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Numerische Simulation auf massiv parallelen Rechnern

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Improvements and Experiments on the Bramble–Pasciak Type CG for Mixed Problems in Elasticity

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Abstract

The authors present a generalization of the preconditioning technique introduced by Bramble and Pasciak. It is shown that this generalized variant in connection with the conjugate gradient method remains well–suited for handling, e.g., linear elasticity problems of nearly incompressible materials. Moreover, it can be efficiently implemented on parallel computers, too.

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1 Introduction

The mathematical formulation of many problems in Mathematical Physics and in the Mechanics of Continuous Media leads to boundary value problems which result in linear systems of equations of the form

$$\begin{pmatrix} K & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

after discretization. In this paper the matrix K will be symmetric and positive definite, C is assumed to be positive semidefinite. Because the coefficient matrix of the whole system is indefinite special methods are necessary for solving the given system. In [6], e.g., the usage of preconditioned Uzawa-type iterative methods is described. A drawback of such methods results from the costs of the action of K^{-1} on various vectors being involved in these algorithms.
Looking for a method without the necessity of computing the action of K^{-1} Bramble and Pasciak in [3] introduced a preconditioning technique which leads to an 'optimally' converging scheme applied to the conjugate gradient method. In this paper we will introduce a generalization of this preconditioning technique. We start with a description of the technique given by Bramble and Pasciak with a slight generalization in Section 2. In Section 3 we present the numerical analysis for our technique, followed by some implementational aspects in Section 4. Finally, the results of numerical experiments using our method are given in Section 5.

2 The Preconditioning Technique of Bramble and Pasciak and a Generalization

We consider the system given in Section 1. Let K be a $(N \times N)$–matrix and C a $(m \times m)$–matrix. As already mentioned the given system requires a special numerical treatment because the system matrix is indefinite. In [3] Bramble and Pasciak describe a way to develop a positive definite system for solving the given problem. They start with assuming K_0 to be a preconditioner for K, that is, there are positive constants $\underline{\gamma}$ and $\overline{\gamma}$ with a small ratio $\overline{\gamma}/\underline{\gamma}$ such that

$$\underline{\gamma}(K_0 u, u) \quad \leq \quad (Ku, u) \quad \leq \quad \overline{\gamma}(K_0 u, u) \qquad \text{for all} \quad u \in \mathbf{R^N}.$$

Under these assumptions they multiply the given system from the left with the matrix

$$\begin{pmatrix} K_0^{-1} & O \\ B^T K_0^{-1} & -I \end{pmatrix}$$

and get the system

$$\begin{pmatrix} K_0^{-1} K & K_0^{-1} B \\ B^T K_0^{-1}(K - K_0) & C + B^T K_0^{-1} B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} K_0^{-1} f \\ B^T K_0^{-1} f - g \end{pmatrix}.$$

It is proved that the system matrix of this system is symmetric and positive definite in a special inner product.
Our main idea consists in introducing an additional Matrix B_0 that shall be a preconditioner for $B^T K_0^{-1} B + C$ in some sense:

$$\underline{\beta}(B_0 u, u) \quad \leq \quad ((B^T K_0^{-1} B + C)u, u) \quad \leq \quad \overline{\beta}(B_0 u, u) \qquad \text{for all} \quad u \in \mathbf{R}^{\mathbf{m}}.$$

We insert B_0^{-1} instead of the unity matrix I within the matrix for multiplication given above. Additionally, we include two scaling factors δ and γ which, to begin with, only have to be positive. Finally we get as new system matrix

$$\mathcal{A} = \begin{pmatrix} K_0^{-1} & O \\ \delta B_0^{-1} B^T K_0^{-1} & -\gamma \delta B_0^{-1} \end{pmatrix} \begin{pmatrix} K & B \\ B^T & -C \end{pmatrix}.$$

In the following section we will show that these generalizations are possible without losing the properties proven by Bramble and Pasciak. Only a restriction for the choice of γ will be necessary. By the way, newer results for the analysis of the technique of Bramble and Pasciak are to be found in [8].

3 Numerical Analysis

It is

$$\mathcal{A} = \begin{pmatrix} K_0^{-1} K & K_0^{-1} B \\ \delta B_0^{-1} B^T (K_0^{-1} K - \gamma I) & \delta B_0^{-1} (B^T K_0^{-1} B + \gamma C) \end{pmatrix}$$

and consequently

$$\mathcal{A}\mathbf{x} = \begin{pmatrix} K^{-1} K_0 \overline{x} + K_0^{-1} B \underline{x} \\ \delta B_0^{-1} B^T (K_0^{-1} K - \gamma I)\overline{x} + \delta B_0^{-1} (B^T K_0^{-1} B + \gamma C)\underline{x} \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} \overline{x} \\ \underline{x} \end{pmatrix}$$

with $\overline{x} \in \mathbf{R}^{\mathbf{N}}$ and $\underline{x} \in \mathbf{R}^{\mathbf{m}}$. In the following let $(K - \gamma K_0)$ be positive definite, what obviously means a condition for the choice of γ:
From the assumption for K_0 we have

$$\underline{\gamma} K_0 \;\leq\; K, \quad \text{i.e.} \quad K - \gamma K_0 \;>\; 0 \quad \text{if} \quad \gamma \;<\; \underline{\gamma}.$$

Then we can define a special inner product by

$$< \mathbf{x}, \mathbf{y} > := ((K - \gamma K_0)\overline{x}, \overline{y}) + ((\delta^{-1} B_0)\underline{x}, \underline{y}),$$

where $(.,.)$ denotes the usual inner product in $\mathbf{R}^{\mathbf{N}}$ or in $\mathbf{R}^{\mathbf{m}}$, respectively. This leads to

$$\begin{aligned} < \mathcal{A}\mathbf{x}, \mathbf{y} > \;&=\; ((K - \gamma K_0)(K_0^{-1} K \overline{x} + K_0^{-1} B \underline{x}), \overline{y}) \\ &\quad + (\delta^{-1} B_0 \delta B_0^{-1} [B^T (K_0^{-1} K - \gamma I)\overline{x} + (B^T K_0^{-1} B + \gamma C)\underline{x}], \underline{y}) \\ &=\; ((K - \gamma K_0)(K_0^{-1} K)\overline{x}, \overline{y}) + ((K K_0^{-1} - \gamma I) B \underline{x}, \overline{y}) \\ &\quad + (B^T (K_0^{-1} K - \gamma I)\overline{x}, \underline{y}) + (B^T K_0^{-1} B + \gamma C)\underline{x}, \underline{y}) \\ &=\; ((K - \gamma K_0)(K_0^{-1} K)\overline{x}, \overline{y}) + ((K K_0^{-1} - \gamma I) B \underline{x}, \overline{y}) \\ &\quad + (\overline{x}, (K K_0^{-1} - \gamma I) B \underline{y}) + (B^T K_0^{-1} B + \gamma C)\underline{x}, \underline{y}). \end{aligned}$$

Thus, $\mathcal{A}$ is symmetric with respect to the inner product defined. In the following we will prove that $\mathcal{A}$ is also positive definite (if $(K-\gamma K_0)$ is positive definite as already stated above). This is a direct consequence of the suceeding lemma.

Lemma. Let $\mathcal{M}$ be defined by

$$\mathcal{M} := \begin{pmatrix} K_0^{-1}K & O \\ O & \delta B_0^{-1}(B^T K_0^{-1} B + \gamma C) \end{pmatrix}.$$

Then the following inequalities hold:

$$\underline{\alpha} < \mathcal{M}\mathbf{x}, \mathbf{x} > \quad \leq \quad < \mathcal{A}\mathbf{x}, \mathbf{x} > \quad \leq \quad \overline{\alpha} < \mathcal{M}\mathbf{x}, \mathbf{x} > . \tag{1}$$

Whereas Bramble and Pasciak use some decomposition to prove a similar result in their paper, we will prove this lemma using considerations on spectra.

Proof. Because

$$\mathcal{M}^{-1} = \begin{pmatrix} K^{-1}K_0 & O \\ O & (B^T K_0^{-1} B + \gamma C)^{-1}\delta^{-1} B_0 \end{pmatrix}$$

we have

$$\mathcal{M}^{-1}\mathcal{A} = \begin{pmatrix} I & K^{-1}B \\ (B^T K_0^{-1} B + \gamma C)^{-1} B^T (K_0^{-1}K - \gamma I) & I \end{pmatrix} = \begin{pmatrix} I & E \\ F^T & I \end{pmatrix}$$

with $\quad E = K^{-1}B \quad$ and $\quad F^T = (B^T K_0^{-1} B + \gamma C)^{-1} B^T (K_0^{-1}K - \gamma I)$.

In the following, $\sigma(A)$ will denote the spectrum of a matrix A and $\rho(A)$ its spectral radius. Obviously, all eigenvalues $\lambda \in \sigma(\mathcal{M}^{-1}\mathcal{A})$ either are equal 1 or have the structure

$$\lambda = 1 + \lambda' = 1 \pm \sqrt{\lambda''} \quad \text{with} \quad \lambda' \in \sigma \begin{pmatrix} O & E \\ F^T & O \end{pmatrix} \quad \text{and} \quad \lambda'' \in \sigma(F^T E),$$

respectively. That's why we have

$$\sigma(\mathcal{M}^{-1}\mathcal{A}) \subset \left[1 - \sqrt{\rho(F^T E)}, 1 + \sqrt{\rho(F^T E)}\right].$$

Consequently, (1) is valid with

$$\underline{\alpha} = 1 - \sqrt{\rho(F^T E)} \quad \text{and} \quad \overline{\alpha} = 1 + \sqrt{\rho(F^T E)}.$$

Let us consider $\rho(F^T E)$ now. It holds

$$\rho(F^T E) \quad = \quad \max_{\underline{x}} \frac{(B^T (K_0^{-1}K - \gamma I) K^{-1} B \underline{x}, \underline{x})}{((B^T K_0^{-1} B + \gamma C)\underline{x}, \underline{x})}$$

$$
\begin{aligned}
&\le \max_{\underline{x}} \frac{(B^T(K_0^{-1} - \gamma K^{-1})B\underline{x}, \underline{x})}{(B^T K_0^{-1} B\underline{x}, \underline{x})} \\
&\le \max_{\overline{y}} \frac{((K_0^{-1} - \gamma K^{-1})\overline{y}, \overline{y})}{(K_0^{-1}\overline{y}, \overline{y})} \quad \text{with} \quad \overline{y} \;=\; B\underline{x} \\
&= \max_{\overline{z}} \frac{((K_0 - \gamma K_0 K^{-1} K_0)\overline{z}, \overline{z})}{(K_0\overline{z}, \overline{z})} \quad \text{where} \quad \overline{y} \;=\; K_0\overline{z} \\
&= \rho((I - \gamma K^{-1} K_0)).
\end{aligned}
$$

Our assumption for the preconditioner K_0 causes

$$
\sigma(K_0^{-1}K) \subset \left[\underline{\gamma}, \overline{\gamma}\right], \quad \text{and therefore} \quad \sigma(K^{-1}K_0) \subset \left[\frac{1}{\overline{\gamma}}, \frac{1}{\underline{\gamma}}\right].
$$

Thus, we have

$$
\rho(F^T E) \le 1 - \frac{\gamma}{\overline{\gamma}}, \quad \text{i.e.} \quad \underline{\alpha} \;=\; 1 - \sqrt{1 - \frac{\gamma}{\overline{\gamma}}} \quad \text{and} \quad \overline{\alpha} \;=\; 1 + \sqrt{1 - \frac{\gamma}{\overline{\gamma}}}.
$$

Remark. The constants $\underline{\alpha}$ and $\overline{\alpha}$ would tend to one (giving a well conditioned system) if γ tends to $\overline{\gamma}$. Because of $\gamma \;<\; \underline{\gamma}$ (to ensure positive definiteness of $K - \gamma K_0$) γ near $\underline{\gamma}$ should be a good choice.

Let us consider the condition number $\kappa(\mathcal{A})$ of the matrix $\mathcal{A}$ now. From our lemma we have

$$
\underline{\alpha}\,\frac{<\mathcal{M}\mathbf{x}, \mathbf{x}>}{<\mathbf{x}, \mathbf{x}>} \quad \le \quad \frac{<\mathcal{A}\mathbf{x}, \mathbf{x}>}{<\mathbf{x}, \mathbf{x}>} \quad \le \quad \overline{\alpha}\,\frac{<\mathcal{M}\mathbf{x}, \mathbf{x}>}{<\mathbf{x}, \mathbf{x}>}, \quad \text{and thus,}
$$

$$
\kappa(\mathcal{A}) \;\le\; \kappa(\mathcal{M})\frac{\overline{\alpha}}{\underline{\alpha}} \;=\; \kappa(\mathcal{M})\frac{1 + \sqrt{1-\xi}}{1 - \sqrt{1-\xi}} \;=\; \kappa(\mathcal{M})\frac{\left(1 + \sqrt{1-\xi}\right)^2}{\xi} \;\le\; \frac{4}{\xi}\kappa(\mathcal{M}),
$$

$$
\text{where} \quad \xi \;=\; \frac{\gamma}{\overline{\gamma}}.
$$

$\mathcal{M}$ is a symmetric operator with respect to the inner product chosen. Thus, from the block structure of $\mathcal{M}$ we have

$$
\kappa(\mathcal{M}) \;=\; \frac{\lambda_{\max}(\mathcal{M})}{\lambda_{\min}(\mathcal{M})} \;=\; \frac{\max(\overline{\gamma}, \delta\overline{\beta})}{\min(\underline{\gamma}, \delta\underline{\beta})}
$$

with λ_{max} and λ_{min} being the largest and the smallest eigenvalue, respectively, and consequently

$$
\kappa(\mathcal{A}) \;\le\; 4\,\frac{\overline{\gamma}}{\gamma}\,\frac{\max(\overline{\gamma}, \delta\overline{\beta})}{\min(\underline{\gamma}, \delta\underline{\beta})}.
$$

Remark. With a suitable choice of δ we can attain that $\underline{\gamma} \;\le\; \delta\underline{\beta} \;\le\; \delta\overline{\beta} \;\le\; \overline{\gamma}$. In this case we have

$$
\kappa(\mathcal{A}) \;\le\; 4\,\frac{\overline{\gamma}}{\underline{\gamma} - \varepsilon}\,\frac{\overline{\gamma}}{\underline{\gamma}}
$$

if $\gamma \;=\; \underline{\gamma} \;-\; \varepsilon$.

4 Some Aspects of Implementation

Because of the properties of $\mathcal{A}$ proved before a number of iterative techniques can be applied to solve the system with this system matrix. For the numerical experiments presented in the following section we use the conjugate gradient method in the inner product introduced above. For the preconditioner K_0 we will choose the hierarchical preconditioner with a coarse grid solver coupled with some Jacobi preconditioning to handle, e.g., the influence of material coefficients. For B_0 a diagonal matrix is used. The matrices are obtained from a finite element discretization of a typical saddlepoint or mixed problem with either triangles or quadrilaterals.
Our algorithm is implemented in such a way that it runs on parallel computers, too. That is sensible for a number of processors up to the number of elements included within the user grid. Then, each processor deals with, at least, one element of the user grid. The number of iterations needed is independent of the number of processors.

5 Numerical Examples

5.1 Linear Elasticity Problem

In this subsection we will present an important application of the method introduced before. We consider the linear elasticity problem. For the state of plane strain we have the following weak formulation for the equilibrium of forces:

$$\text{Find} \quad \vec{u} \in \left(H_0^1(\Omega)\right)^2 : \quad a_\lambda(\vec{u},\vec{v}) = F(\vec{v}) \quad \text{for all} \quad \vec{v} \in \left(H_0^1(\Omega)\right)^2 \quad \text{with}$$

$$\begin{aligned} a_\lambda(\vec{u},\vec{v}) &= \int \left(2\mu\,\varepsilon(\vec{u}) : \varepsilon(\vec{v}) + \lambda \operatorname{div}\vec{u} \operatorname{div}\vec{v}\right) d\,\Omega \quad \text{and} \\ F(\vec{v}) &= \int_\Omega \vec{f}\,\vec{v}\,d\,\Omega + \int_{\Gamma^2} \vec{g}^{\,2}\vec{v}\,d\,\Gamma\,. \end{aligned}$$

Here,

$$\varepsilon(\vec{u}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right)$$

is the linear strain tensor. Furthermore, it denotes

$\vec{u}$	the displacement vector $\vec{u} = (u_1, u_2)^T$,
$\vec{f}$	the vector of the body forces $\vec{f} = (f_1, f_2)^T \in (L_2(\Omega))^2$, and
$\vec{g}^2$	the vector of the specified boundary forces $\vec{g}^2 = (g_1^2, g_2^2)^T \in (H^{1/2}(\Gamma^2))^2$.

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

are LAME's coefficients with YOUNG's modulus E and POISSON's ratio ν ($\nu \in [0, 0.5)$). If ν tends to 0.5 (i.e. if the material becomes nearly incompressible) the coefficient λ tends to infinity. Thus, for this case a special treatment would be necessary. Defining

$$p = \lambda \operatorname{div} \vec{u}$$

we get the problem

$$\text{Find} \quad \vec{u} \in V = \left(H_0^1(\Omega)\right)^2 \quad \text{and} \quad p \in M = L_2(\Omega) :$$

$$\begin{aligned} a_0(\vec{u}, \vec{v}) + b(p, \vec{v}) &= F(\vec{v}) \quad \text{for all} \quad \vec{v} \in V \\ b(q, \vec{u}) - < \frac{1}{\lambda} p, q > &= 0 \quad \text{for all} \quad q \in M \end{aligned}$$

$$\text{with} \quad a_0(\vec{u}, \vec{v}) = \int_\Omega 2\mu\, \varepsilon(\vec{u}) : \varepsilon(\vec{v}) \, d\,\Omega \quad \text{and} \quad b(p, \vec{v}) = \int_\Omega p \operatorname{div} \vec{v} \, d\,\Omega \,.$$

Applying a finite element discretization on this problem we get

$$\text{Find} \quad \vec{u}_h \in V_h \subset V \quad \text{and} \quad p_h \in M_h \subset M :$$

$$\begin{aligned} a_0(\vec{u}_h, \vec{v}_h) + b(p_h, \vec{v}_h) &= F(\vec{v}_h) \quad \text{for all} \quad \vec{v}_h \in V_h \\ b(q_h, \vec{u}_h) - < \frac{1}{\lambda} p_h, q_h > &= 0 \quad \text{for all} \quad q_h \in M_h \,. \end{aligned}$$

Here, among other assumptions, the so–called discrete LBB–condition has to be fulfilled, i.e. there must exist a constant $\beta > 0$ such that

$$\sup_{\vec{v}_h \in V_h} \frac{b(q_h, \vec{v}_h)}{\|\vec{v}_h\|_V} \geq \beta \|q_h\|_M \quad \text{for all} \quad q_h \in M_h \quad (\vec{v}_h \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}).$$

For this, the choice of the finite element spaces V_h and M_h takes a leading part. In [6], e.g., a survey of some combinations of possible finite elements is given. Further information can be found, e.g., in [1, 5, 7, 4]. In our experiments we have used the so–called Taylor–Hood–Elements (cf., e.g., [2]), i.e. we use six node triangles (or the appropriate quadrilaterals, respectively) and take linear (bilinear) elements for p and piecewise linear elements (being linear on subelements we get from dividing the elements into four parts) for $\vec{u}$.
Choosing suitable finite element basis functions $\vec{\Phi}$ on V_h and Ψ on M_h ($\vec{\Phi}$ and Ψ are row–vectors of (vector–/ scalar–)functions) the discrete problem is equivalent to the linear system

$$\begin{pmatrix} K & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix},$$

where $\vec{u}_h = \vec{\Phi}\underline{u}$ and $p_h = \Psi\underline{p}$. Here,

$$
\begin{aligned}
K &= \left(a_0(\vec{\phi}_j, \vec{\phi}_i)\right) \quad i, j = 1 \dots N\,, \\
B &= \left(b(\psi_j, \vec{\phi}_i)\right) \quad i = 1 \dots N,\ j = 1 \dots m\,,\ \text{and} \\
C &= \left(< \frac{1}{\lambda}\psi_j, \psi_i >\right) \quad i, j = 1 \dots m
\end{aligned}
$$

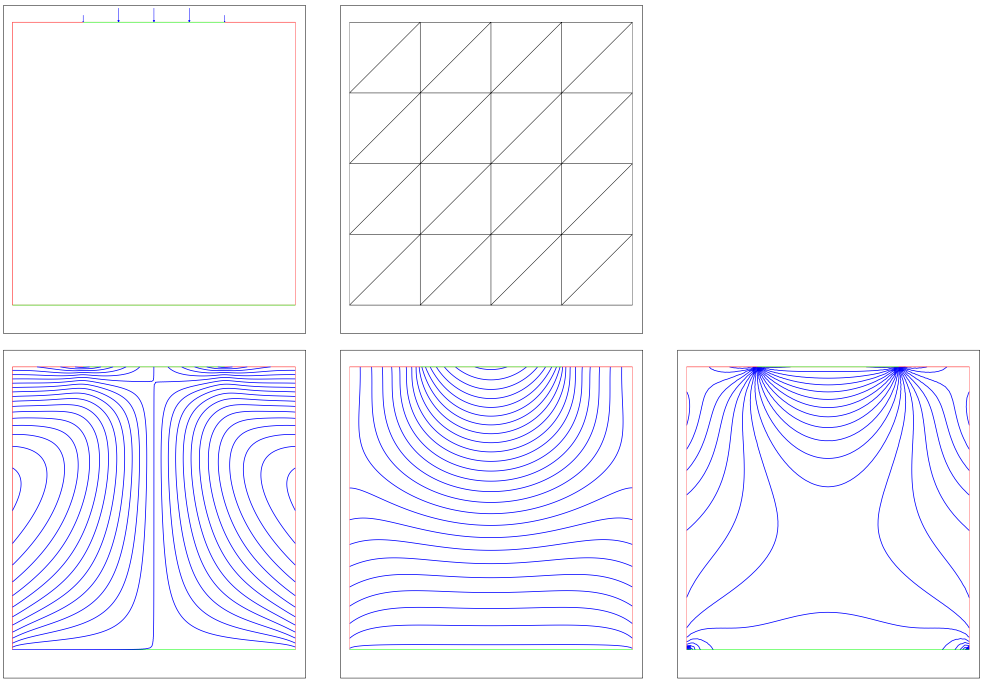
If the ψ_j are constant on each element the matrix C is diagonal, and we can eliminate p from the system (on element level). In our case the ψ_j are piecewise linear on the elements such that a solver for the whole system is required. This system is exactly of the type introduced in section 1.

5.2 Test Examples and numerical results

In the following we present some numerical results of calculations carried out using the algorithm described above. We consider the following three test examples for the linear elasticity problem where the pictures always show the domain Ω, the user grid and calculated isolines for the components u_1 and u_2 as well as for the pressure p:

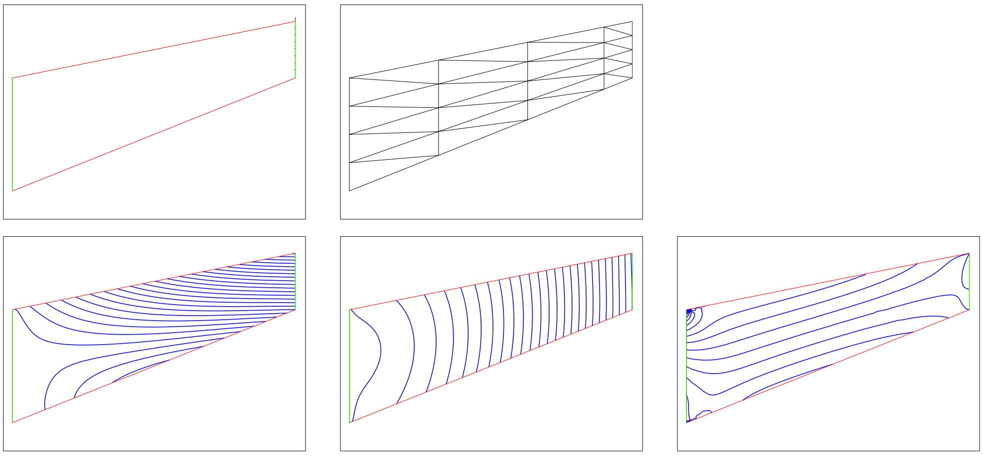
`Example 1:`

We used $\Omega = [0,4]^2$, $E = 20000.$, and some values of ν between 0.3 and 0.5 (cf. the table). As boundary conditions $\vec{u} = \vec{0}$ on $\{y = 0\}$ and $\vec{g}^2 = (0,1)^T$ on $\{x \in [1,3],\ y = 4\}$ were choosen.



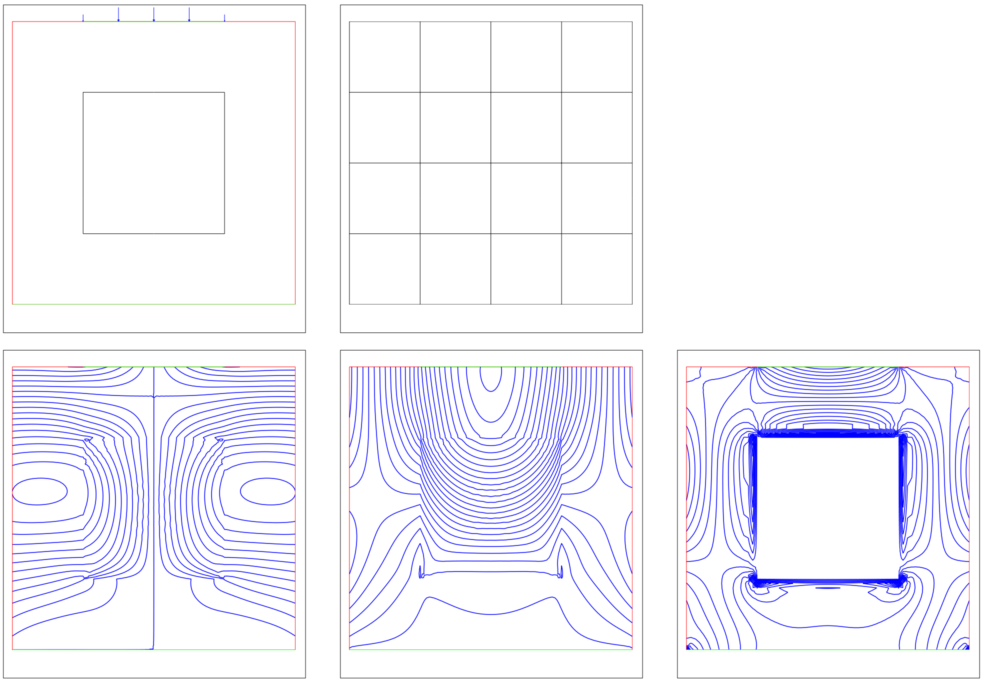
`Example 2:`

We used Ω as illustrated by the picture, $E = 100000.$, and some values of ν as for the first example. For the boundary conditions we put $\vec{u} = \vec{0}$ on $\{x = 0\}$ and $\vec{g}^2 = (0, 1)^T$ on the right boundary ($\{x = 10.\}$).



`Example 3:`

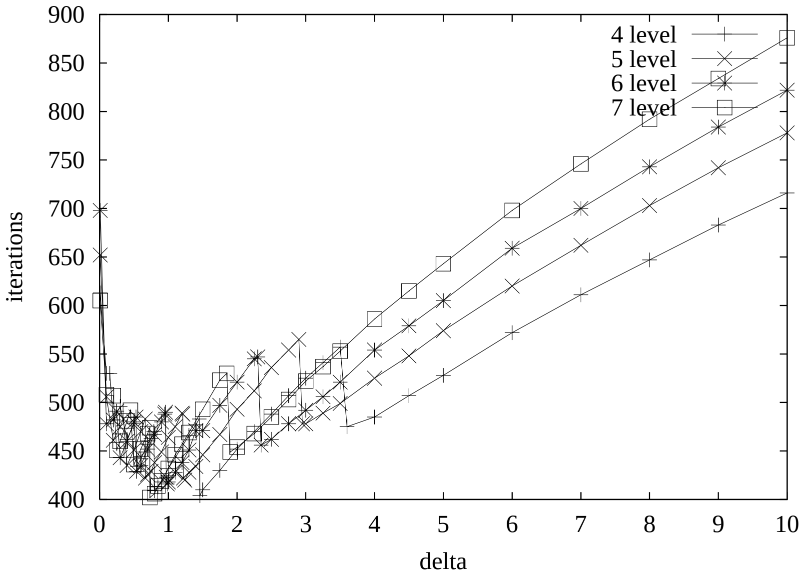
Here we used $\Omega = [0, 4]^2$ again, but this time a nearly incompressible material with $E = 1.$ was simulated in the inner part ($[1, 3]^2$). For the outer part $E = 200000.$ was used; for the values of ν cf. the table again. The boundary conditions were the same as for Example 1.



In the following table the total numbers of degrees of freedom are given for different numbers of refinement levels (valid for all three examples; 0 level means the user grid).

level	0	1	2	3	4	5	6	7
d. of fr.	243	867	3267	12675	49923	198147	789507	3151875

At first we will give some results for the choice of γ and δ. It will be our aim, of course, to get iteration numbers as low as possible to fulfill some error criterion for stopping the iteration process. From our analytical considerations it follows that a parameter γ as big as possible (i.e. near the bound $\underline{\gamma}$) will be a good choice to reach this. Thus, we started with looking for such a value of γ. More precisely, we searched for a γ still yielding to a positive definite system independent of δ. It turned out that small values of δ especially have to be considered for this purpose. In a second step we looked for a value of δ for which a low number of iterations is necessary to solve the resulting system using the parameter γ found out. In addition we repeated this for some smaller values of γ. Indeed, our experiments have shown that our given assumption referring to γ is true at least for the first and the third example. For the second example we could observe that there are relatively low values of γ (typically about one third of the "maximal" value) with iteration numbers being slightly lower for certain choices of δ than those we can reach using the big γ. In the following scheme we present the iteration numbers needed for the second example in dependence on δ using the "maximal" γ for four up to seven levels.



The following table includes the minimal iteration numbers and the attached values of γ and δ found for the first and third example. In the second column "3" means triangular grids, and "4" stands for quadrilaterals.

Example	type	ν	1 level			2 level			3 level			4 level		
			γ	δ	it.	γ	δ	it.	γ	δ	it.	γ	δ	it.
1	3	.3	7.35E-2	0.75	38	4.912E-2	0.66	52	3.09E-2	.52	63	2.16E-2	0.4	73
		.4	7.35E-2	1.29	46	4.912E-2	0.72	61	3.137E-2	.69	75	2.161E-2	0.88	85
		.45	7.35E-2	1.0	53	4.912E-2	0.99	69	3.137E-2	.97	86	2.161E-2	0.77	102
		.5	7.35E-2	0.85	64	4.912E-2	0.97	87	3.137E-2	.90	106	2.161E-2	0.65	125
1	4	.3	1.52E-1	0.82	38	9.94E-2	0.78	44	6.89E-2	.60	49	5.012E-2	0.48	54
		.4	1.523E-1	0.99	44	9.944E-2	0.74	54	6.896E-2	.54	60	5.012E-2	0.84	61
		.45	1.523E-1	1.2	49	9.944E-2	1.04	62	6.896E-2	.86	70	5.012E-2	0.76	76
		.5	1.523E-1	1.12	58	9.944E-2	0.95	73	6.891E-2	.75	83	5.012E-2	1.13	89
3	4	.3, .4	4.1794E-2	0.49	76	3.8695E-2	1.15	76	3.7038E-2	.95	84	3.5791E-2	0.93	87
		.3, .45	4.1793E-2	1.33	78	3.8694E-2	1.08	78	3.7037E-2	.37	86	3.579E-2	0.95	88
		.3, .5	4.1792E-2	0.78	234	3.8693E-2	0.97	279	3.7036E-2	.87	314	3.5789E-2	0.78	331

Example	type	ν	5 level			6 level			7 level		
			γ	δ	it.	γ	δ	it.	γ	δ	it.
1	3	.3	1.57E-2	0.35	75	1.19E-2	.29	79	9.35E-3	.286	85
		.4	1.573E-2	0.745	90	1.194E-2	.712	95	9.357E-2	.717	102
		.45	1.573E-2	0.80	112	1.194E-2	.44	118	9.357E-2	.34	127
		.5	1.573E-2	1.1	138	1.194E-2	.70	141	9.357E-2	.54	145
1	4	.3	3.794E-2	0.30	51	2.96E-2	.24	54	2.37E-2	.26	57
		.4	3.795E-2	0.59	63	2.964E-2	.47	64	2.375E-2	.41	68
		.45	3.795E-2	0.94	74	2.964E-2	.31	76	2.375E-2	.695	80
		.5	3.795E-2	0.745	87	2.964E-2	.49	85	2.375E-2	.365	90
3	4	.3, .4	3.440E-2	0.76	80	2.9575E-2	.64	89	2.3715E-2	.525	93
		.3, .45	3.440E-2	0.52	91	2.9575E-2	.76	94	2.3715E-2	.69	99
		.3, .5	3.4395E-2	0.86	348	2.9432E-2	.77	377	2.3691E-2	.735	414

5.3 Calculations on a PC Cluster

As mentioned above our algorithm was also implemented for use on parallel computers. We tested it on the "Chemnitzer Linux Cluster" (CLiC), a cluster consisting of 528 personal computers each of them having a Intel Pentium III processor (800 MHz). In the table some CPU times are presented showing the efficiency of our parallization. We give two times in each column: the CPU time totally needed for the calculations and the maximal time needed for arithmetics on a single processor. For 6 and 7 levels some calculations could not carried out with low numbers of processors because there was not enough memory available. In these cases we put a dash into the according columns of the table. That means (as to be expected) that using a cluster enables calculations with more unknowns if necessary. — Considering the total CPU times for a fixed number of levels and a increasing number of processors a decrease is to be seen in most cases. Of course, for low numbers of levels the use of many processors is not effective at all because communication times strongly dominate in such cases (compare the two values in the according columns). On the other hand, for higher numbers of unknowns the CPU times approximately are halved if the number of processors is doubled — i.e. our way of parallization is very effective as stated before. This assertion is also proved if we compare the CPU times

Example	Processors	1 level		2 level		3 level		4 level		5 level		6 level		7 level	
2	1	0.2	0.15	1.1	1.1	5.6	5.6	28.5	28.5	120.9	120.9	-	-	-	-
$\nu = 0.4$	2	0.4	0.08	0.9	0.6	3.3	2.7	14.9	14.1	63.5	62.4	-	-	-	-
triangles	4	0.6	0.06	0.8	0.3	2.2	1.3	7.4	6.2	33.1	31.3	141.8	138.5	-	-
	8	0.8	0.05	1.0	0.1	1.7	0.6	4.4	2.9	16.5	14.8	70.3	66.6	-	-
	16	1.0	0.07	1.2	0.1	1.6	0.3	3.0	1.4	8.3	6.4	36.4	33.7	156.0	149.8
3	1	0.1	0.09	0.8	0.8	4.0	4.0	21.9	21.9	88.8	88.8	-	-	-	-
$\nu_1 = 0.3$	2	0.3	0.08	0.7	0.4	2.6	2.0	10.8	10.3	46.4	45.1	191.2	188.6	-	-
$\nu_2 = 0.5$	4	0.5	0.08	0.7	0.2	1.8	1.0	5.6	5.6	23.1	21.9	96.1	93.5	-	-
quadrila-	8	0.7	0.04	0.9	0.1	1.4	0.5	3.4	2.1	11.7	10.1	49.0	46.6	202.8	198.1
terals	16	0.9	0.04	1.1	0.1	1.5	0.2	2.5	1.1	6.2	4.6	25.1	22.9	103.6	99.5
	32			1.3	0.1	1.6	0.2	2.2	0.5	4.2	2.2	12.9	10.5	55.3	51.7

needed or certain numbers of levels and processors with those needed for one more level (i.e. the fourfold number of unknowns) and the fourfold number of processors. In this case we have nearly the same times.

6 Summary

We introduced a generalization of the preconditioning technique of Bramble and Pasciak. It was shown that the choosen way of generalization preserves the properties of the original method. The user of our method can choose the values of two parameters. We illustrated the application of the method to linear elasticity problems especially including problems of nearly incompressible materials. Our algorithm was implemented for use on parallel computers. Results given for calculations on a PC cluster demonstrated the high efficiency of the parallel code.

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