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ERROR ESTIMATES FOR A SEMILINEAR ELLIPTIC CONTROL PROBLEM

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Abstract

We study the numerical approximation of distributed nonlinear optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The analysis of the approximate control problems is carried out. In particular, characterization results for the optimal control and the discretized optimal controls are stated. The uniform convergence of discretized controls to optimal controls is proven under natural assumptions. Finally, error estimates are established.

Keywords: Distributed control, semilinear elliptic equation, numerical approximation, finite element method, error estimates.

AMS subject classification: 49J20, 49K20, 49M05, 65K10

1 Introduction

The paper is concerned with the discretization of the following optimal control problem

$$
\begin{aligned}
(P) \quad \inf \quad & J(u) = \int_\Omega L(x, y_u(x), u(x))\, dx,\\
& \text{subject to } (y_u, u) \in (C(\overline{\Omega}) \cap H^1(\Omega)) \times L^\infty(\Omega),\\
& Ay_u + f(\cdot, y_u) = u \text{ in } \Omega, \qquad y_u = 0 \text{ on } \Gamma, \qquad (1.1)\\
& u \in U^{ad} = \{u \in L^\infty(\Omega) \mid \alpha \le u(x) \le \beta \text{ for a.a. } x \in \Omega\},
\end{aligned}
$$

where Ω is a convex bounded domain, Γ is the boundary of Ω; A denotes a second order elliptic operator of the form $Ay(x) = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_jy(x))$ where D_i denotes the partial derivative with respect to x_i, and α and β are real numbers. Here u is the control while y_u is said to be the associated state.
Under some natural assumptions, we prove the existence of solutions for the problem (P). By using the associated optimality conditions, a characterization of the optimal control is given, and a corresponding regularity result is established.

The second part of the paper is concerned with the full discretization of the control and the state equation by a finite element method. The asymptotic behavior of the corresponding discretized problem (P_h) is studied, and a stability result established. As for

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the continuous problem, we give a characterization result concerning the solutions of (P_h). This enables us to prove the uniform convergence of these solutions to a solution of (P).

Finally, the last and main part is devoted to the approximation errors for the optimal control. Under some natural assumptions, with a second order and a stability condition, we derive some error estimates. Moreover, we show how the error estimates for the state equation and the adjoint equation can be transferred to associated error estimates for the optimal controls.

Let us briefly comment on the relevant literature. There are two early papers on the numerical approximation of linear-quadratic control-constrained elliptic control problems by Falk [11] and Geveci [12]. L^2-error estimates are obtained which reflect the H^1-regularity of the optimal control and the optimal regularity of the state function. Falk considered distributed controls, while Geveci concentrates on Neuman boundary controls. More recently, Arnautu and Neittaanmäki [3] contributed further errors estimates to this clan of problems. Their technique, however, slightly overestimates the order of the error. Moreover, we refer to Arada and Raymond [2], where estimates and convergence results are performed for relaxed optimal control problems governed by semilinear elliptic equations, and Casas [6], where convergence results are proved for optimal control problems governed by linear elliptic equations with controls in the coefficient. We also mention the thesis by Mateos [20], who carefully studies error estimates for semilinear elliptic equations.

In contrast to the elliptic case, quite a number of papers was devoted to parabolic problems, although the associated theory is far from being complete. We refer to Alt and Mackenroth [1], Knowles [14], Lasiecka [15], [16], Mackenroth [17], [18], McKnight and Bosarge [21], Tiba and Tröltzsch [23] and Tröltzsch [24], [25], [26], [27]. The papers [1], [14],[15], [16], [17], [18], [24] consider linear parabolic equations, which are approximated by a semidiscrete Ritz-Galerkin or finite element scheme. Different aspects are investigated. In particular, the (strong) convergence of optimal values and/or optimal controls is shown. In [17] and [18] the final state is required to reach a convex target set, thus a special state constraint is considered. [21] is concerned with the case of unrestricted control for a nonlinear parabolic state equation. Here, the optimal error estimates for parabolic equations extend directly to associated estimates for the controls. The assumption made in [21] on Fréchet-differentiability is only satisfied in particular cases.

In [23], a convex problem with constraints on the control and the state is studied. The state equation is approximated by a fairly general assumption on the approximation in space and an implicit Euler scheme in time. Error estimates are derived, which express the estimate for the optimal control by relevant interpolation errors. Moreover, a semilinear problem without state-constraint is discussed. [24] deals with convergence of switching points for a linear-quadratic parabolic problem. The papers [25]–[27] deal with semilinear equations and constraints on the control. Except [25], where the Fourier method is used to approximate the state equation, the other papers assume a semidiscrete scheme for the parabolic equation under quite abstract assumptions.

Our paper differs from the ideas presented in literature in several points. The equation is semilinear. Due to this, we had to derive L^∞ error estimates in order to deal correctly with the given nonlinearities. We discuss the finite element approximation in more detail than in the papers mentioned above. In particular, the approximation of the given domain Ω by polygonal domains is considered. Moreover, the following ideas are essentially new:

In the first part, the *strong* convergence of subsequences of approximate controls is proven under a fairly weak assumption. In the second part, error estimates are established for such subsequences. Extending an idea due to Malanowski et. al [19], which was used earlier for the case of ordinary differential equations, we are able to improve the error estimates in [3] and [6]. We are not sure that our results express the optimal ones in the nonlinear case. However, they seem to be optimal in the case of linear equations, where L^2-estimates can be used.

2 General assumptions and notation

Throughout the sequel, Ω denotes a convex bounded open subset in $\mathbb{R}^n$ $(n = 2$ or $n = 3)$ of class $C^{1,1}$. The coefficients a_{ij} of the operator A belong to $C^{0,1}(\overline{\Omega})$ and satisfy the ellipticity condition

$$m_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \qquad \forall\ (\xi, x) \in \mathbb{R}^N \times \overline{\Omega}, \qquad m_0 > 0.$$

Moreover, we require:

A1 - The function f is a Carathéodory function from $\overline{\Omega} \times \mathbb{R}$ into $\mathbb{R}$. For every $x \in \Omega$, $f(x, \cdot)$ is of class C^2, and $D_y f(x, \cdot)$ is nonnegative. For all $M > 0$ there exists $C_M > 0$ such that

$$|f(x, y)| + |D_y f(x, y)| + |D_{yy} f(x, y)| \leq C_M,$$

$$|D_{yy} f(x, y_1) - D_{yy} f(x, y_2)| \leq C_M|y_1 - y_2|$$

for all $(x, y, y_1, y_2) \in \Omega \times [-M, +M]^3$.

A2 - L is a Carathéodory function from $\Omega \times \mathbb{R}^2$ into $\mathbb{R}$. For every $x \in \Omega$, $L(x, \cdot, \cdot)$ is of class C^2. For all $M > 0$, and all $(x, x_1, x_2, y, y_1, y_2, u, u_1, u_2) \in \Omega^3 \times [-M, +M]^6$, the following estimates hold

$$|L(x, y, u)| \leq L_M(x), \qquad |D_y L(x, y, u)| \leq L^1_M(x)$$

$$|D_u L(x_1, y, u) - D_u L(x_2, y, u)| \leq C_M|x_1 - x_2|$$

$$|\mathcal{L}''_{(y,u)}(x, y, u)|_{\mathbb{R}^{2\times 2}} \leq C_M$$

$$|\mathcal{L}''_{(y,u)}(x, y_1, u_1) - \mathcal{L}''_{(y,u)}(x, y_2, u_2)|_{\mathbb{R}^{2\times 2}} \leq C_M(|y_1 - y_2| + |u_1 - u_2|),$$

where $L_M \in L^2(\Omega)$, $L^1_M \in L^p(\Omega)$, $p > n$, $C_M > 0$, $\mathcal{L}''_{(y,u)}$ is the Hessian matrix of L with respect to (y,u), and $|\cdot|_{\mathbb{R}^{2\times 2}}$ is any norm of matrices. Moreover, there exists a positive constant m such that the following estimate holds:

$$D_{uu}L(x,y,u) \geq m \qquad \forall\ (x,y,u) \in \Omega \times \mathbb{R}^2.$$

In all the sequel $||\cdot||_{2,\Omega}$ and $||\cdot||_{\infty,\Omega}$ denote the usual norms in $L^2(\Omega)$ and $L^\infty(\Omega)$, respectively, and c will denote a generic constant.

Remark 1 *In particular, the following simple linear-quadratic optimal control problem fits in this setting. We shall refer to this example to illustrate some of the ideas in the Sections 4, 6, and 7.*

$$(E) \quad \inf \quad \frac{1}{2}(\|y - y_d\|_{2,\Omega}^2 + \kappa\|u\|_{2,\Omega}^2),$$

subject to

$$-\Delta y = u \ \ \text{in } \Omega, \qquad\qquad y_u = 0 \ \ \text{on } \Gamma,$$

$$\alpha \leq u(x) \leq \beta \ \ \text{for a.a. } x \in \Omega.$$

Here, $y_d \in L^4(\Omega)$ and $\kappa > 0$ are given, and $L(x,y,u) = \frac{1}{2}((y - y_d(x))^2 + \kappa u^2)$. It is obvious that **A1** *and* **A2** *are satisfied in the example (E).*

3 State equation and Adjoint equation

In this section we derive some useful estimates, which express the Lipschitz continuity of states and adjoint states with respect to the controls.

3.1 State equation

Theorem 1 [4] *Let u be in $L^\infty(\Omega)$ satisfy $\|u\|_{\infty,\Omega} \leq M$. Then equation (1.1) admits a unique solution $y_u \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, for every $p > n$. Moreover, there exists a positive constant $C \equiv C(\Omega, n, p, M)$, independent of u, such that*

$$||y_u||_{W^{2,p}(\Omega)} \leq C.$$

Proposition 1 [4] *Let $a_o \geq 0$ be a function in $L^\infty(\Omega)$ satisfying $||a_o||_{\infty,\Omega} \leq M$. Then, for every $g \in L^p(\Omega)$, the solution y of*

$$Ay + a_o y = g \quad \text{in } \Omega, \quad y_{|\Gamma} = 0,$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for every $p > n$. Moreover, there exists a positive constant $C \equiv C(\Omega, n, p, M)$, independent of a_o, such that

$$||y||_{W^{2,p}(\Omega)} \leq C\ ||g||_{p,\Omega}, \qquad ||y||_{H^2(\Omega)} \leq C\ ||g||_{2,\Omega}.$$

Proposition 2 *Let u_1, u_2 be in $L^\infty(\Omega)$, and let y_1 and y_2 be the associated states, i.e. the corresponding solutions of* (1.1). *Then $y_1 - y_2$ satisfies the estimate*

$$\|y_1 - y_2\|_{H^2(\Omega)} \le C \ \|u_1 - u_2\|_{2,\Omega},$$

where $C > 0$ does not depend on u_1 and u_2.

Proof. The function $y = y_1 - y_2$ satisfies

$$Ay + \tilde{f} y = u_2 - u_1 \ \text{ in } \Omega, \qquad y_{|\Gamma} = 0,$$

where $\tilde{f} = \int_0^1 D_y f(\cdot, \theta y_1 + (1-\theta) y_2, u_1)\, d\theta \ge 0$. The conclusion is a direct consequence of Proposition 1. □

3.2 Adjoint equation

Let u be in $L^\infty(\Omega)$ and y_u denote the corresponding solution of (1.1). The adjoint equation associated with the problem we consider, has the following form:

$$A^*\varphi + D_y f(\cdot, y_u)\varphi = D_y L(\cdot, y_u, u) \ \text{ in } \Omega, \qquad \varphi = 0 \ \text{ on } \Gamma. \tag{3.1}$$

Here A^* is the formal adjoint operator of A. The solution $\varphi = \varphi_u$ is called the adjoint state associated to u. The next theorem follows immediately from Proposition 1.

Theorem 2 *Let $u \in L^\infty(\Omega)$ satisfy $\|u\|_{\infty,\Omega} \le M$. Then equation* (3.1) *admits a unique solution φ_u in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for every $p > n$. Moreover, there exists a positive constant $C \equiv C(\Omega, n, p, M)$, independent of u, such that*

$$\|\varphi_u\|_{W^{2,p}(\Omega)} \le C.$$

Proposition 3 *Let u_1, u_2 be in $L^\infty(\Omega)$ such that $\|u_1\|_{\infty,\Omega} + \|u_2\|_{\infty,\Omega} \le M$, and let φ_1 and φ_2 be the corresponding adjoint states. Then $\varphi_1 - \varphi_2$ satisfies the estimate*

$$\|\varphi_1 - \varphi_2\|_{H^2(\Omega)} \le C \ \|u_1 - u_2\|_{2,\Omega}$$

where $C \equiv C(\Omega, n, M)$ does not depend on u_1 and u_2.

Proof. The function $\varphi = \varphi_1 - \varphi_2$ satisfies $\varphi_{|\Gamma} = 0$ and

$$A^*\varphi + a\varphi = (D_y f(\cdot, y_2) - D_y f(\cdot, y_1))\varphi_2 + D_y L(\cdot, y_1, u_1) - D_y L(\cdot, y_2, u_2) \ \text{ in } \Omega,$$

where y_1 and y_2 are the states associated to u_1 and u_2, respectively, and $a = D_y f(\cdot, y_1)$. Due to assumptions **A1**-**A2**, Theorem 1, and Proposition 2, we obtain

$$\|\varphi_1 - \varphi_2\|_{H^2(\Omega)}$$
$$\le C(\|(D_y f(\cdot, y_1) - D_y f(\cdot, y_2))\varphi_2\|_{2,\Omega} + \|D_y L(\cdot, y_1, u_1) - D_y L(\cdot, y_2, u_2)\|_{2,\Omega})$$

$$\leq C(\|D_y f(\cdot,y_1) - D_y f(\cdot,y_2)\|_{2,\Omega}\|\varphi_2\|_{\infty,\Omega}$$
$$+\|D_y L(\cdot,y_1,u_1) - D_y L(\cdot,y_2,u_1)\|_{2,\Omega} + \|D_y L(\cdot,y_2,u_1) - D_y L(\cdot,y_2,u_2)\|_{2,\Omega})$$
$$\leq C((1+\|\varphi_2\|_{\infty,\Omega})\|y_1-y_2\|_{2,\Omega} + \|u_1-u_2\|_{2,\Omega})$$
$$\leq C(\|y_1-y_2\|_{2,\Omega} + \|u_1-u_2\|_{2,\Omega}) \leq C\|u_1-u_2\|_{2,\Omega}. \qquad \square$$

Remark 2 *Notice that since $n \leq 3$, Propositions 2, 3, and classical imbedding theorems give*

$$\|y_1-y_2\|_{C(\overline{\Omega})} + \|\varphi_1-\varphi_2\|_{C(\overline{\Omega})} \leq C\ \|u_1-u_2\|_{2,\Omega}.$$

This estimate will be intensively used in the sequel.

4 Existence and characterisation of solutions of (P)

4.1 Existence results

We begin this section by a useful continuity result.

Proposition 4 *Suppose that assumption* **A1** *is satisfied. Then the operator $u \mapsto y_u$ is continuous from $L^\infty(\Omega)$, endowed with the weak* topology, into $C(\overline{\Omega})$.*

Proof. Let $(u_\epsilon)_\epsilon$ be a sequence in U^{ad} converging to u in the weak*-$L^\infty(\Omega)$ topology. Let y_ϵ and y_u be the solutions of (1.1) corresponding to u_ϵ and u. We have to show that $(y_\epsilon)_\epsilon$ converges to y_u, uniformly on $\overline{\Omega}$. Due to Theorem 1, the sequence $(y_\epsilon)_\epsilon$ is bounded in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Then there exist a subsequence $(y_{\epsilon_j})_j$ and $y \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, such that $(y_{\epsilon_j})_j$ converges to y in the weak topology of $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is continuously embedded into $C^1(\overline{\Omega})$, it follows that $(y_{\epsilon_j})_j$ converges to y uniformly on $\overline{\Omega}$. Due to this convergence results, passing to the limit in the variational equality satisfied by y_{ϵ_j}, we easily show that $y \equiv y_u$. Finally, since any subsequence $(y_{\epsilon_j})_j$ contains a subsequence tending towards the same limit y_u, the convergence of the whole sequence $(y_\epsilon)_\epsilon$ follows from a standard argument. $\square$

Theorem 3 *Suppose that assumptions* **A1**-**A2** *are satisfied. Then problem (P) admits at least solution.*

Proof. Let $(u_n)_n$ be a minimizing sequence for (P), and let y_n be the state associated to u_n. Since $(u_n)_n$ is bounded in $L^\infty(\Omega)$, there exist a subsequence, still indexed by n, and a function u such that $(u_n)_n$ converges to u in the weak*- $L^\infty(\Omega)$ topology. In addition, u is the weak limit of u_n in $L^k(\Omega)$ (for all $k \geq 1$). Since U^{ad} is convex and closed in $L^k(\Omega)$, it is also weakly closed and $u \in U^{ad}$. Due to Proposition 4, the sequence $(y_n)_n$ converges to y_u uniformly on $\overline{\Omega}$. Therefore, u is admissible for (P), and

$$\inf(P) \leq J(u). \tag{4.1}$$

On the other hand, from **A2** and Theorem 2.1, Chapter 8 in [10], we can prove that

$$J(u) = \int_\Omega L(x, y_u, u)\, dx \le \liminf_{n\to+\infty} \int_\Omega L(x, y_u, u_n)\, dx,$$

expressing the weak*-lower semicontinuity with respect to u. Moreover, by using **A2** and the mean value theorem, we have

$$\lim_{n\to\infty} |\int_\Omega (L(x, y_u, u_n) - L(x, y_n, u_n))\, dx| \le \lim_{n\to\infty} \int_\Omega L_M(x)|y_u - y_n|(x)\, dx = 0.$$

With these continuity results, we easily deduce that

$$J(u) \le \liminf_{n\to\infty} J(u_n) = \inf(P). \tag{4.2}$$

The conclusion follows from (4.1) and (4.2). □

4.2 Characterization of the optimal control

Let us first state for convenience the known first order optimality conditions for problem (P). The classical proof is omitted.

Theorem 4 *If $\bar{u}$ is a solution of (P), then there exists an adjoint state $\varphi_{\bar{u}} \in H_0^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ such that the following conditions hold:*

$$A^*\varphi_{\bar{u}} + D_y f(x, y_{\bar{u}})\varphi_{\bar{u}} - D_y L(x, y_{\bar{u}}, \bar{u}) = 0 \qquad \text{in } \Omega, \tag{4.3}$$

$$\int_\Omega (\varphi_{\bar{u}} + D_u L(x, y_{\bar{u}}, \bar{u}))(u - \bar{u})\, dx \ge 0 \qquad \forall\, u \in U^{ad}. \tag{4.4}$$

To derive a characterization of the optimal control, we first prove two auxiliary results.

Lemma 1 *Suppose that assumptions* **A1**-**A2** *are satisfied. Then, for all $x \in \overline{\Omega}$, the equation*

$$\varphi_{\bar{u}}(x) + D_u L(x, y_{\bar{u}}(x), t) = 0, \tag{4.5}$$

has a unique solution $t = \bar{s}(x)$. Moreover, the mapping $\bar{s} : \overline{\Omega} \longrightarrow I\!R$ is of class $C^{0,1}(\overline{\Omega})$.

Proof. Let us first prove uniqueness of the solution. Suppose that, for $x \in \overline{\Omega}$, equation (4.5) admits two solutions $s_1(x)$ and $s_2(x)$. By Assumption **A2**, we find

$$\begin{aligned} 0 &= |D_u L(x, y_{\bar{u}}(x), s_1(x)) - D_u L(x, y_{\bar{u}}(x), s_2(x))| \\ &= |\int_0^1 D_{uu} L(x, y_{\bar{u}}(x), \theta s_1(x) + (1-\theta)s_2(x))\, d\theta|\ |s_1(x) - s_2(x)| \\ &\ge m\ |s_1(x) - s_2(x)|, \end{aligned}$$

hence $s_1(x) = s_2(x)$ must hold. To prove existence of a solution to (4.5), we consider the function g defined by $g(t) = \varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), t)$. The assumptions on L imply $g \in C^1(\overline{\Omega})$ and $g'(t) \geq m > 0$. It follows that

$$g(t) = g(0) + \int_0^t g'(s)\, ds \begin{cases} \geq g(0) + mt & \text{for } t > 0, \\ \\ \leq g(0) + mt & \text{for } t < 0, \end{cases}$$

and thus $\lim_{t\to -\infty} g(t) = -\infty$, and $\lim_{t\to +\infty} g(t) = +\infty$. Therefore, due to the continuity of g, there exists a solution $t = \bar s_x \equiv \bar s(x)$ of (4.5). Finally, let us prove that $\bar s \in C^{0,1}(\overline{\Omega})$. We observe that, due to the Lipschitz continuity of $\varphi_{\bar u}$, $y_{\bar u}$ and that of $u \mapsto D_u L(\cdot, \cdot, u)$, by **A2** and equation (4.5), we have

$$\begin{aligned}
& m\ |\bar s(x) - \bar s(x_o)| \\
& \leq |\int_0^1 D_{uu} L(x, y_{\bar u}(x), \theta \bar s(x) + (1-\theta)\bar s(x_o))\, d\theta\ (\bar s(x) - \bar s(x_o))| \\
& = |D_u L(x, y_{\bar u}(x), \bar s(x)) - D_u L(x, y_{\bar u}(x), \bar s(x_o))| \\
& = | - \varphi_{\bar u}(x) + \varphi_{\bar u}(x_o) + D_u L(x_o, y_{\bar u}(x_o), \bar s(x_o)) - D_u L(x, y_{\bar u}(x), \bar s(x_o))| \\
& \leq |\varphi_{\bar u}(x) - \varphi_{\bar u}(x_o)| + C_M\{|x - x_o| + |y_{\bar u}(x) - y_{\bar u}(x_o)|\} \leq C|x - x_o|. \qquad \square
\end{aligned}$$

Remark 3 *For the example* (E)*, the variational inequality reads*

$$\int_\Omega (\varphi_{\bar u} + \kappa \bar u)(u - \bar u)\, dx \geq 0 \qquad \forall\ u \in U^{ad}.$$

The equation (4.5) *reads* $\varphi_{\bar u} + \kappa t = 0$*, hence in this case* $\bar s(x) = -\frac{1}{\kappa}\varphi_{\bar u}(x)$.

Lemma 2 *Suppose that the assumptions* **A1**-**A2** *are satisfied. Let* $\bar u$ *be an optimal control for* (P)*, and let* $\bar s$ *be the corresponding solution of* (4.5)*. Then*

$$\varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \alpha) \geq 0 \qquad \textit{iff} \qquad \bar u(x) = \alpha, \tag{4.6}$$

$$\varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \beta) \leq 0 \qquad \textit{iff} \qquad \bar u(x) = \beta. \tag{4.7}$$

$$\begin{aligned} &\textit{If} \quad \varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \alpha) < 0 < \varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \beta) \\ &\textit{then} \ \ \varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \bar u(x)) = 0. \end{aligned} \tag{4.8}$$

Proof. First, let us notice that the optimality condition (4.4) can be rewritten as

$$(\varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \bar u(x)))(v - \bar u(x)) \geq 0 \tag{4.9}$$

for all $v \in [\alpha, \beta]$ and all $x \in \Omega_o$, where $\Omega_o \subset \overline{\Omega}$ and $|\bar\Omega \setminus \Omega_o| = 0$.

- Let $x \in \Omega_o$ be such that $\varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \alpha) \geq \varphi_{\bar u}(x) + D_u L(x, y_{\bar u}(x), \bar s(x)) = 0$.

The monotonicity of D_uL w.r. to u yields $\bar{s}(x) \le \alpha$, and

$$(\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha))(\alpha - \bar{u}(x)) \le 0$$

follows from $\alpha \le \bar{u}(x)$. Moreover, since the function $t \mapsto \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), t)$ is increasing, by taking $v = \alpha$ in (4.9), we obtain

$$(\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha))(\alpha - \bar{u}(x))$$

$$\ge (\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{u}(x)))(\alpha - \bar{u}(x)) \ge 0.$$

Therefore, $(\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha))(\alpha - \bar{u}(x)) = 0$. If $\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{u}(x)) > 0$, the conclusion $\bar{u} = \alpha$ is direct. If not, from the uniqueness of the solution of (4.5), we deduce that $\alpha \le \bar{u}(x) = \bar{s}(x) \le \alpha$, and thus $\bar{u}(x) = \alpha$.
Conversely, if $\bar{u}(x) = \alpha$, then (4.9) implies that $\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha) = \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{u}(x))$ is nonnegative. We have proved (4.6), and assertion (4.7) can be obtained by similar arguments.

- Finally, let us prove (4.8). Let $x \in \Omega_o$ be such that

$$\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha) < 0 < \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \beta).$$

From (4.6) and (4.7), we get $\alpha < \bar{u}(x) < \beta$. Setting $v = \alpha$ and $v = \beta$ in (4.9), we deduce that $\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{u}(x)) = 0$. □

The next result is fundamental for the sequel. It provides a useful characterization of the optimal control, which is well known for linear-quadratic optimal control problems.

Theorem 5 *Suppose that assumptions* **A1**-**A2** *are satisfied. Let $\bar{u}$ be an optimal control, and let $\bar{s}$ be the associated solution of equation* (4.5). *Then*

$$\bar{u}(x) = \mathit{Proj}_{[\alpha,\beta]}(\bar{s}(x)) = \max(\alpha, \min(\beta, \bar{s}(x))),$$

and $\bar{u}$ belongs to $C^{0,1}(\overline{\Omega})$.

Proof. First, suppose that $\bar{s}(x) \le \alpha$. Then

$$0 = \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{s}(x)) \le \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha).$$

From (4.6) we obtain $\bar{u}(x) = \alpha = \mathrm{Proj}_{[\alpha,\beta]}(\bar{s}(x))$. In the same way, the statement follows from (4.7) if $\bar{s}(x) \ge \beta$. Finally, if $\alpha < \bar{s}(x) < \beta$, then

$$\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \alpha) < \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{s}(x)) = 0$$

$$< \varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \beta).$$

Now (4.8) yields $\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{u}(x)) = 0$. Since the solution of (4.5) is unique, it follows that $\bar{u}(x) = \bar{s}(x) = \mathrm{Proj}_{[\alpha,\beta]}(\bar{s}(x))$. The Lipschitz continuity of $\bar{u}$ is a direct consequence, since $\bar{s}$ is Lipschitz (see Lemma 1) and the projection operator $\mathrm{Proj}_{[\alpha,\beta]}$ is Lipschitz continuous with constant 1. □

Remark 4 *In the example (E), the statement of Theorem 5 reduces to the well known characterization*

$$\bar{u}(x) = Proj_{[\alpha,\beta]}(-\frac{1}{\kappa}\ \varphi_{\bar{u}}(x)).$$

Remark 5 *The results of Theorem 5 can be easily extended to the case where α and β are functions of x. In this case, the Lipschitz continuity of the optimal control $\bar{u}$ is obtained under the assumption that α and β are Lipschitz continuous.*

5 Finite-element approximation of (P)

Here we define a finite-element based approximation of the optimal control problem (P). To this aim, we consider a family of triangulations $(\mathcal{T}_h)_{h>0}$ of $\overline{\Omega}$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Define the mesh size of the grid by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(i) - There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \le \sigma, \qquad \frac{h}{\rho(T)} \le \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

(ii) - Let us take $\overline{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\overline{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ. From [22], estimate (5.2.19), we know

$$|\Omega \setminus \Omega_h| \le Ch^2. \tag{5.1}$$

Now, to every boundary triangle T of $\mathcal{T}_h$, we associate another triangle $\hat{T} \subset \overline{\Omega}$ with curved boundary as follows: The edge between the two boundary nodes of T is substituted by the part of Γ connecting these nodes and forming a triangle with the remaining interior sides of T. We denote by $\widehat{\mathcal{T}}_h$ the union of these curved boundary triangles with the interior triangles to Ω of $\mathcal{T}_h$, so that $\overline{\Omega} = \cup_{\hat{T}\in\widehat{\mathcal{T}}_h} \hat{T}$. Let us set

$$U_h = \{u \in L^\infty(\Omega) \mid u_{|\hat{T}} \text{ is constant on all } \hat{T} \in \widehat{\mathcal{T}}_h\}, \qquad U_h^{ad} = U_h \cap U^{ad},$$

$$V_h = \{y_h \in C(\overline{\Omega}) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \overline{\Omega} \setminus \Omega_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. For each $u_h \in U_h$, we denote by $y_h(u_h)$ the unique element of V_h that satisfies

$$a(y_h(u_h), \eta_h) = \int_\Omega (u_h - f(x, y_h(u_h)))\eta_h(x)\, dx \qquad \forall\ \eta_h \in V_h, \tag{5.2}$$

where $a : V_h \times V_h \longrightarrow I\!R$ is the bilinear form defined by

$$a(y,\eta) = \int_\Omega (\sum_{i,j=1}^n a_{ij}(x) D_i y(x) D_j \eta(x))\, dx.$$

In other words, $y_h(u_h)$ is the approximate state associated with u_h. Notice that $y = \eta = 0$ on $\overline{\Omega}\setminus\overline{\Omega}_h$, hence the last integral is equivalent to integration on Ω_h. The finite dimensional approximation of the optimal control problem is defined by

$$(P_h) \quad \inf \ J_h(u_h) = \int_{\Omega_h} L(x, y_h(u_h)(x), u_h(x))\, dx, \qquad u_h \in U_h^{ad}.$$

Existence of a solution for (P_h) follows from the continuity of J_h and the compactness of U_h^{ad}.

Remark 6 *We tacitly assume that we are able to evaluate the integrals in* (5.2) *and* (P_h) *exactly. In general, numerical integration has to be used, which generates another sort of errors. We do not include them in our analysis.*

6 Characterization of solutions of (P_h)

The aim of this section is to characterize solutions of the problem (P_h) similarly to the ideas introduced in Section 4.2 for the characterization of optimal solutions for the continuous problem (P).

Proposition 5 *Suppose that assumptions* **A1**-**A2** *are satisfied. If* $\bar{u}_h$ *is a solution of* (P_h), *then there exists a unique* $\varphi_h(\bar{u}_h) \in H_0^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ *such that the following conditions hold:*

$$\int_\Omega (\sum_{i,j=1}^n a_{ij} D_j \varphi_h(\bar{u}_h) D_i \eta_h)\, dx + \int_\Omega D_y f(x, y_h(\bar{u}_h), \bar{u}_h) \varphi_h(\bar{u}_h) \eta_h \, dx$$
$$= \int_\Omega D_y L(x, y_h(\bar{u}_h), \bar{u}_h) \eta_h \, dx \qquad \forall\ \eta_h \in V_h, \tag{6.1}$$

$$\int_{\Omega_h} (\varphi_h(\bar{u}_h) + D_u L(x, y_h(\bar{u}_h), \bar{u}_h))(u - \bar{u}_h)\, dx \geq 0 \qquad \forall\ u \in U_h^{ad}. \tag{6.2}$$

Throughout the sequel, for v fixed in $L^\infty(\Omega)$, we denote by $y_h(v)$ and $\varphi_h(v)$ respectively the solutions of (5.2) and (6.1) corresponding to v.

Lemma 3 *Suppose that assumptions* **A1**-**A2** *are satisfied, and that* $\bar{u}_h$ *is an optimal solution of* (P_h). *Then there exists a unique function* $\bar{s}_h : \overline{\Omega}_h \longrightarrow I\!R$ *such that* $\bar{s}_h(x) = s_T$ *is constant on each triangle* $T \in \mathcal{T}_h$, *and the equation*

$$\int_T (\varphi_h(\bar{u}_h)(x) + D_u L(x, y_h(\bar{u}_h), s_T))\, dx = 0 \qquad \forall\ T \in \mathcal{T}_h, \tag{6.3}$$

is satisfied.

Proof. Existence of a unique solution of equation

$$\int_T (\varphi_h(\bar{u}_h)(x) + D_u L(x, y_h(\bar{u}_h), t))\, dx = 0,$$

can be proved upon defining $g(t) = \int_T (\varphi_h(\bar{u}_h)(x) + D_u L(x, y_h(\bar{u}_h), t))\, dx$, along the lines of proof of Lemma 1. □

Remark 7 *In the case of* (E)*, equation* 6.3 *is obvious again. We obtain*

$$\int_T (\varphi_h(\bar{u}_h)(x) + \kappa s_T)\, dx = 0,$$

and this equation has the unique solution $s_T = -\frac{1}{\kappa|T|}\int_T \varphi_h(\bar{u}_h)(x)\, dx$.

Theorem 6 *Suppose that* **A1**-**A2** *are satisfied. Let* $\bar{u}_h$ *be an optimal solution of* (P_h)*, and let* $\bar{s}_h$ *be the solution of* (6.3) *corresponding to* $\bar{u}_h$*. Then* $\bar{u}_h$ *is given by*

$$\bar{u}_h(x) = Proj_{[\alpha,\beta]}(\bar{s}_h(x)) = \max(\alpha, \min(\beta, \bar{s}_h(x))) \qquad \text{for a.e. } x \in \Omega_h.$$

Proof. First, let us observe that (6.2) can be rewritten as:

$$\int_T (\varphi_h(\bar{u}_h) + D_u L(x, y_h(\bar{u}_h), \bar{u}_h|_T)\, dx\ (t - \bar{u}_h\ |_T)) \geq 0$$

for all $t \in [\alpha, \beta]$ and all $T \in \mathcal{T}_h$. Following the proof of Lemma 2, we find

$$\int_T (\varphi_h(\bar{u}_h) + D_u L(x, y_h(\bar{u}_h)), \alpha)\, dx \geq 0 \qquad \text{iff} \qquad \bar{u}_h\ |_T = \alpha,$$

$$\int_T (\varphi_h(\bar{u}_h) + D_u L(x, y_h(\bar{u}_h), \alpha))\, dx \leq 0 \qquad \text{iff} \qquad \bar{u}_h\ |_T = \beta.$$

Moreover, if

$$\int_T (\varphi_h(\bar{u}_h) + D_u L(\cdot, y_h(\bar{u}_h), \alpha))\, dx < 0 < \int_T (\varphi_h(\bar{u}_h) + D_u L(\cdot, y_h(\bar{u}_h), \beta))\, dx,$$

then

$$\int_T (\varphi_h(\bar{u}_h) + D_u L(\cdot, y_h(\bar{u}_h), \bar{u}_h\ |_T)))\, dx = 0.$$

The characterization of $\bar{u}_h$ can be derived by proceeding as in the proof of Theorem 5. □

Remark 8 *Let us complete this discussion by the example* (E)*. Here we get*

$$\bar{u}_h|_T = Proj_{[\alpha,\beta]}\Big(-\frac{1}{\kappa|T|}\int_T \varphi_h(\bar{u}_h)(x)\, dx\Big) \qquad \forall\ T \in \mathcal{T}_h.$$

7 Error-estimates for the state and the adjoint state

In this section, we recall some results concerning the finite element approximation of the state equation (1.1) and its adjoint equation (3.1).

Theorem 7 *Let* $(v, v_h) \in L^\infty(\Omega) \times U_h$ *fulfil* $\|v\|_{\infty,\Omega} + \|v_h\|_{\infty,\Omega} \leq M$, *and suppose that* y_v *and* $y_h(v_h)$ *are the solutions of* (1.1) *and* (5.2) *corresponding to* v *and* v_h. *Moreover, let* φ_w *and* $\varphi_h(v_h)$ *be the solutions of* (4.3) *and* (6.1) *corresponding to* v *and* v_h. *Then the following estimates hold*

$$\|y_v - y_h(v_h)\|_{H^1(\Omega)} + \|\varphi_v - \varphi_h(v_h)\|_{H^1(\Omega)} \leq C(h + \|v - v_h\|_{2,\Omega}), \tag{7.1}$$

$$\|y_v - y_h(v_h)\|_{2,\Omega} + \|\varphi_v - \varphi_h(v_h)\|_{2,\Omega} \leq C(h^2 + \|v - v_h\|_{2,\Omega}), \tag{7.2}$$

$$\|y_v - y_h(v_h)\|_{\infty,\Omega} + \|\varphi_v - \varphi_h(v_h)\|_{\infty,\Omega} \leq C(h^\lambda + \|v - v_h\|_{2,\Omega}), \tag{7.3}$$

where $C \equiv C(\Omega, n, M)$ *is a positive constant independent of* h, *and* $\lambda = 2 - n/2$. *Moreover, if the triangulation is of nonnegative type, then*

$$\|y_v - y_h(v_h)\|_{\infty,\Omega_h} + \|\varphi_v - \varphi_h(v_h)\|_{\infty,\Omega_h} \leq (Ch + \|v - v_h\|_{2,\Omega}), \tag{7.4}$$

holds independently of h.

Proof. According to Theorem 8.2.9 in [20], the following estimates hold

$$\|y_{v_h} - y_h(v_h)\|_{H^1(\Omega)} + \|\varphi_{v_h} - \varphi_h(v_h)\|_{H^1(\Omega)} \leq Ch, \tag{7.5}$$

$$\|y_{v_h} - y_h(v_h)\|_{2,\Omega} + \|\varphi_{v_h} - \varphi_h(v_h)\|_{2,\Omega} \leq Ch^2, \tag{7.6}$$

$$\|y_{v_h} - y_h(v_h)\|_{\infty,\Omega} + \|\varphi_{v_h} - \varphi_h(v_h)\|_{\infty,\Omega} \leq Ch^{2-\frac{n}{2}}, \tag{7.7}$$

and if the triangulation is of nonnegative type, then

$$\|y_{v_h} - y_h(v_h)\|_{\infty,\Omega_h} + \|\varphi_{v_h} - \varphi_h(v_h)\|_{\infty,\Omega_h} \leq Ch. \tag{7.8}$$

To prove (7.1), notice that due to (7.5), and Propositions 2, 3, we have

$$\begin{aligned}
&\|y_v - y_h(v_h)\|_{H^1(\Omega)} + \|\varphi_v - \varphi_h(v_h)\|_{H^1(\Omega)} \\
&\leq \|y_v - y_{v_h}\|_{H^1(\Omega)} + \|y_{v_h} - y_h(v_h)\|_{H^1(\Omega)} \\
&+\|\varphi_v - \varphi_{v_h}\|_{H^1(\Omega)} + \|\varphi_{v_h} - \varphi_h(v_h)\|_{H^1(\Omega)} \leq C(h + \|v - v_h\|_{2,\Omega}).
\end{aligned}$$

The estimates (7.2), (7.3), and (7.4), can be obtained by using similar arguments together with (7.6), (7.7), and (7.8). □

Remark 9 *From Theorems* 1, 2, *and* 7, *we can easily see that*

$$\|y_h(v_h)\|_{\infty,\Omega} + \|\varphi_h(v_h)\|_{\infty,\Omega} \leq C,$$

where $C \equiv C(\Omega, n, M)$ *is a positive constant independent of* h.

Remark 10 *In all what follows, let us fix* $\lambda = 2 - n/2$ *for regular triangulations and* $\lambda = 1$, *if the regular triangulation if of nonnegative type.*

The following proposition will be useful for the sequel.

Proposition 6 *Let (v_h, w_h) be in $U_h \times U_h$ satisfy $\|v_h\|_{\infty,\Omega} + \|w_h\|_{\infty,\Omega} \le M$, and let z_{v_h} and $z_h(v_h)$ be the solutions of the following equations*

$$Az + D_y f(x, y_{w_h})z = v_h \quad \text{in } \Omega, \quad z_{|\Gamma} = 0, \tag{7.9}$$

$$a(z_h(v_h), \eta_h) + \int_\Omega D_y f(x, y_h(w_h))\ z_h(v_h)\eta_h\, dx = \int_\Omega v_h \eta_h\, dx \tag{7.10}$$

for all $\eta_h \in V_h$, where y_{w_h} and $y_h(w_h)$ are the solutions of (1.1) *and* (5.2) *corresponding to w_h. Then the following estimates hold*

$$\|z_{v_h} - z_h(v_h)\|_{2,\Omega} \le C\ h^2 \|v_h\|_{2,\Omega}, \tag{7.11}$$

$$\|z_{v_h} - z_h(v_h)\|_{\infty,\Omega_h} \le C\ h^\lambda \|v_h\|_{2,\Omega}. \tag{7.12}$$

Proof. Let $\tilde{z}_{v_h}$ be the solution of

$$Az + D_y f(x, y_h(w_h))z = v_h \quad \text{in } \Omega, \quad z_{|\Gamma} = 0. \tag{7.13}$$

Subtracting (7.9) from (7.13) we see that $z = \tilde{z}_{v_h} - z_{v_h}$ satisfies

$$Az + D_y f(x, y_h(w_h))z = (D_y f(x, y_{w_h}) - D_y f(x, y_h(w_h))z_{v_h} \quad \text{in } \Omega, \quad z_{|\Gamma} = 0.$$

Proposition 1, assumption **A1**, and Theorem 7 yield

$$\begin{aligned}
&\|\tilde{z}_{v_h} - z_{v_h}\|_{\infty,\Omega} \\
&\le C\|(D_y f(\cdot, y_{w_h}) - D_y f(\cdot, y_h(w_h))z_{v_h}\|_{2,\Omega} \le C\|y_{w_h} - y_h(w_h)\|_{2,\Omega}\ \|z_{v_h}\|_{\infty,\Omega} \\
&\le C\|y_{w_h} - y_h(w_h)\|_{2,\Omega}\ \|v_h\|_{2,\Omega} \le Ch^2\|v_h\|_{2,\Omega}.
\end{aligned} \tag{7.14}$$

On the other hand, by arguments similar to those used in the proof of Theorem 7, and due to Proposition 1, we have

$$\|\tilde{z}_{v_h} - z_h(v_h)\|_{2,\Omega} \le Ch^2\|\tilde{z}_{v_h}\|_{H^2(\Omega)} \le Ch^2\|v_h\|_{2,\Omega}, \tag{7.15}$$

$$\|\tilde{z}_{v_h} - z_h(v_h)\|_{\infty,\Omega_h} \le Ch^\lambda\|\tilde{z}_{v_h}\|_{H^2(\Omega)} \le Ch^\lambda\|v_h\|_{2,\Omega}. \tag{7.16}$$

The conclusion follows from (7.14), (7.15) and (7.16). □

8 Convergence results

Lemma 4 *Suppose that assumptions* **A1**-**A2** *are satisfied, and let $v \in L^\infty(\Omega)$ and $v_h \in U_h$ satisfy $\|v_h\|_{\infty,\Omega} + \|v\|_{\infty,\Omega} \le M$. If $\lim_{h\to 0}\|v_h - v\|_{2,\Omega} = 0$, then*

$$\lim_{h\to 0} J_h(v_h) = J(v).$$

Proof. With assumptions on L, Remark 9, (7.2) and (5.1) we have

$$|J(v) - J_h(v_h)| = |\int_\Omega L(x, y_v, v)\, dx - \int_{\Omega_h} L(x, y_h(v_h), v_h)\, dx|$$

$$\leq \int_\Omega |L(x, y_v, v) - L(x, y_v, v_h)|\, dx$$

$$+ \int_\Omega |L(x, y_v, v_h) - L(x, y_h(v_h), v_h)|\, dx + \int_{\Omega \setminus \Omega_h} |L(x, y_h(v_h), v_h)|\, dx$$

$$\leq C(\|v - v_h\|_{2,\Omega} + \|y_v - y_h(v_h)\|_{2,\Omega} + |\Omega \setminus \Omega_h|^{\frac{1}{2}}) \leq C(\|v - v_h\|_{2,\Omega} + h).$$

The last expression tends to zero when $h \to 0$. □

Lemma 5 *Suppose that assumptions* **A1**-**A2** *are satisfied, and let the sequence* $(v_h)_h \subset U_h^{ad}$ *converge weakly* to* v. *Then* $v \in U^{ad}$ *and*

$$J(v) \leq \liminf_{h \to 0} J_h(v_h).$$

Proof. Obviously, v is also the weak limit of $(v_h)_{h>0}$ in $L^k(\Omega)$ (for all $k \geq 1$). Since $U_h^{ad} \subset U^{ad}$ and U^{ad} is convex and closed in $L^k(\Omega)$, it is weakly closed and $v \in U^{ad}$. On the other hand, notice that

$$\begin{aligned} J_h(v_h) &= \int_{\Omega_h} (L(x, y_h(v_h), v_h) - L(x, y_v, v_h))\, dx + \int_{\Omega_h} L(x, y_v, v_h)\, dx \\ &= \int_{\Omega_h} (L(x, y_h(v_h), v_h) - L(x, y_v, v_h))\, dx + \int_\Omega L(x, y_v, v_h)\, dx \\ &- \int_{\Omega \setminus \Omega_h} L(x, y_v, v_h)\, dx. \end{aligned} \tag{8.1}$$

With **A2**, we follow the proof of Theorem 3 to show

$$\int_\Omega L(x, y_v, v)\, dx \leq \liminf_{h \to 0} \int_{\Omega_h} L(x, y_v, v_h)\, dx. \tag{8.2}$$

Moreover, with assumptions on L and (5.1), we easily see that

$$\Big| \int_{\Omega \setminus \Omega_h} L(x, y_v, v_h)\, dx \Big| \leq Ch \longrightarrow 0 \qquad \text{as } h \to 0. \tag{8.3}$$

Finally, **A2**, (7.2), (7.3) and Proposition 4, give

$$\Big| \int_{\Omega_h} (L(x, y_h(v_h), v_h) - L(x, y_v, v_h))\, dx \Big|$$

$$\leq C\|y_h(v_h) - y_v\|_{2,\Omega_h} \leq C(\|y_h(v_h) - y_{v_h}\|_{2,\Omega_h} + \|y_{v_h} - y_v\|_{2,\Omega})$$

$$\leq C(\|y_h(v_h) - y_{v_h}\|_{2,\Omega} + \|y_{v_h} - y_v\|_{\infty,\Omega}) \longrightarrow 0 \qquad \text{if } h \to 0. \tag{8.4}$$

The conclusion follows from (8.1), (8.2), (8.3), and (8.4). □

Proposition 7 *Suppose that* **A1**-**A2** *are satisfied, and let* $(\bar{u}_h)_{h>0}$ *be any sequence of solutions to* (P_h)*. Then there exist weakly*-converging subsequences (still indexed by* h*). If the subsequence* $(\bar{u}_h)_{h>0}$ *is converging weakly* to* $\bar{u}$*, then* $\bar{u}$ *is a solution of* (P)*. Moreover,*

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P). \tag{8.5}$$

Proof. The sequence $(\bar{u}_h)_{h>0}$ is bounded in $L^\infty(\Omega)$. Then there exists a subsequence, still indexed by h, which converges to some element $\bar{u}$ in the weak-$*$ toplogy of $L^\infty(\Omega)$. Lemma 5 implies $\bar{u} \in U^{ad}$ and

$$J(\bar{u}) \le \liminf_{h\to 0} J_h(\bar{u}_h). \tag{8.6}$$

On the other hand, let $\bar{w}$ be a solution of (P), and let Π_h be the interpolation operator defined by

$$\Pi_h v_{|T} = \frac{1}{|T|}\int_T v(x)\,dx \qquad \text{for all } T \in \mathcal{T}_h.$$

Put

$$w_{h|\hat{T}} = \Pi_h \bar{w}_{|T} \qquad \forall\, \hat{T} \in \widehat{\mathcal{T}}_h,$$

where $T \in \mathcal{T}_h$ is the triangle associated with $\hat{T}$. Since $\bar{w} \in W^{1,\infty}(\Omega_h)$, due to Theorem 16.1 in [9], we have

$$\|\bar{w} - w_h\|_{\infty,\Omega_h} \le Ch\|\bar{w}\|_{W^{1,\infty}(\Omega_h)}.$$

Therefore,

$$\|\bar{w} - w_h\|_{2,\Omega} \le C(\|\bar{w} - w_h\|_{\infty,\Omega_h} + |\Omega \setminus \Omega_h|) \le Ch.$$

From Lemma 4, we deduce that

$$\lim_{h\to 0} J_h(w_h) = J(\bar{w}) = \inf(P).$$

Moreover, w_h is obviously admissible for (P_h), and thus

$$J_h(\bar{u}_h) \le J_h(w_h).$$

Passing to the limit in the last inequality, we obtain

$$\liminf_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(w_h) = J(\bar{w}). \tag{8.7}$$

By (8.6) and (8.7), we arrive at

$$\lim_{h\to 0} J_h(\bar{u}_h) = \lim_{h\to 0} \inf(P_h) = J(\bar{u}) = \inf(P). \qquad \square$$

Remark 11 *Throughout the sequel, we fix such a subsequence, still indexed for simplicity by* h*, and we denote by* $\bar{u}$ *its limit, solution of* (P)*.*

Now, we state the main result of this section.

Theorem 8 *If the assumptions* **A1**-**A2** *are satisfied, then*

$$\lim_{h\to 0} \|\bar{u}_h - \bar{u}\|_{\infty,\Omega} = 0. \tag{8.8}$$

Proof. The proof is split into two steps.
Step 1. Let us first prove the convergence result in the L^2-norm:

$$\lim_{h\to 0} \|\bar{u}_h - \bar{u}\|_{2,\Omega} = 0. \tag{8.9}$$

Due to assumptions **A2**, Proposition 4, (7.2), (8.5), and the weak-$*$ convergence of $(\bar{u}_h)_h$ to $\bar{u}$, we have

$$\begin{aligned}
&\frac{m}{2}\|\bar{u}_h - \bar{u}\|_{2,\Omega}^2 \le \frac{1}{2}\int_\Omega \int_0^1 D_{uu}L(x, y_{\bar{u}}, \theta\bar{u}_h + (1-\theta)\bar{u})\, d\theta\ (\bar{u}_h - \bar{u})^2\, dx \\
&= \int_\Omega (L(x, y_{\bar{u}}, \bar{u}_h) - L(x, y_{\bar{u}}, \bar{u}))\, dx + \int_\Omega D_uL(x, y_{\bar{u}}, \bar{u})(\bar{u} - \bar{u}_h)\, dx \\
&= \int_\Omega (L(x, y_{\bar{u}}, \bar{u}_h) - L(x, y_h(\bar{u}_h), \bar{u}_h))\, dx \\
&+ \int_\Omega (L(x, y_h(\bar{u}_h), \bar{u}_h) - L(x, y_{\bar{u}}, \bar{u}))\, dx + \int_\Omega D_uL(x, y_{\bar{u}}, \bar{u})(\bar{u} - \bar{u}_h)\, dx \\
&\le C\|y_{\bar{u}} - y_h(\bar{u}_h)\|_{2,\Omega} + J_h(\bar{u}_h) - J(\bar{u}) + Ch + \int_\Omega D_uL(x, y_{\bar{u}}, \bar{u})(\bar{u} - \bar{u}_h)\, dx \\
&\le C(\|y_{\bar{u}} - y_{\bar{u}_h}\|_{2,\Omega} + \|y_{\bar{u}_h} - y_h(\bar{u}_h)\|_{2,\Omega} + h) + J_h(\bar{u}_h) - J(\bar{u}) \\
&+ \int_\Omega D_uL(x, y_{\bar{u}}, \bar{u})(\bar{u} - \bar{u}_h)\, dx \longrightarrow 0
\end{aligned}$$

when $h \to 0$. Thus, we have shown (8.9).
Step 2. Let us now confirm (8.8). Due to Lemma 1 and Lemma 3, there exist $\bar{s} \in C^{0,1}(\overline{\Omega})$ and $\bar{s}_h \in L^\infty(\Omega_h)$ such that

$$\varphi_{\bar{u}}(x) + D_uL(x, y_{\bar{u}}(x), \bar{s}(x)) = 0 \qquad \forall\, x \in \hat{T} \text{ and } \forall\, \hat{T} \in \hat{\mathcal{T}}_h, \tag{8.10}$$

$$\bar{s}_{h|T} = s_T, \qquad \int_T (\varphi_h(\bar{u}_h) + D_uL(x, y_h(\bar{u}_h), s_T))\, dx = 0 \qquad \forall\, T \in \mathcal{T}_h. \tag{8.11}$$

From (8.11), we deduce that for every $T \in \mathcal{T}_h$, there exists $x_T \in T$ such that

$$\varphi_h(\bar{u}_h)(x_T) + D_uL(x_T, y_h(\bar{u}_h)(x_T), s_T) = 0. \tag{8.12}$$

Suppose that $T \in \mathcal{T}_h$ is given fixed, and select an arbitrary $x \in T$. By making the difference between (8.10) and (8.12), and due to the assumptions on $D_{uu}L$ along with hypothesis **A2**, it follows that

$$m\ |\bar{u}(x) - \bar{u}_h(x)| = m\ |\mathrm{Proj}_{[\alpha,\beta]}(\bar{s}(x)) - \mathrm{Proj}_{[\alpha,\beta]}(\bar{s}_h(x))|$$

$$
\begin{aligned}
&\le m\ |\bar{s}(x) - \bar{s}_h(x)| = m\ |\bar{s}(x) - s_T| \\
&\le |D_u L(x, y_{\bar{u}}(x), \bar{s}(x)) - D_u L(x, y_{\bar{u}}(x), s_T)| \\
&= |(\varphi_{\bar{u}}(x) - \varphi_h(\bar{u}_h)(x_T)) + (D_u L(x, y_{\bar{u}}(x), s_T) - D_u L(x_T, y_h(\bar{u}_h)(x_T), s_T))| \\
&\le |\varphi_{\bar{u}}(x) - \varphi_h(\bar{u}_h)(x_T)| + C\{|x - x_T| + |y_{\bar{u}}(x) - y_h(\bar{u}_h)(x_T)|\}.
\end{aligned}
$$

We know from Theorem 1 and 2 that $y_{\bar{u}}$ and $\varphi_{\bar{u}}$ are Lipschitz, hence

$$
\begin{aligned}
&m\ |\bar{u}(x) - \bar{u}_h(x)| \le C(|x - x_T| + \|\varphi_{\bar{u}} - \varphi_h(\bar{u}_h)\|_{\infty,T} + \|y_{\bar{u}} - y_h(\bar{u}_h)\|_{\infty,T}) \\
&\le C(h + \|\varphi_{\bar{u}} - \varphi_h(\bar{u}_h)\|_{\infty,T} + \|y_{\bar{u}} - y_h(\bar{u}_h)\|_{\infty,T}).
\end{aligned}
$$

Invoking Theorem 7, we get

$$
\begin{aligned}
&\|\bar{u} - \bar{u}_h\|_{\infty,\Omega_h} = \sup_{T \in \mathcal{T}_h} \|\bar{u} - \bar{u}_h\|_{\infty,T} \\
&\le C(h + \|\varphi_{\bar{u}} - \varphi_h(\bar{u}_h)\|_{\infty,\Omega_h} + \|y_{\bar{u}} - y_h(\bar{u}_h)\|_{\infty,\Omega_h}) \le C(h + \|\bar{u} - \bar{u}_h\|_{2,\Omega} + h^\lambda). \quad (8.13)
\end{aligned}
$$

Regard now any $\hat{T} \in \partial\widehat{\mathcal{T}}_h$, and let $T \in \partial\mathcal{T}_h$ be the corresponding boundary triangle (here $\partial\widehat{\mathcal{T}}_h$ and $\partial\mathcal{T}_h$ denote the sets of boundary triangles in $\widehat{\mathcal{T}}_h$ and $\mathcal{T}_h$). For $\hat{x} \in \hat{T} \setminus T$, let x be its projection on the boundary Γ_h of Ω_h. Taking into account the Lipschitz continuity of $\bar{u}$, we obtain

$$
\begin{aligned}
|\bar{u}(\hat{x}) - \bar{u}_h(\hat{x})| &\le |\bar{u}(\hat{x}) - \bar{u}(x)| + |\bar{u}(x) - \bar{u}_h(\hat{x})| = |\bar{u}(\hat{x}) - \bar{u}(x)| + |\bar{u}(x) - \bar{u}_h(x)| \\
&\le C|\hat{x} - x| + \|\bar{u} - \bar{u}_h\|_{\infty,\Omega_h} \le Ch + \|\bar{u} - \bar{u}_h\|_{\infty,\Omega_h}.
\end{aligned}
$$

Hence $\|\bar{u} - \bar{u}_h\|_{\infty,\hat{T}\setminus T} \le Ch + \|\bar{u} - \bar{u}_h\|_{\infty,\Omega_h}$, and

$$
\|\bar{u} - \bar{u}_h\|_{\infty,\Omega\setminus\Omega_h} = \sup_{\hat{T} \in \partial\widehat{\mathcal{T}}_h} \|\bar{u} - \bar{u}_h\|_{\infty,\hat{T}\setminus T} \le Ch + \|\bar{u} - \bar{u}_h\|_{\infty,\Omega_h}. \quad (8.14)
$$

Therefore, (8.13) and (8.14) ensure

$$
\|\bar{u} - \bar{u}_h\|_{\infty,\Omega} \le C(h + \|\bar{u} - \bar{u}_h\|_{2,\Omega} + h^\lambda) \longrightarrow 0 \quad \text{when } h \to 0. \qquad \Box
$$

9 Error-estimates for the optimal control

We start our investigations with a sequence $(\bar{u}_h)_{h>0}$ of solutions of (P_h), $h > 0$, converging to a solution $\bar{u}$ of (P). Given this a priori information, we shall establish error estimates for $\|\bar{u} - \bar{u}_h\|_{2,\Omega}$ and $\|\bar{u} - \bar{u}_h\|_{\infty,\Omega}$. These estimations are performed under the following second order sufficient optimality condition:

(SSC) There is $\delta > 0$ such that

$$
J''(\bar{u})v^2 \ge \delta\ \|v\|_{2,\Omega}^2 \quad (9.1)
$$

holds for all $v \in L^\infty(\Omega)$ satisfying

$$v(x) \begin{cases} \geq 0 & \text{if } \bar{u}(x) = \alpha, \\ \leq 0 & \text{if } \bar{u}(x) = \beta, \\ = 0 & \text{if } |\varphi_{\bar{u}}(x) + D_u L(x, y_{\bar{u}}(x), \bar{u}(x))| \geq \tau > 0. \end{cases} \tag{9.2}$$

Remark 12 *It can be shown that* (SSC) *is equivalent to the condition*

$$J''(\bar{u})v^2 > 0$$

for all $v \in L^\infty(\Omega)$ *satisfying the two first relations in* (9.1) *together with* $v(x) = 0$ *if* $|\varphi_{\bar{u}}(x) + D_u L(x, y_{\bar{u}}(x), \bar{u}(x))| > 0$. *Notice that this condition leaves no gap to the necessary conditions, which require* $J''(\bar{u})v^2 \geq 0$ *for the same set of functions* (*cf.* [20]).

In our analysis, we need an element u_h admissible for (P_h) (so that it can serve as a "test function" in the variational inequality), close to $\bar{u}$, and such that $\bar{u}_h - u_h$ belongs to the cone where our second order sufficient condition applies. A natural choice is given by

$$u_h \in U_h, \quad u_{h|\hat{T}} = \text{Proj}_{[\alpha,\beta]}(\Pi_h \bar{s})_{|T} \qquad \forall\, \hat{T} \in \hat{\mathcal{T}}_h,$$

where $T \in \mathcal{T}_h$ is the triangle associated with $\hat{T} \in \hat{\mathcal{T}}_h$, and $\bar{s}$ is the solution of (4.5) associated with $\bar{u}$. This element is admissible and close to $\bar{u}$, but $\bar{u}_h - u_h$ does not belong to the critical cone. To overcome this difficulty, we introduce a perturbation $\tilde{u}_h$ of u_h defined by

$$\tilde{u}_h(x) = \begin{cases} \bar{u}_h(x) & \text{if } x \in \Omega \setminus \Omega_h, \\ \bar{u}(x) & \text{if } x \in \Omega_h \text{ and } (\bar{u}(x) = \alpha \text{ or } \bar{u}(x) = \beta), \\ u_h(x) & \text{if } x \in \Omega_h \text{ and } \alpha < \bar{u}(x) < \beta. \end{cases}$$

Before we derive some auxiliary results, and to simplify the redaction of this section, let us introduce the following notation:

$$\bar{\mathbf{d}} = \varphi_{\bar{u}} + D_u L(\cdot, y_{\bar{u}}, \bar{u}), \qquad \mathbf{d}_h(u) = \varphi_h(u) + D_u L(\cdot, y_h(u), u), \qquad \bar{\mathbf{d}}_h = \mathbf{d}_h(\bar{u}_h),$$

$$D_{\xi\eta}\mathbf{L}(w) = D_{\xi\eta}L(\cdot, y_w, w), \qquad D_{\xi\eta}\mathbf{L}^h(w) = D_{\xi\eta}L(\cdot, y_h(w), w) \qquad \xi, \eta \in \{y, u\},$$

$$D_{yy}\mathbf{f}(w) = D_{yy}f(\cdot, y_w), \qquad D_{yy}\mathbf{f}^h(w) = D_{yy}f(\cdot, y_h(w)).$$

Lemma 6 *Suppose that assumptions* **A**1-**A**2 *are satisfied and that* $\bar{u}$ *satisfies the second order sufficient condition* (SSC). *Then there exists* $h_o > 0$, *such that for all* $h \leq h_o$

$$J''(\bar{u})(\bar{u}_h - \tilde{u}_h)^2 \geq \delta\, \|\bar{u}_h - \tilde{u}_h\|^2_{2,\Omega}.$$

Proof. We have to show that $v = \bar{u}_h - \tilde{u}_h$ satisfies the relations (9.2). Then the second order condition yields the statement.

• On $\Omega \setminus \Omega_h$, it is clear that $v = \bar{u}_h - \tilde{u}_h = 0$ satisfies (9.2).

• Let $x \in \Omega_h$. If $\bar{u}(x) = \alpha$, then $\tilde{u}_h(x) = \alpha$. Therefore, $v(x) = \bar{u}_h(x) - \tilde{u}_h(x) \geq 0$. Analogously, $\bar{u}(x) = \beta$ implies $v(x) \leq 0$.

• Finally, we prove

$$v(x) = \bar{u}_h(x) - \tilde{u}_h(x) = 0 \quad \text{ on } \Omega_h^\tau = \{x \in \Omega \mid |\bar{\mathbf{d}}(x)| \geq \tau > 0\} \cap \Omega_h,$$

for all sufficiently small $h > 0$. From (7.3) and Corollary 8, we conclude that $\lim_{h\to 0} \|\bar{\mathbf{d}} - \bar{\mathbf{d}}_h\|_{\infty,\Omega} = 0$. Therefore, there exists $h_o > 0$ such that for all $h \leq h_o$, we have $\|\bar{\mathbf{d}} - \bar{\mathbf{d}}_h\|_{\infty,\Omega} \leq \tau/2$, and hence

$$|\bar{\mathbf{d}}_h(x)| \geq |\bar{\mathbf{d}}(x)| - |\bar{\mathbf{d}} - \bar{\mathbf{d}}_h(x)| \geq \frac{\tau}{2} \qquad \forall\, x \in \Omega_h^\tau.$$

It is easy to verify that the functions $\bar{\mathbf{d}}$ and $\mathbf{d}_h$ have the same sign on Ω_h^τ. Let $x \in \Omega_h^\tau$, and suppose that $\bar{\mathbf{d}}(x) > 0$. Then, for all $h \leq h_o$, $|\bar{\mathbf{d}}_h(x)| = \bar{\mathbf{d}}_h(x) \geq \frac{\tau}{2} > 0$. From Lemma 2 and Theorem 6, it follows that $\bar{u}(x) = \bar{u}_h(x) = u_h(x) = \alpha$, and thus $\tilde{u}_h(x) = \alpha$. Therefore, $\bar{u}_h(x) - \tilde{u}_h(x) = 0$. If $\bar{\mathbf{d}}(x) < 0$, we prove in the same way that $\bar{u}_h(x) = \bar{u}(x) = \tilde{u}_h(x) = \beta$.□

Lemma 7 *Suppose that* **A1-A2** *are satisfied, and let $w \in U_h$ fulfil $\|w\|_{\infty,\Omega} \leq M$. Then, for all $v \in U_h$ satisfying $v = 0$ on $\Omega \setminus \Omega_h$, we have*

$$|J''(w)v^2 - J_h''(w)v^2| \leq Ch^\lambda \, \|v\|_{2,\Omega}^2,$$

where $C \equiv C(\Omega, n, M)$ is a positive constant independent of v and h.

Proof. From [8], we know that

$$J''(w)v^2 = \int_\Omega \Big((D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) z_v^2 + 2D_{yu}\mathbf{L}(w) z_v v + D_{uu}\mathbf{L}(w) v^2 \Big)\, dx$$

and

$$\begin{aligned} &J_h''(w)v^2 = \\ &= \int_{\Omega_h} \Big((D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)) z_h(v)^2 + 2D_{yu}\mathbf{L}^h(w) z_h(v) v + D_{uu}\mathbf{L}^h(w) v^2 \Big)\, dx \\ &= \int_{\Omega} \Big((D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)) z_h(v)^2 + 2D_{yu}\mathbf{L}^h(w) z_h(v) v + D_{uu}\mathbf{L}^h(w) v^2 \Big)\, dx \end{aligned}$$

where φ_w, $\varphi_h(w)$ are the solutions of (4.3) and (6.1) corresponding to w, z_v, $z_h(v)$ are the solutions of (7.9) and (7.10) corresponding to (v, w), respectively. It follows that

$$\begin{aligned} &|J''(w)v^2 - J_h''(w)v^2| \\ &\leq \int_\Omega |(D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) z_v^2 - (D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)) z_h(v)^2|\, dx \\ &+ \int_\Omega |D_{uu}\mathbf{L}(w) - D_{uu}\mathbf{L}^h(w)|\; v^2\, dx + 2\int_\Omega |D_{yu}\mathbf{L}(w) z_v v - D_{yu}\mathbf{L}^h(w) z_h(v) v|\, dx \end{aligned}$$

$$= I_1 + I_2 + I_3. \tag{9.3}$$

• First we consider I_1. Due to the assumptions on L and f, and thanks to Proposition 1 we have

$$I_1 \le \int_\Omega |(D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) - (D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w))|\ |z_v^2|\, dx$$

$$+ \int_\Omega |D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)|\ |z_v^2 - z_h(v)^2|\, dx$$

$$\le \|(D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) - (D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w))\|_{\infty,\Omega}\ \|z_v\|_{2,\Omega}^2$$

$$+ \|D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)\|_{\infty,\Omega}\ \|z_v^2 - z_h(v)^2\|_{1,\Omega}$$

$$\le \|(D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) - (D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w))\|_{\infty,\Omega}\ \|v\|_{2,\Omega}^2$$

$$+ \|D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)\|_{\infty,\Omega}\ (\|z_v + z_h(v)\|_{2,\Omega} \|z_v - z_h(v)\|_{2,\Omega}). \tag{9.4}$$

Propositions 1 and 6 permit to estimate

$$\|z_v - z_h(v)\|_{2,\Omega} \le C h^2 \|v\|_{2,\Omega}, \tag{9.5}$$

$$\|z_v + z_h(v)\|_{2,\Omega} \le 2\|z_v\|_{2,\Omega} + \|z_v - z_h(v)\|_{2,\Omega} \le C(1 + h^2)\ \|v\|_{2,\Omega} \le C\ \|v\|_{2,\Omega}. \tag{9.6}$$

Moreover, due to **A1**, **A2**, (7.3), (7.4), and Remark 9, we have

$$\|(D_{yy}\mathbf{L}(w) - \varphi_w D_{yy}\mathbf{f}(w)) - (D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w))\|_{\infty,\Omega}$$

$$\le \|D_{yy}\mathbf{L}(w) - D_{yy}\mathbf{L}^h(w)\|_{\infty,\Omega} + \|\varphi_w\|_{\infty,\Omega} \|D_{yy}\mathbf{f}(w) - D_{yy}\mathbf{f}^h(w)\|_{\infty,\Omega}$$

$$+ \|D_{yy}\mathbf{f}^h(w)\|_{\infty,\Omega} \|\varphi_w - \varphi_h(w)\|_{\infty,\Omega}$$

$$\le C(\|y_w - y_h(w)\|_{\infty,\Omega} + \|\varphi_w - \varphi_h(w)\|_{\infty,\Omega}) \le C\ h^\lambda, \tag{9.7}$$

$$\|D_{yy}\mathbf{L}^h(w) - \varphi_h(w) D_{yy}\mathbf{f}^h(w)\|_{\infty,\Omega} \le C. \tag{9.8}$$

Therefore, from (9.4), (9.5), (9.6), (9.7), and (9.8), we deduce that

$$I_1 \le C(h^2 + h^\lambda)\ \|v\|_{2,\Omega}^2 \le C h^\lambda\ \|v\|_{2,\Omega}^2. \tag{9.9}$$

In the same way we estimate

$$I_2 \le C \|y_w - y_h(w)\|_{\infty,\Omega}\ \|v\|_{2,\Omega}^2 \le C h^\lambda\ \|v\|_{2,\Omega}^2, \tag{9.10}$$

$$I_3 \le C \|y_w - y_h(w)\|_{\infty,\Omega} (\|z_v\|_{\infty,\Omega} + \|z_v - z_h(v)\|_{2,\Omega})\ \|v\|_{2,\Omega} \le C h^\lambda\ \|v\|_{2,\Omega}^2. \tag{9.11}$$

The statement follows from (9.3), (9.9), (9.10) and (9.11). □

Proposition 8 *Suppose that assumptions* **A**1-**A**2 *are satisfied together with the second order condition (SSC). Then there exists $h_1 > 0$, such that for all $h \le h_o$*

$$J_h''(\bar{u})(\bar{u}_h - \tilde{u}_h)^2 \ge \frac{\delta}{2}\ \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2.$$

Proof. This is a direct consequence of Lemma 6 and Lemma 7. □

Lemma 8 *Suppose that* **A**1- **A**2 *are satisfied. Let* w_1 *and* w_2 *be in* $L^\infty(\Omega)$ *such that* $\|w_1\|_{\infty,\Omega} + \|w_2\|_{\infty,\Omega} \leq M$*. Then,*

$$|J_h''(w_1)v^2 - J_h''(w_2)v^2| \leq C(\|w_1 - w_2\|_{\infty,\Omega_h} + h^\lambda)\ \|v\|_{2,\Omega_h}^2 \tag{9.12}$$

for all $v \in U_h$*, where* $C \equiv C(\Omega, n, M)$ *is a constant independent of* v *and* h*.*

Proof. By simple calculations, and using the estimates of the last proof, we can see that

$$\begin{aligned}
&|J_h''(w_1)v^2 - J_h''(w_2)v^2| \\
&\leq \int_{\Omega_h} |(D_{yy}\mathbf{L}^h(w_1) - \varphi_h(w_1)D_{yy}\mathbf{f}^h(w_1)) - (D_{yy}\mathbf{L}^h(w_2) - \varphi_h(w_2)D_{yy}\mathbf{f}^h(w_1))|z_h(v)^2\,dx \\
&\quad + \int_{\Omega_h} |D_{uu}\mathbf{L}^h(w_1) - D_{uu}\mathbf{L}^h(w_2)|\ v^2\,dx \\
&\quad +2\int_{\Omega_h} |D_{yu}\mathbf{L}^h(w_1) - D_{yu}\mathbf{L}^h(w_2)||z_h(v)v|\,dx \\
&\quad \leq C\Big(\|D_{yy}\mathbf{L}^h(w_1) - D_{yy}\mathbf{L}^h(w_2)\|_{\infty,\Omega_h} + \|D_{yy}\mathbf{f}^h(w_1) - D_{yy}\mathbf{f}^h(w_2)\|_{\infty,\Omega_h} \\
&\quad +\|\varphi_h(w_1) - \varphi_h(w_2)\|_{\infty,\Omega_h} + \|D_{yu}\mathbf{L}^h(w_1) - D_{yu}\mathbf{L}^h(w_2)\|_{\infty,\Omega_h} \\
&\quad +\|D_{uu}\mathbf{L}^h(w_1) - D_{uu}\mathbf{L}^h(w_2)\|_{\infty,\Omega_h}\Big)\ \|v\|_{2,\Omega_h}^2 \\
&\quad \leq C(\|y_h(w_1) - y_h(w_2)\|_{\infty,\Omega_h} + \|\varphi_h(w_1) - \varphi_h(w_2)\|_{\infty,\Omega_h} + \|w_1 - w_2\|_{\infty,\Omega_h})\ \|v\|_{2,\Omega_h}^2 \\
&\quad \leq C\Big(\|y_h(w_1) - y_{w_1}\|_{\infty,\Omega_h} + \|y_{w_1} - y_h(w_2)\|_{\infty,\Omega_h} \\
&\quad +\|\varphi_h(w_1) - \varphi_{w_1}\|_{\infty,\Omega_h} + \|\varphi_{w_1} - \varphi_h(w_2)\|_{\infty,\Omega_h} + \|w_1 - w_2\|_{\infty,\Omega_h}\Big)\ \|v\|_{2,\Omega_h}^2 \\
&\quad \leq C(h^\lambda + \|w_1 - w_2\|_{2,\Omega} + \|w_1 - w_2\|_{\infty,\Omega_h})\ \|v\|_{2,\Omega_h}^2 \\
&\quad \leq C\Big(h^\lambda + |\Omega \setminus \Omega_h|^{\frac{1}{2}} + \|w_1 - w_2\|_{\infty,\Omega_h}\Big)\ \|v\|_{2,\Omega_h}^2 \\
&\quad \leq C\Big(h^\lambda + h + \|w_1 - w_2\|_{\infty,\Omega_h}\Big)\ \|v\|_{2,\Omega_h}^2.
\end{aligned}$$

The proof is complete, since $\lambda \leq 1$ holds in all cases. □

By (6.2) and the definition of $\bar{\mathbf{d}}_h$, the approximate optimal control $\bar{u}_h$ satisfies

$$\int_{\Omega_h} \bar{\mathbf{d}}_h(v - \bar{u}_h)(x)\,dx \geq 0 \qquad \forall\ v \in U_h^{ad}.$$

The auxiliary control u_h will not fulfil the analogous inequality

$$\int_{\Omega_h} \mathbf{d}_h(u_h)(v-u_h)(x)\,dx \geq 0 \qquad \forall\ v \in U_h^{ad}.$$

Instead of this, we are able to show that u_h satisfies an associated perturbed variational inequality with perturbation ζ_h. To this aim, we introduce $\zeta_h : \overline{\Omega}_h \longrightarrow I\!R$ by

$$\zeta_{h|T} = \begin{cases} \{-\frac{1}{|T|}\int_T \mathbf{d}_h(u_h)(x)\,dx\}^+ & \text{if } u_{h|T} = \alpha, \\ -\{\frac{1}{|T|}\int_T \mathbf{d}_h(u_h)(x)\,dx\}^+ & \text{if } u_{h|T} = \beta, \\ -\frac{1}{|T|}\int_T \mathbf{d}_h(u_h)(x)\,dx & \text{otherwise,} \end{cases}$$

for all $T \in \mathcal{T}_h$.

Lemma 9 *The auxiliary control u_h satisfies the variational inequality*

$$\int_{\Omega_h} (\mathbf{d}_h(u_h)+\zeta_h)(v-u_h)(x)\,dx \geq 0 \qquad \forall\ v \in U_h^{ad}. \tag{9.13}$$

Proof. First, observe that (9.13) can be equivalently rewritten as

$$(\int_T \mathbf{d}_h(u_h)(x)\,dx + |T|\ \zeta_{h|T})(t-u_{h|T}) \geq 0 \tag{9.14}$$

for all $T \in \mathcal{T}_h$ and all $t \in [\alpha,\beta]$. Let $T \in \mathcal{T}_h$ be given fixed.

- Suppose that $\alpha < u_{h|T} < \beta$. From the definition of $\mathbf{d}_h(u_h)$ and ζ_h, we easily see that $\int_T \mathbf{d}_h(u_h)(x)\,dx + |T|\ \zeta_{h|T} = 0$. Therefore (9.14) is satisfied.
- If $u_{h|T} = \alpha$, then $t - u_{h|T} = t - \alpha \geq 0$ for all $t \in [\alpha,\beta]$, and

$$\int_T \mathbf{d}_h(u_h)(x)\,dx + |T|\ \zeta_{h|T} = \int_T \mathbf{d}_h(u_h)(x)\,dx + (-\int_T \mathbf{d}_h(u_h)(x)\,dx)^+ \geq 0.$$

Therefore (9.14) holds in this case too.

- Finally, if $u_{h|T} = \beta$, then $t - u_{h|T} = t - \beta \leq 0$ for all $t \in [\alpha,\beta]$, and

$$\int_T \mathbf{d}_h(u_h)(x)\,dx + |T|\ \zeta_{h|T} = \int_T \mathbf{d}_h(u_h)(x)\,dx - (\int_T \mathbf{d}_h(u_h)(x)\,dx)^+ \leq 0.$$

We confirm again that (9.14) is true. Since all possible cases have been considered, the proof is complete. □

Lemma 10 *Suppose that* **A**1- **A**2 *are satisfied. Then, there exists a positive constant C, independent of h, such that*

$$||\zeta_h||_{2,\Omega_h} \leq Ch. \tag{9.15}$$

Proof. Let $T \in \mathcal{T}_h$ be fixed.

- Suppose that $\zeta_{h|T} \neq 0$ and $u_{h|T} = \mathrm{Proj}_{[\alpha,\beta]}(\Pi_h \bar{s})_{|T} = \alpha$. Hence,

$$\Pi_h \bar{s}_{|T} = \frac{1}{|T|} \int_T \bar{s}(x)\, dx \leq \alpha. \tag{9.16}$$

Let $x_T \in T$ be such that $\Pi_h \bar{s}_{|T} = \bar{s}(x_T)$. From (9.16), we deduce that

$$\bar{u}(x_T) = \mathrm{Proj}_{[\alpha,\beta]}(\bar{s}(x_T)) = \mathrm{Proj}_{[\alpha,\beta]}(\Pi_h \bar{s}_{|T}) = \alpha = u_{h|T},$$

and

$$\begin{aligned} 0 &= \varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \bar{s}(x_T)) \\ &\leq \varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \alpha) = \bar{\mathbf{d}}(x_T). \end{aligned}$$

Notice that $\zeta_{h|T} \neq 0$ and $u_{h|T} = \alpha$ imply that $\int_T \mathbf{d}_h(u_h)(x)\, dx$ is negative. Therefore,

$$\begin{aligned} &|T|\ |\zeta_{h|T}| = -\int_T \mathbf{d}_h(u_h)(x)\, dx \\ &\leq -\int_T (\mathbf{d}_h(u_h)(x) - \bar{\mathbf{d}}(x_T))\, dx = |\int_T (\mathbf{d}_h(u_h)(x) - \bar{\mathbf{d}}(x_T))\, dx| \\ &= |\int_T \Big((\varphi_h(u_h)(x) + D_u L(x, y_h(u_h)(x), \alpha)) - (\varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \alpha))\Big)\, dx| \\ &\leq \int_T |\varphi_h(u_h)(x) - \varphi_{\bar{u}}(x_T)|\, dx + \int_T |D_u L(x, y_h(u_h)(x), \alpha) - D_u L(x_T, y_{\bar{u}}(x_T), \alpha)|\, dx \\ &\leq \int_T |\varphi_h(u_h)(x) - \varphi_{\bar{u}}(x)|\, dx + \int_T |\varphi_{\bar{u}}(x) - \varphi_{\bar{u}}(x_T)|\, dx \\ &+ \int_T |D_u L(x, y_h(u_h)(x), \alpha) - D_u L(x, y_{\bar{u}}(x), \alpha)|\, dx \\ &+ \int_T |D_u L(x, y_{\bar{u}}(x), \alpha) - D_u L(x_T, y_{\bar{u}}(x), \alpha)|\, dx \\ &+ \int_T |D_u L(x_T, y_{\bar{u}}(x), \alpha) - D_u L(x_T, y_{\bar{u}}(x_T), \alpha)|\, dx \\ &\leq \int_T |\varphi_h(u_h) - \varphi_{\bar{u}}|(x)\, dx + C\ (\int_T |y_h(u_h) - y_{\bar{u}}|(x)\, dx + \int_T |x - x_T|\, dx) \\ &\leq C\ \Big(|T|^{\frac{1}{2}}(\|\varphi_h(u_h) - \varphi_{\bar{u}}\|_{2,T} + \|y_h(u_h) - y_{\bar{u}}\|_{2,T}) + |T|\ h\Big) \end{aligned}$$

After dividing by $|T|$ we find by the Young inequality,

$$|\zeta_{h|T}|^2 \leq C\Big(\frac{1}{|T|}(\|\varphi_h(u_h) - \varphi_{\bar{u}}\|_{2,T}^2 + \|y_h(u_h) - y_{\bar{u}}\|_{2,T}^2) + h^2\Big),$$

and thus, after integration over T,

$$\|\zeta_h\|_{2,T}^2 \leq C(\|\varphi_h(u_h) - \varphi_{\bar{u}}\|_{2,T}^2 + \|y_h(u_h) - y_{\bar{u}}\|_{2,T}^2 + |T|\ h^2). \tag{9.17}$$

- If $\zeta_{h|T} \neq 0$ and $u_{h|T} = \beta$, then

$$\Pi_h \bar{s}_{|T} = \frac{1}{T} \int_T \bar{s}(x)\, dx \geq \beta. \tag{9.18}$$

Let $x_T \in T$ be such that $\Pi_h \bar{s}_{|T} = \bar{s}(x_T)$. From (9.18), we deduce that $\bar{u}(x_T) = \beta$ and

$$\bar{\mathbf{d}}(x_T) = \varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \beta)$$
$$\leq \varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \bar{s}(x_T)) = 0.$$

Moreover, we must have $\int_T \mathbf{d}_h(u_h)(x)\, dx \geq 0$ and hence

$$|T|\ |\zeta_{h|T}| = \int_T \mathbf{d}_h(u_h)(x)\, dx$$
$$\leq \int_T (\mathbf{d}_h(u_h)(x) - \bar{\mathbf{d}}(x_T))\, dx = |\int_T (\mathbf{d}_h(u_h)(x) - \bar{\mathbf{d}}(x_T))\, dx|.$$

Along the lines of the first part we prove (9.17) in this case, too.

- Suppose now $\alpha < u_{h|T} < \beta$, then $u_{h|T} = \Pi_h \bar{s}|_T = \bar{s}(x_T)$. Since

$$\bar{\mathbf{d}}(x_T) = \varphi_{\bar{u}}(x_T) + D_u L(x_T, y_{\bar{u}}(x_T), \bar{s}(x_T)) = 0,$$

we have

$$|T|\ |\zeta_{h|T}| = |\int_T \mathbf{d}_h(u_h)(x)\, dx|$$
$$= |\int_T (\varphi_h(u_h)(x) + D_u L(x, y_h(u_h)(x), \bar{u}(x_T)))\, dx|$$
$$= |\int_T (\varphi_h(u_h)(x) + D_u L(x, y_h(u_h)(x), \bar{u}(x_T)) - \bar{\mathbf{d}}(x_T))\, dx|.$$

Repeating the same arguments, we show (9.17) again. Summarizing up, we have verified (9.17) in all possible cases.

- Summing up the inequality (9.17) over all triangles T yields

$$\begin{aligned} \|\zeta_h\|_{2,\Omega_h} &\leq C(\|\varphi_h(u_h) - \varphi_{\bar{u}}\|_{2,\Omega_h} + \|y_h(u_h) - y_{\bar{u}}\|_{2,\Omega_h} + h) \\ &\leq C(\|\varphi_h(u_h) - \varphi_{\bar{u}}\|_{2,\Omega} + \|y_h(u_h) - y_{\bar{u}}\|_{2,\Omega} + h). \end{aligned}$$

From (7.2), we deduce that

$$\begin{aligned} \|\zeta_h\|_{2,\Omega_h} &\leq C(h^2 + \|u_h - \bar{u}\|_{2,\Omega} + h) \leq C(\|u_h - \bar{u}\|_{2,\Omega_h} + |\Omega \setminus \Omega_h| + h) \\ &\leq C(\|u_h - \bar{u}\|_{2,\Omega_h} + h) \leq C(\|u_h - \bar{u}\|_{\infty,\Omega_h} + h). \end{aligned} \tag{9.19}$$

Since $\bar{s}$ is Lipschitz continuous, we easily see that

$$\begin{aligned} \|u_h - \bar{u}\|_{\infty,\Omega_h} &= \|\mathrm{Proj}_{[\alpha,\beta]}(\Pi_h \bar{s}) - \mathrm{Proj}_{[\alpha,\beta]}(\bar{s})\|_{\infty,\Omega_h} \\ &\leq \|\Pi_h \bar{s} - \bar{s}\|_{\infty,\Omega_h} \leq Ch\ \|\bar{s}\|_{W^{1,\infty}(\Omega_h)}. \end{aligned} \tag{9.20}$$

The conclusion follows from (9.19) and (9.20). □

Remark 13 *In the next proof we shall use the variational inequality* (9.13)*. The function* ζ_h *is constructed such that the auxiliary function* u_h *satisfies the first order necessary optimality condition of the following problem:*

$$(Q_h) \quad \inf \quad \tilde{J}_h(v) = \int_{\Omega_h} (L(x, y_h(v), v) + \zeta_h v)\, dx, \qquad v \in U_h^{ad}.$$

Theorem 9 *Suppose that assumptions* **A**1-**A**2 *are satisfied, and that* $\bar{u}$ *satisfies the second order sufficient condition* (SSC)*. Then for all sufficiently small* $h > 0$

$$\|\bar{u} - \bar{u}_h\|_{2,\Omega} \le Ch,$$

where C *is a positive constant independent of* h*.*

Proof. From the optimality conditions for the problem (P_h) and Remark 13 above, we deduce that

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) \ge 0 \quad \text{ and } \quad J_h'(u_h)(\bar{u}_h - u_h) + \int_{\Omega_h} \zeta_h(x)(\bar{u}_h - u_h)\, dx \ge 0.$$

Therefore,

$$\begin{aligned} &(J_h'(\bar{u}_h) - J_h'(u_h))(\bar{u}_h - u_h) \\ &\le \int_{\Omega_h} \zeta_h(x)(u_h - \bar{u}_h)(x)\, dx \le \|\zeta_h\|_{2,\Omega_h} \|u_h - \bar{u}_h\|_{2,\Omega_h}. \end{aligned} \tag{9.21}$$

On the other hand, with Proposition 8, Lemma 8 and the Young inequality, we have for sufficiently small h

$$\begin{aligned} &(J_h'(\bar{u}_h) - J_h'(u_h))(\bar{u}_h - u_h) = J_h''((1-\theta)\bar{u}_h + \theta u_h)(\bar{u}_h - u_h)^2 \\ &= J_h''(\bar{u})(\bar{u}_h - u_h)^2 + (J_h''((1-\theta)\bar{u}_h + \theta u_h) - J_h''(\bar{u}))(\bar{u}_h - u_h)^2 \\ &= J_h''(\bar{u})(\bar{u}_h - \tilde{u}_h)^2 + J_h''(\bar{u})(\tilde{u}_h - u_h)^2 + 2J_h''(\bar{u})(\bar{u}_h - \tilde{u}_h)(\tilde{u}_h - u_h) \\ &+ (J_h''((1-\theta)\bar{u}_h + \theta u_h) - J_h''(\bar{u}))(\bar{u}_h - u_h)^2 \\ &\ge \frac{\delta}{2}\, \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 - C_1 \|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 - C_2 \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega_h} \|\tilde{u}_h - u_h\|_{2,\Omega_h} \\ &+ (J_h''((1-\theta)\bar{u}_h + \theta u_h) - J_h''(\bar{u}))(\bar{u}_h - u_h)^2 \\ &\ge \frac{\delta}{4}\, \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 - C_3 \|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 + (J_h''((1-\theta)\bar{u}_h + \theta u_h) - J_h''(\bar{u}))(\bar{u}_h - u_h)^2 \\ &\ge \frac{\delta}{4}\, \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 - C_3 \|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 \\ &- C\ (h^\lambda + \|u_h - \bar{u}\|_{\infty,\Omega_h} + \|\bar{u}_h - \bar{u}\|_{\infty,\Omega_h})\ \|\bar{u}_h - u_h\|_{2,\Omega_h}^2 \\ &\ge \frac{\delta}{4}\, \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 - C_3 \|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 - \frac{\delta}{8}\, \|\bar{u}_h - u_h\|_{2,\Omega_h}^2 \end{aligned}$$

with $\theta \in [0, 1]$. The last estimate follows by considering that $\|u_h - \bar{u}\|_{\infty,\Omega_h}$ and $\|\bar{u}_h - \bar{u}\|_{\infty,\Omega_h}$)

tend to zero as $h \searrow 0$. After rewriting $\bar{u}_h - u_h = \bar{u}_h - \tilde{u}_h + \tilde{u}_h - u_h$, we get

$$(J_h'(\bar{u}_h) - J_h'(u_h))(\bar{u}_h - u_h) \geq \frac{\delta}{16} \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 - C_4\|\tilde{u}_h - u_h\|_{2,\Omega_h}^2. \tag{9.22}$$

From (9.21) and (9.22), we obtain

$$\begin{aligned}
&\frac{\delta}{16} \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 \leq C_4\|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 + \|\zeta_h\|_{2,\Omega_h}\|u_h - \bar{u}_h\|_{2,\Omega_h} \\
&\leq C(\|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 + \|\zeta_h\|_{2,\Omega_h}\|\tilde{u}_h - u_h\|_{2,\Omega_h} + \|\zeta_h\|_{2,\Omega_h}^2) + \frac{\delta}{32} \|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2
\end{aligned}$$

after expanding $\|u_h - \bar{u}_h\|_{2,\Omega_h} = \|u_h - \tilde{u}_h + \tilde{u}_h - \bar{u}_h\|_{2,\Omega_h}$ by the Young inequality. Therefore,

$$\begin{aligned}
&\|\bar{u}_h - \tilde{u}_h\|_{2,\Omega}^2 \leq C(\|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 + \|\zeta_h\|_{2,\Omega_h}\|\tilde{u}_h - u_h\|_{2,\Omega_h} + \|\zeta_h\|_{2,\Omega_h}^2) \\
&\leq C(\|\tilde{u}_h - u_h\|_{2,\Omega_h}^2 + \|\zeta_h\|_{2,\Omega_h}^2) \leq C(\|\bar{u} - u_h\|_{2,\Omega_h}^2 + \|\zeta_h\|_{2,\Omega_h}^2),
\end{aligned}$$

since $\|\tilde{u}_h - u_h\|_{2,\Omega_h} \leq \|\bar{u} - u_h\|_{2,\Omega_h}$ as one can easily verify. Consequently,

$$\|\bar{u}_h - \tilde{u}_h\|_{2,\Omega} \leq C(\|\bar{u} - u_h\|_{2,\Omega_h} + \|\zeta_h\|_{2,\Omega_h}). \tag{9.23}$$

By (9.20), (9.23), (7.2), and Lemma 10, we obtain

$$\begin{aligned}
&\|\bar{u} - \bar{u}_h\|_{2,\Omega} \leq \|\bar{u} - \tilde{u}_h\|_{2,\Omega} + \|\tilde{u}_h - \bar{u}_h\|_{2,\Omega} \\
&\leq C(\|\bar{u} - \tilde{u}_h\|_{2,\Omega_h} + |\Omega \setminus \Omega_h|^{\frac{1}{2}} + \|\tilde{u}_h - \bar{u}_h\|_{2,\Omega}) \\
&\leq C(\|\bar{u} - u_h\|_{2,\Omega_h} + h + \|\tilde{u}_h - \bar{u}_h\|_{2,\Omega}) \\
&\leq C(\|\bar{u} - u_h\|_{2,\Omega} + h + \|\zeta_h\|_{2,\Omega_h}) \leq C(h\|\bar{s}\|_{W^{1,\infty}(\Omega)} + h) \leq C\, h.
\end{aligned}$$

This proves the assertion of the lemma. □

Theorem 10 *Suppose that the assumptions of Theorem 9 are satisfied. Then for all sufficiently small h, we have*

$$\|\bar{u} - \bar{u}_h\|_{\infty,\Omega} \leq Ch^{\lambda},$$

where C is a positive constant independent of h.

Proof. With arguments similar to those used in the proof of Theorem 8, we obtain

$$\begin{aligned}
&\|\bar{u} - \bar{u}_h\|_{\infty,\Omega} \\
&\leq C(h + \|\bar{u} - \bar{u}_h\|_{2,\Omega} + \|\varphi_{\bar{u}_h} - \varphi_h(\bar{u}_h)\|_{\infty,\Omega_h} + \|y_{\bar{u}_h} - y_h(\bar{u}_h)\|_{\infty,\Omega_h}).
\end{aligned}$$

The conclusion follows from Theorem 9, and (7.3). □

Remark 14 *We should underline that the error estimates of the Theorems 9 and 10 are derived under the a priori assumption that $(\bar{u}_h)_h$ is converging weakly* to $\bar{u}$. By Theorem*

8, $(\bar{u}_h)_h$ converges strongly to $\bar{u}$. Therefore, these estimates have a local character. This is important to be noticed, since the approximate problem (P_h) may have multiple global solutions $\bar{u}_h$.

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