'-1})u, (Q_j-Q_{j-1})v\rangle|\\
&\quad = \sum_{j,j'=j_0}^{J} \Big| \sum_{k\in\nabla_j}\sum_{k\in\nabla_{j'}} \langle (A-\widetilde{A}_J^\epsilon)\psi_{j',k'},\psi_{j,k}\rangle \langle u,\widetilde{\psi}_{j',k'}\rangle\langle v,\widetilde{\psi}_{j,k}\rangle \Big|\\
&\quad \le \sum_{j,j'=j_0}^{J} \|\mathbf{R}_{j,j'}+\mathbf{S}_{j,j'}+\mathbf{T}_{j,j'}\|_2 \|[\langle u,\widetilde{\psi}_{j',k'}\rangle]_{k'\in\nabla_{j'}}\|_2 \|[\langle v,\widetilde{\psi}_{j,k}\rangle]_{k\in\nabla_j}\|_2.
\end{aligned}
$$

Invoking the inverse estimate (4.3) and the approximation property (4.2), yields

$$
\begin{aligned}
\|[\langle u,\widetilde{\psi}_{j',k'}\rangle]_{k'\in\nabla_{j'}}\|_2 &\sim \|(Q_{j'}-Q_{j'-1})u\|_0 \lesssim 2^{-j't}\|u\|_t\\
\|[\langle v,\widetilde{\psi}_{j,k}\rangle]_{k'\in\nabla_{j'}}\|_2 &\sim \|(Q_j-Q_{j-1})v\|_0 \lesssim 2^{-jt'}\|v\|_{t'}.
\end{aligned}
$$

Further, from Theorem 8.1, Theorem 8.2 and Theorem 8.3, we conclude

$$
\|\mathbf{R}_{j,j'}+\mathbf{S}_{j,j'}+\mathbf{T}_{j,j'}\|_2 \lesssim \epsilon 2^{2Jq}2^{-2d'(J-\frac{j+j'}{2})}.
$$

Inserting these estimates in (9.3), provides

$$
\begin{aligned}
|\langle (A-\widetilde{A}_J^\epsilon)Q_J u, Q_J v\rangle| &\lesssim \epsilon 2^{J(2q-t-t')}\|u\|_t\|v\|_{t'} \sum_{j,j'=j_0}^{J} 2^{-j'(d'-t)}2^{-j(d'-t')}\\
&\lesssim \epsilon 2^{J(2q-t-t')}\|u\|_t\|v\|_{t'}
\end{aligned}
$$

since $t,t' \le d < d'$. □

10. Convergence. With the estimates of Section 8 at hand we can prove that the proposed compression strategy retains the optimal order of convergence of the underlying Galerkin scheme, see [30]. In this context we shall encounter conditions on the parameters a, a', a'' defining ϵ in (9.2). From Theorem 9.1 we deduce

$$
|\langle (A-\widetilde{A}_J^\epsilon)u_J, u_J\rangle| \le \varepsilon \|u_J\|_q^2,
$$

which implies the V_J-ellipticity. Indeed inserting this result into (3.3) we get for $J > J_0$ that

$$
|\langle (\widetilde{A}_J^\epsilon + \widetilde{A}_J^{\epsilon\star})u_J, u_J\rangle| \ge (c-2\varepsilon)\|u_J\|_q^2 \gtrsim \|u_J\|_q^2,
$$

with $c > 0$, if ϵ from (9.2) is sufficiently small.

THEOREM 10.1 (Stability). *Let ϵ from (9.2) be sufficiently small. Then, the matrix $\widetilde{\mathbf{A}}_J^\epsilon$, which arises by the compression according to (7.1) and (7.4), defines a stable scheme, i.e.,*

$$
\|\widetilde{A}_J^\epsilon u_J\|_q \sim \|u_J\|_q,
$$

uniformly in $J > J_0$.

In the above reasoning we have already required that $\widetilde{\gamma} > -q$. It has been shown in [28] that Theorem 10.1 remains valid for $\widetilde{\gamma} = -q$.

THEOREM 10.2 (Convergence). *Let ϵ from (9.2) be sufficiently small to ensure uniform stability of $\widetilde{A}_J^\epsilon$. Then, the solution $u_J = \sum_{j=j_0-1}^{J-1} \sum_{k\in\nabla_j} u_{j,k}\psi_{j,k}$ of the compressed scheme*

$$\widetilde{\mathbf{A}}_J^\epsilon \mathbf{u}_J = \mathbf{f}_J, \qquad \mathbf{u}_J = [u_{j,k}]_{j_0-1\le j<J,\, k\in\nabla_j},$$

differs from the exact solution u, satisfying $Au = f$, in the energy norm only by

$$\|u - u_J\|_q \lesssim 2^{J(q-d)}\|u\|_d$$

uniformly in J.

Proof. Strang's first lemma [4] provides

$$\|u - u_J\|_q \lesssim \inf_{v_J\in V_J}\left\{\|u - v_J\|_q + \sup_{w_J\in V_J}\frac{|\langle (A-\widetilde{A}_J^\epsilon)v_J, w_J\rangle|}{\|w_J\|_q}\right\}.$$

The consistency (Theorem 9.1) implies

$$|\langle (A-\widetilde{A}_J^\epsilon)Q_J u, w_J\rangle| = |\langle (A-\widetilde{A}_J^\epsilon)Q_J u, Q_J w_J\rangle \lesssim 2^{J(q-d)}\|u\|_d\|w_J\|_q$$

for all $u\in H^d(\Gamma)$ and $w_J\in V_J$. Hence, choosing $v_J := Q_J u$, we arrive at

$$\|u - u_J\|_q \lesssim \|u - Q_J u\|_q + \sup_{w_J\in V_J}\frac{|\langle (A-\widetilde{A}_J^\epsilon)Q_J u, Q_J w_J\rangle|}{\|w_J\|_q} \lesssim 2^{J(q-d)}\|u\|_d.$$

□

THEOREM 10.3 (Aubin-Nitsche). *In addition to the assumptions of Theorem 10.2 suppose that $\|A^\star v\|_{t-q} \sim \|v\|_{t+q}$ for all $0\le t\le d-q$, i.e.*

$$A^\star : H^{t+q}(\Gamma) \to H^{t-q}(\Gamma)$$

is an isomorphism. Then the error estimate

$$\|u - u_J\|_{q-t} \lesssim 2^{J(q-d-t)}\|u\|_d$$

holds for all $0\le t\le d-q$.

Proof. Recalling that

$$\|u - u_J\|_{q-t} = \sup_{g\in H^{t-q}(\Gamma)}\frac{\langle u - u_J, g\rangle}{\|g\|_{t-q}}.$$

we obtain for $v\in H^{t+q}(\Gamma)$ with $A^\star v = g$

$$\|u - u_J\|_{q-t} = \sup_{v\in H^{t+q}(\Gamma)}\frac{|\langle A(u-u_J), v\rangle|}{\|v\|_{t+q}}.$$

Utilizing the Galerkin orthogonality $\langle \widetilde{A}_J^\epsilon u_J, Q_J v\rangle = \langle Au, Q_J v\rangle$, we can decompose

$$\begin{aligned}\langle A(u-u_J), v\rangle &= \langle A(u-u_J), v - Q_J v\rangle + \langle A(u-u_J), Q_J v\rangle \\ &= \langle A(u-u_J), v - Q_J v\rangle - \langle (A-\widetilde{A}_J^\epsilon)u_J, Q_J v\rangle.\end{aligned}$$

The first term on the right hand side is estimated by Theorem 10.2 in combination with the approximation property (4.2)

$$|\langle A(u-u_J), v - Q_J v\rangle| \lesssim \|u - u_J\|_q\|v - Q_J v\|_q \lesssim 2^{J(q-d-t)}\|u\|_d\|v\|_{t+q}.$$

For the second term we obtain, on account of Theorem 9.1,

$$\begin{aligned}|\langle (A-\widetilde{A}_J^\epsilon)u_J, Q_J v\rangle| &\le |\langle (A-\widetilde{A}_J^\epsilon)(u_J - Q_J u), Q_J v\rangle| + |\langle (A-\widetilde{A}_J^\epsilon)Q_J u, Q_J v\rangle| \\ &\lesssim 2^{-Jt}\|u_J - Q_J u\|_q\|v\|_{t+q} + 2^{J(q-d-t)}\|u\|_d\|v\|_{t+q}.\end{aligned}$$

Inserting $\|u_J - Q_J u\|_q \le \|u - u_J\|_q + \|u - Q_J u\|_q \lesssim 2^{J(q-d)}\|u\|_d$, yields

$$|\langle (A - \widetilde{A}_J^\epsilon) u_J, Q_J v\rangle| \lesssim 2^{J(q-d-t)}\|u\|_d \|v\|_{t+q}.$$

Therefore, we conclude

$$\|u - u_J\|_{q-t} = \sup_{v \in H^{t+q}(\Gamma)} \frac{\langle A(u-u_J), v\rangle}{\|v\|_{t+q}} \lesssim 2^{J(q-d-t)}\|u\|_d,$$

which finishes the proof. □

Note that in the extreme case $t = d - q$ we obtain the best possible convergence rate of the Galerkin scheme (3.6).

11. Complexity. In this section we present a general theorem which shows that the overall complexity of assembling the compressed system matrix with sufficient accuracy can be kept of the order $\mathcal{O}(N_J)$, even when a computational cost of logarithmic order is allowed for each entry. This theorem will be used later as the essential ingredient in proving that the quadrature strategy proposed in Section 13 scales linearly.

THEOREM 11.1. *Assume that $\mathbf{A}_J^\epsilon$ is obtained by compressing the system matrix $\mathbf{A}_J = [\langle A\psi_{j',k'}, \psi_{j,k}\rangle]_{j_0-1\le j,j'\le J,\ k\in\nabla_j,\ k'\in\nabla_{j'}}$ according to (7.1). The complexity of computing this compressed matrix is $\mathcal{O}(N_J)$ provided that for some $\alpha \ge 0$ at most $\mathcal{O}\big(\big[J - \frac{j+j'}{2}\big]^\alpha\big)$ operations are spent on the approximate calculation of the nonvanishing entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$.*

Proof. (i) We begin with some technical preparations. Recall from the proof of Theorem (8.3) that the cut-off parameter with respect to the first compression is given by

$$\mathcal{B}_{j,j'} \sim \max\Big\{2^{-\min\{j,j,'\}}, 2^{-J}2^{(J-j)M}2^{(J-j')M}\Big\},$$

where, as in the proof of Theorem 8.3, $M = \frac{d'+\widetilde{d}}{2(\widetilde{d}+q)} < 1$. Moreover, we set $M' := \frac{2d'-2q}{\widetilde{d}+d'}$ and $N' := \frac{\widetilde{d}+d'}{\widetilde{d}+2q}$ with d' given by (7.2). Notice that M' and N' satisfy the relations $0 < M' < 1$ and $0 < N'$. As one readily verifies, the cut-off parameter with respect to the second compression may now be rewritten as

$$\mathcal{B}'_{j,j'} \sim \max\Big\{2^{-j}, 2^{-j'}2^{[JM'+(1-M')j'-j]N'}\Big\}, \qquad j \ge j'. \tag{11.1}$$

Further, we make use of the inequality $\log^\alpha x \lesssim 2^{2\delta x}$ which holds for all $x > 0$ and $\delta > 0$. Thus, it suffices to prove the claim for $\mathcal{O}\big(\big[J - \frac{j+j'}{2}\big]^\alpha\big)$ replaced by $\mathcal{O}\big(2^{\delta(J-j)}2^{\delta(J-j')}\big)$ where δ is chosen sufficiently small.

(ii) First, we determine now the complexity $\mathcal{C}^{(1)}$ of computing, within the above cost allowance, all matrix entries found in the block matrices $\mathbf{A}_{j,j'}^\epsilon = [\mathbf{A}_J^\epsilon]_{(j,\nabla_j),(j',\nabla_{j'})}$ with $\mathcal{B}_{j,j'} \sim 2^{-J}2^{(J-j)M}2^{(J-j')M}$. In such blocks, we have to process all coefficients $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ with

$$\operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'}) \lesssim \operatorname{dist}_{j,j'}^{(1)} := 2^{-J}2^{(J-j)M}2^{(J-j')M}. \tag{11.2}$$

In each block, we find only $\mathcal{O}\big(\big[2^{j'}\operatorname{dist}_{j,j'}^{(1)}\big]^n\big)$ entries satisfying (11.2) per row and, hence a total of $\mathcal{O}\big(\big[2^{j+j'}\operatorname{dist}_{j,j'}^{(1)}\big]^n\big)$. Summing over all blocks, yields

$$\begin{aligned}\mathcal{C}^{(1)} &\lesssim \sum_{j,j'=0}^{J} 2^{(j+j')n}2^{-Jn}2^{(J-j)(M+\delta)n}2^{(J-j')(M+\delta)n}\\ &= 2^{Jn}\sum_{j,j'=0}^{J} 2^{(J-j)(M+\delta-1)n}2^{(J-j')(M+\delta-1)n} \lesssim 2^{Jn},\end{aligned}$$

provided that δ is chosen so as to ensure $M + \delta < 1$.

(iii) It remains to show that the complexity for computing the omitted blocks is likewise $\mathcal{O}(N_J)$. Without the loss of generality, we assume $j \geq j'$ in the remainder of this proof, since the roles of j and j' can be reversed. Observing that, because of $0 < M' < 1$, one has $0 < JM' + (1-M')j' \leq J$, we consider first the blocks $\mathbf{A}^{\epsilon}_{j,j'}$ with $(j,j') \in S$, where the index set S is given by

$$S := \{(j,j') : 0 \leq j' \leq J,\ JM' + (1-M')j' \leq j \leq J\}. \tag{11.3}$$

In these blocks, we estimate the complexity $\mathcal{C}^{(2)}$ required for the approximate computation of the matrix entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ satisfying the relation

$$\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}) \lesssim \operatorname{dist}^{(2)}_{j,j'} := 2^{-j'} 2^{[JM' + (1-M')j' - j]N'}, \tag{11.4}$$

where we refer to the expression (11.1) for $\mathcal{B}'_{j,j'}$. Since $\operatorname{dist}^{(2)}_{j,j'} \leq 2^{-j'}$ for all $(j,j') \in S$, in each block one finds only $\mathcal{O}\big(\lceil 2^{jn} 2^{-j'(n-1)} \operatorname{dist}^{(2)}_{j,j'} \rceil\big)$ nontrivial matrix entries per column with (11.4), and thus a total of $\mathcal{O}\big(\lceil 2^{jn} 2^{j'} \operatorname{dist}^{(2)}_{j,j'} \rceil\big)$. Therefore, noting that the set S is equivalent to

$$S = \Big\{(j,j') : JM' \leq j \leq J,\ 0 \leq j' \leq \tfrac{j - JM'}{1-M'}\Big\},$$

the complexity is bounded by

$$\begin{aligned}
\mathcal{C}^{(2)} &\lesssim \sum_{j=JM'}^{J} \sum_{j'=0}^{\frac{j-JM'}{1-M'}} 2^{jn} 2^{[JM' + (1-M')j' - j]N'} 2^{\delta(J-j)} 2^{\delta(J-j')} \\
&= \sum_{j=JM'}^{J} 2^{jn} 2^{[JM' - j]N'} 2^{\delta(J-j)} 2^{\delta J} \sum_{j'=0}^{\frac{j-JM'}{1-M'}} 2^{j'[(1-M')N' - \delta]} \\
&\lesssim 2^{\delta J \frac{2-M'}{1-M'}} \sum_{j=0}^{J} 2^{j(n - \delta\frac{2-M'}{1-M'})} \lesssim 2^{Jn}.
\end{aligned}$$

$\mathcal{C}^{(2)}$ estimates the complexity for those blocks with $(j,j') \in S$ when $\mathcal{B}'_{j,j'} \sim \operatorname{dist}^{(2)}_{j,j'}$. But according to (11.1), the cut-off parameter $\mathcal{B}'_{j,j'}$ is bounded from below by 2^{-j}. In the case of $\mathcal{B}'_{j,j'} \sim 2^{-j}$ we find $\mathcal{O}\big(\lceil 2^{jn} 2^{j'} \operatorname{dist}^{(3)}_{j,j'} \rceil\big)$ matrix entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ with

$$\operatorname{dist}(\Omega'_{j,k}, \Omega_{j',k'}) \lesssim \operatorname{dist}^{(3)}_{j,j'} := 2^{-j}.$$

Arguing analogously as above, summing over all blocks with $(j,j') \in S$, one obtains

$$\begin{aligned}
\mathcal{C}^{(3)} &\lesssim \sum_{j=JM'}^{J} \sum_{j'=0}^{\frac{j-JM'}{1-M'}} 2^{j(n-1)} 2^{j'} 2^{\delta(J-j)} 2^{\delta(J-j')} = \sum_{j=JM'}^{J} 2^{j(n-1)} 2^{\delta(2J-j)} \sum_{j'=0}^{\frac{j-JM'}{1-M'}} 2^{j'(1-\delta)} \\
&\lesssim 2^{\delta J \frac{2-M'}{1-M'}} \sum_{j=0}^{J} 2^{j(n - \delta\frac{2-M'}{1-M'})} \lesssim 2^{Jn}.
\end{aligned}$$

(iv) Finally, we consider the blocks $\mathbf{A}^{\epsilon}_{j,j'}$ with $j \geq j'$ and $(j,j') \notin S$. In view of step *(ii)*, it suffices to consider all entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ which fulfil

$$\operatorname{dist}(\Omega_{j,k}, \Omega_{j',k'}) \lesssim \operatorname{dist}^{(4)} := 2^{-\min\{j',j\}} = 2^{-j'}. \tag{11.5}$$

Each block $\mathbf{A}^{\epsilon}_{j,j'}$ consists of only $\mathcal{O}\big(\lceil 2^{j} 2^{j'} \operatorname{dist}^{(4)} \rceil^n\big)$ entries with (11.5). Hence, according to

(11.3), the complexity $\mathcal{C}^{(4)}$ for the computation of these entries is

$$\mathcal{C}^{(4)} \lesssim \sum_{j'=0}^{J} \sum_{j=j'}^{JM'+(1-M')j'} 2^{jn} 2^{\delta(J-j)} 2^{\delta(J-j')} \lesssim \sum_{j'=0}^{J} 2^{2\delta(J-j')} 2^{j'n} \sum_{j=0}^{(J-j')M'} 2^{j(n-\delta)}$$
$$\lesssim 2^{Jn} \sum_{j'=0}^{J} 2^{(J-j')((M'-1)(n-\delta)+\delta)} \lesssim 2^{Jn},$$

since $(M'-1)(n-\delta)+\delta < 0$. This completes this proof. □

12. Setting up the Compression Pattern. Checking the distance criterion 7.1 for each matrix coefficient, in order to assemble the compressed matrix, would require $\mathcal{O}(N_J^2)$ function calls. To realize linear complexity, we exploit the underlying tree structure with respect to the supports of the wavelets, to predict negligible matrix coefficients. We will call a wavelet $\psi_{j+1,\text{son}}$ a son of $\psi_{j,\text{father}}$ if $\Omega_{j+1,\text{son}} \subseteq \Omega_{j,\text{father}}$. The following observation is an immediate consequence of the relations $\mathcal{B}_{j,j'} \geq \mathcal{B}_{j+1,j'} \geq \mathcal{B}_{j+1,j+1'}$, and $\mathcal{B}'_{j,j'} \geq \mathcal{B}'_{j+1,j'}$ for $j > j'$.

LEMMA 12.1. *For $\Omega_{j+1,\text{son}} \subseteq \Omega_{j,\text{father}}$ and $\Omega_{j'+1,\text{son}} \subseteq \Omega_{j',\text{father}}$ the following statements hold.*

1. $\operatorname{dist}(\Omega_{j,\text{father}}, \Omega_{j',\text{father}'}) > \mathcal{B}_{j,j'}$ *implies* $\operatorname{dist}(\Omega_{j+1,\text{son}}, \Omega_{j',\text{father}'}) > \mathcal{B}_{j+1,j'}$ *as well as* $\operatorname{dist}(\Omega_{j+1,\text{son}}, \Omega_{j'+1,\text{son}'}) > \mathcal{B}_{j+1,j+1'}$.
2. *Suppose that $j > j'$ and* $\operatorname{dist}(\Omega_{j,\text{father}}, \Omega'_{j',\text{father}'}) > \mathcal{B}'_{j,j'}$. *Then one has* $\operatorname{dist}(\Omega_{j+1,\text{son}}, \Omega'_{j',\text{father}'}) > \mathcal{B}'_{j+1,j'}$.

With the aid of this lemma we have to check the distance criteria only for coefficients which stem from subdivisions of calculated coefficients on a coarser level. Therefore, the resulting procedure of checking the distance criteria is still of linear complexity.

13. Computation of Matrix Coefficients. Of course, the significant matrix entries $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ retained by the compression strategy can generally neither be determined analytically nor be computed exactly. Therefore we have to approximate the matrix coefficients by quadrature rules. This causes an additional error which has to be controlled with regard to our overall objective of realizing asymptotically optimal accuracy while preserving efficiency. Theorem 11.1 describes the maximal allowed computational expenses for the computation of the individual matrix coefficients so as to realize still overall linear complexity. From previous studies we already know that sufficient accuracy requires only a level dependent precision of quadrature for computing the retained matrix coefficients, see e.g. [19, 30]. This precision is actually described in Theorem 8.3.

PROPOSITION 13.1. *Let the error of quadrature for computing the relevant matrix coefficients $\langle A\psi_{j',k'}, \psi_{j,k}\rangle$ be bounded by the level dependent accuracy*

$$\varepsilon_{j,j'} = \epsilon \min\left\{2^{-\frac{|j-j'|n}{2}}, 2^{-n(J-\frac{j+j'}{2})\frac{d'-q}{\tilde{d}+q}}\right\} 2^{2Jq} 2^{-2d'(J-\frac{j+j'}{2})} \tag{13.1}$$

for some $\epsilon < 1$ and $d' \in (d, \widetilde{d}+r)$ from (7.2). *Then, the Galerkin scheme is stable and converges with the optimal order* (3.6).

From (13.1) we conclude that the entries on the coarse grids have to be computed with the full accuracy while the entries on the finer grids are allowed to have less accuracy. Unfortunately, the domains of integration are very large on coarser scales.

Since we employ primal multiresolution spaces V_j based on piecewise polynomials, the numerical integration can be reduced to the computation of the interaction of polynomial shape functions on certain elements. Consequently, we have to deal only with integrals of the form

$$I(\Gamma_{i,j,k}, \Gamma_{i',j',k'}) := \int_{\Gamma_{i,j,k}} \int_{\Gamma_{i',j',k'}} k(x,y) p_l(\gamma_i^{-1}(x)) p_{l'}(\gamma_{i'}^{-1}(y))\, d\Gamma_y\, d\Gamma_x \tag{13.2}$$

with p_l denoting the polynomial shape functions. This is quite similar to the traditional Galerkin discretization. The main difference is that in the wavelet approach the elements may appear on different levels due to the multilevel hierarchy of wavelet bases.

Difficulties arise if the domains of integration are very close together relatively to their size. We have to apply numerical integration with some care in order to keep the number of evaluations of the kernel function at the quadrature nodes moderate and to fulfil the requirements of Theorem 11.1. The necessary accuracy can be achieved within the allowed expenses if we employ an exponentially convergent quadrature method.

In [19, 30] a geometrically graded subdivision of meshes is proposed in combination with varying polynomial degrees of approximation in the integration rules, cf. Figure 13.1. Exponential convergence is shown for boundary integral operators under the assumption that the underlying manifolds are piecewise analytic. It is shown in [19] that the combination of tensor product Gauß-Legendre quadrature rules with such a *hp*-quadrature scheme leads to a number of quadrature points satisfying the assumptions of Theorem 11.1 with $\alpha = 2n$. Since the proofs are rather technical we refer to [31, 27, 30, 19, 22] for further details. For that result to be valid we need a slightly stronger assumption on our manifold Γ which should be piecewise analytic. Moreover, the kernels of the operators should satisfy the following condition.

ASSUMPTION: The kernel $k(\widehat{x},\widehat{y})$, $\widehat{x},\widehat{y} \in \Gamma$, is analytically standard of order $2q$, that is, the partial derivatives of the transported kernel function (2.5) are uniformly bounded by

$$|\partial_x^\alpha \partial_y^\alpha \widetilde{k}(x,y)| \lesssim \frac{(|\alpha|+|\beta|)!}{(q\,\mathrm{dist}(\hat{x},\hat{y}))^{n+2q+|\alpha|+|\beta|}},$$

with some $q > 0$.

REMARK: The condition of piecewise analyticity has been relaxed in [28] to a maximal degree piecewise C^k-smoothness of the surface. However these methods are more involved and, although the required smoothness k is finite, it is still relatively large.

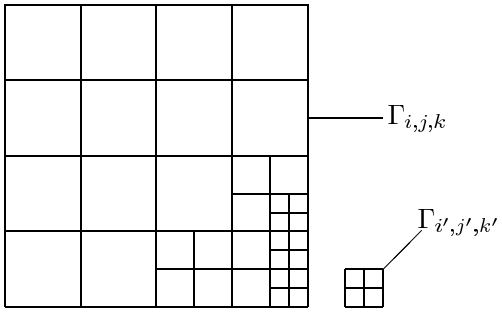

FIG. 13.1. *Adaptive subdivision of the domains of integration.*

Since the kernel function has a singularity on the diagonal we are still confronted with singular integrals if the domains of integration live on the same level and have some points in common. This happens if the underlying elements are identical or share a common edge or vertex. When we do not deal with weakly singular integral operators, the operators can be regularized, e.g. by partial integration [25]. So we end up with weakly singular integrals. Such weakly singular integrals can be treated by the so-called *Duffy-trick* [16, 29]. In this way the singular integrands are transformed into analytical ones.

In summary, from the previous results together with the quoted references it follows that for boundary integral operators arising from Laplace, Stokes and Lamé equation, the required accuracy (13.1) for the computation of an individual matrix coefficient is achieved with $\mathcal{O}\big(\big[J - \frac{j+j'}{2}\big]^{2n}\big)$ function calls. According to Theorem 11.1 the total complexity needed to compute for every dyadic level J a compressed matrix, representing a scheme with the original discretization error accuracy, is $\mathcal{O}(N_J)$. Using nested iteration, where on each level the approximate solution of the previous level is used as initial guess, so that, in view of Theorem 5.1, only a finite uniformly bounded number of iteration is needed to realize the corresponding discretization error accuracy, one has a solution process with asymptotically optimal complexity.

THEOREM 13.2. *Employing nested iteration in combination with the above scheme of computing the compressed stiffness matrices, allows one to compute approximate solutions to* (2.1) *from* V_J*, realizing discretization error accuracy, at the expense of* $\mathcal{O}(N_J)$ *operations, uniformly in* J*.*

14. Numerical Results. In order to demonstrate the efficiency of our method we present here only a representative example which does not only confirm the theoretical results quantitatively, but shows also that our concept can be applied to nontrivial geometries. In fact, it has been already experienced in [19] that we achieve almost the same compression rates as for simple surfaces e.g. a sphere.

We solve an interior Dirichlet problem for the Laplacian by the indirect approach using the single layer potential operator, yielding a Fredholm integral equation of the first kind for an unknown density ρ. Hence, in particular preconditioning is an issue. In our example the domain Ω is a crankshaft of a parallel twin motor (as used in old British motorcycles), cf. Figure 14.1. The surface of this crankshaft is parametrized with the aid of 142 patches. As Dirichlet data we choose the restriction of the harmonical function

$$U(x) = \frac{(a, x-b)}{\|x-b\|^3}, \quad a = (1,2,4), \quad b \in (1.5, 0, 0) \notin \Omega$$

to Γ. Then, U is the unique solution of the Dirichlet problem. We discretize the given boundary integral equation by piecewise constant wavelets with three vanishing moments. For the computation of the potential U we expect a pointwise convergence rate $|U(x) - U_J(x)| \lesssim \|\rho - \rho_J\|_{-2} \lesssim 2^{-3J}\|\rho\|_1$, $x \in \Omega$ [34].

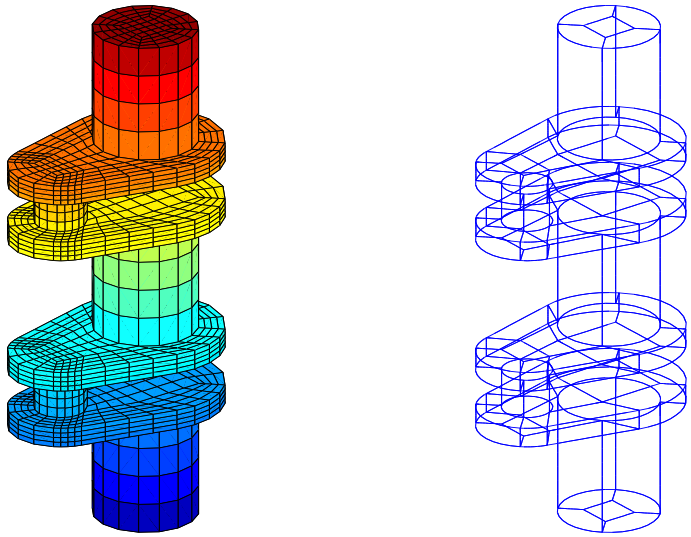

FIG. 14.1. *The surface mesh and the evaluation points* x_i *of the potential.*

In order to measure the error produced by the method, we calculate the approximate solution $U_J = A\rho_J$ in several points x_i inside the domain, plotted in Figure 14.1. The discrete potentials are denoted by

$$\mathbf{U} := [U(x_i)], \qquad \mathbf{U}_J := [(A\rho_J)(x_i)].$$

We list in Table 14.1 the results produced by the wavelet Galerkin scheme. The optimal order of convergence of the discrete potentials is cubic with respect to the l^∞-norm over all points x_i. In fact, for the solution of the compressed scheme this order of convergence can be observed from our numerical results. For $N_J \leq 9088$ we have also computed the solution of the uncompressed scheme. The corresponding absolute errors for this traditional Galerkin scheme are 1.0 if $N_J = 2272$ and

J	N_J	$\|\mathbf{U}_J - \mathbf{U}_J\|_\infty$	time	a-priori (%)	a-posteriori (%)
1	568	15.7	11	31.2	22.5
2	2272	1.0 (16)	62	11.8	8.39
3	9088	2.4e-1 (4.1)	760	4.55	2.11
4	36352	2.2e-2 (19)	5228	1.53	0.48
5	145408	4.0e-3 (3.2)	42785	0.44	0.12

TABLE 14.1
Numerical results with respect to the crankshaft.

2.5e-1 if $N_J = 9088$. This shows that the present compression does not degrade the accuracy of the Galerkin scheme.

We measure the compression via the ratio of the number of nonzero entries of the compressed matrix and N_J^2. For 145408 unknowns, only 0.44% of the entries have to be computed. After the a-posteriori compression even only 0.12% nonzero entries are used for computation of the solution ρ_J. In our wavelet Galerkin scheme we have allocated 1.3 Gigabyte storage for the solution of 145408 unknowns. From our compression rates one can observe that the number of nonzero matrix coefficients grows only linearly with the number of unknowns, as predicted. We have observed that more than 95% of the computing time is consumed by the precomputational steps, namely setting up the matrix pattern and assembling the compressed Galerkin matrix. In comparison with this our preconditioned iteration method for solution of the discrete equations is rather fast.

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