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A preconditioner for solving the inner problem of the p-version of the FEM, Part II- algebraic multi-grid proof

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Abstract

Finding a fast solver for the inner problem in a DD preconditioner for the p-version of the FEM is a difficult question. We discovered, that the system matrix for the inner problem in $2D$ has a similar structure to matrices resulting from discretizations of $-y^2u_{xx} - x^2u_{yy}$ in the unit square using h-version of the FEM or finite differences. Applying multi-grid methods with special smoothers, we have a fast solver for the p-version of the FEM. We give a convergence proof for the multi-grid method and the AMLI-method and present some numerical experiments confirming the theory.

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Contents

1 Introduction

Jensen/Korneev [12] and Ivanov/Korneev [10],[11] developed preconditioners for the $p-$version of the FEM in a two-dimensional domain. They used DD-methods. The unknowns are splitted into 3 groups, the interior, the edge and vertex unknowns

$$A = \begin{pmatrix} A_{vert} & A_{vert,edg} & A_{vert,int} \\ A_{edg,vert} & A_{edg} & A_{edg,int} \\ A_{int,vert} & A_{int,edg} & A_{int} \end{pmatrix}.$$

The vertex unknowns can be solved separately, cf. Lemma 2.3 [10], using

$$C = \begin{pmatrix} A_{vert} & & \\ & A_{edg} & A_{edg,int} \\ & A_{int,edg} & A_{int} \end{pmatrix}.$$

Computing the other unknowns, we factorize the remaining stiffness matrix as follows

$$\begin{pmatrix} A_{edg} & A_{edg,int} \\ A_{int,edg} & A_{int} \end{pmatrix} = \begin{pmatrix} I & A_{edg,int}A_{int}^{-1} \\ & I \end{pmatrix} \begin{pmatrix} S & \\ & A_{int} \end{pmatrix} \begin{pmatrix} I & \\ A_{int}^{-1}A_{int,edg} & I \end{pmatrix}$$

with the Schur-komplement

$$S = A_{edg} - A_{edg,int}A_{int}^{-1}A_{int,edg}.$$

Computing the interior unknowns, we solve a Dirichlet problem on each quadrangle. The vertex unknowns are computed via the Schur-complement S.
We need 3 tools, a preconditioner for the interior problem, a preconditioner for the Schur-komplement and a extension operator from the edges of a quadrangle to the interior. Ivanov/Korneev derived 3 types $C_{i,S}$,$i = 1, \ldots, 3$, of preconditioning the Schur-complement. The condition number for $C_{i,S}^{-1}S$ is in the worst case $\mathcal{O}(log^2 p)$, where p is the polynomial degree. The solution of $C_{i,S}x = y$ costs $\mathcal{O}(p)$ arithmetical operations.
Furthermore, Jensen/Korneev found a spectral equivalent preconditioner for the interior problem, which has $\mathcal{O}(p^2)$ nonzero entries. In the case of parallelogram elements, the element stiffness matrix has $\mathcal{O}(p^2)$ nonzero entries,

too. But, the suggested methods compute the solution in $\mathcal{O}(p^3)$ arithmetical operations. Finding a fast solver for the preconditioner was an open question. This paper is concerned to the construction of a more efficient preconditioner for the interior problem.
We derive a preconditioner for the interior problem, such that the number of iterations of the PCG-method shows an increasing as $\mathcal{O}(\log p)$ or less in numerical experiments and costs of $\mathcal{O}(p^2)$ arithmetical operations. The origin of this preconditioner is the multi-grid method. We give a proof for the convergence of the multi-grid method using the strenghtened Cauchy-inequality.
The paper is organized as follows. In section 2, we consider the stiffness matrix for the model problem and their most important properties. In section 3, we introduce and modify the preconditioner of Jensen/Korneev. Section 4 shows that the modified preconditioner can be obtained by discretizing ellpitic problems with variable coefficients using finite differences or the h-version of the finite element method. In section 5, we give a proof for the convergence of the multi-grid method for this problem with variable coefficients. In section 7, we consider the AMLI-method, [2], [3]. Finally, we consider extensions to the three dimensional case.
Throughout this paper, Ω will denote the unit rectangle $(-1,1)^2$, Ω_1 the rectangle $(0,1)^2$. The integer p is the polynomial degree, $\hat{L}_i$ the $i-$th integrated Legendre polynomial. The real number $\lambda_{max}(A)$ will denote the largest eigenvalue of a matrix A and $\lambda_{min}(A)$ the smallest eigenvalue of A. The parameter c_i will describe a constant, which is independent of p or h.

2 Origin and properties of the stiffness matrix

2.1 Model problem

We try to find a numerical solution of the model problem

$$\begin{aligned} -\triangle u &= f, \text{ in } \Omega = (-1,1)^2 && (2.1)\\ u \mid_{\partial\Omega} &= 0. && (2.2) \end{aligned}$$

Problem (2.1,2.2) is the typical model problem for solving a linear system with the matrix A_{int}.

2.2 Discretization, shape functions

We solve (2.1,2.2) using the $p-$version of the FEM with only one element Ω. As finite element space, we choose

$$M = \{u \in H_0^1(\Omega), u \mid_\Omega \in P^p\},$$

where P^p is the space of all polynomials of degree $\leq p$ in both variables. The discretized problem is: find $u_p \in M$

$$\int_\Omega \nabla u_p \cdot \nabla v_p \mathrm{d}(x,y) = \int_\Omega f v_p \mathrm{d}(x,y)$$

for all $v_p \in M$. As basis in M, we choose the integrated Legendre polynomials, which we define below.
Let for $i = 0, 1, \ldots$

$$L_i(x) = \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}(x^2-1)^i$$

the i-th Legendre polynomial,

$$\tilde{L}_i(x) = \int_{-1}^x L_{i-1}(s)\ \mathrm{d}s$$

the i-th integrated Legendre polynomial, and $\forall i \geq 2$

$$\hat{L}_i(x) = \sqrt{\frac{(2i-3)(2i-1)(2i+1)}{4}}\tilde{L}_i(x) = \gamma_i \tilde{L}_i(x) \tag{2.3}$$

the i-th integrated Legendre polynomial with scaling. By definition,

$$\begin{aligned}\hat{L}_0(x) &= \frac{1+x}{2},\\ \hat{L}_1(x) &= \frac{1-x}{2}.\end{aligned}$$

The properties

$$\int_{-1}^1 L_i(x)L_j(x)\ \mathrm{d}x = \delta_{ij}\frac{2}{2i+1}, \tag{2.4}$$

$$\hat{L}_i(x) = \sqrt{\frac{(2i+1)(2i-3)}{4(2i-1)}}(L_i(x) - L_{i-2}(x)), \tag{2.5}$$

$$\hat{L}_i(1) = 0, \tag{2.6}$$

$$\hat{L}_i(-1) = 0, \tag{2.7}$$

$$(i+1)L_{i+1}(x) + iL_{i-1}(x) = (2i+1)xL_i(x). \tag{2.8}$$

are true for $i \geq 2$, [19].
As basis in M, we choose

$$\hat{L}_{ij}(x,y) = \hat{L}_i(x)\hat{L}_j(y), \tag{2.9}$$

with $p \geq i, j \geq 2$. For satisfying (2.2), the polynomials $\hat{L}_0$ and $\hat{L}_1$ are not used, compare (2.6,2.7). The stiffness matrix K is determined by

$$K = (a_{ij,kl})_{i,j=2;k,l=2}^{p} = \int_{\Omega} \nabla \hat{L}_{ij}(x,y) \cdot \nabla \hat{L}_{kl}(x,y)\ \mathrm{d}(x,y).$$

We get

$$a_{ij,kl} \quad = \quad d_{ik} f_{jl} + f_{ik} d_{jl}, \tag{2.10}$$

where

$$F = \quad (f_{ij})_{i,j=2}^{p} \quad = \int_{-1}^{1} \hat{L}_i(x)\hat{L}_j(x)\ \mathrm{d}x, \tag{2.11}$$

$$D = \quad (d_{ij})_{i,j=2}^{p} \quad = \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\hat{L}_i(x)\frac{\mathrm{d}}{\mathrm{d}x}\hat{L}_j(x)\ \mathrm{d}x. \tag{2.12}$$

Using (2.4,2.5), we determine the entries of the one-dimensional mass matrix, namely

$$F = \begin{pmatrix} 1 & 0 & -c_2 & 0 & \cdots \\ & 1 & 0 & -c_3 & \ddots \\ & & 1 & 0 & \ddots \\ & \text{SYM} & \ddots & \ddots & \ddots \\ & & & & 1 \end{pmatrix}$$

and the one-dimensional stiffness matrix, namely

$$D = \mathrm{diag}(d_i)_{i=2}^{p} = \begin{pmatrix} d_2 & 0 & \cdots \\ 0 & d_3 & \ddots \\ 0 & 0 & \ddots \end{pmatrix}$$

with the coefficients

$$c_i = \sqrt{\frac{(2i-3)(2i+5)}{(2i-1)(2i+3)}},$$
$$d_i = \frac{(2i-3)(2i+1)}{2},$$

[12]. The stiffness matrix for the two-dimensional Laplace can be written using the matrices F and D by

$$K = F \otimes D + D \otimes F,$$

compare (2.10). Applying a permutation P of rows and columns, we get

$$PKP^{-1} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & K_3 & \\ & & & K_4 \end{pmatrix}. \tag{2.13}$$

The first block contains the polynomials $\hat{L}_{2i,2j}$, the second $\hat{L}_{2i+1,2j}$, the third $\hat{L}_{2i,2j+1}$ and the fourth $\hat{L}_{2i+1,2j+1}$. If p is odd, all four blocks have the same size. We wish to find a fast solver for a system of linear equations with the matrix K or equivalently, K_i. This solver should perform the solution in not more than $\mathcal{O}(p^2 \log p)$ arithmetical operations.

3 Deriving a preconditioner for K

In the following, we assume p is odd. We introduce $n = [\frac{p-1}{2}] + 1$. Applying a basis-transformation using the permutation P, (2.13), C_2 and C_5 are block diagonal matrices of 4 identical blocks C_3 and C_6, where

$$C_3 = D_3 \otimes T_3 + T_3 \otimes D_3, \tag{3.1}$$
$$C_6 = D_3 \otimes (T_3 + D_3^{-1}) + (T_3 + D_3^{-1}) \otimes D_3 \tag{3.2}$$

with

$$D_3 = \mathrm{diag}(4i^2)_{i=1}^{n-1}, \tag{3.3}$$
$$T_3 = \frac{1}{2}\mathrm{tridiag}(-1, 2, -1). \tag{3.4}$$

Furthermore, we need the matrices

$$D_4 = 4 \operatorname{diag}\left(i^2 + \frac{1}{6}\right)_{i=1}^{n-1}$$

and

$$C_4 = D_4 \otimes T_3 + T_3 \otimes D_4. \tag{3.5}$$

From [5] and [4], we get

THEOREM 3.1 . *Let $K_i, i = 1, \ldots, 4$ are the 4 blocks of K. The following statements are valid $\forall \underline{v}$ and $i = 1, \ldots, 4$:*

$$\begin{aligned}
c_7(C_3\underline{v}, \underline{v}) &\le (K_i\underline{v}, \underline{v}) \le & c_8(1 + \log p)(C_3\underline{v}, \underline{v}),\\
c_{11}(C_6\underline{v}, \underline{v}) &\le (K_i\underline{v}, \underline{v}) \le & c_{12}(C_6\underline{v}, \underline{v}),\\
c_9(C_4\underline{v}, \underline{v}) &\le (K_i\underline{v}, \underline{v}) \le & c_{10}(1 + \log p)(C_4\underline{v}, \underline{v}).
\end{aligned}$$

4 Similar systems of linear equations for other methods of discretization

4.1 Finite differences

The matrix C_3 is the system matrix for a discretization of

$$\begin{aligned}
-2\left(y^2 u_{xx} - x^2 u_{yy}\right) &= g,\\
u\,|_{\partial\Omega_1} &= 0
\end{aligned} \tag{4.1}$$

in $\Omega_1 = (0, 1)^2$ using finite differences and the grid of Figure 1.

Indeed, we denote the approximation in $\frac{1}{n}(i, j)$ by $u^{i,j}$. We approximate the second derivatives by the usual second order central difference quotient:

$$\begin{aligned}
y^2 u_{xx}\left(\frac{i}{n}, \frac{j}{n}\right) &\approx j^2(u^{i+1,j} + u^{i-1,j} - 2u^{i,j}),\\
x^2 u_{yy}\left(\frac{i}{n}, \frac{j}{n}\right) &\approx i^2(u^{i,j+1} + u^{i,j-1} - 2u^{i,j}).
\end{aligned}$$

If we insert the boundary condition and sort the unknowns in the order $u^{1,1}, u^{1,2}, \ldots, u^{1,n-1}, u^{2,1}, \ldots, u^{n-1,n-1}$, we get the system matrix $\frac{1}{2}C_3$ (3.1).

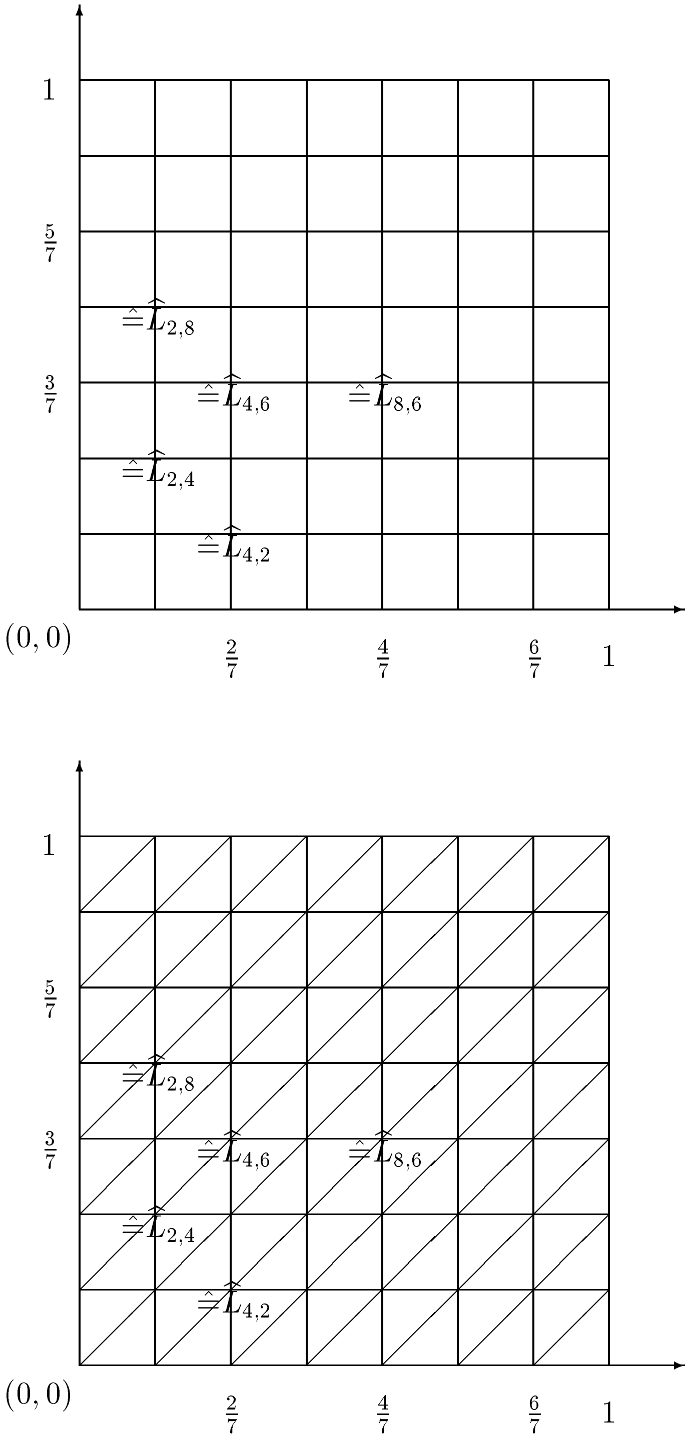

Figure 1: Mesh for h-Version (below), grid (above).

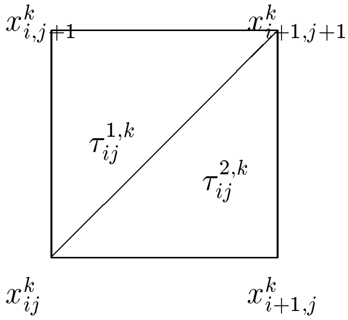

Figure 2: Notation within a cell $\mathcal{E}^k_{ij}$.

REMARK 4.1 *The discretization of*

$$\begin{aligned} -2\left(y^2 u_{xx} - x^2 u_{yy}\right) + \frac{y^2}{x^2} + \frac{x^2}{y^2} u &= g, \\ u\,|_{\partial\Omega_1} &= 0 \end{aligned} \tag{4.2}$$

as above leads to the system matrix C_6 (3.2)

4.2 h-version of the FEM

We consider the following problem: Find $u \in H^1_0(\Omega)$ such that

$$a(u,v) := \int_\Omega y^2 u_x v_x + x^2 u_y v_y \;\mathrm{d}x\mathrm{d}y = \int_\Omega gv \;\mathrm{d}x\mathrm{d}y =: \langle g, v\rangle \tag{4.3}$$

$\forall v \in H^1_0(\Omega)$ holds. The domain Ω is the unit square $(0,1)^2$.

We want to find a numerical solution of (4.3) using finite elements. For this purpose, we introduce some notation. Let k be the level of approximation and $n = 2^k$. Let us introduce $x^k_{ij} = (\frac{i}{n}, \frac{j}{n})$, where $i,j = 0,\ldots,n$. We divide Ω into congruent, isosceles, right triangles $\tau^{s,k}_{ij}$, where $0 \le i,j < n$ and $s = 1,2$, compare Figure 1. The triangle $\tau^{1,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i,j+1}$, $\tau^{2,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i+1,j}$, see Figure 2. Furthermore, let $\mathcal{E}^k_{ij} = \overline{\tau^{1,k}_{ij} \cup \tau^{2,k}_{ij}}$ be the square

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

We use linear finite elements on the mesh

$$T_k = \{\tau_{ij}^{s,k}\}_{i,j=1,s=1}^{n,n,2}$$

and denote by $\mathbb{V}_k$ the subspace of piecewise linear functions ϕ_{ij} with

$$\phi_{ij} \in H_0^1(\Omega), \ \phi_{ij} \mid_{\tau_{lm}^{sk}} \in P^1(\tau_{lm}^{sk}),$$

where P^1 is the space of polynomials of degree ≤ 1. A basis of $\mathbb{V}_k$ is the system of functions $\{\phi_{ij}^k\}_{i,j=1}^{n-1}$ uniquely defined by

$$\phi_{ij}^k(x_{lm}^k) = \delta_{il}\delta_{jm},$$

where δ_{il} is the Kronecker delta.

Now, we can formulate the discretized problem. Find $u^k \in \mathbb{V}_k$ such that

$$a(u^k, v^k) = \langle g, v^k \rangle \ \forall v \in \mathbb{V}_k \tag{4.4}$$

holds. Problem (4.4) is equivalent to solving

$$K_{h,k}\underline{u}_h = \underline{g}_h, \tag{4.5}$$

where

$$\begin{aligned} K_{h,k} &= a(\phi_{ij}^k, \phi_{lm}^k)_{i,j,l,m=1}^{n-1}, \\ \underline{g}_h &= \langle g, \phi_{lm}^k \rangle_{l,m=1}^{n-1}, \\ u_h &= \sum_{i,j=1}^{n-1} u_{ij}\phi_{ij}^k. \end{aligned}$$

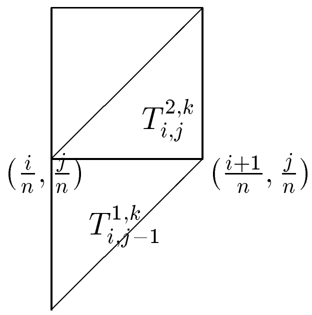

Figure 3: Sketch for calculation the matrix entry between two adjacent nodes.

We determine now $a(\phi_{ij}^k, \phi_{i+1,j}^k)$. We obtain by a simple integration

$$
\begin{aligned}
a(\phi_{ij}^k, \phi_{i+1,j}^k) &= \int_{T_{i,j-1}^{1,k}} \begin{pmatrix} -n \\ n \end{pmatrix} \begin{pmatrix} y^2 & 0 \\ 0 & x^2 \end{pmatrix} \begin{pmatrix} n \\ 0 \end{pmatrix} \mathrm{d}(x,y) \\
&\quad + \int_{T_{ij}^{2,k}} \begin{pmatrix} -n \\ 0 \end{pmatrix} \begin{pmatrix} y^2 & 0 \\ 0 & x^2 \end{pmatrix} \begin{pmatrix} n \\ -n \end{pmatrix} \mathrm{d}(x,y) \\
&= -n^2 \int_{T_{i,j-1}^{1,k} \cup T_{ij}^{2,k}} y^2 \,\mathrm{d}(x,y) \\
&= -n^2 \int_{\frac{j-1}{n}}^{\frac{j}{n}} \int_{\frac{i}{n}}^{y+\frac{i-j+1}{n}} y^2 \,\mathrm{d}x\mathrm{d}y - n^2 \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{y+\frac{i-j}{n}}^{\frac{i+1}{n}} y^2 \,\mathrm{d}x\mathrm{d}y \\
&= -\frac{1}{n^2}\left(\frac{j^2}{2} - \frac{j}{3} + \frac{1}{12}\right) - \frac{1}{n^2}\left(\frac{j^2}{2} + \frac{j}{3} + \frac{1}{12}\right) \\
&= -\frac{1}{n^2}\left(\frac{1}{6} + j^2\right), \qquad (4.6)
\end{aligned}
$$

where $n > i, j$ and $j > 0$, but $i \geq 0$. By symmetry, we have ($i > 0$, $j \geq 0$)

$$a(\phi_{ij}^k, \phi_{i,j+1}^k) = -\frac{1}{n^2}\left(\frac{1}{6} + i^2\right)$$

and

$$a(\phi_{ij}^k, \phi_{ij}^k) = -(a(\phi_{ij}^k, \phi_{i+1,j}^k) + a(\phi_{ij}^k, \phi_{i,j+1}^k) + a(\phi_{ij}^k, \phi_{i,j-1}^k) + a(\phi_{ij}^k, \phi_{i-1,j}^k)).$$

All other matrix entries are zero. Inserting the boundary condition and using (3.5), we arrive after a proper permutation of the unknowns

$$K_{h,k} = \frac{1}{2n^2} C_4. \tag{4.7}$$

5 Multi-grid proof for $-y^2u_{xx} - x^2u_{yy}$ using the strengthened Cauchy-inequality

We are interested in finding a fast solver for (4.4). We will see that we can prove the convergence for a multi-grid algorithm.

REMARK 5.1 *Note, that $a(\cdot,\cdot)$ is on $\mathbb{V}_k$ positive definite.*

5.1 Theory of algebraic multi-grid proofs

In this chapter, we discuss the theory of algebraic multi-grid proofs, [15], [16]. We split

$$\mathbb{V}_k = \mathbb{V}_{k-1} \oplus \mathbb{W}_k.$$

Algebraic multi-grid proofs analyze the angle between the two subspaces $\mathbb{V}_{k-1}$ and $\mathbb{W}_k$, or equivalently, the strengthened Cauchy-inequality

$$(a(v,w))^2 \leq \gamma^2 a(v,v)a(w,w) \; \forall v \in \mathbb{V}_{k-1}, w \in \mathbb{W}_k \tag{5.1}$$

with $\gamma^2 < 1$.

5.1.1 Multi-grid algorithm

We describe in this section the multi-grid method for solving (4.4). Let u_0 be the initial value. We define the iterate u_1 by the recursive process $u_1 = MULT(k, u_0, g)$.

- Set $l = k$.
- If $l = 1$, then solve

  $$a(w,v) = \langle g, v\rangle - a(u_0, v) \; \forall v \in \mathbb{V}_l$$

  exactly. Else, do

- Pre-smoothing on $\mathbb{W}_{l-1}$:
  Solve for $w \in \mathbb{W}_{l-1}$

$$a(w, v) = \langle g, v\rangle - a(u_0, v) := \langle r, v\rangle \ \forall v \in \mathbb{W}_l$$

  using ν steps of a simple iterative method $\tilde{w} = Sr$. Set $u_0^1 = u_0 + \tilde{w}$.

- Coarse grid correction on $\mathbb{V}_{l-1}$:
  Solve for $w \in \mathbb{V}_{l-1}$

$$a(w, v) = \langle g, v\rangle - a(u_0^1, v) = \langle r, v\rangle \ \forall v \in \mathbb{V}_{l-1}$$

  using μ_{l-1} steps of the algorithm $\tilde{w} = MULT(l-1, 0, r)$. Set $u_0^2 = u_0^1 + \tilde{w}$.

- Post-smoothing on $\mathbb{W}_{l-1}$:
  Solve for $w \in \mathbb{W}_{l-1}$

$$a(w, v) = \langle g, v\rangle - a(u_0^2, v) = \langle r, v\rangle \ \forall v \in \mathbb{W}_{l-1}$$

  using ν steps of a simple iterative method $\tilde{w} = Sr$. Set $u_1 = u_0^2 + \tilde{w}$.

- end-if.

5.1.2 Convergence theory for multi-grid

We want to prove the convergence of the multi-grid algorithm for solving (4.4) using $\mu = 3$ and the smoother S, which will be defined in (5.33). The main tool is the theory of algebraic multi-grid proofs, [15], [16]. We formulate only the main theorem.

THEOREM 5.2 *Let us assume that the following assumptions are fulfilled.*

- *Let $a(\cdot, \cdot)$ be a symmetric and positive definite bilinear form on $\mathbb{V}_k$.*

- *Let S be a smoother with*

$$\| S^\nu w \|_a^2 \leq C\rho^{2\nu} \| w \|_a^2 \ \ \forall w \in \mathbb{W}_k, \tag{5.2}$$

  where $0 \leq \rho < 1$ independent of k and $C > 0$.

- *There is a constant* $0 \le \gamma < 1$ *independent of* k *such that*
$$(a(v,w))^2 \le \gamma^2 a(v,v)a(w,w) \; \forall w \in \mathbb{W}_k, \forall v \in \mathbb{V}_{k-1} \qquad (5.3)$$
holds.

- *Let* $u_{j+1,k} = MULT(k, u_{j,k}, g)$, *let* u^* *be the exact solution of* (4.4) *and let*
$$\sigma_k = \sup_{u_{j,k}-u^* \in \mathbb{V}_k} \frac{\| u_{j+1,k} - u^* \|_a}{\| u_{j,k} - u^* \|_a}$$
be the convergence rate of $MULT$ *with* ν *smoothing operations.*

Then, the following recursion formula holds
$$\sigma_k \le \sigma_{k-1}^{\mu_{k-1}} + (1 - \sigma_{k-1}^{\mu_{k-1}})(C\rho^\nu + (1 - C\rho^\nu)\gamma^2). \qquad (5.4)$$
Proof: Theorem 2.2 of [16] with $\rho_1 = \rho_3$, see also Theorem 4 of [15] □
The following lemma of the standard multi-grid theory is helpful for the analysis of the recursion formula (5.4).

LEMMA 5.3 *Let* $\mu_k = \mu \in \mathbb{N}$, $\mu > 1$, *and*
$$\kappa = C\rho^\nu + (1 - C\rho^\nu)\gamma^2 < \frac{\mu - 1}{\mu}.$$
The elements σ_k *of the recursion*
$$\begin{aligned} \sigma_0 &= 0, \\ \sigma_k &= \kappa + \sigma_{k-1}^\mu (1 - \kappa). \end{aligned}$$
are contained in the interval $[0, \sigma)$. *Then, the equation*
$$\sigma = (\kappa + \sigma^\mu (1 - \kappa)$$
has a solution $\sigma \in (0,1)$. *More precisely, the sequence* $\{\sigma_k\}_{k=0}^\infty$ *is monotonically increasing and bounded from above by* 1 *for* $0 < \kappa < 1$. *Especially, we have for* $\mu = 2$
$$\lim_{k\to\infty} = \left\{ \begin{array}{cc} 1 & \text{for} \quad \kappa \ge \frac{1}{2} \\ \frac{\kappa}{1-\kappa} & \text{for} \quad \kappa < \frac{1}{2} \end{array} \right\}$$
and $\mu = 3$
$$\lim_{k\to\infty} = \left\{ \begin{array}{cc} 1 & \text{for} \quad \kappa \ge \frac{2}{3} \\ \sqrt{\frac{1}{4} + \frac{\kappa}{1-\kappa}} - \frac{1}{2} & \text{for} \quad \kappa < \frac{2}{3} \end{array} \right\} \qquad (5.5)$$

Proof: The proof can be found in several papers, see Lemma 3 of [15] and Lemma 3.2 of [16]. □

Using Theorem 5.2 and Lemma 5.3, we can prove the mesh-size independent convergence rate of a symmetric bilinear form a in the case $\mu = 2$ (W-cycle), if $\kappa < \frac{1}{2}$ and $\mu = 3$ if $\kappa < \frac{2}{3}$, if the smoother S satisfies (5.2).

.

REMARK 5.4 *For given values of γ^2 and ρ are needed for $\mu = 3$*

$$\nu > \frac{\ln \frac{\frac{2}{3}-\gamma^2}{1-\gamma^2}}{\ln \rho}$$

smoothing steps if $\gamma^2 < \frac{2}{3}$.

5.2 Hierarchical decomposition of $\mathbb{V}_k$

We want to prove multi-grid convergence for system (4.5) via Theorem 5.2. For this aim, we have to determine bounds for ρ in (5.2) and γ^2 in (5.3). The next subsection dervives some lemmata which are helpful for our aim.

5.2.1 Basic definitions and helpful lemmata of the linear algebra

Let us introduce some more notation. We have

$$\mathbb{V}_k = \operatorname{span}\{\phi_{ij}^k\}_{i,j=1}^{n-1}.$$

We can represent the space $\mathbb{V}_k$ by the space $\mathbb{V}_{k-1}$ and a space $\mathbb{W}_k$, i.e.

$$\mathbb{V}_k = \mathbb{V}_{k-1} \oplus \mathbb{W}_k,$$

where

$$\mathbb{W}_k = \operatorname{span}\{\phi_{ij}^k\}_{(i,j)\in N_k}. \tag{5.6}$$

The subset N_k is given by

$$N_k = \{(i,j) \in \mathbb{N}^2, 1 \le i,j \le n-1, i = 2m+1 \text{ or } j = 2m+1, m \in \mathbb{N}\}. \tag{5.7}$$

For proving a sufficent strengthened Cauchy-inequality

$$(a(v,w))^2 \leq \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V}_{k-1}, w \in \mathbb{W}_k \tag{5.8}$$

with $\gamma^2 < 1$, we split $a(v,w)$ into

$$\begin{aligned} a(v,w) &= \int_\Omega y^2 v_x w_x + x^2 v_y w_y \,\mathrm{d}x\mathrm{d}y \\ &= \sum_{i,j} \int_{\mathcal{E}^k_{i,j}} y^2 v_x w_x + x^2 v_y w_y \,\mathrm{d}x\mathrm{d}y \\ &= \sum_{i,j} a^{\mathcal{E}^k_{i,j}}(v,w). \end{aligned} \tag{5.9}$$

DEFINITION 5.5 *Let $\mathbb{V}$ be a space of functions on Ω. Let $\Omega_1 \subset \Omega$. We denote the restriction of $\mathbb{V}$ on Ω_1 by $\mathbb{V}\mid_{\Omega_1}$.*

LEMMA 5.6 *Let $a(\cdot,\cdot)$ be a symmetric, positive definite bilinear form. Under the assumption that*

$$(a^{\mathcal{E}^k_{i,j}}(v,w))^2 \leq \gamma^2 a^{\mathcal{E}^k_{i,j}}(v,v)a^{\mathcal{E}^k_{i,j}}(w,w) \tag{5.10}$$

for all $v \in \mathbb{V}_k \mid_{\mathcal{E}^k_{ij}}$ and $w \in \mathbb{W}_k \mid_{\mathcal{E}^k_{ij}}$ we have

$$(a(v,w))^2 \leq \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V}_k, w \in \mathbb{W}_k.$$

Proof: [6], [14]. □

We need for some special elements the trivial

LEMMA 5.7 . *Let $a(\cdot,\cdot)$ be any bilinear form. We assume that we have*

$$(a(u,v))^2 \leq \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V},\ \forall w \in \mathbb{W}.$$

Let $\mathbb{V}_0 \subset \mathbb{V}$ and $\mathbb{W}_0 \subset \mathbb{W}$. Then,

$$(a(u,v))^2 \leq \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V}_0,\ \forall w \in \mathbb{W}_0$$

holds.

The following lemma, see [9], [18], relates the constant of the strengthened Cauchy-inequality to the largest eigenvalue of a generalized eigenvalue problem. In order to formulate it, we need 2 definitions.

DEFINITION 5.8 *Let $a(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{R}$ be any bilinear form. We define*

$$\ker\ a = \{v \in \mathbb{V} : a(v,w) = 0\ \forall w \in \mathbb{V}\}.$$

as the kernel of the bilinear form a.

DEFINITION 5.9 *Let $\mathbb{X}$ be a linear (finite dimensional) space, $\mathbb{Y}$ a subspace of $\mathbb{X}$. We define the difference $\mathbb{X} - \mathbb{Y}$ as a linear subspace satisfying*

$$\mathbb{X} = \mathbb{Y} \oplus (\mathbb{X} - \mathbb{Y}).$$

Note that the choice of $\mathbb{X} - \mathbb{Y}$ is not unique.

LEMMA 5.10 *Consider the splitting $\mathbb{V} \oplus \mathbb{W}$. Let*

$$\mathbb{V} = span\{\phi_i\}_{i=1}^n, \mathbb{W} = span\{\psi_i\}_{i=1}^m,$$

$$A = a(\phi_i, \phi_j)_{i,j=1}^n, B^t = a(\phi_i, \psi_j)_{i,j=1}^{n,m}, C = a(\psi_i, \psi_j)_{i,j=1}^m.$$

Furthermore, let

$$\mathbb{V} \cap \mathbb{W} = \{\mathbf{0}\}$$

and

$$\ker\ a \subset \mathbb{V}.$$

The bilinear form $a(\cdot,\cdot)$ is symmetric and positive semidefinite. Then, the minimal constant γ^2 with

$$a(v,w)^2 \leq \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V}, w \in \mathbb{W}$$

is equal to the largest eigenvalue λ of

$$V^t B^t C^{-1} B V \underline{w} = \lambda V^t A V \underline{w}, \tag{5.11}$$

with $V \in \mathbb{R}^{n,q}$, $\mathrm{im}V = \mathbb{R}^n - \ker A$ and $\ker V^t = \mathbf{0}$.

Proof: We have

$$a(v,w)^2 \le \gamma^2 a(v,v)a(w,w) \ \forall v \in \mathbb{V}, w \in \mathbb{W}, \tag{5.12}$$

where γ^2 is as small as possible. For $v \in \ker a$ this inequality is satisfied. Hence, it is equivalent to restrict ourselves to $v \in \mathbb{V} - \ker a$. Because of the positive semidefinitness of a, we can write using $\ker a \subset \mathbb{V}$

$$\frac{a(v,w)^2}{a(v,v)a(w,w)} \le \gamma^2$$

for all $v \in \mathbb{V} - \ker a, w \in \mathbb{W}$. Hence, the inequality (5.12) is equivalent to

$$\sup_{\substack{v \in \mathbb{V} - \ker a \\ w \in \mathbb{W}}} \frac{(a(v,w))^2}{a(v,v)a(w,w)} = \gamma^2. \tag{5.13}$$

Now, we transform the left hand side of (5.13). Using vectors of $\mathbb{R}^n$, we have

$$\gamma^2 = \sup_{\substack{v \in \mathbb{V} - \ker a \\ w \in \mathbb{W}}} \frac{(a(v,w))^2}{a(v,v)a(w,w)} = \sup_{\substack{\underline{v} \in \mathbb{R}^n - \ker A \\ \underline{w} \in \mathbb{R}^m}} \frac{(\underline{w}^t B \underline{v})^2}{\underline{v}^t A \underline{v} \ \underline{w}^t C \underline{w}}$$

Because of our assumptions, the matrix C is a symmetric positive definite matrix. We can substitute $\underline{u} = C^{\frac{1}{2}}\underline{w}$ and we obtain

$$\gamma^2 = \sup_{\substack{\underline{v} \in \mathbb{R}^n - \ker A \\ \underline{u} \in \mathbb{R}^m}} \frac{(\underline{u}^t C^{-\frac{1}{2}} B \underline{v})^2}{\underline{v}^t A \underline{v} \ \underline{u}^t \underline{u}}.$$

The right hand side is maximal, if $\underline{u} = C^{-\frac{1}{2}} B \underline{v}$. Inserting this, we have

$$\begin{aligned} \gamma^2 &= \sup_{\underline{v} \in \mathbb{R}^n - \ker A} \frac{\underline{v} B^t C^{-1} B \underline{v}}{\underline{v} A \underline{v}} \\ &= \sup_{\underline{y} \in \mathbb{R}^q} \frac{\underline{y} V^t B^t C^{-1} B V \underline{y}}{\underline{y} V^t A V \underline{y}}, \end{aligned}$$

which is the largest eigenvalue of the generalized eigenvalue problem

$$V^t B^t C^{-1} B V \underline{y} = \lambda V^t A V \underline{y},$$

i.e. $\lambda_{max} = \gamma^2$. $\square$

DEFINITION 5.11 *Let $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n,n}$. We denote by*

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

the trace of the matrix A.

For estimating the eigenvalues of a 2×2 matrix, we need

LEMMA 5.12 . *Let $M \in \mathbb{R}^{2,2}$ be a matrix with real eigenvalues and K a real number with*

$$p = 2K - \mathrm{trace}(M) \geq 0 \tag{5.14}$$

and

$$q = \det M + \iota^2 - \iota\mathrm{trace}(M) \geq 0. \tag{5.15}$$

Then, we have

$$\lambda_{max}(M) \leq \iota.$$

Proof: The characteristical polynomial of a 2×2 matrix M is given by

$$p_c(x) = x^2 - \mathrm{trace}(M)x + \det M. \tag{5.16}$$

Set $y = x - K$, then

$$\begin{aligned} p_c(x) &= y^2 + (2\iota - \mathrm{trace}(M))y + \det M + \iota^2 - \iota\mathrm{trace}(M), \\ &= y^2 + py + q. \end{aligned} \tag{5.17}$$

Because of our assumption, M has real eigenvalues, and (5.16) and (5.17), this polynomial has 2 real roots. Using (5.14) and (5.15) we can conclude that both are nonpositive. Hence, we have the roots $x_{1,2}$ of p_c fulfill $x_{1,2} \leq K$. $\square$

The following lemma [1] is helpful for the proof of the smoothing property (5.3).

LEMMA 5.13 *Let $\{A_i \in \mathbb{R}^{m_i,m_i}\}_{i=1}^n$ be a finite set of symmetric positive definite matrices. Let*

$$A = \sum_{i=1}^{n} L_i^t A_i L_i,$$

where $L_i \in \mathbb{R}^{m_i,m}$ and $A \in \mathbb{R}^{m,m}$. Furthermore, let C_i a good preconditioner for the matrix A_i, i.e. for all $\underline{w} \in \mathbb{R}^{m_i}$ the relation

$$\lambda_i(C_i\underline{w}, \underline{w}) \leq (A_i\underline{w}, \underline{w}) \leq \lambda^i(C_i\underline{w}, \underline{w}) \tag{5.18}$$

with $0 < \lambda^i$ and $0 \leq \lambda_i$ holds. Let

$$C = \sum_{i=1}^{n} L_i^t C_i L_i.$$

Then, $\forall \underline{v} \in \mathbb{R}^m$

$$\underline{\lambda}(C\underline{v}, \underline{v}) \leq (A\underline{v}, \underline{v}) \leq \overline{\lambda}(C\underline{v}, \underline{v})$$

is valid with

$$\underline{\lambda} = \min_i \lambda_i \;, \overline{\lambda} = \max_i \lambda^i.$$

Proof: Using (5.18) we obtain

$$(C_i\underline{w}, \underline{w}) \geq 0.$$

Now, we can estimate $\forall \underline{v} \in \mathbb{R}^m$

$$\begin{aligned}
(A\underline{v}, \underline{v}) &= (\sum_{i=1}^{n} L_i^t A_i L_i \underline{v}, \underline{v}) \\
&= \sum_{i=1}^{n} (A_i L_i \underline{v}, L_i \underline{v}) \\
&\leq \sum_{i=1}^{n} \lambda^i (C_i L_i \underline{v}, L_i \underline{v}) \\
&\leq \sum_{i=1}^{n} \overline{\lambda}(C_i L_i \underline{v}, L_i \underline{v}) \\
&= \overline{\lambda}(C\underline{v}, \underline{v}).
\end{aligned}$$

The second inequality follows with same arguments. □

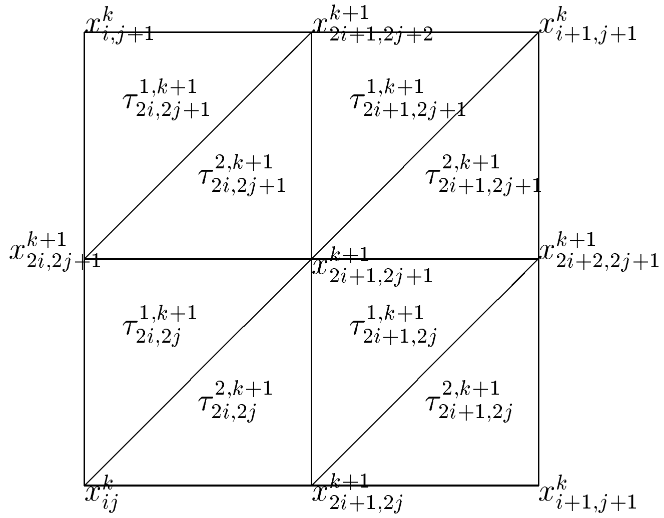

Figure 4: Local numbering of the nodes and sub-triangles of $\mathcal{E}^k_{ij}$

5.2.2 Discussion of the strengthened Cauchy-inequality on subelements $\mathcal{E}_{ij}$

We prove the strengthened Cauchy-inequality (5.1) on the macro-elements $\mathcal{E}^k_{ij}$. It will be done it the case $i, j > 0$ by proving in the triangles $\tau^{1,k}_{ij}$ and $\tau^{2,k}_{ij}$, but for $ij = 0$ in the sub-cells $\mathcal{E}^k_{ij}$.

Consider Figure 4. We want to have the stiffness matrix on the macro-elements $\mathcal{E}^k_{ij}$ with respect to the two level basis. We start with the introduction of the basis functions on $\mathcal{E}^k_{ij}$. Note, that the triangle $\tau^{2,k}_{ij}$ consists of the triangles $\tau^{2,k+1}_{2i,2j}$, $\tau^{1,k+1}_{2i+1,2j}$, $\tau^{2,k+1}_{2i+1,2j}$ and $\tau^{2,k+1}_{2i+1,2j+1}$, the triangle $\tau^{1,k}_{ij}$ consists of the triangles $\tau^{1,k+1}_{2i,2j}$, $\tau^{1,k+1}_{2i,2j+1}$, $\tau^{2,k+1}_{2i,2j+1}$ and $\tau^{1,k+1}_{2i+1,2j+1}$. The nodes x^k_{ij}, $x^k_{i,j+1}$, $x^k_{i+1,j}$ and $x^k_{i+1,j+1}$ are the coarse grid nodes, the nodes $x^{k+1}_{2i+1,2j}$, $x^{k+1}_{2i,2j+1}$, $x^{k+1}_{2i+2,2j+1}$, $x^{k+1}_{2i+1,2j+2}$ and $x^{k+1}_{2i+1,2j+1}$ are new in the $k+1$-st level.

Using this splitting, we have with (5.7)

$$\operatorname{span}\{\phi^k_{l,m}\}_{(l,m)\in N^{\mathbb{V}_k}_{i,j}} = \mathbb{V}_k\mid_{\mathcal{E}^k_{ij}}$$

and

$$\operatorname{span}\{\phi^{k+1}_{l,m}\}_{(l,m)\in N^{\mathbb{W}_{k+1}}_{i,j}} = \mathbb{W}_{k+1}\mid_{\mathcal{E}^k_{ij}},$$

where

$$N_{i,j}^{\mathbb{V}_k} = \{(l,m) \in \mathbb{N}^2, i \le l \le i+1, j \le m \le j+1\}$$

and

$$N_{i,j}^{\mathbb{W}_{k+1}} = \{(l,m) \in \mathbb{N}^2, 2i \le l \le 2i+2, 2j \le m \le 2j+2\} \cap N_k.$$

We have to cancel for sub-cells $\mathcal{E}_{ij}^k$ with $i=0, j=0, i=n-1, j=n-1$ several unknowns because of $\mathbb{V}_k \in H_0^1(\Omega)$. We define the matrices

$$\begin{aligned} A &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^k, \phi_{l,m}^k)_{(r,s),(l,m)\in N_{i,j}^{\mathbb{V}_k}}, \\ B^t &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^k, \phi_{l,m}^{k+1})_{(r,s)\in N_{i,j}^{\mathbb{V}_k},(l,m)\in N_{i,j}^{\mathbb{W}_{k+1}}}, \\ C &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^{k+1}, \phi_{l,m}^{k+1})_{(r,s),(l,m)\in N_{i,j}^{\mathbb{W}_{k+1}}}. \end{aligned}$$

The indices i, j and k are omitted. We introduce the matrices A, B, C in the same way for the element stiffness matrices on $\tau_{ij}^{2,k}$, i.e.

$$\begin{aligned} A &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^k, \phi_{l,m}^k)_{(r,s),(l,m)\in N_{i,j}^{2,\mathbb{V}_k}}, \\ B^t &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^k, \phi_{l,m}^{k+1})_{(r,s)\in N_{i,j}^{2,\mathbb{V}_k},(l,m)\in N_{i,j}^{2,\mathbb{W}_{k+1}}}, \\ C &= a^{\mathcal{E}_{ij}^k}(\phi_{r,s}^{k+1}, \phi_{l,m}^{k+1})_{(r,s),(l,m)\in N_{i,j}^{2,\mathbb{W}_{k+1}}}, \end{aligned}$$

where

$$N_{i,j}^{2,\mathbb{V}_k} = \{(l,m) \in \mathbb{N}^2, i-j \le l-m\} \cap N_{i,j}^{\mathbb{V}_k}$$

and

$$N_{i,j}^{2,\mathbb{W}_{k+1}} = \{(l,m) \in \mathbb{N}^2, i-j \le l-m\} \cap N_{i,j}^{\mathbb{W}_{k+1}}.$$

The ordering of the rows and columns in the matrices A and C corresponds to the ordering of the coarse grid and new nodes written above.

We start with the case $0 < i, j < n-1$.

LEMMA 5.14 *Let $0 < i, j < n-1$. Let*

$$\begin{aligned}
a &= \frac{48i^2+48i+14}{192n^2},\\
b &= \frac{48i^2+16i+2}{192n^2},\\
c &= \frac{48i^2+80i+34}{192n^2},\\
d &= \frac{48j^2+48j+14}{192n^2},\\
e &= \frac{48j^2+16j+2}{192n^2},\\
f &= \frac{48j^2+80j+34}{192n^2}.
\end{aligned}$$

Then, we have on $\mathcal{E}^k_{ij}$

$$A = \begin{pmatrix} a+b+d+e & -d-e & -a-b & 0 \\ -d-e & a+c+d+e & 0 & -a-c \\ -a-b & 0 & a+b+d+f & -d-f \\ 0 & -a-c & -d-f & a+c+d+f \end{pmatrix}, \tag{5.19}$$

,

$$\begin{aligned}
B^t &= 2\begin{pmatrix} 0 & 0 & -d & -a & a+d \\ a & 0 & d & 0 & -a-d \\ 0 & d & 0 & a & -a-d \\ -a & -d & 0 & 0 & a+d \end{pmatrix},\\
C &= 4\begin{pmatrix} a+e & 0 & 0 & 0 & -a \\ 0 & b+d & 0 & 0 & -d \\ 0 & 0 & c+d & 0 & -d \\ 0 & 0 & 0 & a+f & -a \\ -a & -d & -d & -a & 2a+2d \end{pmatrix}.
\end{aligned} \tag{5.20}$$

In the case of matrices on the triangle $\tau_{ij}^{2,k}$, we have

$$\begin{aligned}
A &= \begin{pmatrix} d+e & -d-e & 0 \\ -d-e & a+c+d+e & -a-c \\ 0 & -a-c & a+c \end{pmatrix}, \\
B^t &= 2\begin{pmatrix} 0 & -d & d \\ a & d & -a-d \\ -a & 0 & a \end{pmatrix}, \\
C &= 4\begin{pmatrix} a+e & 0 & -a \\ 0 & c+d & -d \\ -a & -d & a+d \end{pmatrix}.
\end{aligned}$$

Proof: The proof is a simple calculation. □

REMARK 5.15 *In the case of elements laying on the boundary, the equations hold, but we have to cancel all rows and columns on A, B and C, which correspond to boundary nodes.*

COROLLARY 5.16 *We have* $\ker A \subset \ker B$ *in both cases.*

Proof: We have in the case of $\mathcal{E}_{ij}^k$

$$\ker A = \operatorname{span}\{(1,1,1,1)^t\}$$

and for τ_{ij}^2 we have

$$\ker A = \operatorname{span}\{(1,1,1)^t\}.$$

□

Now, we try to determine the constant $\gamma_{\tau_{ij}^{2,k}}$. For this purpose is

LEMMA 5.17 . *We have for $\tau_{ij}^{2,k}$, $1 \le i,j \le n-2$*

$$(a^{\tau_{ij}^2}(v,w))^2 \le \gamma_{\tau_{ij}^2}^2 a^{\tau_{ij}^2}(v,v) a^{\tau_{ij}^2}(w,w) \ \forall v \in \mathbb{V}_k \mid_{\tau_{ij}^2}, w \in \mathbb{W}_{k+1} \mid_{\tau_{ij}^2} \qquad (5.21)$$

with $\gamma_{\tau_{ij}^{2,k}}^2 = \frac{95}{176}$. The constant is optimal in the case $i = j = 1$.

Proof: Corollary states5.16 $\ker A \subset \ker B$ and Lemma 5.12 states that $\ker C$ is trivial. Hence, we can apply Lemma 5.10. We know

$$\ker A = \mathrm{span}\{(1,1,1)^t\}.$$

Thus, we choose

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix V^tAV is symmetric and positive definite, the matrix $V^tB^tC^{-1}BV$ is symmetric. Therefore, the generalized 2×2 eigenvalue problem has real eigenvalues and is equivalent to the eigenvalue problem

$$(V^tAV)^{-1}V^tB^tC^{-1}BV\underline{x} = \lambda\underline{x}.$$

This is a 2×2 eigenvalue problem, for which we can apply Lemma 5.12. We build using the help of a computer algebra system the matrix

$$M = (V^tAV)^{-1}V^tB^tC^{-1}BV$$

and show with $\gamma^2_{\tau^{2,k}_{ij}} = \frac{95}{176}$

$$p = 2\gamma^2_{\tau^{2,k}_{ij}} - \mathrm{trace}(M) \geq 0 \tag{5.22}$$

and

$$q = \det M + \gamma^4_{\tau^{2,k}_{ij}} - \gamma^2_{\tau^{2,k}_{ij}}\mathrm{trace}(M) \geq 0. \tag{5.23}$$

Using Lemmata 5.10 and 5.12 we have (5.21). □

REMARK 5.18 1. *We obtain the constant* $\gamma^2_{\tau^{2,k}_{ij}} = \frac{95}{176}$ *for* $i = j = 1$ *by a direct calculation.*

2. *By symmetry, the relation* (5.21) *is valid for* $\tau^{1,k}_{ij}$, $0 < i, j < n-1$.

3. *Using the arguments of Lemma 5.6 we can prove* (5.21) *for* $\mathcal{E}^k_{ij}$, $0 < i, j < n-1$.

$$(a^{\mathcal{E}^k_{ij}}(v,w))^2 \leq \gamma^2_{\mathcal{E}^k_{ij}} a^{\mathcal{E}^k_{ij}}(v,v)a^{\mathcal{E}^k_{ij}}(w,w) \forall v \in \mathbb{V}_k\mid_{\mathcal{E}^k_{ij}}, w \in \mathbb{W}_k\mid_{\mathcal{E}^k_{ij}} \tag{5.24}$$

4. *The values p (5.22) and q (5.23) are broken rational functions in i and j. We give the exact values in the appendix.*

COROLLARY 5.19 *Let $ij > 0$. The inequality*

$$(a^{\mathcal{E}_{ij}^k}(v,w))^2 \leq \gamma^2_{\mathcal{E}_{ij}^k} a^{\mathcal{E}_{ij}^k}(v,v) a^{\mathcal{E}_{ij}^k}(w,w) \ \forall v \in \mathbb{V}_k \mid_{\mathcal{E}_{ij}^k}, w \in \mathbb{W}_{k+1} \mid_{\mathcal{E}_{ij}^k} \qquad (5.25)$$

is valid for $i = n-1$ or $j = n-1$ with $\gamma^2_{\mathcal{E}_{ij}^k} \leq \frac{95}{176}$.

Proof: We consider the case $i = n-1$ and $0 < j < n-1$. We omit the unknowns corresponding to $\phi^k_{i+1,j}$, $\phi^k_{i+1,j+1}$ and $\phi^{k+1}_{2i+2,2j+1}$. We have to cancel the second and last row and column in (5.19) and the third in (5.20). We do not use the assumption $i < n-1$ in the proof of Lemma 5.17. Hence, the estimate is valid for $i = n-1$ and $0 < j < n-1$. By Lemma 5.7, we can conclude that a Dirichlet boundary condition does not increase the constant of the strengthened Cauchy inequality. The cases $j = n-1$, $0 < i < n-1$ and $i = j = n-1$ follow with same arguments or by symmetry. □

We consider now the case $0 < i < n-1$ and $j = 0$. We cannot split $\mathcal{E}_{ij}^k$ into $\tau_{ij}^{1,k}$ and $\tau_{ij}^{2,k}$ in the case $j = 0$. On the triangle $\tau_{ij}^{1,k}$ we have no influence of the Dirichlet boundary condition. We would obtain a constant $\gamma_{\tau_{i,0}^{1,k}}$ which is closer to 1. To avoid this phenomenon, we determine $\gamma_{\mathcal{E}_{ij}^k}$ directly. We omit the unknowns corresponding to $\phi^k_{i+1,0}$, $\phi^k_{i,0}$ and $\phi^{k+1}_{2i+1,0}$ corresponding to the first two rows and columns in (5.19) and the first in (5.20), and the corresponding rows and columns in B^t.

We obtain from Lemma 5.10

$$\begin{aligned}
A &= \begin{pmatrix} a+b+48C_1 & -48C_1 \\ -48C_1 & a+c+48C_1 \end{pmatrix}, \\
B &= 2\begin{pmatrix} 14C_1 & 0 & a & -a-14C_1 \\ -14C_1 & 0 & 0 & a+14C_1 \end{pmatrix}, \\
C &= 4\begin{pmatrix} b+14C_1 & 0 & 0 & -14C_1 \\ 0 & c+14C_1 & 0 & -14C_1 \\ 0 & 0 & a+34C_1 & -a \\ -14C_1 & -14C_1 & -a & 2a+28C_1 \end{pmatrix}
\end{aligned}$$

with $C_1 = \frac{1}{192n^2}$. From $\ker A = \{\mathbf{0}\}$ follows that the identity matrix is a possible choice for V. Using a computer algebra program we get with the same arguments as in the proof of Lemma 5.17

LEMMA 5.20 . *We have*

$$\gamma^2_{\mathcal{E}_{i,0}} < \frac{95}{176} \tag{5.26}$$

for $0 < i < n-1$ *and*

$$\gamma^2_{\mathcal{E}_{0,j}} < \frac{95}{176} \tag{5.27}$$

for $0 < j < n-1$.

REMARK 5.21 *The estimates (5.26) and (5.27) can be extended to* $i = n-1$ *and* $j = n-1$ *using the same arguments as in the proof of Corollary 5.19.*

The last case is $i = j = 0$. We have

$$a = d = 14C_1\,, b = e = 2C_1,\; c = f = 34C_1$$

with $C_1 = \frac{1}{192n^2}$.

We have only the shape functions $\phi^k_{1,1}$, $\phi^{k+1}_{1,0}$, $\phi^{k+1}_{0,1}$ and $\phi^{k+1}_{1,1}$. We obtain from Lemma 5.14 by canceling the first three rows and columns in (5.19) and the first two rows and columns in (5.20),

$$\begin{aligned} A &= (2(a+c)), \\ B &= \begin{pmatrix} 0 & 0 & 4a \end{pmatrix}, \\ C &= 4\begin{pmatrix} a+c & 0 & -a \\ 0 & a+c & -a \\ -a & -a & 4a \end{pmatrix}. \end{aligned}$$

A is regular. Thus, we choose $V = 1$ and a short computation shows using Lemma 5.10

LEMMA 5.22 . *It holds*

$$\gamma^2_{\mathcal{E}_{00}} = (V^tAV)^{-1}V^tB^tC^{-1}BV = \frac{a}{a+2c} = \frac{7}{41}. \tag{5.28}$$

Now, we can formulate

THEOREM 5.23 . *The inequality*

$$(a(v,w))^2 \le \gamma^2 a(v,v)a(w,w)\ \forall v \in \mathbb{V}_k, w \in \mathbb{W}_{k+1}.$$

is valid with $\gamma^2 = \frac{95}{176}$.

Proof: The proof is done by Lemmata 5.6, 5.20 and 5.22, Remarks 5.21 and 5.18, Corollary 5.19 and inequality (5.24). □

5.2.3 Construction of the smoother

We need a good smoother for applying multi-grid to the linear system (4.5). This smoother will be contructed by the local behaviour of the differential operator. An idea of [1] for anisotropic problems is extended to the problem (4.3). This smoother operates only on the space $\mathbb{W}_{k+1}$. Consider the triangle $\tau_{ij}^{2,k}$. For our discussion is needed only the sub-matrix C, which corresponds to the nodal basis functions on $\mathbb{W}_{k+1}$. We discuss the two cases $i < j$ and $i \geq j$. We start with $i < j$. We have from Lemma 5.14

$$C_{2,ij} = 4 \begin{pmatrix} a+e & 0 & -a \\ 0 & c+d & -d \\ -a & -d & a+d \end{pmatrix}.$$

The index k is omitted. Let for $i < j$

$$\tilde{C}_{2,ij} = 4 \begin{pmatrix} a+e & 0 & 0 \\ 0 & c+d & -d \\ 0 & -d & a+d \end{pmatrix}.$$

We prove now

LEMMA 5.24 . *It holds for* $0 \leq i < j < n$

$$\begin{aligned} \lambda_{min}(\tilde{C}_{2,ij}^{-1} C_{2,ij}) &\geq 1 - \frac{1}{3}\sqrt{3} \text{ and} \\ \lambda_{max}(\tilde{C}_{2,ij}^{-1} C_{2,ij}) &\leq 1 + \frac{1}{3}\sqrt{3}. \end{aligned}$$

Proof: Let

$$\beta = ac + ad + cd.$$

Then, we have

$$\tilde{C}_{2,ij}^{-1} C_{2,ij} = \begin{pmatrix} 1 & 0 & \frac{-a}{a+e} \\ \frac{-ad}{\beta} & 1 & 0 \\ \frac{-ac-ad}{\beta} & 0 & 1 \end{pmatrix}.$$

We get the characteristical polynomial

$$\det(\lambda I - \tilde{C}_{2,ij}^{-1} C_{2,ij}) = (1-\lambda)\left((1-\lambda)^2 - \frac{a}{a+e}\frac{ac+ad}{ac+ad+cd}\right).$$

We can estimate using $a < c$

$$\frac{ac+ad}{ac+ad+cd} \le \frac{ac+ad}{ac+2ad} = \frac{c+d}{c+2d} = \frac{1}{1+\frac{1}{\frac{c}{d}+1}}.$$

We have for $i \le j-1$ that $\frac{c}{d} \le 1$ and $\frac{e}{a} \ge 1$. Therefore, we obtain

$$\frac{ac+ad}{ac+ad+cd} \le \frac{2}{3} \tag{5.29}$$

and

$$\frac{a}{a+e} \le \frac{1}{2}. \tag{5.30}$$

The roots of the charactertistical polynomial are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_{2,3} &= 1 \pm \sqrt{\rho}, \end{aligned}$$

where

$$\rho = \frac{a}{a+e}\frac{ac+ad}{ac+ad+cd}.$$

Inserting the estimates (5.30) and (5.29), we obtain

$$1 - \sqrt{\frac{1}{3}} \le \lambda_3 \le \lambda_2 \le 1 + \sqrt{\frac{1}{3}}.$$

Hence, the assertion follows immediately. □

We consider now $i \ge j$. Let for $i \ge j$

$$\tilde{C}_{2,ij} = 4 \begin{pmatrix} a+e & 0 & -a \\ 0 & c+d & 0 \\ -a & 0 & a+d \end{pmatrix}.$$

We prove now

LEMMA 5.25 . *It holds*

$$\begin{aligned} \lambda_{min}(\tilde{C}_{2,ij}^{-1} C_{2,ij}) &\ge 1 - \frac{1}{10}\sqrt{3}5 \text{ and} \\ \lambda_{max}(\tilde{C}_{2,ij}^{-1} C_{2,ij}) &\le 1 + \frac{1}{10}\sqrt{3}5. \end{aligned}$$

for $n > i \ge j \ge 0$.

Proof: We start with the case $i < n-1$ and $j > 0$. The proof is similar to the proof of Lemma 5.24. A short calculation yields

$$\det(\lambda I - \tilde{C}_{2,ij}^{-1} C_{2,ij}) = (\lambda - 1)\left((\lambda-1)^2 - \frac{d}{d+c}\frac{ad+ed}{ae+ad+ed}\right).$$

We have from $i \geq j$

$$\frac{c}{d} > 1$$

and using $j \geq 1$

$$\frac{d}{e} \leq \frac{5}{3}.$$

Hence, we can estimate

$$\frac{d}{d+c}\frac{ad+ed}{ae+ad+ed} < \frac{7}{20}.$$

The assertion follows as in the proof of Lemma 5.24.

We consider now $i = n-1$. Then, we can cancel the second row and column of $\tilde{C}_{2,ij}$ and $C_{2,ij}$. These matrices are identical and we obtain

$$\lambda_1(\tilde{C}_{2,n-1,j}^{-1} C_{2,n-1,j}) = \lambda_2(\tilde{C}_{2,n-1,j}^{-1} C_{2,n-1,j}) = 1.$$

The last case is $j = 0$. We have to omit the first row and column. A short calculation shows

$$\det(\lambda I - \tilde{C}_{2,i,0}^{-1} C_{2,i,0}) = (\lambda-1)^2 - \frac{14C_1}{c+14C_1}\frac{14C_1}{a+14C_1}.$$

We have $a \geq 14C_1$ and $c \geq 14C_1$ with $C_1 = \frac{1}{192n^2}$. Hence, we get for the roots of the charcateristical polynomial the estimates

$$\frac{1}{2} \leq \lambda_2 < \lambda_1 \leq \frac{3}{2}.$$

□

REMARK 5.26 *We define matrices* $\tilde{C}_{1,ij}$ *in the same way:*

$$\tilde{C}_{1,ij} = 4\begin{pmatrix} b+d & 0 & -d \\ 0 & a+f & 0 \\ -d & 0 & a+d \end{pmatrix} \quad \text{for } i \leq j$$

and

$$\tilde{C}_{1,ij} = 4 \begin{pmatrix} b+d & 0 & 0 \\ 0 & a+f & -a \\ 0 & -a & a+d \end{pmatrix} \text{ for } i > j$$

By the symmetry of the differential operator, we obtain the same results for the triangles $\tau_{ij}^{1,k}$ as in Lemmata 5.24 and 5.25.

Now, we define a global preconditioner C_w using the local matrices $\hat{C}_{s,ij}$ and $\tilde{C}_{s,ij}$. We know that

$$K_{\mathbb{W}_{k+1}} = a(\phi_{ij}^{k+1}, \phi_{lm}^{k+1})_{(i,j),(l,m)\in N_{k+1}},$$

is the stiffness matrix K restricted to the space $\mathbb{W}_k$ compare (5.6), (5.7). The matrix $K_{\mathbb{W}_{k+1}}$ is the result of assembling the local stiffness matrices $C_{s,ij}$, $s = 1, 2$ and $i, j = 0, \ldots, n-1$, i.e.

$$K_{\mathbb{W}_{k+1}} = \sum_{s=1}^{2} \sum_{i,j=0}^{n-1} L_{s,ij}^t C_{s,ij} L_{s,ij}. \tag{5.31}$$

The matrices $L_{sij} \in \mathbb{R}^{3\cdot 4^{k-1}-2^k,3}$ are the usual finite element assembling matrices, because

$$(2^k - 1)^2 - (2^{k-1} - 1)^2 = 3 \cdot 4^{k-1} - 2^k.$$

DEFINITION 5.27 *We define the matrix $C_{\mathbb{W}_{k+1}}$ by*

$$C_{\mathbb{W}_{k+1}} = \sum_{s=1}^{2} \sum_{i,j=0}^{n-1} L_{s,ij}^t \tilde{C}_{s,ij} L_{s,ij}. \tag{5.32}$$

We formulate now the main theorem of this section.

THEOREM 5.28 *It holds*

$$\begin{aligned} \lambda_{min}(C_{\mathbb{W}_{k+1}}^{-1} K_{\mathbb{W}_{k+1}}) &\geq 1 - \frac{1}{10}\sqrt{35}, \\ \lambda_{max}(C_{\mathbb{W}_{k+1}}^{-1} K_{\mathbb{W}_{k+1}}) &\leq 1 + \frac{1}{10}\sqrt{35}. \end{aligned}$$

Proof: Use lemmata 5.13, 5.24 and 5.25, Remark 5.26 and relations (5.31) and (5.32). □

COROLLARY 5.29 *Let*

$$S = I - \omega C_{\mathbb{W}_{k+1}}^{-1} K_{\mathbb{W}_{k+1}} \tag{5.33}$$

be a ω-Jacobi- like smoother on Level $k+1$. Let

$$\| w \|_a^2 = a(w,w).$$

Then, for all $w \in \mathbb{W}_{k+1}$

$$\| S^\nu w \|_a \leq \rho^\nu \| w \|_a$$

holds, where

$$\omega = 1$$

and

$$\rho = \frac{1}{10}\sqrt{3}5. \tag{5.34}$$

Proof: We have by calculation

$$\begin{aligned}
\rho^2 &= \sup_{w \in \mathbb{W}_{k+1}, w \neq \mathbf{0}} \frac{\| Sw \|_a^2}{\| w \|_a^2} \\
&= \sup_{\underline{w}} \frac{(K_{\mathbb{W}_{k+1}} S\underline{w}, S\underline{w})}{(K_{\mathbb{W}_{k+1}} \underline{w}, \underline{w})} \\
&= \sup_{\underline{u}} \frac{(K_{\mathbb{W}_{k+1}}^{-\frac{1}{2}} S^t K_{\mathbb{W}_{k+1}} S K_{\mathbb{W}_{k+1}}^{-\frac{1}{2}} \underline{u}, \underline{u})}{(\underline{u}, \underline{u})} \\
&= \lambda_{max}(K_{\mathbb{W}_{k+1}}^{-\frac{1}{2}} S^t K_{\mathbb{W}_{k+1}} S K_{\mathbb{W}_{k+1}}^{-\frac{1}{2}}) = (\lambda_{max}(S^2)) = (\lambda_{max}(S))^2.
\end{aligned}$$

The assertion follows using (5.33) and Theorem 5.28. □

We have defined a relatively difficult preconditioner $C_{\mathbb{W}_{k+1}}$ for the matrix $K_{\mathbb{W}_{k+1}}$. The question is why do we not use simple diagonal preconditioner? Consider the matrix C_4 (3.5), which is

$$K_{h,k} = \frac{1}{2n^2} C_4, .$$

compare (4.7). Furthermore,

$$K_{\mathbb{W}_k} = V^t K_{h,k} V$$

with $V = \mathrm{diag}(v_{ij})_{i,j=1}^{n-1}$ and

$$v_{ij} = \left\{ \begin{array}{ll} 1 & (i,j) \in N_k \\ 0 & (i,j) \notin N_k \end{array} \right\}.$$

We have from (3.5)

$$K_{h,k} = \frac{1}{2n^2}(D_4 \otimes T_3 + T_3 \otimes D_4).$$

The main diagonal of this matrix is given by

$$D_h = \frac{1}{2n^2}(D_4 \otimes I + I \otimes D_4), \tag{5.35}$$

where I denotes the identity matrix. Then

$$D_{\mathbb{W}_k} = V^t D_h V$$

is the main diagonal of $K_{\mathbb{W}_{k+1}}$. Evidently,

$$\lambda_{min}(D_{\mathbb{W}_k}^{-1} K_{\mathbb{W}_k}) = \min_{\underline{v} \notin \ker V} \frac{(K_{\mathbb{W}_k}\underline{v}, \underline{v})}{(D_{\mathbb{W}_k}\underline{v}, \underline{v})} \tag{5.36}$$

and

$$\lambda_{max}(D_{\mathbb{W}_k}^{-1} K_{\mathbb{W}_k}) = \max_{\underline{v} \notin \ker V} \frac{(K_{\mathbb{W}_k}\underline{v}, \underline{v})}{(D_{\mathbb{W}_k}\underline{v}, \underline{v})}. \tag{5.37}$$

Consider now the vector

$$\underline{v}_l = \underline{x} \otimes \underline{e}_l, \tag{5.38}$$

where $\underline{x}$ is some vector and $\underline{e}_l$ is the k-th unit vector. For $l = 1, 3, \ldots, n-1$, we have

$$V \underline{v}_l = \underline{v}_l \notin \ker V.$$

Using the properties of the Kronecker product and (5.35) and (5.38), we obtain

$$\frac{(K_{\mathbb{W}_k}\underline{v}_l,\underline{v}_l)}{(D_{\mathbb{W}_k}\underline{v}_l,\underline{v}_l)}=\frac{(D_4\underline{x},\underline{x})(T_3\underline{e}_l,\underline{e}_l)+(T_3\underline{x},\underline{x})(D_4\underline{e}_l,\underline{e}_l)}{(D_4\underline{x},\underline{x})(I\underline{e}_l,\underline{e}_l)+(I\underline{x},\underline{x})(D_4\underline{e}_l,\underline{e}_l)}.$$

A simple calculation shows

$$\begin{aligned}(D_4\underline{e}_l,\underline{e}_l) &= 4(l^2+\frac{1}{6}),\\ (T_3\underline{e}_l,\underline{e}_l) &= 1,\\ (I\underline{e}_l,\underline{e}_l) &= 1.\end{aligned}$$

Inserting this, we obtain, setting $l=n-1$,

$$\frac{(K_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}{(D_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}=\frac{(D_4\underline{x},\underline{x})+(T_3\underline{x},\underline{x})4((n-1)^2+\frac{1}{6})}{(D_4\underline{x},\underline{x})+(I\underline{x},\underline{x})4((n-1)^2+\frac{1}{6})}. \tag{5.39}$$

We introduce the matrix

$$D_7=\frac{1}{4((n-1)^2+\frac{1}{6})}D_4=\frac{1}{4((n-1)^2+\frac{1}{6})}\mathrm{diag}\left(4(l^2+\frac{1}{6})\right)_{l=1}^{n-1}. \tag{5.40}$$

Inserting (5.40) into (5.39), we have

$$\frac{(K_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}{(D_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}=\frac{(D_7\underline{x},\underline{x})+(T_3\underline{x},\underline{x})}{(D_7\underline{x},\underline{x})(I\underline{x},\underline{x})}=\frac{((D_7+T_3)\underline{x},\underline{x})}{((D_7+I)\underline{x},\underline{x})}. \tag{5.41}$$

The matrix D_7+I is spectral equivalent to the unity matrix I, i.e.

$$(\underline{x},\underline{x})\leq((D_7+I)\underline{x},\underline{x})\leq 2(\underline{x},\underline{x})$$

for all $\underline{x}\in\mathbb{R}^{n-1}$. Therefore, we obtain

$$\frac{(K_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}{(D_{\mathbb{W}_k}\underline{v}_{n-1},\underline{v}_{n-1})}\asymp\frac{((D_7+T_3)\underline{x},\underline{x})}{(\underline{x},\underline{x})},$$

where $\underline{x}$ is any vector of $\mathbb{R}^{n-1}$. We calculated the maximal and minimal eigenvalue of the matrix D_7+T_3 using a matlab routine for several values of n. Table 1 displays the results. We see that the eigenvalue $\lambda_{max}(T_3+D_7)\leq 3$, this estimate can be proven:

$$1<\lambda_{max}(T_3)<\lambda_{max}(T_3+D_7)\leq\lambda_{max}(T_3)+\lambda_{max}(D_7)\leq 2+1=3.$$

n	λ_{min}	λ_{max}	$\frac{\lambda_{max}}{\lambda_{min}}$
4	0.6908	2.3547	3.4125
8	0.2995	2.5319	8.4546
16	0.1407	2.6710	18.9827
32	0.0683	2.7755	40.6542
64	0.0336	2.8497	84.7278
128	0.0167	2.9009	173.7657
256	0.0083	2.9353	352.9433
512	0.0042	2.9581	712.6717
1024	0.0021	2.9731	1433.8

Table 1: Eigenvalues of the matrix $T_3 + D_7$

The calculation shows, that $\lambda_{min}(T_3 + D_7)$ goes to zero with order 1. Hence, we can deduce that

$$\begin{aligned} \lambda_{min}(D^{-1}_{\mathbb{W}_k} K_{\mathbb{W}_k}) &\preceq \frac{c_{15}}{n}, \\ \lambda_{max}(D^{-1}_{\mathbb{W}_k} K_{\mathbb{W}_k}) &\geq c_{16}. \end{aligned}$$

Thus using (5.36) and (5.37), we cannot expect that a simple Jacobi-method with diagonal preconditioning fulfills the smoothing property of Corollary 5.29.

REMARK 5.30 *A simple Jacobi-method with diagonal scaling does not fulfill the assertion of Corollary 5.29.*

5.2.4 Application of the multi-grid theory to $-x^2 u_{yy} - y^2 u_{xx} = g$

We apply now the theory of 5.1 to the problem (4.4). With Theorem 5.23 assumption (5.3) is fulfilled with $\gamma^2 \leq \frac{95}{176}$. The second assumption, (5.2), of Theorem 5.2 is fulfilled for the smoother S defined in (5.33). Hence, we can prove a convergence rate $0 \leq \sigma < 1$ of the multi-grid algorithm for $\mu \geq 3$ if we do sufficiently many smoothing steps. The parameter σ does not depend on the level k. No mesh-size independent convergence rate can be proven for $\mu \leq 2$ because $\gamma^2 > \frac{1}{2}$. We summarize the results in

THEOREM 5.31 . *Consider* (4.4) *with the exact linear system solution* u^* . *We solve this linear system using the multi-grid algorithm* $u_{j+1,k} =$

ν	σ
< 2	1
3	0.88063
4	0.79639
8	0.70453
∞	0.69283

Table 2: Estimates for convergence rates σ for $\mu = 3$.

$MULT(k, u_{j,k}, g)$ with $\mu \geq 3$ and ν smoothing steps. The rate of convergence

$$\sigma_k = \sup_{u_{j,k}-u^* \in \mathbb{V}_k} \frac{\| u_{j+1,k} - u^* \|_a}{\| u_{j,k} - u^* \|_a}$$

on level k can be bounded by

$$\sigma_k \leq \sigma < 1.$$

Using Lemma 5.3, we can analyze the number of smoothing steps ν, which are necessary for a convergence rate $\sigma < 1$. We have

$$\kappa = C\rho^\nu + (1 - C\rho^\nu)\gamma^2,$$

with $C = 1$, $\gamma^2 = \frac{95}{176}$ and from (5.34) $\rho = \frac{1}{10}\sqrt{35}$. Using Remark 5.4, we have for

$$\nu \geq 2\frac{\ln 67 - \ln 243}{\ln 7 - \ln 20} \approx 2.45$$

a mesh-size independent convergence rate $\sigma < 1$. Table 2 displays the theoretical convergence rates σ for several values of ν obtained by Lemma 5.3 for $\mu = 3$.

5.3 Implementional details

During this subsection the procedure of solving the linear system

$$C_{\mathbb{W}_{k+1}}\underline{w} = \underline{r} \tag{5.42}$$

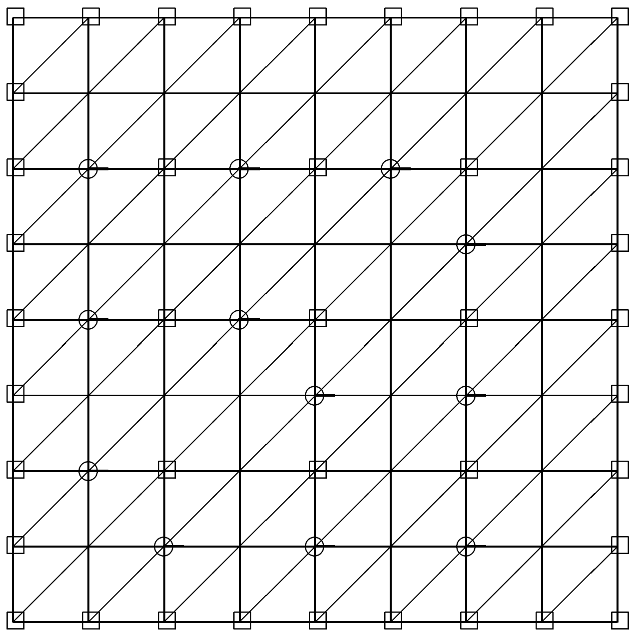

Figure 5: En-coupling of the nodes.

is discussed. We consider the sub-cell $\mathcal{E}_{ij}^k$ with $i < j$. Here, we have the matrix

$$C = \begin{pmatrix} a+e & 0 & 0 & 0 & 0 \\ 0 & b+d & 0 & 0 & -d \\ 0 & 0 & c+d & 0 & -d \\ 0 & 0 & 0 & a+f & 0 \\ 0 & -d & -d & 0 & 2a+2d \end{pmatrix}.$$

Using the notation of Figure 4. The nodes $x_{2i+1,2j}^{k+1}$ and $x_{2i+1,2j+2}^{k+1}$ on $\mathcal{E}_{ij}^k$ are decoupled from the remaining unknowns. The basis function $\phi_{2i+1,2j}$ associated to this node has a support contained in $\overline{\mathcal{E}_{ij}^k \cup \mathcal{E}_{i,j-1}^k}$. The node $x_{2i+1,2j}^{k+1}$ plays on the macro-element $\mathcal{E}_{i,j-1}$ the same rule as $x_{2i+1,2j+2}^{k+1}$ on $\mathcal{E}_{ij}^k$ and is decoupled on the macro-element $\mathcal{E}_{i,j-1}$. Hence, the functions $\phi_{2i+1,j}$ are decoupled for $i < j$. The functions $\phi_{2i,2j+1}$ are decoupled for $i < j$ by symmetry. We consider now Figure 5. The nodes marked with $\square$ are coarse grid nodes or nodes on the boundary and do not exist for the matrices $C_{\mathbb{W}_{k+1}}$

Level	$\mu = 1$		$\mu = 2$		$\mu = 3$		$\mu = 4$	
	It	σ	It	σ	It	σ	It	σ
2	18	0.4070	18	0.4070	18	0.4070	18	0.4070
3	32	0.6017	24	0.4997	22	0.4778	22	0.4722
4	50	0.7239	25	0.5221	22	0.4698	21	0.4583
5	72	0.7974	27	0.5449	22	0.4770	21	0.4582
6	97	0.8463	30	0.5755	24	0.5035	22	0.4719
7	128	0.8814	34	0.6201	25	0.5156	22	0.4788
8	176	0.9123	37	0.6432	26	0.5282	23	0.4838
9	247	0.9373	41	0.6724	26	0.5339	23	0.4847
10	300	0.9545	44	0.6901	26	0.5380	23	0.4841

Table 3: Convergence rates of multi-grid algorithm $MULT$ using smoother S ($\nu = 1$).

and $K_{\mathbb{W}_{k+1}}$. The nodes marked with $\circ$ are decoupled for the preconditioner. The remaining nodes are coupled. The remaining functions $\phi_{2i+1,2j+1}$ are coupled in the groups $2m+1$, where

$$\max_{i,j} = m,\ m = 1, 2, \dots, \frac{n}{2}.$$

Hence, $C_{\mathbb{W}_{k+1}}$ is after a proper permutation a block diagonal matrix of diagonal and tridiagonal blocks. Therefore, we can solve the system (5.42) using Cholesky decomposition in $\mathcal{O}(n^2)$ flops. Hence, the operation $S\underline{w}$ is arithmetically optimal.
The unknowns of the system (4.4) increase per level to the factor 4. Furthermore, we can choose $\nu = 3$ on each level. So, using Theorem 5.31 the multi-grid algorithm $MULT$ for $\mu = 3$ is an arithmetical optimal method.

5.4 Numerical results

5.4.1 Convergence rate of multi-grid

Table 3 shows the convergence rates of the multi-grid algorithm $MULT$ for solving (4.4) with $g \equiv 1$ used for a several kind of cycles. We stop the algorithm, if the relative error in the energy norm is lower than 10^{-7}. The V-cycle has clearly growing numbers of iterations, but for $\mu \geq 3$ we have mesh-independent convergence rates. We can not say what happens for the W-cycle. The bad numbers of iterations for the V-cycle are not satisfactory.

The reason is the smoother S which operates only on the nodes on $\mathbb{W}_k$. We define now a better smoother on $\mathbb{V}_k$. We consider Figure 4. We introduce the matrix

$$\tilde{A}_{ij} = \begin{pmatrix} a+b+d+e & -d-e & 0 & 0 \\ -d-e & a+c+d+e & 0 & 0 \\ 0 & 0 & a+b+d+f & -d-f \\ 0 & 0 & -d-f & a+c+d+f \end{pmatrix}$$

for $i < j$, the matrix

$$\tilde{A}_{ij} = \begin{pmatrix} a+b+d+e & 0 & -a-b & 0 \\ 0 & a+c+d+e & 0 & -a-c \\ 0 & 0 & a+b+d+f & 0 \\ 0 & -a-c & 0 & a+c+d+f \end{pmatrix}$$

for $i > j$ and

$$\tilde{A}_{ij} = \begin{pmatrix} a+b+d+e & 0 & 0 & 0 \\ 0 & a+c+d+e & 0 & -a-c \\ 0 & 0 & a+b+d+f & -d-f \\ 0 & -a-c & -d-f & a+c+d+f \end{pmatrix}$$

for $i = j$. The matrix K_h is the result of assembling local stiffness matrices, i.e.

$$K_h = \sum_{i,j=0}^{n-1} L_{ij}^t A_{ij} L_{ij}$$

with some matrices L_{ij} and the matrices A_{ij} of Lemma 5.14. We define now the matrix

$$\tilde{K}_h = \sum_{i,j=0}^{n-1} L_{ij}^t \tilde{A}_{ij} L_{ij}$$

and the Jacobi-smoother

$$S_1 = I - \omega \tilde{K}_h^{-1} K_h \tag{5.43}$$

Level	$\mu = 1$		$\mu = 2$	
	It	σ	It	σ
2	9	0.1611	9	0.1611
3	11	0.2290	10	0.1951
4	13	0.2723	12	0.2522
5	15	0.3250	14	0.2941
6	16	0.3517	15	0.3192
7	16	0.3619	15	0.3331
8	17	0.3680	15	0.3392
9	17	0.3720	16	0.3429
10	17	0.3750	16	0.3442

Table 4: Convergence rates of multi-grid algorithm $MULT$ using smoother S_1 ($\nu = 1$).

on Level k. This smoother is very similar to S. The matrix $\tilde{K}_h$ is block tridiagonal. One block corresponds to nodal basis functions ϕ_{ij}^k with

$$\max_{i,j} = m, \; m = 1, 2, \ldots, n.$$

Therefore,

$$S_1 \underline{w} = \underline{r}$$

can be done using Cholesky decompostion in $\mathcal{O}(n^2)$ flops. This smoother operates on the space $\mathbb{V}_k$ and we expect better convergence rates of the multi-grid algorithm $MULT$. Table 4 displays the convergence rates for $MULT$ using the smoother S_1. We solve (4.4) with $g \equiv 1$ and we stop the algorithm, if the error in the energy norm is lower than 10^{-7}. We choose $\omega = 0.8$, which shows the best convergence rates. We see for V and W-cycle mesh-independent convergence rates, but the convergence rates are not satisfactory.

We expect to obtain better results by using a preconditioned conjugate gradient method with one multi-grid cycle as preconditioner. Table 5 shows the number of iterations to reduce the error in the preconditioned energy norm up to a factor 10^{-9}. We choose $f \equiv 1$. We see constant number of iterations in two cases, V-cycle with smoother S_1 and $\mu = 3$ with smoother S, but nonconstant number of iterations for the V-cycle and smoother S.

Level	S			S_1
	$\mu = 1$	$\mu = 2$	$\mu = 3$	$\mu = 1$
2	7	8	7	7
3	12	12	11	9
4	15	13	13	10
5	16	14	13	10
6	18	14	13	11
7	21	15	13	11
8	23	16	14	11
9	25	16	14	11

Table 5: Number of iterations of the PCG-method using a multi-grid preconditioner with smoother S ($\omega = 1$) and S_1 ($\omega = 0.8$) and $\nu = 1$.

We cannot say if the number of iterations for the W-cycle with smoother S are constant.

6 AMLI-method

6.1 Convergence theory for AMLI

We discuss the possibility of applying the Algebraic Multi-Level preconditioner (AMLI), derived by Axelsson and Vassilevski [2], [3]. Consider the stiffness matrix for (4.5). We introduce the block structure of the stiffness matrix

$$K_{h,k} = \begin{pmatrix} K_{11,k} & K_{12,k} \\ K_{21,k} & K_{22,k} \end{pmatrix},$$

where $K_{22,k} = K_{\mathbb{W}_k}$ corresponds to the nodal basis functions in $\mathbb{W}_k$ and

$$K_{11,k} = a(\phi^k_{2i,2j}, \phi^k_{2l,2m})^{\frac{n}{2}-1}_{i,j,l,m=1}$$

corresponds to nodal basis functions of nodes on the coarse grid. Let $C_{22,k}$ be a matrix satisfying

$$(K_{22,l}\underline{v}, \underline{v}) \leq (C_{22,l}\underline{v}, \underline{v}) \leq (1 + b)(K_{22,l}\underline{v}, \underline{v}) \tag{6.1}$$

for all $\underline{v}$, $l = 1, \ldots, k$ and $b \geq 0$. Let

$$\hat{K}_{h,k} = \begin{pmatrix} \hat{K}_{11,k} & \hat{K}_{12,k} \\ \hat{K}_{21,k} & K_{22,k} \end{pmatrix},$$

be the stiffness matrix with respect to the two level basis

$$\{\phi_{ij}^{k-1}\}_{i,j=1}^{\frac{n}{2}-1} \in \mathbb{V}_{k-1}$$

and

$$\{\phi_{ij}^{k}\},\ \phi_{ij}^{k} \in \mathbb{W}_k,$$

corresponding to the splitting $\mathbb{V}_k = \mathbb{V}_{k-1} \oplus \mathbb{W}_k$. Thus, we have

$$\hat{K}_{11,k} = K_{h,k-1}.$$

Obviously, we have

$$\hat{K}_{h,k} = J_k K_{h,k} J_k^t,$$

with the interpolation matrix

$$J_k = \begin{pmatrix} I & J_{12,k} \\ \mathbf{0} & I \end{pmatrix}.$$

We define now, see [3],[13], the preconditioning matrix $C_{h,k}$.

DEFINITION 6.1 *Let P_μ be a polynomial of degree μ satisfying*

$$P_\mu(0) = 1 \tag{6.2}$$

and

$$0 < P_\mu(t) < 1 \text{ for } 0 < t \leq 1.$$

Let $C_{22,k}$ a matrix which fulfilles (6.1). *Then, we define preconditioning matrix $C_{h,k}$ by*

$$\begin{aligned} C_{h,k} \;=\; & \begin{pmatrix} C_{k-1}^c & K_{12,k} + J_{12,k}(K_{22,k} - C_{22,k}) \\ \mathbf{0} & C_{22,k} \end{pmatrix} \\ & \begin{pmatrix} I & \mathbf{0} \\ C_{22,k}^{-1}(K_{21,k} + (K_{22,k} - C_{22,k})J_{12,k}^t) & I \end{pmatrix}, \end{aligned} \tag{6.3}$$

with

$$(C_{k-1}^c)^{-1} = (I - P_\mu(C_{h,k-1}^{-1} K_{h,k})) K_{h,k}^{-1}. \tag{6.4}$$

Examples for the choice of P_μ are given in [2], [3], we consider

$$P_\mu(t) = \left(T_\mu\left(\frac{1+\alpha-2t}{1-\alpha}\right)+1\right) \Big/ \left(T_\mu\left(\frac{1+\alpha}{1-\alpha}\right)+1\right) \tag{6.5}$$

with some $0 < \alpha < 1$. $T_\mu(x)$ denotes the μ-th Chebyshev-polynomial

$$T_\mu(x) = \cos(\mu \arccos(x)).$$

The following theorem holds.

THEOREM 6.2 *Consider the preconditioner* $C_{h,k}$ (6.3) *with the polynomial* (6.5). *Let us assume, that*

$$\mu > \frac{1}{\sqrt{1-\gamma^2}}. \tag{6.6}$$

Thus, the inequality

$$c_{17}(C_{h,k}\underline{v}, \underline{v}) \leq (K_{h,k}\underline{v}, \underline{v}) \leq (C_{h,k}\underline{v}, \underline{v})$$

holds for all $\underline{v}$, *where*

$$c_{17} = (1-\gamma^2)\left(b + \left(\frac{(1+\sqrt{\alpha})^\mu + (1-\sqrt{\alpha})^\mu}{(1+\sqrt{\alpha})^\mu - (1-\sqrt{\alpha})^\mu}\right)^2\right)^{-1}.$$

The constant γ *is the constant of the strengthened Cauchy-inequality* (5.1), *the constant* b *the constant of* (6.1). *The parameter* α *is the smallest positive solution of the polynomial equation*

$$1-\gamma^2 = tb + \left(\frac{(1+\sqrt{t})^\mu + (1-\sqrt{t})^\mu}{2\sum_{s=1}^{\mu}(1+\sqrt{t})^{\mu-s}(1-\sqrt{t})^{s-1}}\right)^2. \tag{6.7}$$

Proof: [3]. □

We describe now the algorithm for solving a linear system with the matrix C^c_{k-1} (6.4). From (6.2), we can deduce

$$P_\mu(t) = \sum_{j=0}^{\mu} a_j t^j,$$

where $a_0 = 1$. Hence, we obtain

$$\begin{aligned}
(C^c_{k-1})^{-1} &= (I - P_\mu(C^{-1}_{h,k-1}K_{h,k-1}))K^{-1}_{h,k-1} \\
&= (I - \sum_{j=0}^{\mu} a_j(C^{-1}_{h,k-1}K_{h,k-1})^j)K^{-1}_{h,k-1} \\
&= -\sum_{j=1}^{\mu} a_j(C^{-1}_{h,k-1}K_{h,k-1})^j K^{-1}_{h,k-1} \\
&= -C^{-1}_{h,k-1}(a_1 + K_{h,k-1}C^{-1}_{h,k-1}(a_2 + \ldots \\
&\quad \ldots + K_{h,k-1}C^{-1}_{h,k-1}(a_{\mu-1} + a_\mu K_{h,k-1}C^{-1}_{h,k-1})\ldots)).
\end{aligned}$$

Thus, a linear system with the matrix C^c_{k-1} can be solved by solving μ linear systems with the matrix $C_{h,k-1}$.

6.2 Application to $-x^2u_{yy} - y^2u_{xx}$

We apply now this theory to problem (4.5). We have from Theorem 5.23, that the constant in the strengthened Cauchy-inequality (5.1)

$$\gamma^2 = \frac{95}{176}.$$

Thus, we have

$$\frac{1}{\sqrt{1-\gamma^2}} = \frac{4\sqrt{11}}{9} < 2.$$

Using (6.6), we can choose $\mu = 2$. Hence, we obtain

$$T_\mu(x) = T_2(x) = 2x^2 - 1$$

and

$$P_2(t) = (1 - \frac{2t}{1+\alpha})^2. \tag{6.8}$$

Furthermore, we have to ensure (6.1). Using Theorem 5.28, we have for all $k \in \mathbb{N}$

$$c_{18}(C_{\mathbb{W}_k}\underline{v}, \underline{v}) \leq (K_{\mathbb{W}_k}\underline{v}, \underline{v}) \leq c_{19}(C_{\mathbb{W}_k}\underline{v}, \underline{v}),$$

where $c_{18} = 1 - \frac{1}{10}\sqrt{35}$ and $c_{19} = 1 + \frac{1}{10}\sqrt{35}$, or equivalently

$$c_{19}^{-1}(K_{\mathbb{W}_k}\underline{v}, \underline{v}) \leq (C_{\mathbb{W}_k}\underline{v}, \underline{v}) \leq c_{18}^{-1}(K_{\mathbb{W}_k}\underline{v}, \underline{v}).$$

DEFINITION 6.3 *We define now*

$$C_{22,l} = c_{19} C_{\mathbb{W}_l},$$

for $l = 1, \ldots, k$.

Hence, we obtain

$$(K_{22,l}\underline{v}, \underline{v}) = (K_{\mathbb{W}_l}\underline{v}, \underline{v}) \le (C_{22,l}\underline{v}, \underline{v}) \le \frac{c_{19}}{c_{18}}(K_{22,l}\underline{v}, \underline{v}),$$

e.g. (6.1) is satisfied with

$$\tilde{b} = -1 + \frac{c_{19}}{c_{18}} = \frac{4}{13}\sqrt{35} + \frac{14}{13} < \frac{3119}{1056}. \tag{6.9}$$

With $b = \frac{3119}{1056}$ and $\gamma^2 = \frac{95}{176}$, we obtain as smallest positive solution of (6.7)

$$\alpha = \frac{2}{33}.$$

Thus, we choose

$$P_{2,\frac{66}{35}}(t) = (1 - \frac{66}{35}t)^2. \tag{6.10}$$

We summarize these observations in

THEOREM 6.4 . *Consider the matrix* $C_{h,k}$ *of Definition 6.1 with* $C_{22,l}$, $l = 1, \ldots, k$ *of Definition 6.3 and the polynomial* $P_{2,\frac{66}{35}}(t)$ *(6.10). Then*

$$c_{17}(C_{h,k}\underline{v}, \underline{v}) \le (K_{h,k}\underline{v}, \underline{v}) \le (C_{h,k}\underline{v}, \underline{v})$$

holds $\forall \underline{v} \in \mathbb{R}^{n^2}$ *with*

$$c_{17} = (1 - \gamma^2)\frac{4\alpha^2}{\alpha^2(4b+1) + 1 + 2\alpha} = \frac{324}{309095} \approx 0.001048.$$

6.3 Numerical results

We consider (4.5) and solve this linear system with the preconditioned conjugate gradient method. As preconditioner we choose $C_{h,k}$ of Definition 6.1.

Level	$P_{1,1}(t)$	$P_{2,1}(t)$	$P_{2,\frac{52}{35}}(t)$	$P_{2,\frac{66}{35}}(t)$	$P_{3,1}(t)$
2	8	8	8	8	8
3	17	16	16	16	16
4	23	17	17	18	17
5	28	18	17	19	18
6	33	19	17	21	18
7	39	20	18	21	18
8	46	21	18	21	18
9	52	22	17	22	18

Table 6: Number of iterations of the PCG-method with AMLI-preconditioners.

The relative accuracy is 10^{-9} in the preconditioned energy norm, and $g \equiv 1$ is chosen. We consider for the choice of the polynomial $P_\mu(t)$ the cases

$$\begin{aligned} P_{\mu,1}(t) &= (1-t)^\mu \text{ for } \mu = 1,2,3, \\ P_{2,r}(t) &= (1-rt)^2 \text{ for } r = \frac{52}{35}, \frac{66}{35}. \end{aligned} \tag{6.11}$$

The matrix $C_{22,l}$ of Definition 6.3 is chosen. Note, that we have proved Theorem 6.4 for $P_{2,\frac{66}{35}}(t)$. But, the estimates of the maximal and minimal eigenvalue of $C^{-1}_{\mathbb{W}_{k+1}} K_{\mathbb{W}_{k+1}}$ in Theorem 5.28 are estimates and we do not know the exact values, which can be better. The polynomial $P_{2,\frac{52}{35}}(t)$ is that polynomial $(1-rt)^2$ with the lowest number of iterations in Level 9 for $r = \frac{36}{35}, \frac{38}{35}, \ldots, \frac{66}{35}$.

Table 6 displays the number of iterations for the AMLI-preconditioners with the several polynomials. The number of iterations are constant for $P_{2,\frac{52}{35}}(t)$, $P_{2,\frac{66}{35}}(t)$, $P_{3,1}(t)$ and grow proportional to the number of levels for $P_{1,1}(t)$ and $P_{2,1}(t)$..

7 Preconditioning for the p-version matrix K

7.1 Estimates for the multi-grid method $MULT$ and the AMLI-method as preconditioner

We are interested in a good preconditioner for the matrix K, the element stiffness matrix for the interior unknowns on $(-1,1)^2$ with respect to the basis of the integrated Legendre polynomials $\hat{L}_{ij}$, $2 \le i, j \le p$. From Theorem 3.1, we have that a matrix C_4 is a good preconditioner for each block K_i of the matrix K. From (4.7) we can conclude, the matrix C_4 can be interpreted as the stiffness matrix for $-x^2 u_{yy} - y^2 u_{xx}$ using piecewise linear shape functions on isosceles, right and congruent triangles on the domain $(0,1)^2$ with Dirichlet boundary conditions, i.e.

$$K_{h,k} = \frac{1}{2n^2} C_4.$$

We have proved in Theorem 5.31, that the multi-grid algorithm $MULT$ brings a mesh-independent convergence rate $\sigma < 1$ for $\mu = 3$ and the smoother S. Therefore using Theorem 6.5. of [8], we have proved

THEOREM 7.1 . *Let M_μ^S the preconditioner resulting from 1 iteration multi-grid algorithm $MULT$ with $\mu = 3$ and the smoother S. Let $K_i, i = 1, \dots, 4$ the 4 blocks of K. The statement*

$$c_{13}(M_\mu^S \underline{v}, \underline{v}) \le \frac{1}{p^2}(K_i \underline{v}, \underline{v}) \le c_{14}(1 + \log p)(M_\mu^S \underline{v}, \underline{v}).$$

is valid for all $\underline{v}$ and $i = 1, \dots 4$. The constants do not depend on p.

Hence, we have found a nearly assymptotical optimal method. But, we have $\mu = 3$. The next theorem considers the application of $C_{h,k}$ of Definition 6.1 with $C_{22,k}$ of Definition 6.3 and the polynomial $P_\mu(t)$ (6.10) as preconditioner for K_i, $i = 1, \dots, 4$.

THEOREM 7.2 *Let K_i, $i = 1, \dots, 4$ be the 4 blocks of the matrix K. The statement*

$$c_{11}c_{17}(C_{h,k}\underline{v}, \underline{v}) \le \frac{1}{2p^2}(K_i \underline{v}, \underline{v}) \le c_{12}(1 + \log p)(C_{h,k}\underline{v}, \underline{v}).$$

is valid for all $\underline{v}$ and $i = 1, \dots 4$. The constants of Theorems 3.1 and 6.4 do not depend on p.

Proof: We observe from (4.7)

$$K_{h,k} = \frac{1}{2n^2} C_4.$$

Using Theorems 3.1 and 6.4, the assertion follows immediately. □

Thus, we have found a second nearly assymptotically optimal method, but we have chosen a polynomial of degree $\mu = 2$.

7.2 Numerical results

7.2.1 Multi-grid preconditioner

We solve the system

$$K\underline{u}_p = \underline{f}_p \tag{7.1}$$

using the PCG-method with the preconditioner M on each block K_i. We choose

$$\underline{f}_p = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^t.$$

All calculations are done on a Pentium-III 800 MHz. Table 7 displays the number of iterations and time to reduce the error in the preconditioned energy norm up to a factor 10^{-9}. We see in the two cases $M_1^{S_1}$, V-cycle with smoother S_1, and M_3^S a slight increase of the number of iterations and for M_1^S a stronger increasing number of iterations. The method using the preconditioner $M_1^{S_1}$ is the fastest method.

7.2.2 AMLI-preconditioner

We solve the system (7.1) using the PCG-method with the preconditioner $C_{h,k}$ on each block K_i. We choose

$$\underline{f}_p = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^t$$

as before. All calculations are done on a Pentium-III 800 MHz. Table 8 displays the number of iterations and time to reduce the error in the preconditioned energy norm up to a factor 10^{-9} for the polynomials (6.11) and $C_{22,l}$ of Definition 6.3. We see in the two cases $P(t) = (1 - \frac{12}{7}t)^2$ and $P(t) = (1 - \frac{66}{35}t)^2$

p	$M_1^{S_1}$		M_1^S		M_2^S		M_3^S	
	It	time [sec]	It	time [sec]	It	time [sec]	It	time [sec]
7	15	0.008	16	0.008	16	0.012	16	0.008
15	17	0.035	20	0.035	20	0.066	20	0.062
31	20	0.148	26	0.171	23	0.203	23	0.301
63	21	0.637	31	0.844	24	0.855	24	1.238
127	22	2.988	36	4.301	26	3.887	25	5.520
255	23	13.855	42	22.457	28	18.145	26	24.508
511	24	64.539	50	121.793	29	84.406	27	112.371
1023	24	265.621	59	595.727	30	368.695	28	496.777

Table 7: Number of iterations for the PCG-method for K using several multi-grid preconditioners M_μ^{Smooth}.

p	$P_{1,1}$		$P_{2,1}$		$P_{2,\frac{66}{35}}$		$P_{2,\frac{12}{7}}$	
	It	time [sec]	It	time [sec]	It	time [sec]	It	time [sec]
7	16	0.004	16	0.008	18	0.004	17	0.008
15	22	0.035	22	0.039	23	0.039	22	0.039
31	28	0.184	25	0.203	26	0.211	26	0.215
63	34	0.941	28	0.992	29	1.031	28	1.004
127	43	5.273	31	4.859	31	4.855	29	4.625
255	51	29.086	33	23.633	33	23.637	30	21.887
511	61	162.699	35	117.761	34	114.437	31	106.195
1023	73	815.035	37	537.500	34	493.477	31	458.015

Table 8: Number of iterations for the PCG-method for K using several AMLI-preconditioners M_μ^{Smooth}.

a slight increase of the number of iterations. For $P(t) = (1 - t)$, similar to the V-cycle of multi-grid, there is a stronger increasing number of iterations. The method using the preconditioner $P(t) = (1 - \frac{12}{7}t)^2$ is the fastest AMLI-preconditioner.
But, the comparison of the results for the AMLI-preconditioners of Table 8 with the multi-grid preconditioners of Table 7 shows significantelly lower the number of iterations for some multi-grid preconditioners ($M_1^{S_1}$ than for all AMLI-preconditioners. And, less time to reduce the error is needed.
If we compare the preconditioners M_3^S of Theorem 7.1 and $C_{h,k}$ with $P(t) = (1 - \frac{66}{35}t)^2$ of Theorem 7.2, we see that the number of iterations is slightly lower for M_3^S, but the time to reduce the error is about the same for both preconditioners.

8 Further remarks

8.1 Improvement for rectangular elements

Let us assume that in (2.1,2.2) Ω is the rectangle $(-a, a) \times (-b, b)$. Thus, we have the element stiffness matrix

$$K_{a,b} = \frac{a}{b}(F \otimes D) + \frac{b}{a}(D \otimes F).$$

We should obtain a faster method if we use a multi-grid preconditioner resulting from

$$-\frac{a}{b}y^2 u_{xx} - \frac{b}{a}x^2 u_{yy} = g$$

instead of

$$-y^2 u_{xx} - x^2 u_{yy} = g.$$

8.2 Extensions to the three-dimensional case

We consider

$$\begin{aligned} -\triangle u &= f, \text{ in } \tilde{\Omega} = (-1,1)^3 \\ u\,|_{\partial\tilde{\Omega}} &= 0. \end{aligned} \qquad (8.1)$$

We solve (8.1) using the p-Version of the FEM with only one element. Defining the space M as in 2.2, we obtain: Find $u_p \in M$, such that

$$\tilde{a}(u_p, v_p) := \int_\Omega \nabla u_p \cdot \nabla v_p \mathrm{d}(x,y) = \int_\Omega f v_p \mathrm{d}(x,y)$$

holds $\forall v_p \in M$. As basis in M, we choose

$$\hat{L}_{ijk}(x,y,z) = \hat{L}_i(x)\hat{L}_j(y)\hat{L}_k(z)$$

with the integrated Legendre-polynomial $\hat{L}_l$ (2.3), $2 \le i,j,k \le p$. With same arguments as in 2.2, we have

$$\begin{aligned} K_{3D} &= \tilde{a}(\hat{L}_{ijk}, \hat{L}_{lmn})_{i,j,k=2;l,m,n=2}^{p} \\ &= F \otimes F \otimes D + F \otimes D \otimes F + D \otimes F \otimes F, \end{aligned} \tag{8.2}$$

with the one dimensional mass matrix F (2.11) and the one dimensional stiffness matrix D (2.12). Applying a permutation P of rows and columns, we have as in (2.13)

$$PK_{3D}P^{-1} = \mathrm{diag}(K_{3D,i})_{i=1}^{8}.$$

The theory of chapter 3 can now be generalized. Using the arguments of Theorem 3.1, we can prove

THEOREM 8.1 . *Let*

$$C_7 = T_3 \otimes T_3 \otimes D_3 + T_3 \otimes D_3 \otimes T_3 + D_3 \otimes T_3 \otimes T_3, \tag{8.3}$$

$$\begin{aligned} C_8 = {} & (T_3 + D_3^{-1}) \otimes (T_3 + D_3^{-1} \otimes D_3 + (T_3 + D_3^{-1}) \otimes D_3 \otimes (T_3 + D_3^{-1}) \\ & + D_3 \otimes (T_3 + D_3^{-1}) \otimes (T_3 + D_3^{-1}) \end{aligned} \tag{8.4}$$

with the matrices D_3 (3.3) *and* T_3 (3.4). *Let* $K_{3D,i}, i = 1,\ldots,8$ *are the 8 blocks of* K_{3D}. *The following statements are valid* $\forall \underline{v}$ *and* $i = 1,\ldots,8$*:*

$$\begin{aligned} c_{21}(C_7\underline{v},\underline{v}) &\le (K_{3D,i}\underline{v},\underline{v}) \le c_{23}(1+\log p)^2(C_7\underline{v},\underline{v}), \\ c_{22}(C_8\underline{v},\underline{v}) &\le (K_{3D,i}\underline{v},\underline{v}) \le c_{24}(C_8\underline{v},\underline{v}). \end{aligned}$$

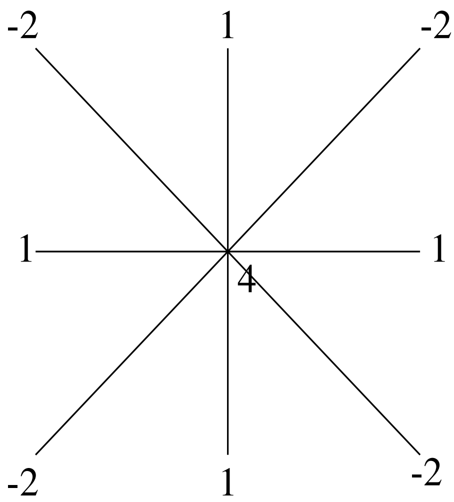

Figure 6: Stencils for discretizing u_{xxyy}.

Analogously to chapter 4, we obtain that C_7 is the discretization matrix of

$$\begin{aligned} z^2 u_{xxyy} + y^2 u_{xxzz} + x^2 u_{yyzz} &= g, \\ u \,|_{\partial\tilde{\Omega}_1} &= 0, \\ \frac{\partial u}{\partial n} \,|_{\partial\tilde{\Omega}_1} &= 0. \end{aligned}$$

in $\tilde{\Omega}_1 = (0,1)^3$ using finite differences and an equidistant grid. Let $u^{i,j,k}$ be the approximation of u in $(\frac{i}{n}, \frac{j}{n}, \frac{k}{n})$. These fourth order derivatives are discretized by the stencil of Figure 6,

$$\begin{aligned} z^2 u_{xxyy}(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \;\approx\; & k^2(4u^{ijk} - 2u^{i,j-1,k} - 2u^{i,j+1,k} - 2u^{i-1,jk} - 2u^{i+1,jk} \\ & + u^{i-1,j-1,k} + u^{i+1,j-1,k} + u^{i-1,j+1,k} + u^{i+1,j+1,k}). \end{aligned}$$

For C_8, we have to consider

$$\begin{aligned} & z^2 u_{xxyy} + y^2 u_{xxzz} + x^2 u_{yyzz} \\ & -2(\frac{y^2}{z^2} + \frac{z^2}{y^2})u_{xx} - 2(\frac{x^2}{z^2} + \frac{z^2}{x^2})u_{yy} - 2(\frac{x^2}{y^2} + \frac{y^2}{x^2})u_{zz} \\ & +4(\frac{x^2}{y^2 z^2} + \frac{y^2}{x^2 z^2} + \frac{z^2}{x^2 y^2})u \;=\; g. \end{aligned}$$

A Remarks to the estimate of the strengthened Cauchy-inequality

We give here the exact values for the paramters p (5.22) and q (5.23). We set

$$\begin{aligned} r &= i-1, \\ s &= j-1. \end{aligned}$$

Then, we obtain the following results for p and q.

$$\begin{aligned} p := \frac{1}{704}&(5857266360\,s + 4407665790\,r + 1508755050 + 146252736\,s^6 \\ &+ 1111426560\,s^5 + 27808704\,r^6 + 302620032\,r^5 + 9324984713\,s^2 \\ &+ 5434977449\,r^2 + 3923127840\,s^4 + 7936810608\,s^3 \\ &+ 3647255568\,r^3 + 1415409600\,r^4 + 9269249088\,s^4\,r \\ &+ 8601027360\,s^4\,r^2 + 20130620928\,s^3\,r + 20920075392\,s^3\,r^2 \\ &+ 17559686400\,s^2\,r^3 + 6376566048\,s^2\,r^4 + 12919365888\,s\,r^3 \\ &+ 4918733952\,s\,r^4 + 124830720\,s^2\,r^6 + 1326974976\,s^2\,r^5 \\ &+ 3982219776\,s^4\,r^3 + 3786647040\,s^3\,r^4 + 11609339904\,s^3\,r^3 \\ &+ 277115904\,s^6\,r^2 + 328872960\,s^6\,r + 2493112320\,s^5\,r \\ &+ 999364608\,s^4\,r^4 + 2094465024\,s^5\,r^2 + 151621632\,s^4\,r^5 \\ &+ 735657984\,s^3\,r^5 + 69672960\,s^3\,r^6 + 108158976\,s^5\,r^4 \\ &+ 779452416\,s^5\,r^3 + 14432256\,s^4\,r^6 + 14432256\,s^6\,r^4 \\ &+ 103514112\,s^6\,r^3 + 1047619584\,s\,r^5 + 97625088\,s\,r^6 \\ &+ 28493849120\,s^2\,r^2 + 25194885712\,s^2\,r + 19809599216\,s\,r^2 \\ &+ 16586949280\,s\,r)/((20\,r + 17 + 6\,r^2)(82016\,s + 76846\,r \\ &+ 65589\,s^2 + 58245\,r^2 + 47232\,s^2\,r^2 + 93456\,s^2\,r + 93168\,s\,r^2 \\ &+ 139936\,s\,r + 4896\,s^4 + 26112\,s^3 + 21120\,r^3 + 3168\,r^4 \\ &+ 5760\,s^4\,r + 1728\,s^4\,r^2 + 30720\,s^3\,r + 9216\,s^3\,r^2 + 11520\,s^2\,r^3 \\ &+ 1728\,s^2\,r^4 + 30720\,s\,r^3 + 4608\,s\,r^4 + 39930)(6\,s^2 + 16\,s + 11)) \end{aligned}$$

$$
\begin{aligned}
q := {} & \frac{1}{123904}(3175524000\, s + 10752404850\, r + 925888320\, s^6 \\
& + 3527193600\, s^5 + 153679680\, r^6 + 2180787840\, r^5 \\
& + 6123829635\, s^2 + 25829259555\, r^2 + 5339341800\, s^4 \\
& + 5845588560\, s^3 + 24034055760\, r^3 + 10651944600\, r^4 \\
& + 18162835680\, s^4\, r + 24937019664\, s^4\, r^2 + 42653867520\, s^3\, r \\
& + 81996584832\, s^3\, r^2 + 120359893824\, s^2\, r^3 + 52045531152\, s^2\, r^4 \\
& + 90435290880\, s\, r^3 + 39407913600\, s\, r^4 + 742404096\, s^2\, r^6 \\
& + 10535067648\, s^2\, r^5 + 17602460928\, s^4\, r^3 + 29858095872\, s^3\, r^4 \\
& + 70165140480\, s^3\, r^3 + 1735243776\, s^6\, r^2 + 2071802880\, s^6\, r \\
& + 7892582400\, s^5\, r + 6669527040\, s^4\, r^4 + 6610452480\, s^5\, r^2 \\
& + 1260582912\, s^4\, r^5 + 5998067712\, s^3\, r^5 + 422682624\, s^3\, r^6 \\
& + 338411520\, s^5\, r^4 + 2450718720\, s^5\, r^3 + 88833024\, s^4\, r^6 \\
& + 88833024\, s^6\, r^4 + 643313664\, s^6\, r^3 + 8005662720\, s\, r^5 \\
& + 564157440\, s\, r^6 + 136254292064\, s^2\, r^2 + 65646211760\, s^2\, r \\
& + 100770474640\, s\, r^2 + 46577704800\, s\, r)/((20\, r + 17 + 6\, r^2)(\\
& 82016\, s + 76846\, r + 65589\, s^2 + 58245\, r^2 + 47232\, s^2\, r^2 \\
& + 93456\, s^2\, r + 93168\, s\, r^2 + 139936\, s\, r + 4896\, s^4 + 26112\, s^3 \\
& + 21120\, r^3 + 3168\, r^4 + 5760\, s^4\, r + 1728\, s^4\, r^2 + 30720\, s^3\, r \\
& + 9216\, s^3\, r^2 + 11520\, s^2\, r^3 + 1728\, s^2\, r^4 + 30720\, s\, r^3 + 4608\, s\, r^4 \\
& + 39930)(6\, s^2 + 16\, s + 11))
\end{aligned}
$$

Obviously, $p > 0$ and $q \geq 0$ for $i, j \geq 1$. But, we can conclude

$$q = 0 \iff i = 1 \wedge j = 1.$$

Hence, the estimate of Lemma 5.17 is sharp.

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