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An adaptive regularization by projection for noisy pseudodifferential equations of negative order

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Abstract

It is well known, that pseudodifferential equations of negative order considered in the Sobolev space with a small smoothness index are ill–posed. On the other hand, it is known that efficient discretization schemes with properly chosen discretization parameter allow to obtain a regularization effect for such equations. The main accomplishment of the present paper is the principle for the adaptive choice of the discretization parameter directly from noisy discrete data. We argue that the combination of this principle with wavelet–based matrix compression techniques leads to algorithms which are order–optimal in the sense of complexity.

1. Introduction

Let $L_2 = L_2(\Omega)$ be the space of square summable functions on a given bounded domain or manifold Ω which admits the definition of Sobolev spaces $H^t, t \in \mathbb{R}$, equipped with the usual norms $\|\cdot\|_t$. Thus, in particular $H^0 = L_2$ and H^{-t} is the dual of H^t relative to the inner product $\langle \cdot, \cdot \rangle$ in L_2, i.e. $(H^t)^* = H^{-t}$. When Ω is a bounded domain the definition of H^t may incorporate (homogeneous) boundary conditions.

The operator A is called an elliptic pseudodifferential operator of negative order $-r$, $r > 0$, if for any $t \in \mathbb{R}$ it is a bounded linear operator $A : H^t \to H^{t+r}$. The bilinear form $\langle Au, v \rangle$, $u, v \in H^{-\frac{r}{2}}$ is symmetric and $\langle Au, u \rangle \sim \|u\|^2_{-\frac{r}{2}}$. Here and in the following $a \lesssim b$ means that b can be bounded by some constant times a uniformly with respect to any parameters on which a and b depend. We will write $a \sim b$ if there holds $a \lesssim b$ and $b \lesssim a$.

The typical examples included in the above assumptions are given by integral operators A arising from the reformulation of elliptic boundary value problems as boundary integral equations, when the boundary is smooth, see e.g. [McL], [At], [YS], [SS], [DPS], [BPV], [PP]. In most of these examples the order is -1, i.e., $r = 1$.

By duality, an elliptic pseudodifferential operator of negative order $-r$, $r > 0$, defines an isomorphism from $H^{-\frac{r}{2}}$ onto $H^{\frac{r}{2}}$. Thus, the equation

$$Au = f \tag{1.1}$$

has a unique solution $u = A^{-1}f$ for any $f \in H^t$, $t \geq \frac{r}{2}$. At this point it is convenient to note that, usually, in practice the right hand side f reflecting boundary conditions cannot be given exactly but only in discretized and noisy form.

The noise may be caused e.g. by rounding errors preparing the problem to a discretization, measurement errors and modeling errors. As it has been indicated in [BPV], the rounding errors cause perturbations of order $10^{-10}\%$, the measurement and modeling errors may cause much larger perturbations, say of order 0.1–1%. As a result, instead of $Au = f$ we have at our disposal an equation $Au = f_\delta$ with some $f_\delta \in H^\theta$, where $\theta < \frac{r}{2}$ and δ characterizes the level of the noise in our data. Considered as an equation in H^θ for $\theta < \frac{r}{2}$, the last pseudodifferential equation is ill–posed. Small perturbations of the data may cause dramatic changes in its solution considered as an element from $H^{-\frac{r}{2}}$. On the other hand, it is known [Nat], [VH], [BPV], [DM], [PP], [MP] that effective discretization schemes with properly chosen discretization parameter allow to obtain a regularization effect for pseudodifferential equation of a negative order; no special regularization of the problem is needed. This phenomenon is sometimes called the self–regularization of the ill–posed problem through its discretization, or the regularization by projection.

The main accomplishment of present paper is the principle for a posteriori adaptive choice of the discretization parameter directly from noisy discrete data. We argue that the combination of this principle with wavelet–based matrix compression technique proposed in [Sch] leads to the algorithm having order–optimal complexity among all thresholding type adaptive methods.

2. Wavelets and Multiresolution analysis

Multiresolution is by now a well–studied notion. There are many excellent accounts about it, we refer the reader to the survey paper [D1] and the references therein. Here we focus only on those aspects which are useful for our purpose. In general, a multiresolution analysis consists of a nested family of finite dimensional subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \cdots \subset \cdots \subset L_2(\Omega), \tag{2.1}$$

such that $\dim V_j \sim 2^{nj}$ and

$$\overline{\bigcup_{j\in\mathbb{N}_0} V_j} = L_2(\Omega), \quad \mathbb{N}_0 = \{0, 1, \cdots\}.$$

Each space V_j is defined by a single scale basis $\Phi_j = \{\varphi_k^j\}$, i.e., $V_j = \text{span}\ \{\varphi_k^j : k \in \Delta_j\}$, where Δ_j denotes a suitable index set with cardinality $\#(\Delta_j) \sim 2^{nj}$. A final requirement is that these bases are uniformly stable, i.e., for any vector $c = \{c_k,\ k \in \Delta_j\}$

$$\|c\|_{l_2(\Delta_j)} \sim \left\| \sum_{k\in\Delta_j} c_k \varphi_k^j \right\|_0 \tag{2.2}$$

holds uniformly in j. Furthermore, the bases should satisfy a locality condition

$$\text{diam supp } (\varphi_k^j) \sim 2^{-j}. \tag{2.3}$$

The wavelets $\Psi_j = \{\psi_k^j : k \in \nabla_j = \Delta_{j+1}/\Delta_j\}$ are the bases of complementary spaces W_j of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \oplus W_j, \quad V_j \cap W_j = \{0\}, \quad W_j = \text{span}\{\psi_k^j : k \in \nabla_j\}.$$

It is supposed that the collections $\Phi_j \cup \Psi_j$ are also uniformly stable bases of V_{j+1}. Furthermore, we suppose that the wavelet basis

$$\Psi = \bigcup_{j=-1}^{\infty} \Psi_j,$$

where $\Psi_{-1} = \Phi_0$, is a Riesz–basis of $L_2(\Omega)$. Then, there exists a biorthogonal, or dual, Riesz–basis

$$\tilde{\Psi} = \{\tilde{\psi}_k^j : k \in \nabla_j, j = -1, 0, 1, \cdots\}$$

such that $\langle \tilde{\psi}_k^j, \psi_l^i \rangle = \delta_{k,l}\delta_{ij}$ and every $v \in L_2$ has a representation

$$v = \sum_{j=-1}^{\infty} \sum_{k\in\nabla_j} \langle v, \psi_k^j \rangle \tilde{\psi}_k^j = \sum_{j=-1}^{\infty} \sum_{k\in\nabla_j} \langle v, \tilde{\psi}_k^j \rangle \psi_k^j \tag{2.4}$$

and that

$$\|v\|_0^2 \sim \sum_{j=-1}^{\infty} \sum_{k\in\nabla_j} |\langle v, \psi_k^j \rangle|^2 \sim \sum_{j=-1}^{\infty} \sum_{k\in\nabla_j} |\langle v, \tilde{\psi}_k^j \rangle|^2.$$

We refer to [D1] for further details.

If one is going to use the spaces V_j and

$$\tilde{V}_j = \text{span}\{\tilde{\psi}_k^i : k \in \nabla_i, \quad i = -1, 0, 1, \cdots, j-1\}$$

as trial spaces for the approximate solution of (1.1) then additional properties are required. Usually it is assumed that the following Jackson and Bernstein type estimates hold for $t \leq \tau \leq \gamma_1$, $t \leq s \leq \gamma_0$ and uniformly in j

$$\inf_{v \in V_j} \|u - v\|_t \leq b_0 2^{-j(\tau - t)} \|u\|_\tau, \quad u \in H^\tau, \tag{2.5}$$

and

$$\|v\|_s \leq b_1 2^{j(s-t)} \|v\|_t, \quad v \in V_j, \tag{2.6}$$

where $\gamma_0, \gamma_1 > 0$ are fixed constants given by

$$\begin{aligned} \gamma_0 &= \sup\ \{s \in \mathbb{R} : V_j \subset H^s\}, \\ \gamma_1 &= \sup\ \{s \in \mathbb{R} : \inf_{v \in V_j} \|u - v\|_0 \leq b_0 2^{-js} \|u\|_s\}. \end{aligned}$$

Usually, γ_1 is the maximal degree of polynomials which are locally contained in V_j and is referred to the order of exactness of the multiresolution analysis $\{V_j\}$. The parameter γ_0 denotes the regularity or smoothness of the functions in the spaces V_j. We will assume that $\gamma_0 \leq \gamma_1$, which is the case in all known examples of wavelet functions. Analogous estimates are valid for the dual multiresolution analysis $\{\tilde{V}_j\}$ with constants $\tilde{\gamma}_0, \tilde{\gamma}_1$.

The assumptions that (2.5), (2.6) hold with some constants γ_0, $\tilde{\gamma}_0$ relative to $\{V_j\}, \{\tilde{V}_j\}$. They provide a convenient device for switching between the norms $\|\cdot\|_t$ and corresponding sums of weighted wavelet coefficients from the representation (2.4). Namely, the following celebrated norm equivalence

$$\|v\|_t^2 \sim \sum_{j=-1}^{\infty} 2^{2jt} \sum_{k \in \nabla_j} |\langle v, \phi_k^j \rangle|^2 \tag{2.7}$$

holds for $\phi_k^j = \tilde{\psi}_k^j, t \in (-\tilde{\gamma}_0, \gamma_0)$ and $\phi_k^j = \psi_k^j, t \in (-\gamma_0, \tilde{\gamma}_0)$, see, e.g., [D] and [Sch] for the details.

In conclusion of the section we note that at first glance it would be more convenient to deal with a single orthonormal system of wavelets, say $\{\psi_k^j\}$, but full orthonormality in this sense, in conjunction with compact support (2.3), is hard to realize in a non–tensor product setting. Examples of compactly supported biorthogonal wavelets meeting above assumptions, however, are easier to obtain, see e.g. [D1].

3. Stability and convergence of wavelet–based Galerkin schemes

Popular methods for solving (1.1) numerically are projection schemes. For the case when the right hand side f is given without any noise these schemes have been investigated in [DPS], [DPS1], [Sch] in the context of multiresolution analysis. Here we confine ourselves to the case of the Galerkin scheme, i.e., we seek for a solution $u^j \in V_j$ of the variational problem

$$\langle Au^j, v_j\rangle \;=\; \langle f, v_j\rangle$$

for all $v_j \in V_j$. It is convenient to reformulate the Galerkin method as a projection method

$$Q_j^* A Q_j u = Q_j^* f, \tag{3.1}$$

where Q_j^* is the L_2 adjoint of

$$Q_j = \sum_{i=-1}^{j-1} \sum_{k\in\nabla_i} \langle \tilde{\psi}_k^i, \cdot\rangle \psi_k^i.$$

Under the assumptions $\gamma_0, \tilde{\gamma}_0 > 0$, Q_j and Q_j^* are uniformly bounded projectors from L_2 onto V_j and $\tilde{V}_j$ respectively, which are dual to each other.

It is clear that for finding such u^j one should solve a system of no more than $N \sim 2^{nj}$ linear algebraic equations. The scheme (3.1) is called (t,r)–stable if

$$\|Q_j^* A Q_j v\|_{t+r} \geq b_2 \|v\|_t \tag{3.2}$$

for all $v \in V_j$ uniformly in j. Clearly, (3.2) means that the finite dimensional operators $A_j = Q_j^* A Q_j$ have left inverses $A_j^{-1} : H^{t+r} \to H^t$ which are bounded uniformly in j. An establishment of the (t,r)–stability for wavelet–based projection schemes and elliptic pseudodifferential equations have been given in [DPS], [DPS1], [Sch].

Combining stability (3.2) with the approximation properties (2.5), (2.6) yields the following canonical error estimates for the unique solution u^j of (4.1).

Theorem 3.1 [DPS], [Sch]. *Suppose that for $t \in [-\gamma_1 - r, \gamma_0]$, $0 < \frac{r}{2} < \tilde{\gamma}_0$, the scheme (4.1) is (t,r)–stable. Furthermore, assume that for some $\tau \geq t$ such that $\tau \leq \gamma_1$ the solution $u = A^{-1} f$ belongs to H^τ. Then*

$$\|u - u^j\|_t \leq c 2^{-j(\tau - t)} \|u\|_\tau,$$

where c depends only on the constants from (2.5), (2.6) and (3.2).

4. Self–regularization for known smoothness of the solution

We now turn to the case of the noisy equation

$$Au = f_\delta, \tag{4.1}$$

where f_δ can be any element from $H^\theta, \theta < \frac{r}{2}$, such that $\|f - f_\delta\|_\theta \leq \delta$. Projection scheme (3.1) can be directly applied to (4.1) if Q_j^* is defined on H^θ. In such a case it is natural to assume that the norms of $Q_j^* : H^\theta \to H^\theta$ are bounded uniformly in j, i.e.,

$$\|Q_j^*\|_{H^\theta \to H^\theta} \leq b_3. \tag{4.2}$$

Then from (t, r)–stability (3.2) it follows that there is always a unique solution u_δ^j of the equation

$$Q_j^* A Q_j u = Q_j^* f_\delta. \tag{4.3}$$

Moreover, keeping in mind that $u^j, u_\delta^j \in V_j$, from (2.6) and (3.2) we have

$$\begin{aligned} \|u^j - u_\delta^j\|_t &\leq b_2^{-1} \left\| Q_j^* A Q_j (u^j - u_\delta^j) \right\|_{t+r} = b_2^{-1} \|Q_j^*(f - f_\delta)\|_{t+r} \\ &\leq b_1 b_2^{-1} 2^{j(t+r-\theta)} \left\| Q_j^*(f - f_\delta) \right\|_\theta \leq b_1 b_2^{-1} b_3 \delta 2^{j(t+r-\theta)}. \end{aligned}$$

Thus, under the conditions of Theorem 3.1 the following error bound holds

$$\begin{aligned} \|u - u_\delta^j\|_t &\leq \|u - u^j\|_t + \|u^j - u_\delta^j\|_t \\ &= c_0 2^{-j(\tau - t)} + c_1 \delta 2^{j(t+r-\theta)}, \end{aligned} \tag{4.4}$$

where c_0 depends on the norm of the unknown solution $\|u\|_\tau$ and b_0, b_1, b_2, while $c_1 = b_1 b_2^{-1} b_3$. It is easy to see that the best result in (4.4) will be obtained for $j = j_{opt}$ such that $2^{-j(\tau-t)}$ and $\delta 2^{j(t+r-\theta)}$ are of the same size, i.e., for $\theta < \tau + r$

$$2^{j_{opt}} \sim \delta^{-\frac{1}{\tau + r - \theta}} \tag{4.5}$$

resulting to

$$\|u - u_\delta^{j_{opt}}\|_t \leq c \delta^{\frac{\tau - t}{\tau + r - \theta}}. \tag{4.6}$$

The estimates (4.5), (4.6) characterize the self–regularization of problem (4.1) through its discretizations (4.3). These estimates are order optimal for the case when the Sobolev smoothness of the solution $u = A^{-1}f$ is known to be τ. To show this we should compare the scheme (4.3) with all methods for solving (4.1) which use the same amount, say N, of discrete information about f_δ. Each such a method must be based on some collection of elements $\Lambda_N = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in H^{-\theta}$ describing the way we obtain noisy discrete information $\Lambda_N, (f_\delta) = (\langle\lambda_1, f_\delta\rangle, \langle\lambda_2, f_\delta\rangle, \ldots, \langle\lambda_N, f_\delta\rangle) \in \mathbb{R}^N$ about f_δ. The resulting approximation u_δ based on such information may be given by any mapping $Q : \mathbb{R}^N \to H^t$, hence $u_\delta = Q \circ \Lambda_N(f_\delta)$. For a given method (Q, Λ_N) its uniform error over the class of solutions $u \in H^\tau$ with the same norm bound, say $\|u\|_\tau \le M$, is determined as

$$e_\delta(A, Q, \Lambda_N) = \sup_{\substack{u:\\ \|u\|_\tau \le M}} \sup_{\substack{f_\delta:\\ \|Au - f_\delta\|_\theta \le \delta}} \|u - Q \circ \Lambda_N(f_\delta)\|_t.$$

Thus, we should measure the quality of any method against the lower bound

$$r_{N,\delta}(A; \theta, t, \tau) = \inf_{\Lambda_N : H^\theta \to \mathbb{R}^N} \inf_{Q : \mathbb{R}^N \to H^t} e_\delta(A, Q, \Lambda_N).$$

For a fixed noise level $\delta > 0$ the sequence $\{r_{N,\delta}\}$ will decrease as $N \to \infty$, but there will be a positive limit $r_\delta(A; \theta, t, \tau) = \lim_{N\to\infty} r_{N,\delta}(A; \theta, t, \tau)$ that cannot be beaten by any approximate method. In particular, from (4.6) it follows that under the conditions of Theorem 3.1

$$r_\delta(A; \theta, t, \tau) \le c\delta^{\frac{\tau - t}{\tau + r - \theta}}. \tag{4.7}$$

As in [Nat1] we have

$$r_\delta(A; \theta, t, \tau) \ge \sup\left\{\|u\|_t : \|Au\|_\theta \le \delta, \|u\|_\tau \le M\right\}.$$

To estimate the last quantity we take into account that the finite section of the Sobolev scale $\{H^t\}, |t| \le \tau + r$, can be considered as a part of some Hilbert scale with an equivalence of corresponding norms. Moreover, for our pseudodifferential operator A of order $-r$ one has $\|A\|_{H^{\theta - r} \to H^\theta} \le c_\theta$. Then for $\theta - r \le t \le \tau$ using strict interpolation property of the Hilbert scales we can continue

$$\begin{aligned}
\sup&\left\{\|u\|_t : \|Au\|_\theta \le \delta,\ \|u\|_\tau \le M\right\} \\
&\ge \sup\left\{\|u\|_t : \|u\|_{\theta - r} \le c_\theta^{-1}\delta,\ \|u\|_\tau \le M\right\} \\
&\sim (c_\theta^{-1}\delta)^{\frac{\tau - t}{\tau + r - \theta}} M^{\frac{t + r - \theta}{\tau + r - \theta}} \\
&\sim \delta^{\frac{\tau - t}{\tau + r - \theta}}.
\end{aligned}$$

Keeping in mind (4.7) one can see that

$$r_\delta(A;\theta,t,\tau) \sim \delta^{\frac{\tau-t}{\tau+r-\theta}}. \tag{4.8}$$

From (4.6) and (4.8) it becomes evident that for a given noise level $\delta > 0$ the projection scheme (4.3) with discretization parameter $j = j_{opt}$ provides the regularization of the ill–posed problem (4.1) with the best possible order of accuracy.

Let us discuss now the relation (4.5). This relation allows to estimate the number $\#(Q_j^* f_\delta)$ of noisy wavelet coefficients of f_δ used to construct a regularized approximate solution u_δ^j. Keeping in mind that $\dim V_j = \dim \tilde{V}_j \sim 2^{nj}$, we have

$$\#(Q_j^* f_\delta) \sim \delta^{-\frac{n}{\tau+r-\theta}}. \tag{4.9}$$

On the other hand, the minimal amount of discrete information which allows to reach the best possible accuracy up to some constant c is measured by the quantity

$$N(A;\theta,t,\tau,\delta) = \inf\left\{N : r_{N,\delta}(A;\theta,t,\tau) \le c r_\delta(A;\theta,t,\tau)\right\}.$$

From [MP] it follows that in our case

$$r_{N,\delta}(A;\theta,t,\tau) \ge cN^{-\frac{\tau-t}{n}}, \tag{4.10}$$

where the last quantity is just the order of the N–th Gelfand number of the canonical embedding $J_{t,\tau} : H^\tau \to H^t$. Then as in [MP] from (4.8) we can obtain

$$N(A;\theta,t,\tau,\delta) \ge \inf\left\{N : N^{-\frac{(\tau-t)}{n}} \le c\delta^{\frac{\tau-t}{\tau+r-\theta}}\right\} \sim \delta^{-\frac{n}{\tau+r-\theta}}.$$

Combining this estimate with (4.9) yields

$$N(A;\theta,t,\tau,\delta) \sim \#(Q_{j_{opt}}^* f_\delta) \sim \delta^{-\frac{n}{\tau+r-\theta}}. \tag{4.11}$$

In summation we arrive at the following statement.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be fulfilled. Then projection scheme (4.3) with discretization parameter chosen as in (4.5) has the self–regularization property and allows to reach the best possible order of accuracy (4.8) with order–optimal amount of noisy discrete information (4.11).*

5. Adaptation to unknown smoothness

The order–optimal choice of the discretization parameter (4.5) requires a priori information on the Sobolev smoothness index τ of the unknown solution. But usually in practice one can indicate only some interval $(\tau_0, \tau_1]$ containing this index. For example, within the framework of Theorem 3.1 such an interval can be chosen as $(t, \gamma - r]$. Thus, the problem naturally arises how to adapt the discretization parameter j to unknown smoothness in such a way that the optimal order of accuracy (4.8) would be reached automatically. To the best of our knowledge this problem was studied only in [VH] and [K] where the residual principle was discussed. In accordance with this principle the discretization parameter $j = j(\delta)$ is chosen as the minimal one for which $\|Au_\delta^j - f_\delta\|_\theta \le c\delta$, where c is some constant. It is clear that such a choice is possible under accessing infinitely many data. Moreover, in [VH] only convergence $u_\delta^{j(\delta)} \to u = A^{-1}f$, as $\delta \to 0$, has been proved, and in a recent paper [K] it has been shown that using a residual principle one can also reach the accuracy of order $\delta^{\frac{1}{2}}$. This order is still far from (4.8) with $\tau = 1$, $t = -\frac{1}{2}$, $\theta = 0$, for example. One of the goals of the present paper is to develop a new principle for the adaptive choice of j based on the finite amount of noisy discrete data that is minimal in a certain sense. It will solve the problem mentioned in the beginning of the section.

The idea of this principle has its origin in the paper [L], devoted to statistical estimation from direct white noise observations that corresponds to (4.1) with identity operator A, but with random noisy data. In the context of ill–posed problems of the form (4.1) with compact operators A acting along some Hilbert scale, but still with random noise, this idea has been realized in [GP] for adaptive estimating the value of a linear functional on the solution of (4.1). If, as it is usual for statisticians, we will treat the terms in the right hand side of (4.4) as bias and variance, respectively, then the idea is to choose the minimal j for which the bias is still dominated by the variance.

If τ_0 is the minimal expected Sobolev smoothness of the solution $u = A^{-1}f$ then in view of (4.5) it is natural to choose the discretization parameter j for the scheme (4.3) from the finite set

$$\mathbb{N}(\delta, \tau_0) = \left\{ j : 2^j \le b_4 \delta^{-\frac{1}{\tau_0 + r - \theta}} \right\},$$

where b_4 is some design parameter and we assume that δ is small enough such that $\mathbb{N}(\delta, \tau_0)$ is not empty. Let us consider one more finite set

$$\Pi_\delta = \left\{ j \in \mathbb{N}(\delta, \tau_0) : \|u_\delta^j - u_\delta^i\|_t \leq 4c_1 \delta 2^{i(t+r-\theta)} \ \forall i \geq j, i \in \mathbb{N}(\delta, \tau_0) \right\},$$

where c_1 is the constant from (4.4). We would like to stress that the exact Sobolev smoothness τ of the unknown solution $u = A^{-1} f$ and $\|u\|_\tau$ are not involved in the construction of Π_δ. As to the constant c_1 it can be calculated as $c_1 = b_1 b_2^{-1} b_3$ or can be estimated effectively. Now our adaptive choice of the discretization parameter j for the scheme (4.3) is $j_* = \min\{j : j \in \Pi_\delta\}$.

Theorem 5.1 *Assume that the conditions of Theorem 3.1 hold with $\tau > \tau_0 \geq t \geq \theta - r$. Then for $u = A^{-1} f \in H^\tau$ there holds*

$$\|u - u_\delta^{j_*}\|_t \leq c \delta^{\frac{\tau - t}{\tau + r - \theta}},$$

where the constant c does not depend on δ.

Proof. Let us consider

$$\hat{j} = \min \left\{ j \in \mathbb{N}(\delta, \tau_0) : c_0 2^{-j(\tau - t)} < c_1 \delta 2^{j(t+r-\theta)} \right\},$$

where c_0 is the constant from (4.4). It follows immediately from definition that

$$\begin{aligned} & c_0 2^{-(\hat{j}-1)(\tau - t)} \geq c_1 \delta 2^{(\hat{j}-1)(t+r-\theta)} \\ & \qquad \Longrightarrow 2^{\hat{j}} \leq \left[2\delta^{-1} \left(\frac{c_0}{c_1} \right) \right]^{\frac{1}{\tau + r - \theta}} = c_2 \delta^{-\frac{1}{\tau + r - \theta}}. \end{aligned} \tag{5.1}$$

Let us show that $\hat{j} \in \Pi_\delta$. Indeed, for any $i \in \mathbb{N}(\delta, \tau_0)$ such that $i > \hat{j}$

$$c_0 2^{-i(\tau - t)} < c_0 2^{-\hat{j}(\tau - t)} < c_1 \delta 2^{\hat{j}(t+r-\theta)}.$$

Then from (4.4) is follows that

$$\begin{aligned} \|u_\delta^{\hat{j}} - u_\delta^i\|_t &\leq \|u - u_\delta^{\hat{j}}\|_t + \|u - u_\delta^i\|_t \\ &\leq c_0 2^{-\hat{j}(\tau - t)} + c_1 \delta 2^{\hat{j}(t+r-\theta)} + c_0 2^{-i(\tau - t)} + c_1 \delta 2^{i(t+r-\theta)} \\ &\leq 3c_1 \delta 2^{\hat{j}(t+r-\theta)} + c_1 \delta 2^{i(t+r-\theta)} \\ &\leq 4c_1 \delta 2^{i(t+r-\theta)}. \end{aligned}$$

It means that $\hat{j}$ belongs to Π_δ by definition. Then $j_* \leq \hat{j}$, and taking into account (5.1) we obtain

$$\begin{aligned}
\|u - u_\delta^{j_*}\|_t &\leq \|u - u_\delta^{\hat{j}}\|_t + \|u_\delta^{\hat{j}} - u_\delta^{j_*}\|_t \\
&\leq c_0 2^{-\hat{j}(\tau - t)} + c_1 \delta 2^{\hat{j}(t+r-\theta)} + 4c_1 \delta 2^{\hat{j}(t+r-\theta)} \\
&\leq 6c_1 \delta 2^{\hat{j}(t+r-\theta)} \\
&\leq 6c_1 c_2^{(t+r-\theta)} \delta^{\frac{\tau - t}{\tau + r - \theta}} \\
&= c\delta^{\frac{\tau - t}{\tau + r - \theta}},
\end{aligned}$$

as claimed.

6. Adaptive projection scheme with matrix compression

The adaptive principle for choosing the discretization parameter proposed in the previous section can be applied not only for wavelet–based projection schemes. The advantage of wavelets consists mainly in two issues. First, the norm equivalence (2.7) makes the norms which are required to achieve the solution given by Theorem 5.1 directly computable, even if $t \neq 0$. A second advantage is the possibility to use special matrix compression technique developed in [DPS], [DPS1], [Sch] which improves the efficiency drastically. The basic strategy for compressing is to decompose first

$$A_j = Q_j^* A Q_j = \sum_{k,l=-1}^{j-1} (Q_{k+1}^* - Q_k^*) A (Q_{l+1} - Q_l),$$

$Q_{-1}^* = Q_{-1} = 0$, into different components $A_{k,l} = (Q_{k+1}^* - Q_k^*) A (Q_{l+1} - Q_l)$ each of which is then to be compressed appropriately. The key idea is that the matrices corresponding to the operators $A_{k,l}$ exhibit fast decay away from the diagonals. Namely, in the case that $2^{\min(i,k)} \lesssim \operatorname{dist}(\operatorname{supp}(\psi_\mu^i), \operatorname{supp}(\psi_\nu^k))$, for a large class of elliptic operators A one has that the coefficients in the Galerkin matrix can be estimated by

$$\left|\langle A\psi_\mu^i, \psi_\nu^k \rangle\right| \lesssim \frac{2^{-|i+k|(\frac{n}{2}+\tilde{\gamma}_1)}}{[\operatorname{dist}(\operatorname{supp}(\psi_\mu^i), \operatorname{supp}(\psi_\nu^k))]^{n-r+\tilde{\gamma}_1}}.$$

In general one has a slightly weaker estimate

$$\left|\langle A\psi^i_\mu, \psi^k_\nu\rangle\right| \lesssim 2^{-|i-k|\sigma}\frac{2^{-|i+k|\frac{r}{2}}}{[1+2^{\min(i,k)}\mathrm{dist}(\mathrm{supp}(\psi^i_\mu),\mathrm{supp}(\psi^k_\nu))]^{n-r+\tilde{\gamma}_1}},$$

where $\frac{n}{2} < \sigma < \frac{n+r}{2} + \gamma_0$ and $r > 0$ is the negative order of the operator as before. Estimates of such a type are known to hold for a wide range of cases (see e.g. [D1],[Sch]). In particular, pseudodifferential operators of negative order fall into this category. This suggests discarding the vast majority of entries of the stiffness matrix, i.e., to approximate the operators A_j by the operators $A_j^{\epsilon_j} : V_j \to \tilde{V}_j$ corresponding to the matrices with only $\mathcal{O}(2^{nj})$ nonvanishing entries, where ϵ_j are some small parameters governing the bandwidth of non–zero entries around the main diagonals of the blocks corresponding to $A_{k,l}, k, l = -1, 0, \cdots, j-1$. However, the convergence rate of the Galerkin scheme given by Theorem 3.1 will not be violated. This can be done on the basis of the second estimate if $\tau < \gamma_0$. For $\gamma_0 \le \tau \le \gamma_1$ the first estimate yields a compression strategy with $\mathcal{O}(j^2\, 2^{nj})$ nonvanishing matrix entries. The latter logarithmic terms can also be removed, for $\gamma_0 \le \tau < \gamma_1$ using a second compression, see [Sch]. We refer to [DPS], [DPS1], [Sch] for the details. As a result, one obtains the sequence of operators $A_j^{\epsilon_j}, j \in \mathbb{N}(\delta, \tau_0)$, with the stiffness matrices being the parts of the matrix corresponding to $A_{j_0}^{\epsilon_{j_0}}, j_0 = \max\{j \in \mathbb{N}(\delta, \tau_0)\},\ 2^{j_0} \sim \delta^{-\frac{1}{\tau_0+r-\theta}}$.

In e.g. [Sch] it has been shown that under the conditions of Theorem 3.1 the compressed scheme $A_j^{\epsilon_j} u = Q_j^* f$ is (t, r)–stable and has a unique solution $u^j_{\epsilon_j}$ satisfying

$$\|u - u^j_{\epsilon_j}\|_t \le c\|u\|_\tau 2^{-j(\tau-t)}$$

uniformly in j. Then using an argument like that in the proof of Theorem 5.1 we get the estimate

$$\|u - u^{j_*}_{\epsilon_{j_*},\delta}\|_t \le c\delta^{\frac{\tau-t}{\tau+r-\theta}},$$

where $u^{j_*}_{\epsilon_{j_*},\delta}$ is the solution of the equation

$$A_j^{\epsilon_j} u = Q_j^* f_\delta \tag{6.1}$$

for $j = j_*$ and j_* is chosen in accordance with principle presented in Section 5, where one should put $u^j_{\epsilon_j,\delta}$ instead of u^j_δ and slightly change the constant c_1 because of changing the constant b_2 in the stability condition (3.2) for compressed scheme.

Note that to construct $u^{j_*}_{\epsilon_{j_*},\delta}$ we should realize a finite iteration procedure, within the framework of which we begin with $j = j_0$ and go from $j+1$ to j until $j = j_*$. On each step we solve the equation (6.1) and check the condition

$$\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_{j},\delta}\|_t \le 4c_1\delta 2^{(j+1)(t+r-\theta)}. \tag{6.2}$$

Keeping in mind that both approximate solutions in (6.2) belong to V_{j+1}, one can switch the norm $\|\cdot\|_t$ of their difference to the corresponding finite sum of $\mathcal{O}(2^{nj})$ weighted wavelet coefficients with respect to the elements $\tilde{\psi}^{\lambda}_{k}$, i.e.

$$\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_{j},\delta}\|_t \sim \left(\sum_{\lambda=-1}^{j} 2^{2\lambda t} \sum_{k\in\nabla_\lambda} |\langle u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_{j},\delta}, \tilde{\psi}^{\lambda}_{k}\rangle|^2\right)^{\frac{1}{2}},$$

because of the norm equivalence (2.7).

Note that the numbers $\langle u^{j}_{\epsilon_{j},\delta}, \tilde{\psi}^{\lambda}_{k}\rangle$, $j = j_0, j_0-1, ..., j_*$, are exactly the wavelet coefficients of the solutions of the system of linear algebraic equations corresponding to (6.1). Using the fact that the stiffness matrix of $A^{\epsilon_j}_{j}$ has only $\mathcal{O}(2^{nj})$ nonzero entries, these solutions can be found for $\mathcal{O}(2^{nj})$ arithmetic operations. Here we refer to [Sch] for the details. Thus, j–th iteration step can be done for $\mathcal{O}(2^{nj})$ operations. Then the computational cost of the whole iteration procedure has the order of

$$\sum_{j=j_*}^{j_0} 2^{nj} \sim 2^{nj_0} \sim \delta^{-\frac{n}{\tau_0+r-\theta}} \tag{6.3}$$

Using the estimates from Section 4. we argue now that in the sense of asymptotical order the cost (6.3) cannot be reduced for any so–called thresholding type adaptive algorithms.

It should be noted that the Information–Based Complexity theory [TWW] is dominated now by two models of adaptive algorithms. One of them is connected with the notion of "oracle" and suppose that in the process of adaptation one has the possibility to put the questions to oracle concerning the values of some information functionals that can be chosen depending on previous answers [Nov]. Typical example of such an adaptive algorithm is well–known bisection method for solving nonlinear scalar equations. In our opinion such oracle model does not fit well to the case of noisy data, because the oracle providing noisy information looks slightly artificially.

Another model covers the adaptive algorithms like thresholding, see e.g. [DeV], or adaptive wavelet methods for elliptic equations, proposed in [CDD]. Within the framework of this model it is assumed that there is some surplus of discrete information and it is possible to select the most important part of it using some criterion applied to each unit of information (value of informational functional) from above mentioned surplus. The application of the selection criterion is connected with a fixed number of arithmetic operations, like subtraction and comparison with zero, determining the computational cost of the adaptive procedure.

The adaptive algorithms covered by this second model are called here as thresholding type adaptive algorithms. In the class of such algorithms the method determined by (6.1), (6.2) is order–optimal in the sense of complexity measured by the number of executed operations. Indeed, if we know only that the Sobolev smoothness of the solution $u = A^{-1}f$ belongs to the interval $(\tau_0, \tau_1]$ then the minimal amount of above mentioned surplus of discrete information required to reach the best possible order of accuracy (4.8) is

$$\begin{aligned} & N\,(A; \theta, t, (\tau_0, \tau_1], \delta) \\ & \qquad = \inf\left\{N : \forall \tau \in (\tau_0, \tau_1]\ \ r_{N,\delta}(A; \theta, t, \tau) \le c r_\delta(A; \theta, t, \tau)\right\}. \end{aligned}$$

On the other hand, form (4.10) it follows that

$$\begin{aligned} & N\,(A; \theta, t, (\tau_0, \tau_1], \delta) \\ & \qquad \ge \inf\left\{N : \forall \tau \in (\tau_0, \tau_1]\ \ N^{-\frac{(\tau-t)}{n}} \le c\delta^{\frac{\tau-t}{\tau+r-\theta}}\right\} \\ & \qquad = \inf\left\{N : \forall \tau \in (\tau_0, \tau_1]\ \ N \ge c^{-\frac{n}{\tau-t}}\delta^{-\frac{n}{\tau+r-\theta}}\right\} \\ & \qquad \sim \delta^{-\frac{n}{\tau_0+r-\theta}} \end{aligned}$$

It means that within the framework of any thresholding type adaptive algorithm dealing with such an amount of discrete information at least $N \sim \delta^{-\frac{n}{\tau_0+r-\theta}}$ operations should be executed, because each unit of this information should be involved in the adaptive procedure at least once. Then the comparison with (6.3) gives us.

Theorem 6.1. *Assume that the conditions of Theorem 3.1 and Theorem 5.1 hold. Then for unknown Sobolev smoothness index $\tau \in (\tau_0, \tau_1]$ of the solution $u = A^{-1}f$ the algorithm (6.1), (6.2) automatically provides the best possible order of accuracy (4.8) and has the order–optimal complexity among all thresholding type adaptive algorithms giving the accuracy of the same order.*

7. Numerical Results

To give an numerical example in order to demonstrate the performance of our method, we consider the operator A as the single layer operator defined on the analytical boundary $\Gamma = \partial\Omega$ of a domain $\Omega \in \mathbb{R}^2$

$$(Au)(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|u(y)dy_\Gamma, \quad x \in \Gamma.$$

This operator defines an operator of the order $-r = -1$, i.e., $A : H^t(\Gamma) \to H^{t+1}(\Gamma)$ for all $t \in \mathbb{R}$. We consider two cases, $t = -\frac{1}{2}$, this is the energy norm for the corresponding operator and useful for certain applications, and, $t = 0$ which denotes the L^2-norm. The right hand side f is chosen analytically. We generate a random vector r such that for $f_\delta := f + r$ holds $\|f - f_\delta\|_\theta = \delta$ with $\delta = 10^{-3}$ and $\theta = -\frac{1}{2}$. The integral equation of first kind $Au = f$ is discretized by piecewise constant wavelets. In this case the highest smoothness is $\tau_1 = \tau_{max} = \gamma_1 = 1$. To demonstrate our algorithm we do not use the precise information about the smoothness of f and get an upper limit $\tau_1 = 1$. We assume a minimal smoothness $\tau_0 = -\frac{1}{2}$. To be on safe ground and for demonstration we choose $j_0 = 15$. The present computations made heavy use of the wavelet matrix compression as an efficient method for solving integral equations. Otherwise, we have not been able to solve such large linear equation systems with dense matrices on a single workstation. We compute the solution of the original system $A_j^{\epsilon_j} u_{\epsilon_j}^j = Q_j^* f$, the solution with respect to the noisy data $A_j^{\epsilon_j} u_{\epsilon_j,\delta}^j = Q_j^* f_\delta$, and their difference $v_{\epsilon_j}^j := u_{\epsilon_j}^j - u_{\epsilon_j,\delta}^j$. Since we do not know the exact solution u, we take $u_{\epsilon_{16}}^{16}$ as a reference solution instead of u.

First, we consider the case $t = -1/2$. In Table 1 we list the norms $\|u_{\epsilon_{j+1},\delta}^{j+1} - u_{\epsilon_j,\delta}^j\|_{-1/2}$ which we need for our algorithm together with the (approximated) error of the solution $\|u_{\epsilon_{16}}^{16} - u_{\epsilon_j,\delta}^j\|_{-1/2}$ for comparison reasons. These resluts are depicted in Figure 1. The computation of these norms is performed by exploiting the norm equivalences of the wavelet basis.

The case $t = 0$ is listed in Table 2. In Figure 2 we depict the quantity $\|u_{\epsilon_{j+1},\delta}^{j+1} - u_{\epsilon_j,\delta}^j\|_0$ which enters our algorithm and compare it with the error of the numerical solution $\|u_{\epsilon_{16},\delta}^{16} - u_{\epsilon_j,\delta}^j\|_0$.

From this comparisons we see that the present algorithm detects the optimal solutions quite well. We observe also a small shift of the optimal level j_*

j	n	$\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_j,\delta}\|_{-1/2}$	$\|u^{16}_{\epsilon_{16}} - u^{j}_{\epsilon_j,\delta}\|_{-1/2}$
6	64	27.746	28.547
7	128	7.4226	7.9766
8	256	2.7359	2.9264
9	512	9.7128e-01	1.0352
10	1024	3.4666e-01	3.6792e-01
11	2048	1.7615e-01	1.4989e-01
12	4096	2.5878e-01	1.5267e-01
13	8192	5.0581e-01	2.9549e-01
14	16384	1.0080	5.8781e-01
15	32768	2.0359	1.1718

Table 1: Numerical results with respect to $t = -1/2$.

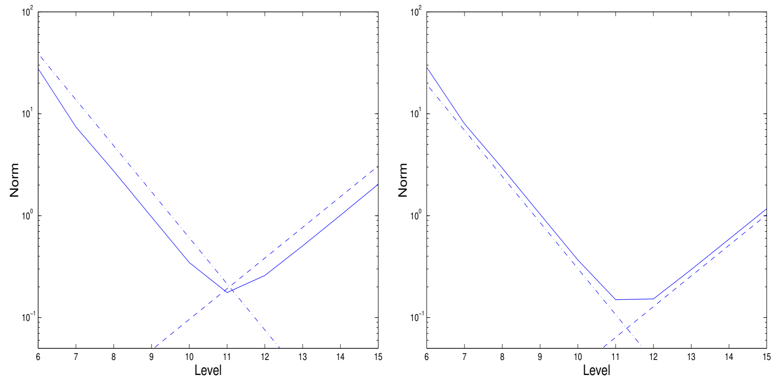

Figure 1: $\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_j,\delta}\|_{-1/2}$ (left) and $\|u^{16}_{\epsilon_{16}} - u^{j}_{\epsilon_j,\delta}\|_{-1/2}$ (right).

according to the theory. Since j_* is an integer number this behaviour does not affect the actual choice in the present example. The numerical example exhibits the correlation between the computational quantity $\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_j,\delta}\|_t$ and the true error $\|u - u^{j}_{\epsilon_j,\delta}\|_t$ which is in fact less than the worst case estimation in the proof of Theorem 5.1.

Let us remark that the left plots in Figure 1 and Figure 2 have also been used in practice as a rule of thumb taking the optimal level at the minimum of the curve. Due to the shape of the curve this rule of thumb is sometimes called the L-method. Whereas in the present paper we present a concrete algorithm together with the rigorous proof of order optimal accuracy.

j	n	$\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_{j},\delta}\|_0$	$\|u^{16}_{\epsilon_{16}} - u^{j}_{\epsilon_{j},\delta}\|_0$
6	64	156.58	169.52
7	128	59.339	69.264
8	256	30.954	35.758
9	512	15.541	17.888
10	1024	7.7617	8.9602
11	2048	5.5642	4.7181
12	4096	11.547	4.9334
13	8192	32.209	12.310
14	16384	92.292	34.535
15	32768	259.47	98.721

Table 2: Numerical results with respect to $t = 0$.

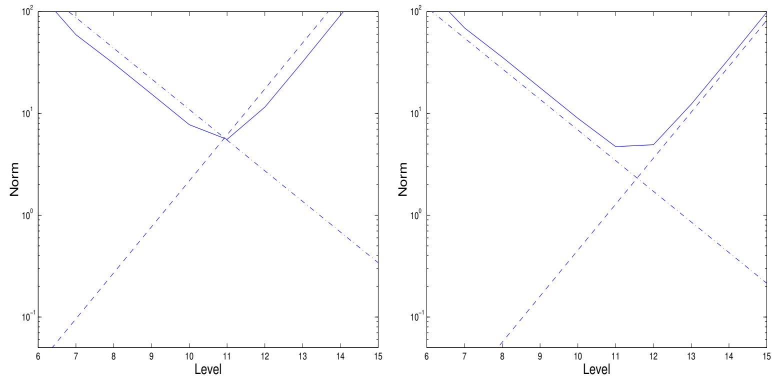

Figure 2: $\|u^{j+1}_{\epsilon_{j+1},\delta} - u^{j}_{\epsilon_{j},\delta}\|_0$ (left) and $\|u^{16}_{\epsilon_{16}} - u^{j}_{\epsilon_{j},\delta}\|_0$ (right).

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