# Technische Universität Chemnitz
Sonderforschungsbereich 393
Numerische Simulation auf massiv parallelen Rechnern

Gerd Kunert

A posteriori H^1 error estimation for a singularly perturbed reaction diffusion problem on anisotropic meshes

Preprint SFB393/01-21

Abstract

The paper deals with a singularly perturbed reaction diffusion model problem. The focus is on reliable *a posteriori* error estimators for the H^1 seminorm that can be applied to anisotropic finite element meshes. A residual error estimator and a local problem error estimator are proposed and rigorously analysed. They are locally equivalent, and both bound the error reliably. Furthermore three modifications of these estimators are introduced and discussed. Numerical experiments for all estimators complement and confirm the theoretical results.

Keywords: error estimator, H^1 seminorm, anisotropic mesh, reaction diffusion equation, singularly perturbed problem

AMS (MOS): 65N15, 65N30, 35B25

Preprint-Reihe des Chemnitzer SFB 393

SFB393/01-21 **August 24, 2001**

Contents

Gerd Kunert
TU Chemnitz, Fakultät für Mathematik, D-09107 Chemnitz, Germany
gkunert@mathematik.tu-chemnitz.de
http://www.tu-chemnitz.de/∼gku

1 Introduction

Singularly perturbed problems play an important role in the numerical simulation of physical phenomena. Here we consider a singularly perturbed reaction diffusion model problem which generically gives rise to solutions with boundary layers. When discretizing problems with such solutions by the finite element method, it can be advantageous to employ *anisotropic* finite elements. By this we understand elements whose aspect ratio can be arbitrarily large, i.e. the ratio of the diameters of the circumscribed and inscribed spheres can be unbounded.

The focus of this paper is on *a posteriori* error estimators that form an indispensable ingredient of any self–adaptive, reliable solution algorithm. By now the theory of error estimation for *isotropic* finite element meshes is well understood; we refer to the overview textbooks by Verfürth [Ver96] and Ainsworth/Oden [AO00] and the citations therein. For *anisotropic* meshes the theory of error estimation is much less developed but has attracted some attention recently, see [Sie96, Kun99, Kun00, KV00, Kun01a, DGP99]. Similarly, only lately error estimators have been proposed that are suitable for singularly perturbed diffusion–(convection)–reaction problems (on isotropic meshes), cf. [Ver98a, Ver98b, SK01, San01, FPZ01].

Here we concentrate on the combination of the previous two challenges, namely robust error estimation for a singularly perturbed reaction diffusion problem on anisotropic meshes. Recently the author succeeded to derive and investigate two kinds of error estimators for the *energy norm* [Kun01b, Kun01c]. The present paper, however, is devoted to the error measurement in the H^1 *seminorm*. Forthcoming research will extend the results obtained here to diffusion convection problems. Indeed, our exposition here has been inspired partially by [SK01] where exactly the latter problem is treated (on isotropic meshes).

Our main results provide error estimates in the H^1 seminorm for a singularly perturbed reaction diffusion problem on anisotropic meshes. We propose and analyse a residual error estimator and a local problem error estimator, both of which are presented in an element based form as well as in a face based version. Moreover we state and discuss three modifications of the error estimators. The results show that a proper definition of the estimators is far from obvious. Furthermore it turns out that there is some relation to estimators for the energy norm although there are also distinct differences. Hence we are able to isolate effects that are due to the H^1 seminorm.

The remainder of the paper is organized as follows. After presenting the model problem in Section 2, we introduce in Section 3 some notation as well as main tools for the subsequent analysis. Section 4 is devoted to the residual error estimator and its modification. A local problem error estimator and two modifications are given and examined in Section 5. Finally Section 6 investigates the numerical performance of all estimators.

2 The reaction diffusion model problem

For a small, positive parameter $\varepsilon \ll 1$ we consider the singularly perturbed reaction diffusion model problem with mixed boundary conditions

$$\left.\begin{aligned} -\varepsilon\Delta u + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \varepsilon \cdot \partial_n u &= g && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}\right\} \tag{1}$$

The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is assumed to be polyhedral with Lipschitz boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. Such a problem frequently gives rise to boundary layers.

Denote by $H^1_o(\Omega)$ the usual Sobolev space of functions from $H^1(\Omega)$ whose trace vanishes on Γ_{D}. The variational formulation related to (1) then becomes

$$\text{Find } u \in H^1_o(\Omega): \qquad a(u,v) \;=\; \langle \mathfrak{f}, v\rangle \qquad \forall\, v \in H^1_o(\Omega) \quad , \tag{2}$$

with $a(u,v) := \int_\Omega \varepsilon\nabla u \cdot \nabla v + uv$ and $\langle \mathfrak{f}, v\rangle := \int_\Omega fv + \int_{\Gamma_{\mathrm{N}}} gv$. It admits a unique solution provided that $f \in L_2(\Omega)$, $g \in L_2(\Gamma_{\mathrm{N}})$ and $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) > 0$.

In order to solve problem (2) approximately with the finite element method, introduce a family $\mathcal{F} = \{\mathcal{T}_h\}$ of triangulations $\mathcal{T}_h$ of Ω. Let $V_{o,h} \subset H^1_o(\Omega)$ be the finite element space of continuous functions that are piecewise linear over $\mathcal{T}_h$, and that vanish on Γ_{D}. The finite element formulation corresponding to (2) becomes

$$\text{Find } u_h \in V_{o,h}: \qquad a(u_h,v_h) \;=\; \langle \mathfrak{f}, v_h\rangle \qquad \forall\, v_h \in V_{o,h} \quad . \tag{3}$$

Again it admits a unique solution due to the Lax–Milgram Lemma.

3 Notation and analytical ingredients

This section introduces the notation and important ingredients for the subsequent analysis. The presentation is given for the three dimensional (3D) case. The 2D analogies can be derived easily.

Let $\omega \subset \mathbb{R}^d$, $d = 2, 3$, be some domain, and denote by $|\omega| := \mathrm{meas}_d(\omega)$ its measure. Let $(v,w)_\omega$ be the usual $L_2(\omega)$ scalar product of functions v and w, and let $\|v\|_\omega := (v,v)^{1/2}_\omega$ be the associated L_2 norm. The energy norm related to the bilinear form becomes $|||v|||_\omega := (\varepsilon\|\nabla v\|^2_\omega + \|v\|^2_\omega)^{1/2}$. Let $\mathbb{P}^k(\omega)$ be the space of polynomials of order k or less over the domain ω. Finally, for terms x and y we use the shorthand notation $x \lesssim y$ or $x \sim y$ if there exist positive constants (independent of x, y, ε, and $\mathcal{T}_h$) such that $x \le cy$ or $c_1 x \le y \le c_2 x$, respectively.

3.1 Tetrahedra – Subdomains – Mesh requirements

Tetrahedron: The four vertices of an arbitrary tetrahedron $T \in \mathcal{T}_h$ are denoted by $P_0, \ldots, P_3$ such that P_0P_1 is the longest edge of T, $\mathrm{meas}_2(\triangle P_0P_1P_2) \ge \mathrm{meas}_2(\triangle P_0P_1P_3)$, and $\mathrm{meas}_1(P_1P_2) \ge \mathrm{meas}_1(P_0P_2)$.

Additionally define three pairwise orthogonal vectors $\mathbf{p}_i$ with lengths $h_{i,T} := |\mathbf{p}_i|_{\mathbb{R}^3}$, see Figure 1. Trivially one gets $h_{1,T} > h_{2,T} \geq h_{3,T}$. Set $h_{min,T} := h_{3,T}$ and define the matrix

$$C_T := (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \in \mathbb{R}^{3\times 3} \qquad .$$

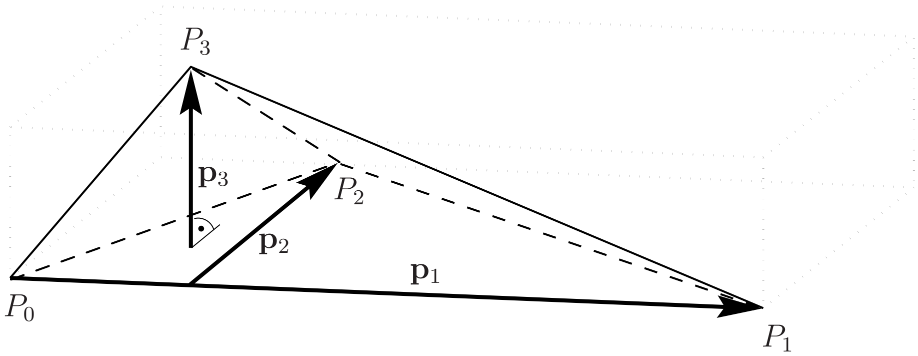

Figure 1: Notation of tetrahedron T

Tetrahedra are denoted by T, T' or T_i. Faces of a tetrahedron are denoted by E. Set $|T| = \text{meas}_3(T)$, $|E| = \text{meas}_2(E)$, and let

$$h_{E,T} := 3|T|/|E|$$

be the length of the *height* over a face E. Note that $h_{E,T}$ is *not* the diameter of E, as in the usual convention. A closer investigation of the geometrical properties of the tetrahedron yields

$$h_{E,T} > h_{min,T}/2 \qquad \forall\, E \subset \partial T \qquad . \tag{4}$$

When deriving the error estimates, one often encounters a term

$$\alpha_T := \min\{1, \varepsilon^{-1/2} \cdot h_{min,T}\} \qquad . \tag{5}$$

This factor is closely related to the singular character of the reaction diffusion problem. We remark its similarity to the Peclet number for *convection* diffusion problems.

Squeezed tetrahedron $T_{E,\delta}$: Since we are dealing with a singularly perturbed problem, we can employ advantageously a sub–tetrahedron $T_{E,\delta} \subset T$ which depends on a face E of T and a real number $\delta \in (0,1]$. Such a sub–tetrahedron has been introduced first in [Ver98b] (in a simpler form there) and subsequently improved in [Kun01b]. This squeezed tetrahedron will be utilized to define the squeezed face bubble functions of Section 3.2.

For a precise definition, let T be an arbitrary but fixed tetrahedron, and enumerate temporarily its vertices such that $E = Q_1Q_2Q_3$ and $T = OQ_1Q_2Q_3$, cf. Figure 2. With S_E being the midpoint (i.e. center of gravity) of the face E, introduce the point P that lies on the line S_EO such that $|\vec{S_EP}| = \delta \cdot |\vec{S_EO}|$. Then the *squeezed tetrahedron* $T_{E,\delta}$ is the tetrahedron with vertices P and Q_1, Q_2, Q_3, i.e. $T_{E,\delta}$ has the same face E as T but

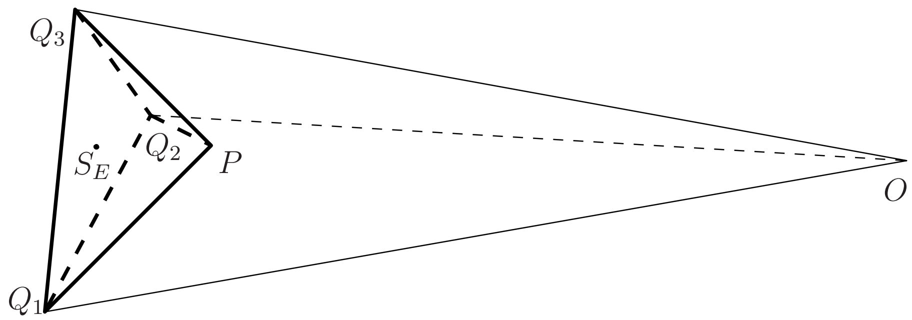

Figure 2: Tetrahedra $T = OQ_1Q_2Q_3$ and $T_{E,\delta} = PQ_1Q_2Q_3$

the fourth vertex is moved towards E with the rate δ. Note that for $\delta = 1$ one obtains $T_{E,\delta} \equiv T$ whereas in the limiting case $\delta \to 0$ the tetrahedron $T_{E,\delta}$ collapses to the face E.

Auxiliary subdomains: We repeat the standard definitions of [Ver96, AO00]. For an arbitrary element T define the subdomain ω_T that consists of T itself and all elements having a common face with it. Note that ω_T consists of less than five tetrahedra if T has a boundary face. For an interior face E denote by ω_E the union of both elements with that face. For a boundary face E modify ω_E accordingly.

Mesh requirements: In addition to the usual conformity conditions of the mesh (see Ciarlet [Cia78], Chapter 2) we assume the following two requirements.

1. The number of tetrahedra containing a node x_j is bounded uniformly.
2. The dimensions of adjacent tetrahedra must not change rapidly, i.e.

$$h_{i,T'} \sim h_{i,T} \qquad \forall\, T, T' \text{ with } T \cap T' \neq \emptyset\,,\ i = 1 \ldots d \quad .$$

Remark 3.1 Occasionally we do not want to employ *element based* quantities (such as $h_{min,T}$) but use *face related* terms instead. To this end consider an interior face $E = T_1 \cap T_2$, and define the terms

$$h_E := \frac{h_{E,T_1} + h_{E,T_2}}{2} \quad , \qquad h_{min,E} := \frac{h_{min,T_1} + h_{min,T_2}}{2} \quad , \qquad \alpha_E := \frac{\alpha_{T_1} + \alpha_{T_2}}{2} \quad .$$

The mesh assumptions from above imply $h_E \sim h_{E,T_i}$ as well as $h_{min,E} \sim h_{min,T_i}$ and $\alpha_E \sim \alpha_{T_i}$. For a boundary face $E \subset \partial T \cap \Gamma$ define similarly $h_E := h_{E,T}$, $h_{min,E} := h_{min,T}$, $\alpha_E := \alpha_T$.

Note that $h_{min,E}$ is *not* the minimal size of a face E, as the notation might suggest. $\square$

3.2 Bubble functions

Bubble functions play an important role in the analysis of residual error estimators and in the definition and investigation of local problem error estimators. Partly we can follow the standard techniques as presented e.g. in [Ver96, AO00]. However, the singularly perturbed problem considered here leads to certain modifications of the face bubble functions. For these we follow the lines of [Kun01b, Ver98b].

Let $\lambda_{T,1}, \ldots, \lambda_{T,4}$ be the barycentric coordinates of an arbitrary tetrahedron T. The *element bubble function* b_T is given by

$$b_T := 4^4 \cdot \lambda_{T,1} \cdot \lambda_{T,2} \cdot \lambda_{T,3} \cdot \lambda_{T,4} \in \mathbb{P}^4(T) \qquad \text{on } T \qquad . \tag{6}$$

Next we require face bubble functions. Let $E = T_1 \cap T_2$ be an interior face (triangle) of $\mathcal{T}_h$. The vertices of T_1 and T_2 are enumerated such that the vertices of E are numbered first. Now define the *standard face bubble function* $b_E \in C^0(\omega_E)$. It acts on $\omega_E = T_1 \cup T_2$ and is given in a piecewise fashion by

$$b_E|_{T_i} := 3^3 \cdot \lambda_{T_i,1} \cdot \lambda_{T_i,2} \cdot \lambda_{T_i,3} \qquad \text{on } T_i,\ i = 1, 2 \qquad .$$

Both bubble functions are extended by zero outside of their original domain of definition. Note that $0 \le b_T(\mathbf{x}), b_E(\mathbf{x}) \le 1$ and $\|b_T\|_\infty = \|b_E\|_\infty = 1$.

By standard scaling arguments one obtains the anisotropic equivalences below. Note that they are originally called ‘inverse inequalities’.

Lemma 3.1 (Inverse equivalences I) *Assume* $\varphi_T \in \mathbb{P}^1(T)$ *and* $\varphi_E \in \mathbb{P}^0(E)$*. Then*

$$\|b_T^{1/2} \cdot \varphi_T\|_T \sim \|\varphi_T\|_T, \tag{7}$$

$$\|\nabla(b_T \cdot \varphi_T)\|_T \sim h_{min,T}^{-1} \cdot \|\varphi_T\|_T, \tag{8}$$

$$\|b_E^{1/2} \cdot \varphi_E\|_E \sim \|\varphi_E\|_E. \tag{9}$$

Proof: See e.g. [Kun99]. ■

As mentioned above we have to modify the face bubble functions to analyse successfully the singularly perturbed problem. Following [Kun01b], we start with some interior face E. Let T_1, T_2 be its two neighbouring tetrahedra, i.e. $\omega_E = T_1 \cup T_2$. For an arbitrary real number $\delta \in (0, 1]$ consider both squeezed tetrahedra $T_{1,E,\delta} \subset T_1$ and $T_{2,E,\delta} \subset T_2$, cf. Figure 2. Define the *squeezed face bubble function* $b_{E,\delta}$ to be the standard face bubble function on the *squeezed* tetrahedra $T_{i,E,\delta}$, i.e.

$$\text{supp } b_{E,\delta} = T_{1,E,\delta} \cup T_{2,E,\delta} \qquad .$$

To facilitate the understanding, Figure 3 depicts the subdomain ω_E and the squeezed face bubble function $b_{E,\delta}$ (in the 2D case).

For clarity of notation we also introduce a trivial extension operator $F_{ext} : \mathbb{P}^0(E) \to \mathbb{P}^0(\omega_E)$ that maps a constant function over some face E to the same constant function acting on ω_E. If E is a boundary face then F_{ext} as well as b_E and $b_{E,\delta}$ are obviously defined only on the single tetrahedron $T \supset E$.

As before, inverse inequalities are sought for the squeezed face bubble functions.

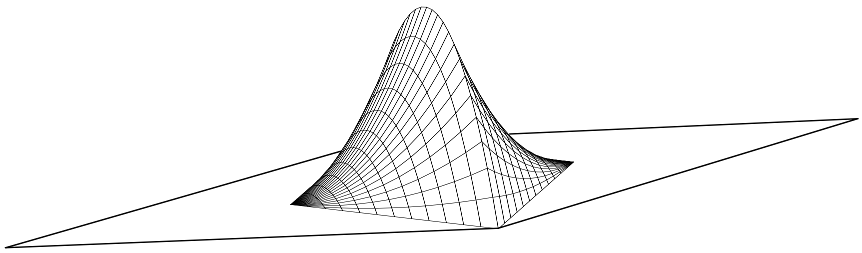

Figure 3: Subdomain ω_E and squeezed face bubble function $b_{E,\delta}$ (2D case)

Lemma 3.2 (Inverse equivalences II) *Let E be an arbitrary face of T, assume $\varphi_E \in \mathbb{P}^0(E)$, and let $\delta \in (0,1]$ be arbitrary. Then one has*

$$\|b_{E,\delta} \cdot F_{ext}(\varphi_E)\|_T \;\sim\; \delta^{1/2} \cdot h_{E,T}^{1/2} \cdot \|\varphi_E\|_E, \tag{10}$$

$$\|\nabla(b_{E,\delta} \cdot F_{ext}(\varphi_E))\|_T \;\sim\; \delta^{1/2} \cdot h_{E,T}^{1/2} \cdot \min\{\delta \cdot h_{E,T}\ ,\ h_{min,T}\}^{-1} \cdot \|\varphi_E\|_E \quad . \tag{11}$$

Proof: The proof employs refined scaling arguments (for the squeezed tetrahedron). Details are given in [Kun01c]. ■

3.3 Interpolation estimates and matching function

Interpolation error estimates are crucial ingredients to derive residual error estimates, cf. [Ver96, AO00]. When one tries to adopt the standard interpolation estimates to anisotropic meshes, one discovers an additional dependence on the anisotropic function to be interpolated. More precisely, the desired estimates are only valid when the anisotropic mesh and the anisotropic function correspond in a certain way. The anisotropic element should be stretched in that direction where the anisotropic function exhibits little variation.

For a rigorous mathematical description we define a measure of the alignment of the anisotropic mesh and function. For this purpose a so–called *matching function* has been introduced and discussed in [Kun99, Kun00].

Definition 3.1 (Matching function) *Let $v \in H^1(\Omega)$, and $\mathcal{T}_h \in \mathcal{F}$ be a triangulation of Ω. Define the* matching function *$m_1 : H^1(\Omega) \times \mathcal{F} \mapsto \mathbb{R}$ by*

$$m_1(v, \mathcal{T}_h) \;:=\; \Big(\sum_{T \in \mathcal{T}_h} h_{min,T}^{-2} \cdot \|C_T^\top \nabla v\|_T^2 \Big)^{1/2} \Big/ \ \|\nabla v\|_\Omega \qquad ,$$

with C_T given in Section 3.1. This implies $m_1(v, \mathcal{T}_h) \geq 1$.

The influence of the matching function can be observed in Theorem 3.3 below, or in the error estimates of Theorems 4.1 and 5.3. To enhance the insight into the matching function, consider first *isotropic* meshes. There the matching function is always $\mathcal{O}(1)$; thus m_1 merges with other constants and becomes invisible.

On *anisotropic* meshes that are *well–aligned* with the anisotropic function one still obtains $m_1 \sim 1$. In numerical experiments one observes a range of about $1.5 \ldots 4$. In contrast to this, *mis–aligned anisotropic* meshes may lead to arbitrarily large values of m_1, which is confirmed numerically in [Kun01a]. For further discussion see Remark 4.2.

Next we consider the Clément interpolation operator and the corresponding interpolation inequalities. As it will turn out later, we can utilize exactly the same interpolation estimates that have been derived previously in [Kun01b, Kun01c]. Hence we present here only the result. To this end recall the definition of α_T and α_E from (5) and Remark 3.1.

Lemma 3.3 *Let $v \in H^1_o(\Omega)$. The Clément interpolation operator $R_o : H^1_o(\Omega) \mapsto V_{o,h}$ satisfies the inequalities below:*

$$\sum_{T \in \mathcal{T}_h} \alpha_T^{-2} \cdot \|v - R_o v\|_T^2 \;\lesssim\; m_1(v, \mathcal{T}_h)^2 \cdot |||v|||_\Omega^2 \tag{12}$$

$$\varepsilon^{1/2} \sum_{E \subset \bar{\Omega} \setminus \Gamma_{\mathrm{D}}} \alpha_E^{-1} \cdot \|v - R_o v\|_E^2 \;\lesssim\; m_1(v, \mathcal{T}_h)^2 \cdot |||v|||_\Omega^2 \qquad , \tag{13}$$

where the sum over $E \subset \bar{\Omega} \setminus \Gamma_{\mathrm{D}}$ includes all interior faces and Neumann boundary faces.

Proof: The proof is given in [Kun01b]. ■

4 Residual error estimation

4.1 Residual error estimator

Residual error estimators are obtained by measuring and weighting the residuals. As it is common [Ver96], one first replaces the input data f and g by approximations f_h and g_h that are piecewise polynomial over the elements of $\mathcal{T}_h$ and the faces of Γ_{N}, respectively. Here we use piecewise constant approximations.

Define now the *element residual* $r_T \in \mathbb{P}^1(T)$ over an element T by

$$r_T := f_h \;-\; (-\varepsilon \Delta u_h + u_h) \qquad \text{on } T.$$

For $x \in E$ define the *face residual* $r_E \in \mathbb{P}^0(E)$ by

$$r_E(x) := \begin{cases} \varepsilon \cdot \lim\limits_{t \to +0} \left[\partial_{n_E} u_h(x + t n_E) - \partial_{n_E} u_h(x - t n_E)\right] & \text{if } E \subset \Omega \setminus \Gamma \\ g_h - \varepsilon \cdot \partial_n u_h & \text{if } E \subset \Gamma_{\mathrm{N}} \\ 0 & \text{if } E \subset \Gamma_{\mathrm{D}} \quad . \end{cases}$$

Here $n_E \perp E$ is any of the two unitary normal vectors whereas $n \perp E \subset \Gamma_{\mathrm{N}}$ denotes the outer unitary normal vector. Hence r_E is the ε scaled gradient jump for interior faces. With the help of these residuals the error estimator is defined now, and the corresponding error estimates are stated and proven.

Definition 4.1 (Element based residual error estimator)
For a tetrahedron T, define the element based residual error estimator *by*

$$\eta_{H^1,\mathrm{R},T} := \Big(\varepsilon^{-1}\alpha_T^2 \cdot \|r_T\|_T^2 + \varepsilon^{-3/2}\alpha_T \cdot \sum_{E\subset\partial T\setminus\Gamma_{\mathrm{D}}} \|r_E\|_E^2\Big)^{1/2}. \tag{14}$$

To shorten the notation, introduce the local approximation term

$$\zeta_{H^1,T} := \Big(\varepsilon^{-1}\alpha_T^2 \cdot \|f - f_h\|_{\omega_T}^2 \ + \ \varepsilon^{-3/2}\alpha_T \cdot \sum_{E\subset\partial T\cap\Gamma_{\mathrm{N}}} \|g - g_h\|_E^2\Big)^{1/2}. \tag{15}$$

Finally, define the global *terms*

$$\eta_{H^1,\mathrm{R}}^2 := \sum_{T\in\mathcal{T}_h} \eta_{H^1,\mathrm{R},T}^2 \qquad \textit{and} \qquad \zeta_{H^1}^2 := \sum_{T\in\mathcal{T}_h} \zeta_{H^1,T}^2 \qquad .$$

Theorem 4.1 (Residual error estimation)
The error is bounded locally from below for all $T \in \mathcal{T}_h$ by

$$\eta_{H^1,\mathrm{R},T} \lesssim \|\nabla(u-u_h)\|_{\omega_T} \ + \ \varepsilon^{-1/2}\alpha_T \cdot \|u-u_h\|_{\omega_T} \ + \ \zeta_{H^1,T} \qquad . \tag{16}$$

The error is bounded globally from above by

$$\|\nabla(u-u_h)\|_\Omega \lesssim m_1(u-u_h,\mathcal{T}_h) \cdot \left[\eta_{H^1,\mathrm{R}}^2 + \zeta_{H^1}^2\right]^{1/2} \qquad . \tag{17}$$

Both error bounds are uniform in ε.

Proof: The methodology of the proof is analogous to that of known residual error estimators, cf. [AO00, Ver96, Ver98b]. In order to treat *anisotropic* elements, we require modified tools and a refined analysis. Since similar ingredients have already been applied in our previous works [Kun00, Kun01b], we present only major steps in our exposition here.

Start with the lower error bound (16) for an arbitrary but fixed tetrahedron T, and consider the norm $\|r_T\|_T$ of the element residual $r_T = f_h + \varepsilon\cdot\Delta u_h - u_h$. Since we use linear ansatz functions there holds $r_T \equiv f_h - u_h \in \mathbb{P}^1(T)$. For $x \in T$ let

$$w(x) := r_T(x)\cdot b_T(x) \qquad \in \mathbb{P}^5(T)\cap H_o^1(T) \qquad ,$$

with b_T being the element bubble functions of (6). Integration by parts yields

$$\begin{aligned}
\int_T r_T\cdot w \ &= \ \int_T \varepsilon\nabla(u-u_h)\cdot\nabla w + \int_T (u-u_h)w \ + \ \int_T (f_h - f)w \\
|(r_T,w)_T| \ &\le \ \varepsilon\cdot\|\nabla(u-u_h)\|_T\cdot\|\nabla w\|_T \ + \ \|u-u_h\|\cdot\|w\|_T \ + \ \|f-f_h\|_T\cdot\|w\|_T.
\end{aligned}$$

The inverse inequalities (7), (8) and $0 \le b_T \le 1$ readily imply the bounds

$$|(r_T,w)_T| \sim \|r_T\|_T^2 \quad , \quad \|\nabla w\|_T \lesssim h_{min,T}^{-1}\cdot\|r_T\|_T \quad , \quad \|w\|_T \le \|r_T\|_T \quad .$$

In conjuction with (5) one obtains

$$\varepsilon^{-1}\alpha_T^2 \cdot \|r_T\|_T^2 \;\lesssim\; \|\nabla(u-u_h)\|_T^2 \,+\, \varepsilon^{-1}\alpha_T^2 \cdot (\|u-u_h\|_T^2 + \|f-f_h\|_T^2) \quad . \tag{18}$$

Next we derive a bound of the norm $\|r_E\|_E$ of the face residual for some interior face $E \subset \partial T$. The linear ansatz functions imply $r_E \in \mathbb{P}^0(E)$. Denote temporarily by $T_1 \equiv T$ and T_2 the two tetrahedra that E belongs to. Define the function

$$w := b_{E,\delta_E} \cdot F_{ext}(r_E) \in H_o^1(\omega_E) \qquad ,$$

with F_{ext} being the trivial extension operator (cf. Section 3.2) and b_{E,δ_E} being the squeezed face bubble functions of Section 3.2. The real number δ_E will be chosen later. Integration by parts then yields

$$\begin{aligned} -\int_E r_E \cdot w \;&=\; \varepsilon \sum_{i=1}^{2} \int_{\partial T_i} w \cdot \frac{\partial u_h}{\partial n} \;=\; \varepsilon \sum_{i=1}^{2} \int_{T_i} (\nabla u_h \cdot \nabla w \;+\; \Delta u_h \, w) \\ &=\; \sum_{i=1}^{2} \int_{T_i} (\varepsilon \nabla(u_h-u)\cdot\nabla w \;+\; (u_h-u)w \;+\; (r_{T_i}+f-f_h)w) \quad . \end{aligned}$$

From $\int_E r_E\, w = \|b_E^{1/2} \cdot r_E\|_E^2 \sim \|r_E\|_E^2$ one infers

$$\|r_E\|_E^2 \;\leq\; \sum_{i=1}^{2} \Big(\varepsilon\, \|\nabla(u-u_h)\|_{T_i} \cdot \|\nabla w\|_{T_i} + \Big[\|u-u_h\|_{T_i} + \|r_{T_i}\|_{T_i} + \|f-f_h\|_{T_i} \Big] \cdot \|w\|_{T_i} \Big).$$

Apply the inverse inequalities (10), (11) to bound $\|w\|_{T_i}$ and $\|\nabla w\|_{T_i}$, respectively. In order to obtain the desired bound, we choose now the parameter

$$\delta_E \;:=\; \min\{1, \varepsilon^{1/2}/h_E \,,\, h_{min,E}/h_E\} \;\sim\; \varepsilon^{1/2} \cdot h_E^{-1} \cdot \alpha_E \qquad . \tag{19}$$

This implies in particular $\min\{\delta_E \cdot h_{E,T_i}, h_{min,T_i}\} \sim \varepsilon^{1/2} \cdot \alpha_{T_i}$, cf. (11). Finally insert the previous estimate (18) which provides a bound of $\|r_{T_i}\|_{T_i}$, and note that h_{min,T_i}, α_{T_i} and h_{E,T_i} do not change rapidly across adjacent tetrahedra. Eventually this leads to

$$\varepsilon^{-3/2}\alpha_T \cdot \|r_E\|_E^2 \;\lesssim\; \|\nabla(u-u_h)\|_{\omega_E}^2 \,+\, \varepsilon^{-1}\alpha_T^2 \cdot (\|u-u_h\|_{\omega_E}^2 + \|f-f_h\|_{\omega_E}^2) \qquad .$$

For a Neumann face $E \subset \Gamma_{\mathrm{N}} \cap \partial T$ one proceeds similarly and infers

$$\varepsilon^{-3/2}\alpha_T \cdot \|r_E\|_E^2 \;\lesssim\; \|\nabla(u-u_h)\|_T^2 + \varepsilon^{-1}\alpha_T^2 \cdot (\|u-u_h\|_T^2 + \|f-f_h\|_T^2) + \varepsilon^{-3/2}\alpha_T \cdot \|g-g_h\|_E^2 \quad .$$

Summing up over all faces E of T, recalling the definition of $\eta_{H^1,T}$ and employing (18) finishes the proof of the lower error bound (16).

The upper error bound (17) is a consequence of the results of [Kun01b], save for the treatment of Neumann boundary conditions. For self–containment we repeat major steps of the proof. Recall first that R_o denotes the Clément interpolation operator. The Galerkin orthogonality and integration by parts imply for all $v \in H_o^1(\Omega)$

$$
\begin{aligned}
a(u-u_h,v) &= a(u-u_h, v-R_o v) \\
&= \sum_{T\in\mathcal{T}_h} (r_T + f - f_h, v - R_o v)_T + \sum_{E\subset\bar\Omega\setminus\Gamma_{\mathrm{D}}} (r_E, v-R_o v)_E + \sum_{E\subset\Gamma_{\mathrm{N}}} (g-g_h, v-R_o v)_E \\
&\le \sum_{T\in\mathcal{T}_h} \alpha_T(\|r_T\|_T + \|f-f_h\|_T)\cdot \alpha_T^{-1}\|v-R_o v\|_T + \\
&\quad + \sum_{E\subset\bar\Omega\setminus\Gamma_{\mathrm{D}}} \varepsilon^{-1/4}\alpha_E^{1/2}\|r_E\|_E \cdot \varepsilon^{1/4}\alpha_E^{-1/2}\|v-R_o v\|_E + \\
&\quad + \sum_{E\subset\Gamma_{\mathrm{N}}} \varepsilon^{-1/4}\alpha_E^{1/2}\|g-g_h\|_E \cdot \varepsilon^{1/4}\alpha_E^{-1/2}\|v-R_o v\|_E\,.
\end{aligned}
$$

The discrete Cauchy-Schwarz inequality and the interpolation estimates (12), (13) yield

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \alpha_T(\|r_T\|_T + \|f-f_h\|_T)\cdot\alpha_T^{-1}\|v-R_o v\|_T \lesssim \\
&\qquad \overset{(12)}{\lesssim} \left(\sum_{T\in\mathcal{T}_h} \alpha_T^2\Big(\|r_T\|_T^2 + \|f-f_h\|_T^2\Big)\right)^{1/2} \cdot m_1(v,\mathcal{T}_h)\cdot |||v|||_\Omega \\
&\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\setminus\Gamma_{\mathrm{D}}} \varepsilon^{-1/4}\alpha_E^{1/2}\|r_E\|_E \cdot \varepsilon^{1/4}\alpha_E^{-1/2}\|v-R_o v\|_E \lesssim \\
&\qquad \overset{(13)}{\lesssim} \left(\varepsilon^{-1/2}\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\setminus\Gamma_{\mathrm{D}}} \alpha_E\|r_E\|_E^2\right)^{1/2} \cdot m_1(v,\mathcal{T}_h)\cdot |||v|||_\Omega\,.
\end{aligned}
$$

Combining all estimates implies

$$
a(u-u_h,v) \lesssim \left(\sum_{T\in\mathcal{T}_h} \varepsilon\cdot\left[\eta_{H^1,\mathrm{R},T}^2 + \zeta_{H^1,T}^2\right]\right)^{1/2} \cdot m_1(v,\mathcal{T}_h)\cdot |||v|||_\Omega \quad .
$$

Substituting $v := u-u_h \in H_o^1(\Omega)$ and recalling $\varepsilon^{1/2}\|\nabla v\|_\Omega \le |||v|||_\Omega$ finishes the proof. ∎

Remark 4.1 The lower error bound (16) contains the additional L_2 error term $\varepsilon^{-1/2}\alpha_T\cdot\|u-u_h\|_{\omega_T}$ that is not present in the upper error bound (17). Hence both bounds do not correspond completely. We note that a very similar situation is seen for error estimators for *convection* diffusion problems [Ver98a, SK01].

On the other hand the upper and lower error bound will be of the same quality only if the L_2 error term $\varepsilon^{-1/2}\alpha_T\cdot\|u-u_h\|_{\omega_T}$ is dominated by the H^1 error term $\|\nabla(u-u_h)\|_{\omega_T}$. In analogy to convection diffusion problems this requires suitable meshes, i.e. correct control on the factor $\varepsilon^{-1/2}\alpha_T$ of the L_2 error term.

We believe that the additional L_2 error term is mainly due to the H^1 seminorm. In contrast, for the energy norm (which is naturally associated with the differential equation) the upper and lower error bounds contain the same terms [Kun01b]. □

Remark 4.2 The upper error bound (17) contains the matching function $m_1(u - u_h, \mathcal{T}_h)$ which cannot be computed. For a comprehensive discussion of m_1 we refer to [Kun00, Kun01b] since the same matching function occurs there, and has been treated there. Here two remarks should suffice.

Firstly, although $m_1(u - u_h, \mathcal{T}_h)$ cannot be computed exactly, it can be approximated quite well, e.g. by means of a recovered gradient. Secondly, our numerical experience tells that m_1 ranges from about $1.5 \ldots 4$ for sensible anisotropic meshes. Summarizing, the upper error bound could theoretically be regarded as useless. From a practical point of view, however, it is a reliable and important result. □

Using the same ideas as before, one can easily derive a *face based* estimator.

Definition 4.2 (Face based residual error estimator)
For a face E, define the face based residual error estimator *by*

$$\eta_{H^1,\mathrm{R},E} := \Big(\varepsilon^{-1}\alpha_E^2 \sum_{T\subset\omega_E} \|r_T\|_T^2 \; + \varepsilon^{-3/2}\alpha_E \cdot \|r_E\|_E^2 \Big)^{1/2} .$$

To shorten the notation, introduce the local approximation term

$$\zeta_{H^1,E} := \varepsilon^{-1/2}\alpha_E \cdot \|f - f_h\|_{\omega_E} \; + \; \varepsilon^{-3/4}\alpha_E^{1/2} \cdot \|g - g_h\|_{E\cap\Gamma_{\mathrm{N}}} \qquad .$$

With these definitions the following error estimates can be proven.

Theorem 4.2 (Residual error estimation)
The error is bounded locally from below for all faces E of $\mathcal{T}_h$ by

$$\eta_{H^1,\mathrm{R},E} \lesssim \|\nabla(u - u_h)\|_{\omega_E} \; + \; \varepsilon^{-1/2}\alpha_E \cdot \|u - u_h\|_{\omega_E} \; + \; \zeta_{H^1,E} \qquad .$$

The error is bounded globally from above by

$$\|\nabla(u - u_h)\|_\Omega \lesssim m_1(u - u_h, \mathcal{T}_h) \cdot \left(\sum_{E\in\mathcal{T}_h} \eta_{H^1,\mathrm{R},E}^2 + \zeta_{H^1,E}^2 \right)^{1/2} \qquad ,$$

where the sum over $E \in \mathcal{T}_h$ includes interior and boundary faces of the triangulation. Both error bounds are uniform in ε.

Proof: The derivation of the error bounds is completely analogous to the proof of Theorem 4.1 and thus omitted. ■

4.2 Modified residual error estimator

The previous residual error estimator can be modified slightly which implies error bounds of a similar structure but with different scaling factors.

Definition 4.3 (Modified residual error estimator)
For a tetrahedron T, define the modified residual error estimator *by*

$$\tilde{\eta}_{H^1,\mathrm{R},T} := \left(\varepsilon^{-2} h_{min,T}^2 \cdot \|r_T\|_T^2 + \varepsilon^{-2} h_{min,T} \cdot \sum_{E\subset\partial T\setminus\Gamma_\mathrm{D}} \|r_E\|_E^2 \right)^{1/2} . \tag{20}$$

To shorten the notation, introduce the modified local approximation term

$$\tilde{\zeta}_{H^1,T} := \left(\varepsilon^{-2} h_{min,T}^2 \cdot \|f - f_h\|_{\omega_T}^2 \; + \; \varepsilon^{-2} h_{min,T} \cdot \sum_{E\subset\partial T\cap\Gamma_\mathrm{N}} \|g - g_h\|_E^2 \right)^{1/2} .$$

Define again the global terms $\tilde{\eta}_{H^1,\mathrm{R}}^2 := \sum_{T\in\mathcal{T}_h} \tilde{\eta}_{H^1,\mathrm{R},T}^2$ *and* $\tilde{\zeta}_{H^1}^2 := \sum_{T\in\mathcal{T}_h} \tilde{\zeta}_{H^1,T}^2$.

Theorem 4.3 (Modified residual error estimation)
The error is bounded locally from below for all $T \in \mathcal{T}_h$ by

$$\tilde{\eta}_{H^1,\mathrm{R},T} \lesssim \|\nabla(u-u_h)\|_{\omega_T} \; + \; \varepsilon^{-1} h_{min,T} \cdot \|u-u_h\|_{\omega_T} \; + \; \tilde{\zeta}_{H^1,T} \qquad . \tag{21}$$

The error is bounded globally from above by

$$\|\nabla(u-u_h)\|_\Omega \lesssim m_1(u-u_h, \mathcal{T}_h) \cdot \left[\tilde{\eta}_{H^1,\mathrm{R}}^2 + \tilde{\zeta}_{H^1}^2 \right]^{1/2} \qquad . \tag{22}$$

Both error bounds are uniform in ε.

Proof: Let us start by comparing the original and the modified residual error estimator. One has

$$\varepsilon^{-2} h_{min,T}^2 = \varepsilon^{-1} \alpha_T^2 \cdot \max\{1, \varepsilon^{-1} h_{min,T}^2\} \quad , \quad \varepsilon^{-2} h_{min,T} = \varepsilon^{-3/2} \alpha_T \cdot \max\{1, \varepsilon^{-1/2} h_{min,T}\}.$$

Recalling the definitions of $\eta_{H^1,\mathrm{R},T}$ and $\zeta_{H^1,\mathrm{R},T}$ from (14) and (15), this implies

$$\begin{array}{rcccl} \max\{1, \varepsilon^{-1/2} h_{min,T}\} \cdot \eta_{H^1,\mathrm{R},T}^2 & \le & \tilde{\eta}_{H^1,\mathrm{R},T}^2 & \le & \max\{1, \varepsilon^{-1} h_{min,T}^2\} \cdot \eta_{H^1,\mathrm{R},T}^2 \\ \max\{1, \varepsilon^{-1/2} h_{min,T}\} \cdot \zeta_{H^1,\mathrm{R},T}^2 & \le & \tilde{\zeta}_{H^1,\mathrm{R},T}^2 & \le & \max\{1, \varepsilon^{-1} h_{min,T}^2\} \cdot \zeta_{H^1,\mathrm{R},T}^2 \end{array} .$$

In conjunction with (17) this proves immediately the upper error bound (22).

In order to derive the lower error bound (21), proceed analogously to the proof of Theorem 4.1. The main difference is now the choice of δ_E to define the face bubble functions, cf. (19). Here we have to use $\delta_E := \min\{1, h_{min,E}/h_E\} \sim h_{min,E}/h_E$ (cf. (4)) to obtain the desired result. The rest of the proof is omitted. ∎

Remark 4.3 Just by comparing the results of Theorems 4.1 and 4.3 it is not clear whether the original or the modified residual error estimator should be favoured. One distinct difference is that the original estimator implies an equivalence with some local problem error estimator (see Theorem 5.2 below) which could not be established for the modified residual estimator. Furthermore the original estimator has a smaller L_2 error term and smaller data approximation term which is a slight advantage. □

5 Local problem error estimation

The key idea consists in solving the local problem with *higher accuracy* but only on a small *local subdomain*. The norm of the difference between this (hopefully more accurate) local solution and the original (piecewise linear) solution u_h serves as *local problem error estimator*, cf. the textbooks [AO00, Ver96]. Furthermore the underlying ideas have been adapted successfully to *anisotropic elements* [Kun01a, Kun01c].

Here we present three approaches that try to estimate the error in the H^1 seminorm. The first approach in Section 5.1 provides a local problem error estimator that is equivalent to the residual error estimator $\eta_{H^1,\mathrm{R},T}$ of Section 4.1. Almost immediately the actual error bounds follow.

In Section 5.2 two further approaches are presented that differ from the first estimator either by the local problem or by the definition of the estimator. One of them is even the seemingly 'natural' choice for defining the estimator. Unfortunately only suboptimal results are achieved. This illustrates the difficulties in finding an appropriate local problem error estimator.

5.1 Local problem error estimator

Consider an arbitrary but fixed element T. The local problem will be posed over the local subdomain ω_T. The local, finite dimensional space V_T is spanned by a single element bubble function and some squeezed face bubble functions,

$$V_T := \mathrm{span}\{b_T, b_{E,\delta_E} : E \subset \partial T \setminus \Gamma_\mathrm{D}\} \qquad .$$

The 'squeezing' parameters δ_E of the squeezed face bubble functions (cf. Section 3.2) is chosen exactly as in the proof of the residual error estimation, namely

$$\delta_E \;:=\; \min\{1, \varepsilon^{1/2}/h_E\,,\, h_{min,E}/h_E\} \qquad , \tag{23}$$

cf. (19). Now the estimator can be defined.

Definition 5.1 (Element based local problem error estimator)
Find a solution $e_T \in V_T$ of the local variational problem:

$$a(e_T, v_T) \;=\; \int\limits_{\omega_T} f_h v_T + \int\limits_{\partial\omega_T \cap \Gamma_\mathrm{N}} g_h v_T - \int\limits_{\omega_T} \varepsilon \nabla u_h \cdot \nabla v_T - \int\limits_{\omega_T} u_h v_T \tag{24}$$

for all $v_T \in V_T$. The local *and* global error estimators *then become*

$$\eta_{H^1,\mathrm{D},T} := \varepsilon^{-1/2} \cdot |||e_T|||_{\omega_T} \qquad \textit{and} \qquad \eta^2_{H^1,\mathrm{D}} := \sum_{T\in\mathcal{T}_h} \eta^2_{H^1,\mathrm{D},T} \qquad . \tag{25}$$

Two equivalent formulations of the local problem are derived by partial integration.
Equivalent descriptions: Find $e_T \in V_T$ such that

$$a(e_T, v_T) = a(u - u_h, v_T) - \int_{\omega_T} (f - f_h)\, v_T - \int_{\partial T \cap \Gamma_{\rm N}} (g - g_h)\, v_T \qquad \forall\, v_T \in V_T \quad (26)$$

$$a(e_T, v_T) = \sum_{T' \in \omega_T} \int_{T'} r_{T'} \cdot v_T + \sum_{E \subset \partial T \setminus \Gamma_{\rm D}} \int_E r_E \cdot v_T \qquad \forall\, v_T \in V_T \quad . \quad (27)$$

The local problem solved here is exactly the same as for the energy norm error estimator of [Kun01c]. The difference is the choice of the norm that defines the error estimator.

Not surprisingly, the techniques for proving the error estimates here are similar to that of [Kun01c] but of course adapted to the error measurement in the H^1 seminorm. For this reason we present the major steps only in our exposition. We start with an essential lemma.

Lemma 5.1 *The following relations hold for all $v_T \in V_T$.*

$$\|v_T\|_{\omega_T} \lesssim h_{min,T} \cdot \|\nabla v_T\|_{\omega_T} \quad (28)$$

$$\|v_T\|_E \lesssim h_E^{-1/2}\, \delta_E^{-1/2} \cdot \min\{h_{min,T}, \delta_E\, h_E\} \cdot \|\nabla v_T\|_{\omega_T} \qquad \forall\, E \subset \partial T \quad . \quad (29)$$

The inequalities are uniform in the squeezing parameter $\delta_E \in (0, 1]$.

If T has at least two Neumann boundary faces then the constants in (28), (29) can depend on the shape of the Neumann boundary (but do not depend on the triangulation $\mathcal{T}_h$ nor on T). More precisely, the constants depend on the angle between the Neumann boundary faces. The smaller this angle, the worse the constants may be.

Proof: The technical proof is given in [Kun01c, Kun01c]. ∎

Next a certain local equivalence of the residual error estimator and the local problem error estimator is established. For simplicity of notation introduce

$$\eta^2_{H^1,\mathrm{R},\omega_T} := \sum_{T' \subset \omega_T} \eta^2_{H^1,\mathrm{R},T'} \qquad , \qquad \eta^2_{H^1,\mathrm{D},\omega_T} := \sum_{T' \subset \omega_T} \eta^2_{H^1,\mathrm{D},T'} \qquad .$$

Theorem 5.2 (Equivalence with residual error estimator) *The local problem error estimator $\eta_{H^1,\mathrm{D},T}$ is equivalent to the residual error estimator $\eta_{H^1,\mathrm{R},T}$ in the following sense:*

$$\eta_{H^1,\mathrm{D},T} \lesssim \eta_{H^1,\mathrm{R},\omega_T} \quad (30)$$

$$\eta_{H^1,\mathrm{R},T} \lesssim \eta_{H^1,\mathrm{D},\omega_T} \qquad . \quad (31)$$

Both inequalities are uniform in ε.

If T has at least two Neumann boundary faces then the constant in (30) can depend on the shape of the Neumann boundary (but does not depend on $\mathcal{T}_h$ nor on T).

Note that the equivalences hold only for the original residual error estimator $\eta_{H^1,\mathrm{R},T}$. Similar relations for the modified residual error estimator $\tilde{\eta}_{H^1,\mathrm{R},T}$ could not be achieved.
Proof: Start with the equivalent formulation (27) of the local problem giving

$$\begin{aligned}\|\!|e_T|\!\|^2_{\omega_T} &\overset{(27)}{=} \sum_{T'\in\omega_T}\int_{T'} r_{T'}\cdot e_T + \sum_{E\subset\partial T\setminus\Gamma_\mathrm{D}}\int_E r_E\cdot e_T \\ &\leq \left(\sum_{T'\subset\omega_T}\|r_{T'}\|^2_{T'}\right)^{1/2}\cdot\|e_T\|_{\omega_T} + \sum_{E\subset\partial T\setminus\Gamma_\mathrm{D}}\|r_E\|_E\cdot\|e_T\|_E .\end{aligned}$$

Next we aim at bounds of $\|e_T\|_{\omega_T}$ and $\|e_T\|_E$, $E\subset\partial T$. Apply Lemma 5.1 and recall the definition of α_T and δ_E from (5) and (23) to obtain

$$\|e_T\|_{\omega_T} \lesssim \alpha_T\cdot\|\!|e_T|\!\|_{\omega_T} \tag{32}$$

$$\|e_T\|_E \overset{(29)}{\lesssim} \varepsilon^{-1/4}\,\alpha_T^{1/2}\,\|\!|e_T|\!\|_{\omega_T} \quad , \tag{33}$$

cf. also [Kun01c, Theorem 4.3]. Both inequalities (in conjunction with $\alpha_T\sim\alpha_{T'}$ for neighbouring tetrahedra) result in

$$\|\!|e_T|\!\|^2_{\omega_T} \lesssim \left(\sum_{T'\subset\omega_T}\alpha^2_{T'}\cdot\|r_{T'}\|^2_{T'} + \varepsilon^{-1/2}\,\alpha_T\cdot\sum_{E\subset\partial T\setminus\Gamma_\mathrm{D}}\|r_E\|_E\right)^{1/2}\cdot\|\!|e_T|\!\|_{\omega_T}$$

which, together with $\eta_{H^1,\mathrm{D},T}=\varepsilon^{-1/2}\,\|\!|e_T|\!\|_{\omega_T}$, proves (30).

In order to derive (31) one has to bound $\eta_{H^1,\mathrm{R},T}$, and thus $\|r_T\|_T$ and $\|r_E\|_E$. The proof is similar to our analysis in [Kun01c]. Let us start with the term $\|r_T\|_T$. Set $v_T := b_T\cdot r_T\in V_T$, with b_T being the element bubble function of (6). The local problem (27) and equivalence (7) imply

$$\|r_T\|^2_T \overset{(7)}{\sim} \|b_T^{1/2}\cdot r_T\|^2_T \overset{(27)}{=} a(e_T,v_T) \leq \|\!|e_T|\!\|_T\cdot\|\!|v_T|\!\|_T \quad .$$

The inverse inequality (8) yields

$$\|\!|v_T|\!\|^2_T = \varepsilon\,\|\nabla(b_T\cdot r_T)\|^2_T + \|b_T\cdot r_T\|^2_T \overset{(8)}{\sim} \alpha_T^{-2}\,\|r_T\|^2_T \quad .$$

Both relations together result in

$$\|r_T\|_T \lesssim \alpha_T^{-1}\cdot\|\!|e_T|\!\|_T \leq \alpha_T^{-1}\cdot\varepsilon^{1/2}\cdot\eta_{H1,\mathrm{D},T} \quad . \tag{34}$$

Analogously one bounds the norm of $r_E\in\mathbb{P}^0(E)$ for an interior face $E\subset\partial T\setminus\Gamma$. Recall the definition of the squeezed face bubble function $b_{E,\delta}$, and set $v_E := b_{E,\delta}\cdot F_{ext}(r_E)\in V_T$.

The local problem (27) implies

$$\begin{aligned}\|r_E\|_E^2 \overset{(9)}{\sim} \|b_E^{1/2}\cdot r_E\|_E^2 &\overset{(27)}{=} a(e_T, v_E) - \sum_{T'\subset\omega_E}\int_{T'} r_{T'}\, v_E \\ &\leq |||e_T|||_{\omega_E}\cdot |||v_E|||_{\omega_E} + \sum_{T'\subset\omega_E}\|r_{T'}\|_{T'}\cdot\|v_E\|_{T'} \quad .\end{aligned}$$

The inverse inequalities provide bounds of the norms of v_E. Furthermore utilize the specific value of δ_E from (19), leading to

$$\begin{aligned}\|v_E\|_{T'} &= \|b_{E,\delta}\cdot F_{ext}(r_E)\|_{T'} \overset{(10)}{\lesssim} \delta_E^{1/2}\cdot h_{E,T'}^{1/2}\cdot\|r_E\|_E \overset{(19)}{\sim} \varepsilon^{1/4}\,\alpha_T^{1/2}\cdot\|r_E\|_E \\ |||v_E|||_{\omega_E} &= (\varepsilon\|\nabla(v_E)\|_{\omega_E}^2 + \|v_E\|_{\omega_E}^2)^{1/2} \overset{(11),(19)}{\lesssim} \varepsilon^{1/4}\,\alpha_T^{-1/2}\cdot\|r_E\|_E \quad ,\end{aligned}$$

cf. also [Kun01c]. In conjunction with the previous bound (34) of $\|r_{T'}\|_{T'}$ for both tetrahedra $T'\subset\omega_E$ we infer

$$\|r_E\|_E \lesssim \varepsilon^{3/4}\,\alpha_T^{-1/2}\cdot\sum_{T'\subset\omega_E}\eta_{H^1,\mathrm{D},T'} \qquad \forall\, E\subset\partial T\setminus\Gamma \quad .$$

The norm of $r_E\in\mathbb{P}^0(E)$ for a Neumann boundary face $E\subset\partial T\cap\Gamma_\mathrm{N}$ is bounded similarly and gives the corresponding result $\|r_E\|_E \lesssim \varepsilon^{3/4}\,\alpha_T^{-1/2}\cdot\eta_{H^1,\mathrm{D},T}$. Combining all bounds of $\|r_T\|_T$ and $\|r_E\|_E$ establishes (31). ∎

Utilizing the previous theorem and its proof, we can easily derive the error bounds for the local problem error estimator.

Theorem 5.3 (Local problem error estimation)
The error is bounded locally from below for all $T\in\mathcal{T}_h$ by

$$\eta_{H^1,\mathrm{D},T} \leq \|\nabla(u-u_h)\|_{\omega_T} + \varepsilon^{-1/2}\cdot\|u-u_h\|_{\omega_T} + c\,\zeta_{H^1,T} \quad . \tag{35}$$

The error is bounded globally from above by

$$\|\nabla(u-u_h)\|_\Omega \lesssim m_1(u-u_h,\mathcal{T}_h)\cdot\left[\eta_{H^1,\mathrm{D}}^2+\zeta_{H^1}^2\right]^{1/2} \quad . \tag{36}$$

Both inequalities are uniform in ε.

The lower error bound (35) is a strict inequality where the only constant c is at the data approximation term $\zeta_{H^1,T}$. As always, this constant c is independent of ε, T, u and u_h. However, if T has at least two Neumann boundary faces then c can depend on the shape of the Neumann boundary (but does not depend on the triangulation $\mathcal{T}_h$ nor on T).

Proof: In order to show (35), utilize formulation (26) of the local problem and obtain

$$\begin{aligned}|||e_T|||_{\omega_T}^2 &\overset{(26)}{=} a(u-u_h,e_T) - \int_{\omega_T}(f-f_h)\cdot e_T - \int_{\Gamma_\mathrm{N}\cap\partial T}(g-g_h)\cdot e_T \\ &\leq |||u-u_h|||_{\omega_T}\cdot|||e_T|||_{\omega_T} + \|f-f_h\|_{\omega_T}\cdot\|e_T\|_{\omega_T} + \|g-g_h\|_{\Gamma_\mathrm{N}\cap\partial T}\cdot\|e_T\|_{\Gamma_\mathrm{N}\cap\partial T}.\end{aligned}$$

The bounds (32), (33) as well as $\eta_{H^1,\mathrm{D},T} = \varepsilon^{-1/2}\,|||e_T|||_{\omega_T}$ provide the lower error bound.

The upper error bound is an immediate consequence of the residual error estimate (17) and the relation (31) between $\eta_{H^1,\mathrm{R},T}$ and $\eta_{H^1,\mathrm{D},T}$. ■

Remark 5.1 The lower error bound can be rephrased slightly such that it has the same structure as the residual error bound (16), i.e.

$$\eta_{H^1,\mathrm{D},T} \;\le\; \|\nabla(u-u_h)\|_{\omega_T} \;+\; c\cdot\varepsilon^{-1/2}\alpha_T\cdot\|u-u_h\|_{\omega_T} \;+\; c\,\zeta_{H^1,T} \qquad .$$

The difference is only at the L_2 error term. □

Remark 5.2 Since the local problem here is the same as for the energy norm error estimation, we refer to [Kun01c] for a discussion of implementational aspects. This includes for example the choice of a stable basis for the local problem and the fast generation and solution of the local problem. □

Similar to the exposition at the end of Section 4.1 one can derive a *face based* local problem estimator. To this end let the local space associated with a face E be

$$V_E := \mathrm{span}\{b_{E,\delta_E} \text{ if } E \not\subset \Gamma_{\mathrm{D}}\ ,\ b_T\ \forall\, T\subset\omega_E\} \qquad ,$$

where the squeezed face bubble functions are as above (in particular with the same squeezing parameter δ_E).

Definition 5.2 (Face based local problem error estimator)
Find the solution $e_E\in V_E$ of the local problem

$$a(e_E,v_E) \;=\; \int\limits_{\omega_E} f_h v_E + \int\limits_{E\cap\Gamma_{\mathrm{N}}} g_h v_E - \int\limits_{\omega_E} \varepsilon\nabla u_h\cdot\nabla v_E - \int\limits_{\omega_E} u_h v_E$$

for all $v_E\in V_E$. The face based local error estimator *is then given by*

$$\eta_{H^1,\mathrm{D},E} := \varepsilon^{-1/2}\cdot|||e_E|||_{\omega_E} \qquad .$$

With the same techniques as above one infers the following error estimates.

Theorem 5.4 (Face based local problem error estimation)
The face based residual error estimator and local problem error estimator are equivalent:

$$\eta_{H^1,\mathrm{R},E} \;\sim\; \eta_{H^1,\mathrm{D},E} \qquad \forall\, E\in\mathcal{T}_h \qquad .$$

The error is bounded locally from below for all faces E of $\mathcal{T}_h$ by

$$\eta_{H^1,\mathrm{D},E} \;\le\; \|\nabla(u-u_h)\|_{\omega_E} \;+\; \varepsilon^{-1/2}\cdot\|u-u_h\|_{\omega_E} \;+\; c\,\zeta_{H^1,E} \qquad ,$$

with the constant c at the data approximation term being as in Theorem 5.3.
The error is bounded globally from above by

$$\|\nabla(u-u_h)\|_\Omega \lesssim m_1(u-u_h,\mathcal{T}_h)\cdot\left(\sum_{E\in\mathcal{T}_h}\eta^2_{H^1,\mathrm{D},E}+\zeta^2_{H^1,E}\right)^{1/2} \qquad .$$

All relations are uniform in ε.

Proof: The proofs are similar to the ones above and therefore omitted. ■

5.2 Two further, modified local problem error estimators

This section shows that the choice of an appropriate local problem error estimator is far from obvious. To this end we present two suboptimal estimators with different features.

Start with the local problem estimator of Section 5.1. In our opinion the corresponding local problem does not seem to be the 'natural' choice. The 'natural' choice would be to solve the original reaction diffusion problem with higher accuracy, and then measure the local solution in the *same* H^1 *seminorm* in which we seek to bound the error. This leads to the definition below. The only difference to the original estimator of (25) is the different norm.

Definition 5.3 (First modified local problem error estimator)
Solve the same local problem (24) as before but define the local *and* global error estimator *by*

$$\tilde{\eta}_{H^1,\mathrm{D},T} := \|\nabla e_T\|_{\omega_T} \quad , \quad \tilde{\eta}^2_{H^1,\mathrm{D}} := \sum_{T\in\mathcal{T}_h} \tilde{\eta}^2_{H^1,\mathrm{D},T} \quad , \quad \tilde{\eta}^2_{H^1,\mathrm{D},\omega_T} := \sum_{T'\subset\omega_T} \tilde{\eta}^2_{H^1,\mathrm{D},T'} \quad .$$

Unfortunately, however, this approach does not yield an equivalence to the residual error estimator $\eta_{H^1,\mathrm{R},T}$.

Theorem 5.5 (Comparison with residual error estimator) *The local problem error estimator $\tilde{\eta}_{H^1,\mathrm{D},T}$ is related to the residual error estimator $\eta_{H^1,\mathrm{R},T}$ in the following sense:*

$$\tilde{\eta}_{H^1,\mathrm{D},T} \lesssim \eta_{H^1,\mathrm{R},\omega_T} \tag{37}$$

$$\eta_{H^1,\mathrm{R},T} \lesssim \max\{1, \varepsilon^{-1/2} h_{min,T}\} \cdot \tilde{\eta}_{H^1,\mathrm{D},\omega_T} \quad . \tag{38}$$

If T has at least two Neumann boundary faces then the constant in (37) can depend on the shape of the Neumann boundary.

The proof employs the same techniques as before and is thus omitted. As a consequence of (38) and (17) the desired *upper* error bound is not obtained while a lower error bound (corresponding to (35)) still holds.

For the second approach we do not solve a local reaction diffusion problem but only the corresponding (ε scaled) Poisson part of it.

Definition 5.4 (Second modified local problem error estimator)
Find a solution $\check{e}_T \in V_T$ of the local variational problem:

$$\varepsilon \int_{\omega_T} \nabla \check{e}_T \nabla v_T = \int_{\omega_T} f_h \cdot v_T + \int_{\partial\omega_T\cap\Gamma_\mathrm{N}} g_h \cdot v_T - \int_{\omega_T} \varepsilon \nabla u_h \nabla v_T - \int_{\omega_T} u_h v_T$$

for all $v_T \in V_T$. The local *and* global error estimators *then become*

$$\check{\eta}_{H^1,\mathrm{D},T} := \|\nabla \check{e}_T\|_{\omega_T} \quad , \quad \check{\eta}^2_{H^1,\mathrm{D}} := \sum_{T\in\mathcal{T}_h} \check{\eta}^2_{H^1,\mathrm{D},T} \quad , \quad \check{\eta}^2_{H^1,\mathrm{D},\omega_T} := \sum_{T'\subset\omega_T} \check{\eta}^2_{H^1,\mathrm{D},T'} \quad .$$

This leads to the following theorem whose proof is again omitted.

Theorem 5.6 (Comparison with residual error estimator) *The local problem error estimator $\check{\eta}_{H^1,\mathrm{D},T}$ is related to the residual error estimator $\eta_{H^1,\mathrm{R},T}$ in the following sense:*

$$\begin{aligned} \check{\eta}_{H^1,\mathrm{D},T} &\lesssim \max\{1, \varepsilon^{-1/2} h_{min,T}\} \cdot \eta_{H^1,\mathrm{R},\omega_T} && (39)\\ \eta_{H^1,\mathrm{R},T} &\lesssim \check{\eta}_{H^1,\mathrm{D},\omega_T} \quad . && (40) \end{aligned}$$

If T has at least two Neumann boundary faces then the constant in (39) can depend on the shape of the Neumann boundary.

Hence the usual upper error bound holds (cf. Theorem 5.3). In contrast, the *lower* error bound changes and becomes

$$\check{\eta}_{H^1,\mathrm{D},T} \lesssim \|\nabla(u-u_h)\|_{\omega_T} + \max\{1, \varepsilon^{-1/2} h_{min,T}\} \cdot \left[\varepsilon^{-1/2}\alpha_T \cdot \|u-u_h\|_{\omega_T} + \zeta_{H^1,T}\right] \quad .$$

Here, however, we can draw a different conclusion. If the element T has lengths such that $\varepsilon^{-1/2} h_{min,T} < 1$ then this lower bound coincides with the usual lower bound (as in (16) or Remark 5.1). Hence one may solve the reduced problem of Definition 5.4 for such elements.

Finally we remark that other choices of the local space V_T or different parameters δ_E to define the squeezed face bubble functions b_{E,δ_E} do not improve the results.

6 Numerical experiments

Here we aim to verify our theoretical results. To this end we discuss in detail the numerical performance of the residual error estimator $\eta_{H^1,\mathrm{R},T}$ (Theorem 4.1) and of the local problem error estimator $\eta_{H^1,\mathrm{D},T}$ (Theorem 5.3). Additionally we present the results of the modified estimators graphically. This should give enhanced insight although we are aware that it is impossible to demonstrate every feature of the modified estimators on just a single example.

Starting with the problem description, we solve

$$-\varepsilon\Delta u + u = 0 \quad \text{in } \Omega := (0,1)^3 \qquad , \qquad u = u_\mathrm{D} \quad \text{on } \Gamma_\mathrm{D} := \partial\Omega \qquad .$$

The exact solution is prescribed to be

$$u = e^{-x/\sqrt{\varepsilon}} + e^{-y/\sqrt{\varepsilon}} + e^{-z/\sqrt{\varepsilon}}, \qquad \varepsilon = 10^{-4},$$

with u_D chosen accordingly. This anisotropic solution exhibits three distinct boundary layers. We utilize a sequence of tetrahedral meshes $\mathcal{T}_k$, $k = 1\ldots6$, that are the tensor product of three 1D Bakhvalov type meshes, each having 2^k intervals and the transition point $\tau := \sqrt{\varepsilon}|\ln\sqrt{\varepsilon}|$, cf. Figure 4 and [Kun01b]. Strictly speaking these meshes do not satisfy our mesh requirements since the dimensions of neighbouring tetrahedra may change

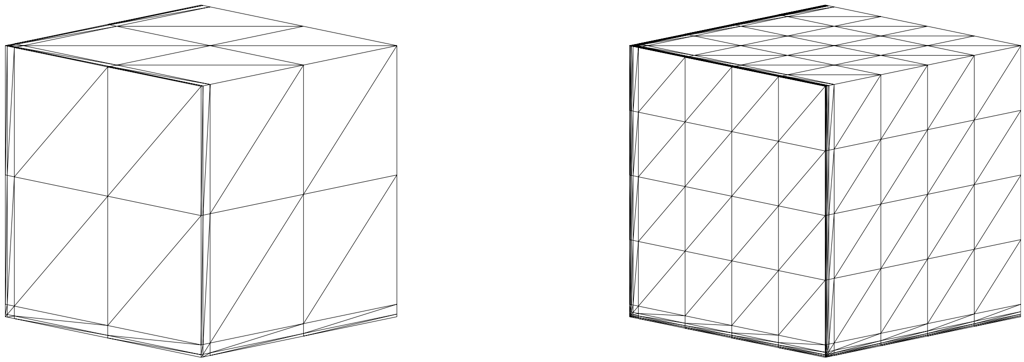

Figure 4: Mesh 2 – Mesh 3

heavily. This happens, however, only at the transition point. Since the solution is well resolved there, the adverse effect of the significantly different element sizes can be neglected.

The following table provides details of the meshes and of the error.

Mesh $\mathcal{T}_k$	# Elements	$\max\limits_{T\in\mathcal{T}_k} \frac{h_{1,T}}{h_{3,T}}$	$\|\nabla(u-u_h)\|_\Omega$	$m_1(u-u_h,\mathcal{T}_k)$
1	48	29.4	9.23E+0	1.55
2	384	69.5	4.57E+0	1.62
3	3 072	82.6	2.18E+0	1.69
4	24 576	88.6	1.08E+0	1.88
5	196 608	91.5	5.46E−1	2.37
6	1 572 864	92.9	2.79E−1	3.04

The numerical convergence rate of approximately $N^{-0.33}$ is close to the optimal value. In conjunction with the comparatively small values of the matching function this confirms that the anisotropic meshes $\mathcal{T}_k$ are well–suited to discretize our problem.

6.1 The original estimators $\eta_{H^1,\mathrm{R},T}$ and $\eta_{H^1,\mathrm{D},T}$

Let us start with the main results and consider the residual error estimator $\eta_{H^1,\mathrm{R},T}$ and the local problem error estimator $\eta_{H^1,\mathrm{D},T}$. Note that the data approximation terms vanish, $\zeta_{H^1,T} = \zeta_{H^1} = 0$. Denote the error by $e := u - u_h$ for the remainder of this section.

In order to investigate the *upper error bounds* (cf. (17) and (36)), we compute the terms $\|\nabla e\|_\Omega/(m_1\,\eta_{H^1,*})$ which have to bounded from above. The table below confirms this behaviour:

	Relation	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$
$\|\nabla e\|_\Omega/(m_1\,\eta_{H^1,\mathrm{R}})$	(17)	0.259	0.141	0.111	0.094	0.073	0.057
$\|\nabla e\|_\Omega/(m_1\,\eta_{H^1,\mathrm{D}})$	(36)	0.581	0.692	0.694	0.620	0.488	0.383

These results are also given graphically in Figure 5. We remark that the error is increasingly overestimated, and the residual estimator overestimates more that the local problem estimator. Such a behaviour has been experienced before for different problems, see [Kun00, Kun01a] and [Kun01b, Kun01c].

Secondly, to verify the *lower error bounds* (16) and (35), we proceed similarly by computing the corresponding element–wise ratios, cf. the next table below. These ratios have to bounded from above uniformly in $T \in \mathcal{T}_k$. This is confirmed by the next table. As before, the results are also found in Figure 6.

	Relation	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$
$\max\limits_{T\in\mathcal{T}_k} \dfrac{\eta_{H^1,\mathrm{R},T}}{\|\nabla e\|_{\omega_T} + \varepsilon^{-1/2}\alpha_T\|e\|_{\omega_T}}$	(16)	1.026	2.699	4.229	4.186	4.085	4.010
$\max\limits_{T\in\mathcal{T}_k} \dfrac{\eta_{H^1,\mathrm{D},T}}{\|\nabla e\|_{\omega_T} + \varepsilon^{-1/2}\|e\|_{\omega_T}}$	(35)	0.449	0.509	0.549	0.550	0.542	0.532

Thirdly, Theorem 5.2 states the *equivalence* of both local estimators. Hence the related ratios have to bounded from above uniformly in $\mathcal{T}_k$ which is verified by the next table as well as by Figures 7 and 8.

	Relation	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$
$\max\limits_{T\in\mathcal{T}_k} \eta_{H^1,\mathrm{R},T}/\eta_{H^1,\mathrm{D},\omega_T}$	(31)	1.611	4.385	4.971	4.864	4.721	4.781
$\max\limits_{T\in\mathcal{T}_k} \eta_{H^1,\mathrm{D},T}/\eta_{H^1,\mathrm{R},\omega_T}$	(30)	0.291	0.284	0.351	0.327	0.256	0.210

6.2 The modified estimators

After having examined the main results for the original estimators $\eta_{H^1,\mathrm{R},T}$ and $\eta_{H^1,\mathrm{D},T}$, we now turn to the modified residual estimator $\tilde{\eta}_{H^1,\mathrm{R},T}$ and both modified local problem estimators $\tilde{\eta}_{H^1,\mathrm{D},T}$ and $\check{\eta}_{H^1,\mathrm{D},T}$. As indicated before, we intend to get some impression about their particularities but cannot expect to observe all features with our single example. Altogether we compare five estimators, i.e. the two original ones and three modifications.

Starting with the *upper error bound* again, we compute the ratio $\|\nabla e\|_\Omega/(m_1\,\eta)$ as before, where η is one of the five estimators. The ratios have to be bounded from above for all estimators except the first modified local problem estimator $\tilde{\eta}_{H^1,\mathrm{D}}$. This is confirmed by the results of Figure 5. Note that even the ratios for $\tilde{\eta}_{H^1,\mathrm{D}}$ are bounded from above. We believe that this behaviour is due to our particular problem and the well–adapted meshes. The error is (locally) overestimated on those elements T with a large minimal size $h_{min,T}$. There, however, the error is small, and the global influence of the overestimation should be neglectable.

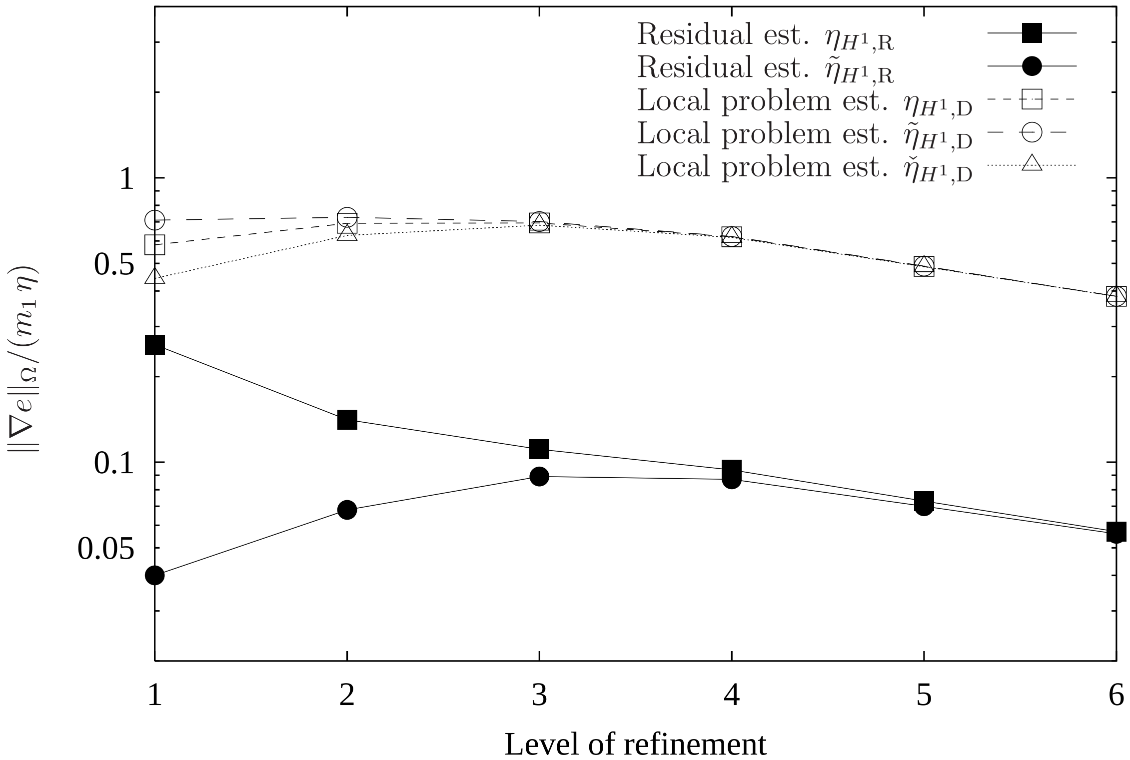

Figure 5: Upper error bound

In order to analyse the *lower error bounds*, proceed analogously as before and compute the ratios $\eta_T/(\|\nabla e\|_{\omega_T}+\gamma\cdot\|e\|_{\omega_T})$, where η_T is one of the five local estimators. The factor γ equals $\varepsilon^{-1/2}\alpha_T$ for the residual estimator and both modified local problem estimators; the factor γ is $\varepsilon^{-1/2}$ for the local problem estimator, and γ is $\varepsilon^{-1}h_{min,T}$ for the modified residual estimator, cf. (16), (21), (35) and Theorems 5.5, 5.6. The ratios have to be bounded from above for all estimators except the second modified local problem estimator $\check{\eta}_{H^1,\mathrm{D},T}$. This can be observed in Figure 6. We even notice that $\check{\eta}_{H^1,\mathrm{D},T}$ performs differently than the other two local problem error estimators. As to be expected, this behaviour occurs on the coarse triangulations where $h_{min,T}$ is large in comparison with $\varepsilon^{1/2}$, cf. Theorem 5.6.

Referring to Remark 5.1, the local problem estimator $\eta_{H^1,\mathrm{D},T}$ performs very similar if the factor $\varepsilon^{-1/2}$ in the ratio above is replaced by $\varepsilon^{-1/2}\alpha_T$.

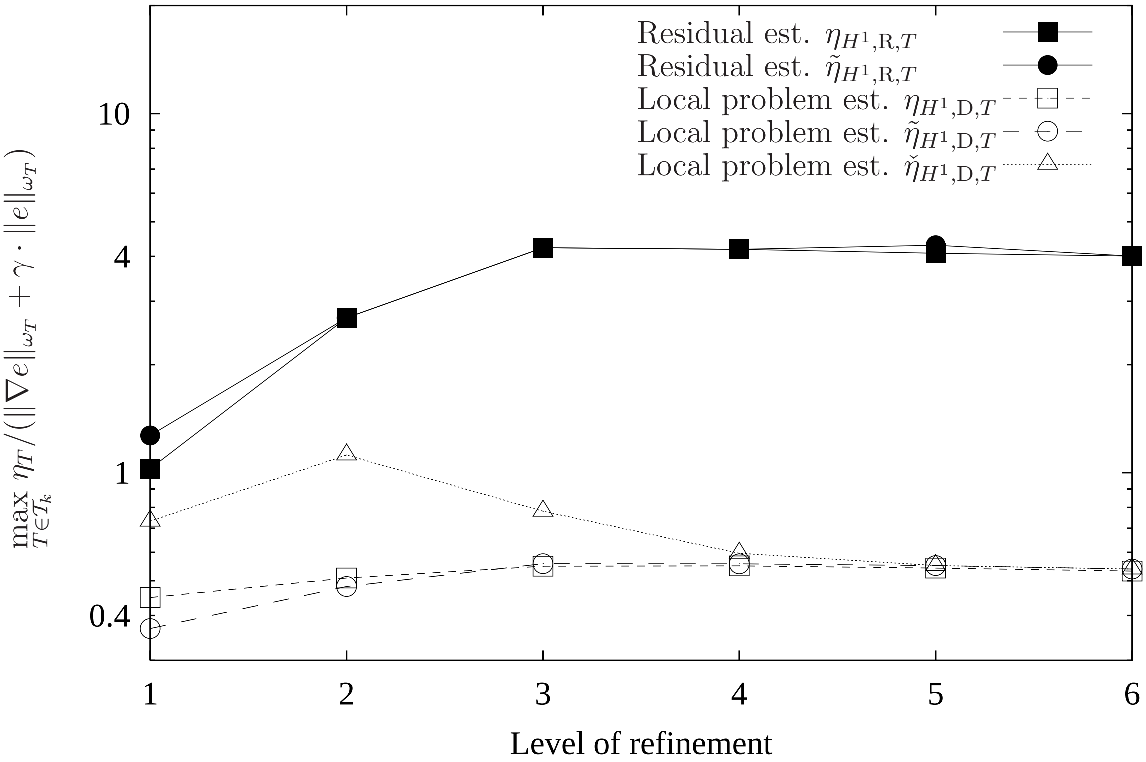

Figure 6: Lower error bound

Finally we explore the local equivalence of estimators. In addition to the results from above we now also compare both modified local problem estimators with the original residual estimator, and the modified residual estimator with the original local problem estimator. The theoretical results for the first pairs of estimators are given in Theorems 5.2, 5.5 and 5.6. For the last pair of estimators we could not establish an equivalence, see Remark 4.3.

Let us start with inequality (31), i.e. $\eta_{H^1,\mathrm{R},T}/\eta_{H^1,\mathrm{D},\omega_T}$ is bounded uniformly on $\mathcal{T}_k$. A corresponding inequality has been proven for the second modified local problem estimator $\check{\eta}_{H^1,\mathrm{D},\omega_T}$ but not for the modified residual estimator $\tilde{\eta}_{H^1,\mathrm{R},T}$ and the first modified local problem estimator $\tilde{\eta}_{H^1,\mathrm{D},\omega_T}$. The anticipated behaviour is partially seen in Figure 7. Note, however, that the first modified local problem estimator $\tilde{\eta}_{H^1,\mathrm{D},\omega_T}$ does not fail for our example.

For the converse inequality (30) one computes the ratio $\eta_{H^1,\mathrm{D},T}/\eta_{H^1,\mathrm{R},\omega_T}$ and the corresponding terms for the modified estimators. Similarly, the ratio has been proven to be uniformly bounded for the first modified local problem estimator $\tilde{\eta}_{H^1,\mathrm{D},T}$. Utilizing the proof of Theorem 4.3 one easily obtains that $\eta_{H^1,\mathrm{D},T}/\tilde{\eta}_{H^1,\mathrm{R},\omega_T}$ is uniformly bounded as well (this ratio corresponds to the modified residual estimator). In contrast to this, $\check{\eta}_{H^1,\mathrm{D},T}/\eta_{H^1,\mathrm{R},\omega_T}$ (for the second modified local problem estimator) need not be bounded. The expected results are clearly visible in Figure 8.

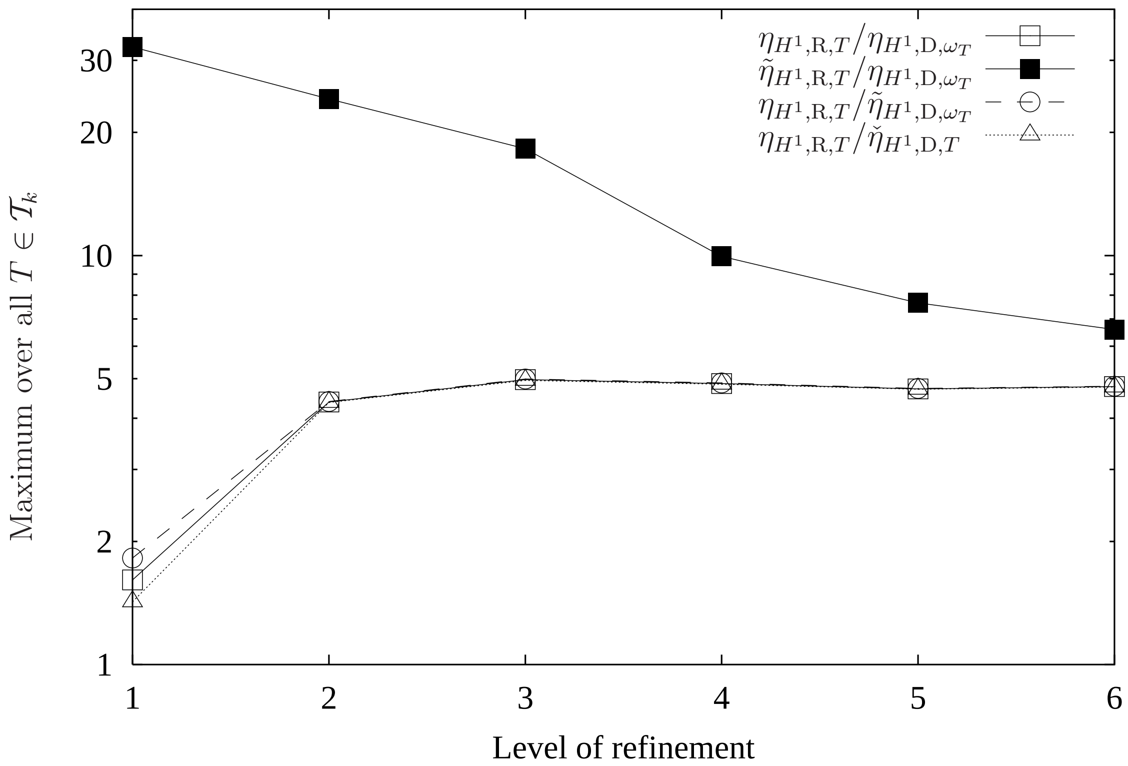

Figure 7: Residual estimator on $T \lesssim$ local problem estimator on ω_T.

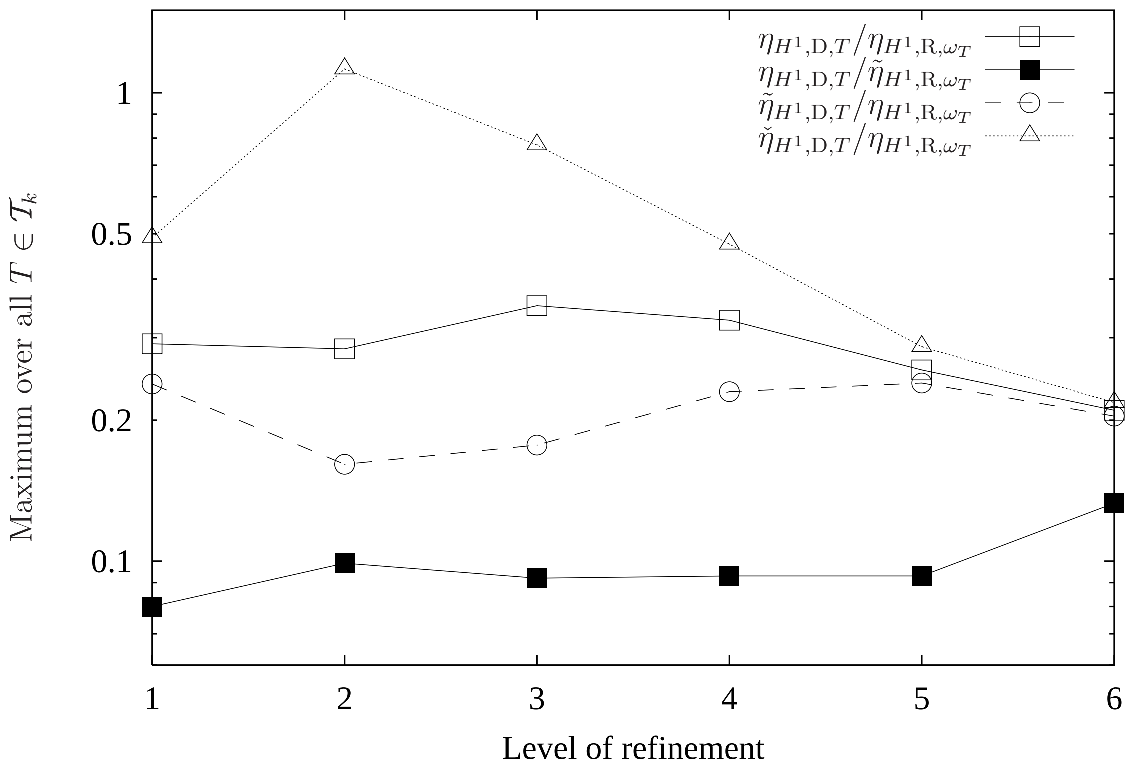

Figure 8: Local problem estimator on $T \lesssim$ residual estimator on ω_T.

7 Summary

For the H^1 seminorm we have investigated *a posteriori* error estimation that is applicable to singularly perturbed reaction diffusion problems on anisotropic meshes. A residual error estimator and a local problem error estimator have been proposed and analysed. They are locally equivalent and bound the error reliably from above provided the anisotropic mesh is sufficiently aligned with the anisotropic solution.

The lower error bound contains an additional L_2 error term. Hence efficient error control is achieved for suitable meshes where some local term is small enough. This local term can be viewed as some analogy to a local mesh Peclet number for convection diffusion problems. The similarity to such convection diffusion problems is also seen in the *structure* of the error bounds, cf. Remark 4.1 or [Ver98a, SK01].

Three further, modified error estimators have been suggested and discussed. Partially they are equivalent to the previous, original versions. Finally, numerical experiments for all estimators complement and confirm the theoretical results.

References

[AO00] M. Ainsworth and J.T. Oden. *A posteriori error estimation in finite element analysis*. Wiley, 2000.

[Cia78] P. G. Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[DGP99] M. Dobrowolski, S. Gräf, and C. Pflaum. On a posteriori error estimators in the finite element method on anisotropic meshes. *Electronic Transactions Num. Anal.*, 8:36–45, 1999.

[FPZ01] L. Formaggia, S. Perotto, and P. Zunino. An anisotropic a-posteriori error estimate for a convection-diffusion problem. *submitted to Comput. Vis. Sci.*, 2001.

[Kun99] G. Kunert. *A posteriori error estimation for anisotropic tetrahedral and triangular finite element meshes*. Logos Verlag, Berlin, 1999. Also PhD thesis, TU Chemnitz, http://archiv.tu-chemnitz.de/pub/1999/0012/index.html.

[Kun00] G. Kunert. An a posteriori residual error estimator for the finite element method on anisotropic tetrahedral meshes. *Numer. Math.*, 86(3):471–490, 2000. DOI 10.1007/s002110000170.

[Kun01a] G. Kunert. A local problem error estimator for anisotropic tetrahedral finite element meshes. *SIAM J. Numer. Anal.*, 39(2):668–689, 2001.

[Kun01b] G. Kunert. Robust a posteriori error estimation for a singularly perturbed reaction–diffusion equation on anisotropic tetrahedral meshes. *to appear in*

Adv. Comp. Math., 2001.
Also http://archiv.tu-chemnitz.de/pub/2000/0086/index.html.

[Kun01c] G. Kunert. Robust local problem error estimation for a singularly perturbed problem on anisotropic finite element meshes. Preprint SFB393/01-01, TU Chemnitz, January 2001. submitted to Math. Model. Numer. Anal.,
Also http://archiv.tu-chemnitz.de/pub/2001/0001/index.html.

[KV00] G. Kunert and R. Verfürth. Edge residuals dominate a posteriori error estimates for linear finite element methods on anisotropic triangular and tetrahedral meshes. *Numer. Math.*, 86(2):283–303, 2000. DOI 10.1007/s002110000152.

[San01] G. Sangalli. A robust a posteriori estimator for the Residual–free Bubbles method applied to advection-diffusion problems. *Numer. Math.*, 89(2):379–399, 2001.

[Sie96] K. G. Siebert. An a posteriori error estimator for anisotropic refinement. *Numer. Math.*, 73(3):373–398, 1996.

[SK01] D. Silvester and D. Kay. The reliability of local error estimators for convection–diffusion equations. *IMA J. Numer. Anal.*, 21(1):107–122, 2001.

[Ver96] R. Verfürth. *A review of a posteriori error estimation and adaptive mesh–refinement techniques*. Wiley-Teubner, Chichester; Stuttgart, 1996.

[Ver98a] R. Verfürth. A posteriori error estimators for convection–diffusion equations. *Numer. Math.*, 80(4):641–663, 1998.

[Ver98b] R. Verfürth. Robust a posteriori error estimators for singularly perturbed reaction–diffusion equations. *Numer. Math.*, 78(3):479–493, 1998.