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Optimal control of a linear elliptic equation with a supremum-norm functional

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Optimal control of a linear elliptic equation with a supremum-norm functional

Thomas Grund and Arnd Rösch

Abstract

We consider an optimal control problem of a linear elliptic equation with a functional containing a supremum-norm term. The control acts on the boundary. Necessary first order optimality conditions are derived for problems with pointwise control and state constraints. For this purpose the original problem is substituted by an equivalent problem with a differentiable functional. In a second part we discuss a numerical approach to such problems. The control problem is transformed into a linear (resp. quadratic) programming problem. In a particular situation we can compare the numerical results with the analytic solutions.

Keywords: Optimal control, supremum norm, minimax, state constraints, numerical solution

AMS subject classification: 49K35, 49K20, 90C90

1 Introduction

In this paper, we consider an optimal control problem with a cost functional of the form

$$J(y,u) = \|y - y_d\|_{C(\overline{\Omega})} + \frac{\kappa}{2}\int_\Gamma u^2 \, ds,$$

where $y_d \in C(\overline{\Omega})$, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with boundary Γ of class $C^{1,1}$, and κ is a nonnegative real number. The pair (y,u) satisfies the following linear equation

$$\begin{aligned} -\Delta y + y &= 0 \quad \text{in } \Omega, \\ \partial_\nu y &= u \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\partial_\nu y$ denotes the outward normal derivative of y on Γ. The admissible sets for the state and the control variables are given by

$$U_{ad} = \{u \in L^t(\Gamma) : u_1 \leq u(s) \leq u_2 \quad \text{a.e. on } \Gamma\},$$

$$Y_{ad} = \{y \in C(\overline{\Omega}) : y_1 \leq y(x) \leq y_2 \quad \text{in } \overline{\Omega}\},$$

with $t \geq n$ and real numbers u_1, u_2, y_1, y_2. The paper is concerned with the following control problem

$$(\mathcal{P}) \qquad \text{Minimize} \quad J(y,u), \quad \text{subject to (1) and } (y,u) \in Y_{ad} \times U_{ad}.$$

Functionals including a C-norm are not differentiable. For that reason such problems are less investigated. The necessary optimality conditions for different classes of optimal control problems with a supremum-norm functional have been studied by several authors, see for instance Glashoff and Weck [7], Tröltzsch [13], Li and Yong [9], Arada and Raymond [1]. Taking advantage of the special structure of our cost functional, and using a standard technique in the finite optimization, we substitute problem $(\mathcal{P})$ by an equivalent control problem with a differentiable functional and additional state constraints. More precisely, by using the equivalence

$$\|y - y_d\|_{C(\overline{\Omega})} \leq \delta \quad \Longleftrightarrow \quad -\delta \leq y - y_d \leq \delta \quad \text{in } \overline{\Omega}, \tag{2}$$

we can prove that $(\mathcal{P})$ is equivalent (in a sense to specify later) to the control problem

$(\mathcal{P}_d)$ Minimize $J_d(u, \delta) = \delta + \dfrac{\kappa}{2} \displaystyle\int_\Gamma u^2 \, ds$

subject to (1), and

$$y - \delta \leq y_d \qquad \text{in } \overline{\Omega}, \tag{3}$$

$$-y - \delta \leq -y_d \quad \text{in } \overline{\Omega}, \tag{4}$$

$$(y, u) \in Y_{ad} \times U_{ad}. \tag{5}$$

This reformulation enables us to establish optimality conditions by applying a classical Lagrangian multiplier rule, see Tröltzsch [14], Casas [4].

In Section 4 we handle the particular case of $(\mathcal{P})$ corresponding to $\kappa = 0$, in the absence of control and state constraints. Since the functional is not coercive and U_{ad} is not bounded in this case, classical existence results can not be applied. Under some natural assumptions on y_d, we prove the existence of an optimal control, and give a detailed characterization of the corresponding state.

Section 5 deals with the numerical solution of $(\mathcal{P}_d)$. For the numerical treatment of other elliptic control problems with control and state constraints we refer to Bergounioux and Kunisch [2], Casas [3], Maurer and Mittelmann [10].

Due to the particular structure of the cost functional we are able to apply methods of linear (resp. quadratic) programming, such as simplex or interior point methods. In our paper, particular emphasis is laid on a comparison of different techniques to build up the programming problem. The control problem is fully discretized in the first case, see also [10]. In a second case we eliminate the variables corresponding to the state functions by using the linearity of the state equation. Numerical examples are given in Section 6.

The approach presented in this paper can be applied to more general classes of optimal control problems. However we confine ourselves to a simplified linear equation to demonstrate the techniques to handle the supremum-norm functional.

2 Preliminary results

2.1 State and adjoint equation

In this section we recall some results concerning the state equation and the adjoint equation, see Casas [4] for more general results.

Theorem 1 ([4], Theorem 3.1) *For all $u \in L^t(\Gamma)$, $t \geq n-1$, there exists a unique solution y_u of the state equation* (1) *belonging to $H^1(\Omega) \cap C(\overline{\Omega})$. Moreover, there exists a constant C_1 independent of u such that*

$$\|y_u\|_{H^1(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \leq C_1 \|u\|_{L^t(\Gamma)}.$$

In the sequel, the unique solution of (1) for a control $u \in L^t(\Gamma)$ is denoted by y_u.

To derive optimality conditions we need the so called adjoint equation. As we see below, the adjoint equation of the optimality conditions for the problem (P) has measures as data in Ω and on Γ. More precisely, we deal with the following problem

$$\begin{aligned} -\Delta p + p &= \mu_\Omega \quad \text{in } \Omega, \\ \partial_\nu p &= \mu_\Gamma \quad \text{on } \Gamma, \end{aligned} \tag{6}$$

where $\mu = \mu_\Omega + \mu_\Gamma$ is a Radon measure on $\overline{\Omega}$, μ_Ω is the restriction of μ to Ω, μ_Γ is the restriction of μ to Γ. We denote by $\mathcal{M}(\overline{\Omega})$ the space of Radon measures on $\overline{\Omega}$. $\mathcal{M}(\overline{\Omega})$ is the dual space of $C(\overline{\Omega})$. The duality pairing in $\mathcal{M}(\overline{\Omega}) \times C(\overline{\Omega})$ is denoted by $\langle \cdot, \cdot \rangle_{\overline{\Omega}}$.

Definition 1 *A weak solution of* (6) *is a function $p \in W^{1,1}(\Omega)$ such that*

$$\int_\Omega (\nabla p(x) \nabla \varphi(x) + p\varphi) \; dx = \langle \mu, \varphi \rangle_{\overline{\Omega}} \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Theorem 2 ([4], Theorem 4.3) *For every $\mu \in \mathcal{M}(\overline{\Omega})$, there exists a unique weak solution p of* (6) *satisfying*

$$\int_\Omega p \; (-\Delta y + y) \; dx + \int_\Gamma p \; \partial_\nu y \; ds = \langle \mu, y \rangle_{\overline{\Omega}} \qquad \text{for all } y \in W^{2,r}(\Omega), \quad r > n.$$

Moreover, p belongs to $W^{1,s}(\Omega)$ for every $s \in [1, n/(n-1))$, and there exists a positive constant C_2, not depending on μ, such that

$$\|p\|_{W^{1,s}(\Omega)} \leq C_2 \|\mu\|_{\mathcal{M}(\overline{\Omega})}.$$

2.2 Lagrangian multipliers for control problems

In this section, we state a Lagrangian multiplier theorem for an abstract control problem applicable to (P_d). For this, suppose that $\mathcal{V}$ and $\mathcal{Z}$ are real Banach spaces, and $K \subset \mathcal{V}$ is a closed convex set. Let $P \subset \mathcal{Z}$ be a convex closed cone, $f : K \to \mathbb{R}$ be a functional and $G : K \to \mathcal{Z}$ be an operator. In $\mathcal{Z}$ we have a partial ordering by $z \geq 0 \Leftrightarrow z \in P$. If $P \subset \mathcal{Z}$ is a convex cone, then its dual cone P^+ is defined by $P^+ = \{z \in \mathcal{Z}^* : \langle z, p\rangle_{\mathcal{Z}^* \times \mathcal{Z}} \geq 0 \quad \text{for all } p \in P\}$, where $\mathcal{Z}^*$ denotes the dual space of $\mathcal{Z}$ and $\langle \cdot, \cdot \rangle_{\mathcal{Z}^* \times \mathcal{Z}}$ the duality pairing. Consider the abstract control problem $(\mathcal{CP})$,

$$(\mathcal{CP}) \qquad \text{Minimize } f(v), \quad G(v) \leq 0, \quad v \in K.$$

An element $v_0 \in K$ is called regular if

$$\text{there exists } v \in K \text{ such that } G(v_0) + G'(v_0)(v - v_0) \in -\operatorname{int} P \tag{7}$$

holds, $\operatorname{int} P$ being the interior of P. The following optimality conditions hold.

Theorem 3 ([14], Theorem 1.2.4) *Let v_0 be a regular solution of $(\mathcal{CP})$. Let us suppose that*

- *f is Fréchet-differentiable,*
- *G is continuous Fréchet-differentiable.*

Then there is a Lagrangian multiplier λ at v_0, that is a $\lambda \in P^+$ such that

$$\langle f'(v_0) + G'(v_0)^* \lambda, v - v_0 \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq 0 \quad \text{for all } v \in K \tag{8}$$

holds. Moreover, the complementary slackness condition

$$\langle \lambda, G(v_0) \rangle_{\mathcal{Z}^* \times \mathcal{Z}} = 0 \tag{9}$$

is fulfilled.

3 Study of the optimal control problem $(\mathcal{P})$

3.1 Existence result

Theorem 4 *If there exists a control $u \in U_{ad}$ satisfying $y_u \in Y_{ad}$, then $(\mathcal{P})$ has at least one solution. This solution is unique, if in addition $\kappa > 0$ holds.*

The proof is standard, see for instance Casas [4]. □

3.2 Equivalence of $(\mathcal{P})$ and $(\mathcal{P}_d)$

Theorem 5 *If $(\overline{u}, \overline{\delta})$ is a solution of $(\mathcal{P}_d)$, then $(y_{\overline{u}}, \overline{u})$ solves $(\mathcal{P})$. Conversely, if $(y_{\overline{u}}, \overline{u})$ solves $(\mathcal{P})$ then $(\overline{u}, \overline{\delta})$, with $\overline{\delta} = \|y_{\overline{u}} - y_d\|_{C(\overline{\Omega})}$, solves $(\mathcal{P}_d)$.*

Proof. Let $(y_{\overline{u}}, \overline{u})$ be a solution of $(\mathcal{P})$. We prove that $(\overline{u}, \overline{\delta})$ with $\overline{\delta} = \|y_{\overline{u}} - y_d\|_{C(\overline{\Omega})}$ is a solution of $(\mathcal{P}_d)$. Argue by contradiction, suppose that $(\hat{u}, \hat{\delta})$ is a solution of $(\mathcal{P}_d)$ with $J_d(\hat{u}, \hat{\delta}) < J_d(\overline{u}, \overline{\delta})$. From (2) it follows that

$$J(y_{\hat{u}}, \hat{u}) = \|y_{\hat{u}} - y_d\|_{C(\overline{\Omega})} + \frac{\kappa}{2} \int_\Gamma \hat{u}^2 \, ds \leq \hat{\delta} + \frac{\kappa}{2} \int_\Gamma \hat{u}^2 \, ds = J_d(\hat{u}, \hat{\delta}),$$

and therefore $J(y_{\hat{u}}, \hat{u}) \leq J_d(\hat{u}, \hat{\delta}) < J_d(\overline{u}, \overline{\delta}) = J(y_{\overline{u}}, \overline{u})$, a contradiction to the optimality of $(y_{\overline{u}}, \overline{u})$.

• For the opposite let $(\overline{u}, \overline{\delta})$ be a solution of $(\mathcal{P}_d)$ and $(y_{\hat{u}}, \hat{u})$ a solution of $(\mathcal{P})$ with $J(y_{\hat{u}}, \hat{u}) < J(y_{\overline{u}}, \overline{u})$. With $\hat{\delta} = \|y_{\hat{u}} - y_d\|_{C(\overline{\Omega})}$ it follows that

$$J_d(\hat{u}, \hat{\delta}) = J(y_{\hat{u}}, \hat{u}) < J(y_{\overline{u}}, \overline{u}) \leq J_d(\overline{u}, \overline{\delta}),$$

a contradiction which completes the proof. The last inequality is a consequence of (2). □

3.3 Optimality system for $(\mathcal{P}_d)$

Contrary to the problem $(\mathcal{P})$, the problem $(\mathcal{P}_d)$ possesses some interesting properties (Frechet-differentiability of the corresponding cost functional). This enables us to establish the optimality conditions by using the classical Lagrangian multipliers theorem.

Theorem 6 *Let $(\overline{u}, \overline{\delta})$ be a regular solution of $(\mathcal{P}_d)$. Then there exist elements $\overline{p} \in W^{1,s}(\Omega)$, for all $s \in [1, n/(n-1))$, and $\overline{\mu}_i \in \mathcal{M}(\overline{\Omega})$, $i = 1, \ldots, 4$, $\mu_i \geq 0$, satisfying*

- *the adjoint equation*

$$\begin{aligned} -\Delta \overline{p} + \overline{p} &= \overline{\mu}_{1_\Omega} - \overline{\mu}_{2_\Omega} + \overline{\mu}_{3_\Omega} - \overline{\mu}_{4_\Omega} \quad \text{in } \Omega, \\ \partial_\nu \overline{p} &= \overline{\mu}_{1_\Gamma} - \overline{\mu}_{2_\Gamma} + \overline{\mu}_{3_\Gamma} - \overline{\mu}_{4_\Gamma} \quad \text{on } \Gamma; \end{aligned} \tag{10}$$

- *complementary slackness conditions*

$$\int_{\overline{\Omega}} (y_{\overline{u}} - \overline{\delta} - y_d) \, d\overline{\mu}_1 = 0, \qquad \int_{\overline{\Omega}} (y_{\overline{u}} - y_2) \, d\overline{\mu}_3 = 0, \tag{11}$$

$$\int_{\overline{\Omega}} (-y_{\overline{u}} - \overline{\delta} + y_d) \, d\overline{\mu}_2 = 0, \qquad \int_{\overline{\Omega}} (-y_{\overline{u}} + y_1) \, d\overline{\mu}_4 = 0; \tag{12}$$

- *the variational inequality*

$$\int_\Gamma (\overline{p} + \kappa \overline{u})(u - \overline{u}) \, ds \geq 0 \quad \text{for all } u \in U_{ad}, \tag{13}$$

- *and*

$$\int_{\overline{\Omega}} d(\overline{\mu}_1 + \overline{\mu}_2) = 1. \tag{14}$$

Proof. To apply Theorem 3 we set

$$\begin{aligned}
\mathcal{V} &= L^t(\Gamma) \times \mathbb{R}, \\
\mathcal{Z} &= C(\overline{\Omega})^4 = C(\overline{\Omega}) \times C(\overline{\Omega}) \times C(\overline{\Omega}) \times C(\overline{\Omega}), \\
K &= \{(u,\delta) : u \in U_{ad}, \delta \in \mathbb{R}\}, \\
P &= \{(z_1, z_2, z_3, z_4) \in \mathcal{Z} : z_i \geq 0, \quad i = 1, \dots, 4\}, \\
f(u,\delta) &= J_d(u,\delta) = \delta + \frac{\kappa}{2} \int_\Gamma u^2 \, ds, \\
G(u,\delta) &= (y_u - \delta - y_d, -y_u - \delta + y_d, y - y_2, -y + y_1).
\end{aligned}$$

The assumptions of Theorem 3 are obviously fulfilled, the set P^+ is given by

$$P^+ = \{(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathcal{M}(\overline{\Omega})^4 : \mu_i \geq 0, \quad i = 1, \dots, 4\}.$$

From Theorem 3 we deduce the existence of measures $\overline{\mu}_i$, $i = 1, \dots, 4$, $\overline{\mu}_i \geq 0$, satisfying (11) and (12). Setting $v = (u,\delta)$ for arbitrary $u \in L^t(\Gamma)$ and $\delta \in \mathbb{R}$, $\overline{v} = (\overline{u}, \overline{\delta})$, and $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4)$ we have

$$\begin{aligned}
\langle f'(\overline{v}) + [G'(\overline{v})]^* \overline{\mu}, v \rangle &= \delta + \kappa \int_\Gamma \overline{u} u \, ds + \int_{\overline{\Omega}} (y_u - \delta) \, d\overline{\mu}_1 + \int_{\overline{\Omega}} (-y_u - \delta) \, d\overline{\mu}_2 \\
&\quad + \int_{\overline{\Omega}} y_u \, d\overline{\mu}_3 + \int_{\overline{\Omega}} (-y_u) \, d\overline{\mu}_4 \\
&= \delta \left(1 - \int_{\overline{\Omega}} d(\overline{\mu}_1 + \overline{\mu}_2) \right) + \kappa \int_\Gamma \overline{u} u \, ds \\
&\quad + \int_{\overline{\Omega}} y_u \, d(\overline{\mu}_1 - \overline{\mu}_2 + \overline{\mu}_3 - \overline{\mu}_4),
\end{aligned}$$

Following the proof of Theorem 5.3 in Casas [4] we can show that

$$\int_{\overline{\Omega}} y_u \, d(\overline{\mu}_1 - \overline{\mu}_2 + \overline{\mu}_3 - \overline{\mu}_4) = \int_\Gamma \overline{p} \, u \, ds$$

holds, where $\overline{p}$ is the unique solution of the adjoint equation (10). Together with (8) it follows that

$$(\delta - \overline{\delta}) \left(1 - \int_{\overline{\Omega}} d(\overline{\mu}_1 + \overline{\mu}_2) \right) + \int_\Gamma (\kappa \overline{u} + \overline{p}) \, (u - \overline{u}) \, ds \geq 0 \quad \text{for all } (u,\delta) \in K.$$

Because δ is free we deduce (14) and finally (13). $\square$

Remark 1 Consider the case of $(\mathcal{P}_d)$ with absence of state constraints. Then the regularity condition (7) is fulfilled for each pair $(u_0, \delta_0) \in U_{ad} \times \mathbb{R}$: In this case, (7) is equivalent to the existence of a pair $(u,\delta) \in U_{ad} \times \mathbb{R}$ and a real number $\varepsilon > 0$ satisfying

$$\begin{aligned}
y_u - \delta - y_d &\leq -\varepsilon \quad \text{in } \overline{\Omega}, \\
\text{and } -y_u - \delta + y_d &\leq -\varepsilon \quad \text{in } \overline{\Omega},
\end{aligned}$$

which is satisfied for δ sufficiently large.

4 A particular case

In this section we study the control problem

$$(\mathcal{P}_1) \qquad \text{Minimize} \quad J_1(y,u) = \|y - y_d\|_{C(\overline{\Omega})},$$

$$\text{subject to the state equation (1) and } u \in L^t(\Gamma),\ t \geq n,$$

with $y_d \in W^{2,n}(\Omega)$. Notice that control and state are unconstrained. Our goal is to prove the existence of a solution provided that y_d satisfies the following inequality

$$-\Delta y_d + y_d \leq 0 \quad \text{in } \Omega. \tag{15}$$

In this case we derive a characterization of the state corresponding to this optimal control.

4.1 Auxiliary problem

Let us start by considering the auxiliary optimization problem

$$(\mathcal{P}_{aux}) \qquad \text{Minimize} \quad F(y) = \|y - y_d\|_{C(\overline{\Omega})},$$

subject to $y \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ and the equation

$$-\Delta y + y = 0 \quad \text{in } \Omega. \tag{16}$$

The following lemmas are useful for the sequel.

Lemma 1 ([6], Theorem 9.6) *If $y \in W^{2,n}_{\mathrm{loc}}(\Omega)$ satisfies $-\Delta y + y \leq 0$ in Ω, then y cannot achieve a nonnegative maximum in Ω unless it is a constant.*

The next result is a consequence of Lemma 1.

Lemma 2 (Comparison principle) *Let $y \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be a function satisfying $-\Delta y + y \geq 0$ in Ω and $y(s) \geq 0$ for all $s \in \Gamma$. Then $y \geq 0$ in $\overline{\Omega}$.*

With these results we can prove the following theorem.

Theorem 7 *If the function y_d satisfies inequality (15), then problem $(\mathcal{P}_{aux})$ admits at least one solution $\overline{y}$. Moreover, this solution satisfies*

$$\overline{y}(s) = y_d(s) - \overline{\delta} \quad \textit{for all } s \in \Gamma, \tag{17}$$

where

$$\overline{\delta} = \inf\{F(y) : y \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega}) \text{ satisfies (16)}\}. \tag{18}$$

Proof. The infimum in (18) exists because the functional F is bounded from below by zero. We have to show that the infimum $\overline{\delta}$ is attained.

Since $y_d \in W^{2,n}(\Omega)$, we can prove the existence of a unique solution $\overline{y} \in W^{2,n}(\Omega)$ of the Dirichlet problem

$$-\Delta y + y = 0 \quad \text{in } \Omega, \tag{19}$$

$$y = y_d - \overline{\delta} \quad \text{on } \Gamma, \tag{20}$$

see Grisvard [8], Theorem 2.4.2.5. From (20) it follows

$$\max_{s\in\Gamma} (y_d(s) - \overline{y}(s)) = \overline{\delta}. \tag{21}$$

If we assume $\max\limits_{x\in\overline{\Omega}} (y_d(x) - \overline{y}(x)) > \max\limits_{s\in\Gamma} (y_d(s) - \overline{y}(s))$, then the function $y_d - \overline{y}$ admits a positiv maximum in Ω, due to (21) and the continuity of $y_d - \overline{y}$. From Lemma 1, together with (15) and (19), it follows that $y_d - \overline{y}$ is constant, a contradiction to our assumption. Since $\Gamma \subset \overline{\Omega}$ it follows that

$$\max_{x\in\overline{\Omega}} (y_d(x) - \overline{y}(x)) = \max_{s\in\Gamma} (y_d(s) - \overline{y}(s)). \tag{22}$$

Combining (21) and (22) leads to $\overline{y} \geq y_d - \overline{\delta}$ in $\overline{\Omega}$.

- It remains to prove the inequality

$$\overline{y} \leq y_d + \overline{\delta} \quad \text{in } \overline{\Omega}. \tag{23}$$

Let ε be a positive real number. By the definition of $\overline{\delta}$,

$$\overline{\delta} = \inf\{\|y - y_d\|_{C(\overline{\Omega})} : y \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \text{ satisfies } -\Delta y + y = 0\},$$

there exists a function $y_\varepsilon \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfying $-\Delta y_\varepsilon + y_\varepsilon = 0$ and

$$\|y_\varepsilon - y_d\|_{C(\overline{\Omega})} \leq \overline{\delta} + \varepsilon. \tag{24}$$

The function $v = y_\varepsilon + \varepsilon - \overline{y}$ fulfills

$$-\Delta v + v = -\Delta y_\varepsilon + y_\varepsilon - \Delta\varepsilon + \varepsilon - \Delta\overline{y} + \overline{y} = \varepsilon \geq 0 \quad \text{in } \Omega,$$

and, by (20) and (24),

$$v(s) = y_\varepsilon(s) + \varepsilon - \overline{y}(s) = y_\varepsilon(s) + \varepsilon - y_d(s) + \overline{\delta} \geq 0 \quad \text{for all } s \in \Gamma.$$

From Lemma 2 it follows $v \geq 0$ in $\overline{\Omega}$ and therefore $\overline{y} \leq y_\varepsilon + \varepsilon$. Finally, (24) yields $y_\varepsilon + \varepsilon \leq y_d + \overline{\delta} + 2\varepsilon$ in $\overline{\Omega}$. Since ε is arbitrary, (23) and the statement of the theorem are direct conclusions. $\square$

4.2 Result for $(\mathcal{P}_1)$

Now we can state the main result of this section.

Corollary 1 *Let $y_d \in W^{2,n}(\Omega)$ be a function satisfying* (15)*. Then there exists an optimal control $\overline{u}$ of the control problem $(\mathcal{P}_1)$. The corresponding state function $y_{\overline{u}}$ satisfies equation* (17)*.*

Proof. The solution y_u of (1) for $u \in L^t(\Gamma)$ is in $C(\overline{\Omega})$, see Theorem 1. Therefore $y_u|_\Gamma \in C(\Gamma)$ and $y_u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, see Gilbarg and Trudinger [6], Corollary 9.18. From the definition of $\overline{\delta}$ we conclude

$$\overline{\delta} \leq \inf\{J_1(y_u, u) : u \in L^t(\Gamma)\}.$$

Applying Theorem 7, we find a solution $\overline{y}$ for problem $(\mathcal{P}_{aux})$ which is an element of $W^{2,n}(\Omega)$. The function $\overline{u} = \partial_\nu \overline{y}$ is an element of $L^t(\Gamma)$, due to an imbedding theorem. Therefore $\overline{u}$ is an optimal control for $(\mathcal{P}_1)$. $\square$

Remark 2 The corollary remains true if we substitute (15) by $-\Delta y_d + y_d \geq 0$ in Ω, and (17) by $\overline{y}(s) = y_d(s) + \overline{\delta}$ for all $s \in \Gamma$.

5 Numerical approach

In this section, we regard different ways to solve $(\mathcal{P}_d)$ numerically. Thus, we are looking for functions $\overline{u}_h$ resp. $\overline{y}_h$, which approximate the optimal control function $\overline{u}$ resp. the corresponding state function $y_{\overline{u}}$.

Using a finite element method, control functions u resp. state functions y are replaced by finite vectors of real numbers $\mathbf{u}$ resp. $\mathbf{y}$, and the state equation by a system of linear equations $\mathbf{E}\mathbf{y} = \mathbf{B}\mathbf{u}$. Since this discretization procedure is standard, the details are described in the Appendix, see also Remark 3 below.

In a second step the problem $(\mathcal{P}_d)$ is substituted by a linear (resp. linear-quadratic) programming problem. We concentrate on two approaches:

Complete discretization: We regard a control function with its corresponding state function as independent. The discretized form of the state equation is part of the arising programming problem.

Reduced problem: Exploiting the linearity of the state equation, we eliminate the variables belonging to the state function. To do so, we calculate the solutions of the discretized state equation for basis control functions in advance.

We shall compare the efficiency and reliability of both methods. The optimality conditions are not used during the numerical approach. We will verify them later numerically. An error analysis for the numerical solutions is not given, since this would go far beyond the scope of this paper.

Remark 3 For simplicity we choose Ω to be the two dimensional open unit square $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in (0,1)\}$. However, the methods described in this section extend to other choices of Ω as well. Ω in our case does clearly not have a boundary Γ of class $C^{1,1}$, but the theory is confirmed by the numerical solutions.

Remark 4 As mentioned above, some of the notation used in the sequel, is introduced in the Appendix.

5.1 Complete discretization – programming problem $(\mathcal{O}_C)$

The vectors $\mathbf{u}$ and $\mathbf{y}$ are considered as independent variables. The vector of the unknowns is $\mathbf{x} = (\mathbf{u}^T, \mathbf{y}^T, \delta)^T$. The linear-quadratic programming problem $(\mathcal{O}_C)$ is

$$(\mathcal{O}_C) \qquad \text{Minimize} \quad \delta + \frac{\kappa}{2}\mathbf{u}^T\mathbf{A}\mathbf{u}$$

$$\text{subject to} \quad \mathbf{E}\mathbf{y} = \mathbf{B}\mathbf{u} \quad \text{and}$$

$$\begin{aligned}
y_{i,j} - \delta &\le y_d(x_{i,j}) & (i,j) &\in I(\overline{\Omega}), \\
-y_{i,j} - \delta &\le -y_d(x_{i,j}) & (i,j) &\in I(\overline{\Omega}), \\
u_{i,j} &\le u_2 & (i,j) &\in I(\Gamma), \\
-u_{i,j} &\le -u_1 & (i,j) &\in I(\Gamma), \\
y_{i,j} &\le y_2 & (i,j) &\in I(\overline{\Omega}), \\
-y_{i,j} &\le -y_1 & (i,j) &\in I(\overline{\Omega}).
\end{aligned} \tag{25}$$

The size of the optimization problem is given by

Number of unknowns:	$\mathcal{O}(N^2)$,
Number of inequalities:	$\mathcal{O}(N^2)$,
Number of equations:	$\mathcal{O}(N^2)$,
Number of coefficients:	$\mathcal{O}(N^2)$,

where the discretization parameter N is defined in the Appendix. For the number of coefficients, write down all equality and inequality constraints in matrix notation, $\mathbf{Q}\mathbf{x} = \mathbf{q}$, $\mathbf{N}\mathbf{x} \le \mathbf{n}$, with matrices $\mathbf{Q}$, $\mathbf{N}$, and vectors $\mathbf{q}$, $\mathbf{n}$. In this notation the number of coefficients is defined as the number of nonzero elements in $\mathbf{Q}$ and $\mathbf{N}$.

5.2 Reduced problem – programming problem $(\mathcal{O}_R)$

Here we regard $\mathbf{y}$ as depending on $\mathbf{u}$. To the control basis functions $e^{i,j}$, $(i,j) \in I(\Gamma)$, we calculate the corresponding state functions $y_h^{i,j}$ via (32), which are represented by vectors $\mathbf{y}^{i,j}$. The elements of a vector $\mathbf{y}^{i,j}$ are denoted by $y_{k,l}^{i,j}$, $(k,l) \in I(\overline{\Omega})$. Having this basis solutions, the solution $\mathbf{y}$ to (32) for a given vector $\mathbf{u}$ can be written as

$$\mathbf{y} = \sum_{(i,j)\in I(\Gamma)} u_{i,j}\mathbf{y}^{i,j},$$

due to the linearity of our state equation. With this we build up the programming problem $(\mathcal{O}_R)$,

$$
\begin{aligned}
(\mathcal{O}_R) \qquad & \text{Minimize} \quad \delta + \frac{\kappa}{2}\mathbf{u}^T\mathbf{A}\mathbf{u} \\
& \text{subject to} \\
& \sum_{(i,j)\in I(\Gamma)} u_{i,j}\; y_{k,l}^{i,j} - \delta \le y_d(x_{k,l}) && (k,l) \in I(\overline{\Omega}), \qquad (26)\\
& -\sum_{(i,j)\in I(\Gamma)} u_{i,j}\; y_{k,l}^{i,j} - \delta \le -y_d(x_{k,l}) && (k,l) \in I(\overline{\Omega}),\\
& u_{i,j} \le u_2 && (i,j) \in I(\Gamma),\\
& -u_{i,j} \le -u_1 && (i,j) \in I(\Gamma),\\
& \sum_{(i,j)\in I(\Gamma)} u_{i,j}\; y_{k,l}^{i,j} \le y_2 && (k,l) \in I(\overline{\Omega}),\\
& -\sum_{(i,j)\in I(\Gamma)} u_{i,j}\; y_{k,l}^{i,j} \le -y_1 && (k,l) \in I(\overline{\Omega}).
\end{aligned}
$$

Number of unknowns:	$\mathcal{O}(N)$,
Number of inequalities:	$\mathcal{O}(N^2)$,
Number of equations:	0,
Number of coefficients:	$\mathcal{O}(N^3)$.

The number of coefficients highly effects the amount of time and memory which is needed to solve the programming problem. We expect that $(\mathcal{O}_C)$ will be solved much faster then $(\mathcal{O}_R)$.

5.3 Optimization codes

There exists a large number of software packages for linear-quadratic programming problems. Surveys and decision trees for optimization software can be found in [11] and [5]. We utilized three packages, Loqo [15], the Matlab Optimization Toolbox, and Mosek [12].

5.4 Verification of the necessary optimality conditions

In order to verify the optimality conditions (see Theorem 6), especially (13), we have to approximate $\overline{\mu}_i$, $i = 1, \dots, 4$, and the adjoint state $\overline{p}$.

Approximation of $\overline{\mu}_i$

The Lagrangian multipliers associated with inequalities (25) of $(\mathcal{O}_C)$ resp. (26) of $(\mathcal{O}_R)$ are denoted by $\lambda_{i,j}$, $(i,j) \in I(\overline{\Omega})$. We approximate $\overline{\mu}_1$ by

$$\overline{\mu}_1^h = \sum_{(i,j)\in I(\overline{\Omega})} \lambda_{i,j} \delta x_{i,j},$$

where $\delta x_{i,j}$ is the Dirac measure concentrated in $x_{i,j}$. In an analogous way, the measures $\overline{\mu}_2$, $\overline{\mu}_3$, and $\overline{\mu}_4$ are approximated by measures $\overline{\mu}_2^h$, $\overline{\mu}_3^h$, and $\overline{\mu}_4^h$.

Approximation of $\overline{p}$

The function $\overline{p}$ is approximated by the solution p_h of the discretized variational equation,

Find $p_h \in V_h$, such that

$$\int_\Omega (\nabla\varphi \cdot \nabla p_h + \varphi p_h)\, dx = \int_\Gamma \varphi \; d(\overline{\mu}_1^h - \overline{\mu}_2^h + \overline{\mu}_3^h - \overline{\mu}_4^h) \tag{27}$$

holds for every $\varphi \in V_h$. For the definition of V_h see the Appendix.

Verification of (13)

Let $\overline{u}_h = \sum_{(i,j)\in I(\Gamma)} \overline{u}_{i,j} e^{i,j}$ be a numerical solution of $(\mathcal{P}_d)$. We can not expect that $\overline{u}_h$ satisfies (13) in its original form. However, we will see numerically that it fulfills the discretized variational inequality

$$\sum_{(i,j)\in I(\Gamma)} (\overline{p}_{i,j} + \kappa \overline{u}_{i,j})(u_{i,j} - \overline{u}_{i,j}) \geq 0 \text{ for all } u_{i,j} \in [u_1, u_2],\ (i,j) \in I(\Gamma),$$

with $\overline{p}_h = \sum_{(i,j)\in I(\overline{\Omega})} \overline{p}_{i,j} \eta^{i,j}$ being the solution of (27). From the last inequality it follows for the case $\kappa = 0$:

$$\overline{u}_{i,j} = \begin{cases} u_1 & \text{if } \overline{p}_{i,j} > 0, \\ u_2 & \text{if } \overline{p}_{i,j} < 0, \end{cases} \tag{28}$$

and for the case $\kappa > 0$:

$$\overline{u}_{i,j} = \mathrm{Pr}_{[u_1,u_2]} \left(-\frac{\overline{p}_{i,j}}{\kappa} \right), \tag{29}$$

for $(i,j) \in I(\Gamma)$ and $\mathrm{Pr}_{[u_1,u_2]}$ being the projection from $\mathbb{R}$ onto the interval $[u_1, u_2]$.

6 Numerical examples

6.1 Comparison of $(\mathcal{O}_C)$ and $(\mathcal{O}_R)$

Consider
Example 1:

$$\begin{aligned} y_d &= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 + 3, & -10 \le u(s) \le 10 \quad &\text{for all } s \in \Gamma, \\ \kappa &= 0, & -10 \le y(x) \le 10 \quad &\text{for all } y \in \Omega. \end{aligned}$$

The numerical solution of $(\mathcal{O}_C)$ for $N = 50$ is shown in Figure 1. The optimal control agrees on all four edges of Γ and is therefore shown only along one edge. This is the case for all the following examples too.

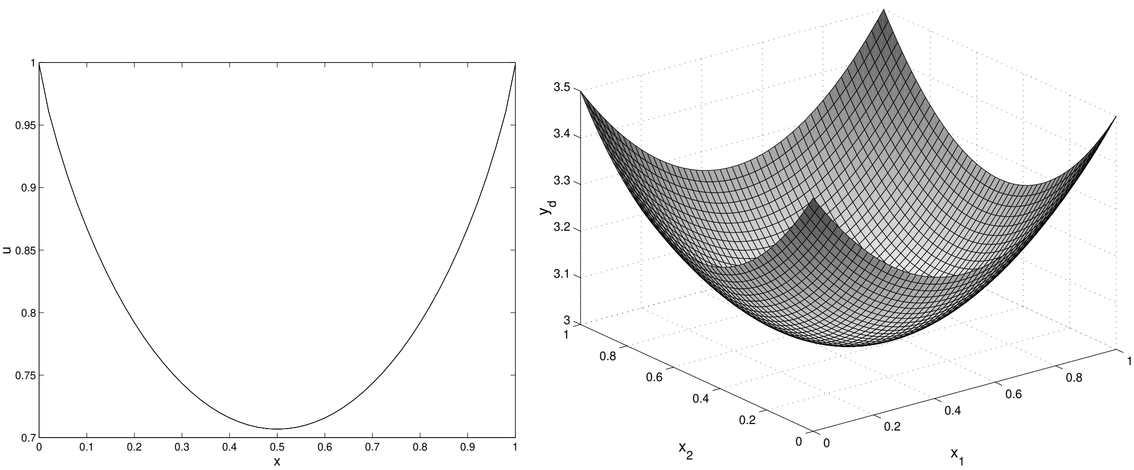

Figure 1: Optimal control and state for Example 1 with $N = 50$

In Table 1 the results are collected for $(\mathcal{O}_C)$ and $(\mathcal{O}_R)$ and for different mesh sizes.

Problem $(\mathcal{O}_C)$ was solved up to $N = 120$ with the same amount of memory and using Mosek, whereas $(\mathcal{O}_R)$ was solved only up to $N = 80$. Up to $N = 50$ the solutions of $(\mathcal{O}_C)$ and $(\mathcal{O}_R)$ agreed within a small tolerance. For $N > 50$ the numerical solution of $(\mathcal{O}_C)$ became unstable, see Figure 2, while the solution for $(\mathcal{O}_R)$ remained stable.

In the solution of Example 1 the constraints on the control are not active. Therefore from (28) it follows that the adjoint state has to be zero on Γ in this case. This is true as one can see in Figure 3. In Figure 4 the measures $\overline{\mu}_1^h$ and $\overline{\mu}_2^h$ are shown.

In the sequel we do not distinguish between the exact and the numerical solutions, that means we drop the 'h' in the notation. All solutions are calculated via $(\mathcal{O}_C)$ and $N = 50$.

Table 1: Results for different mesh sizes

N	$(\mathcal{O}_c)$ time in s	$(\mathcal{O}_c)$ $\lVert\overline{y}_h - y_d\rVert_{C(\overline{\Omega})}$	$(\mathcal{O}_R)$ time in s	$(\mathcal{O}_R)$ $\lVert\overline{y}_h - y_d\rVert_{C(\overline{\Omega})}$
10	1	0.0328879	1	0.0328879
20	2	0.0333261	9	0.0333261
30	7	0.0333989	48	0.0333989
40	20	0.0334234	164	0.0334234
50	46	0.0334345	424	0.0334345
60	121	0.0334406	1019	0.0334404
70	401	0.0334440	1994	0.0334440
80	405	0.0334470	4056	0.0334463
90	653	0.0334479		
100	1184	0.0334490		
110	1759	0.0334591		
120	3509	0.0334504		

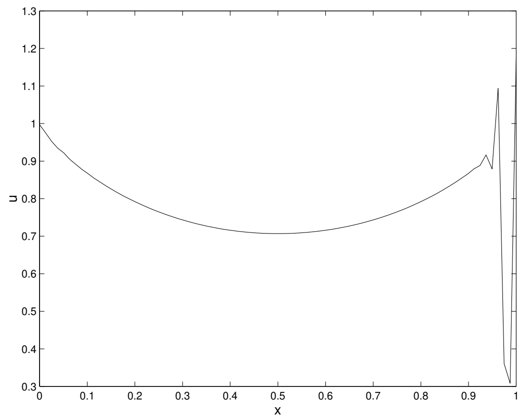

Figure 2: Optimal control for $(\mathcal{O}_C)$ and $N = 60$

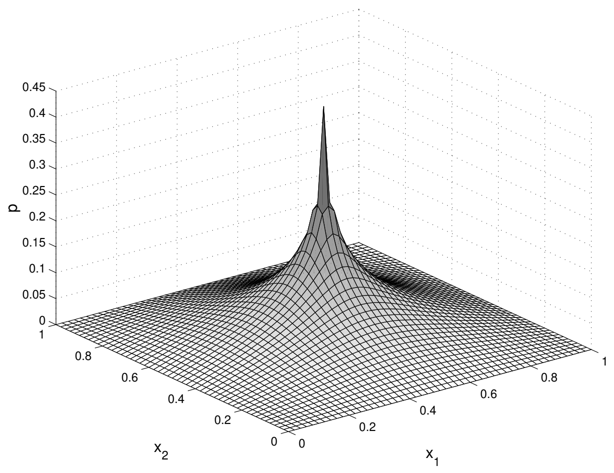

Figure 3: Adjoint state for Example 1

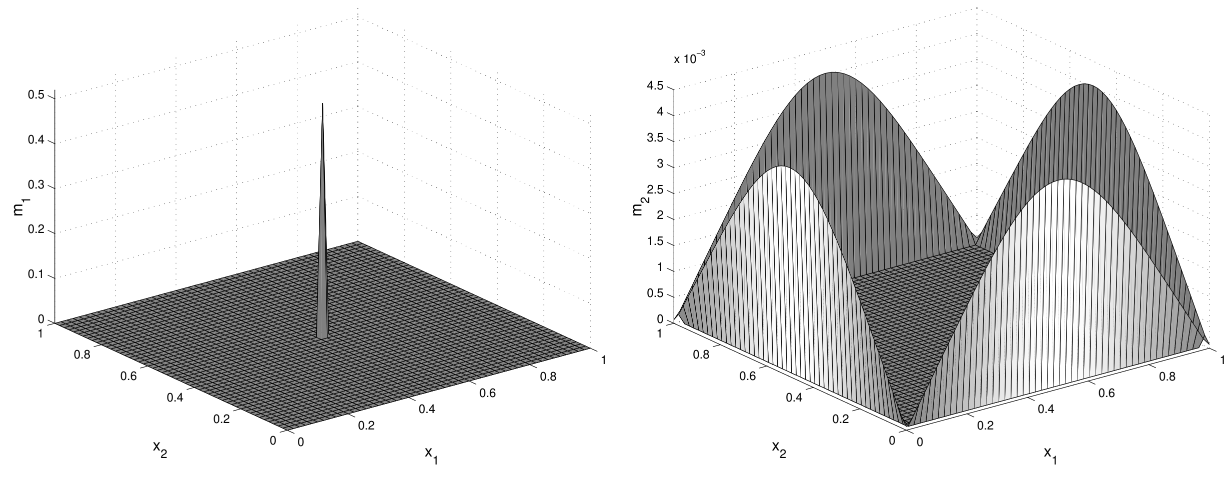

Figure 4: Measures $\overline{\mu}_1^h$ and $\overline{\mu}_2^h$ for Example 1

6.2 Illustration of Corollary 1

Example 2:

$$
\begin{aligned}
&y_d = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 + a, && -10 \le u(s) \le 10 \quad \text{for all } \ s \in \Gamma, \\
&\kappa = 0, && -10 \le y(x) \le 10 \quad \text{for all } \ y \in \Omega,
\end{aligned}
$$

with a real number a, $a \le 3.5$ or $a \ge 4$. The function y_d satisfies

$$
-\Delta y_d + y_d \begin{cases} \le 0 & \text{if } a \le 3.5 \\ \ge 0 & \text{if } a \ge 4, \end{cases}
$$

and therefore the assumptions of Corollary 1 hold. The numerical solutions fulfills the analytically predicted properties, especially (17). As examples we illustrate the functions $y_{\overline{u}}$, y_d, and $y_d \pm \overline{\delta}$ for $a = 3$ (resp. $a = 5$) in Figure 5 along $x_1 = 0.5$. The values of the functional are $J_d(\overline{u}, \overline{\delta}) = 0.0335$ (resp. $J_d(\overline{u}, \overline{\delta}) = 0.0389$).

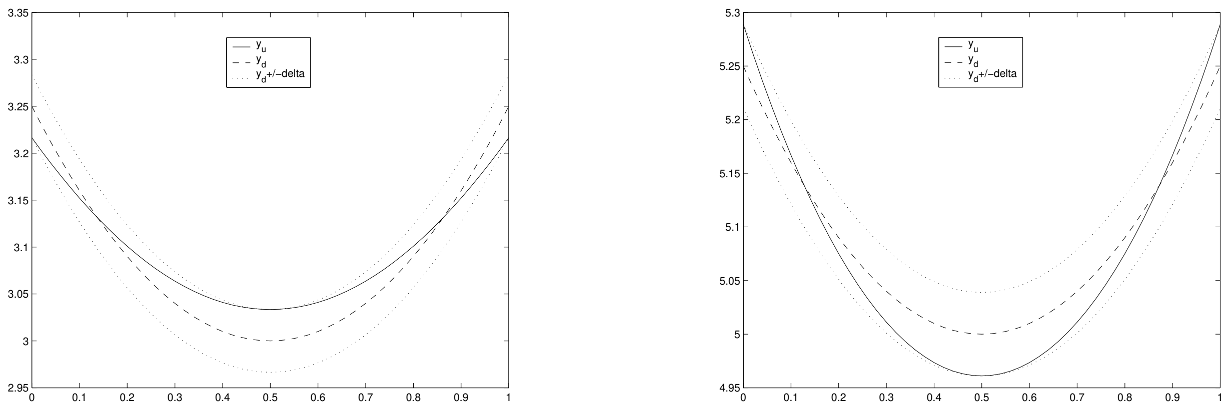

Figure 5: $y_{\overline{u}}$, y_d and $y_d \pm \overline{\delta}$ for Example 2 and $a = 3$ (resp. $a = 5$) along $x_1 = 0.5$

6.3 Further examples

Example 3:

$$
\begin{aligned}
&y_d = \sin 3\pi x_1 + \sin 3\pi x_2, && -1.5 \le u(s) \le 1 \quad \text{for all } \ s \in \Gamma, \\
&\kappa = 0.01, && -10 \le y(x) \le 10, \quad \text{for all } \ y \in \Omega.
\end{aligned}
$$

This example is to illustrate relationship (28). The function y_d is shown in Figure 6 and the numerical solution in Figure 7.

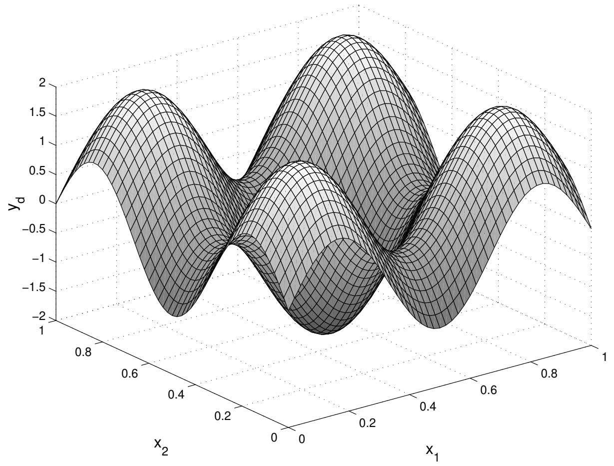

Figure 6: Function y_d for Example 3

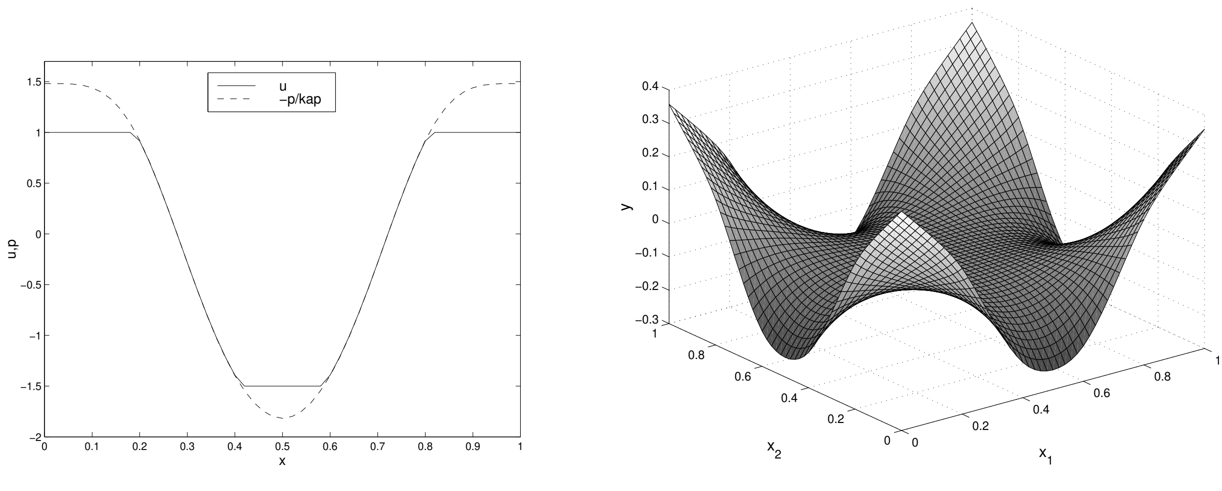

Figure 7: $\overline{u}$, $-\dfrac{\overline{p}}{\kappa}$, and $y_{\overline{u}}$ for Example 3

In addition, Figure 7 shows the function $-\overline{p}/\kappa$ on one edge of Γ. The adjoint state also agrees on all four edges of Γ. For the functional we get $J_d(\overline{u},\overline{\delta}) = 1.96$.

In order to demonstrate relationship (29), consider

Example 4:

$$y_d = \sin 3\pi x_1 + \sin 3\pi x_2, \qquad -5 \le u(s) \le 5 \quad \text{for all } s \in \Gamma,$$
$$\kappa = 0, \qquad -0.5 \le y(x) \le 0.5 \quad \text{for all } y \in \Omega.$$

In Figure 8 the optimal control together with the adjoint state on Γ and the corresponding state function are illustrated; $J_d(\overline{u},\delta) = 1.87$.

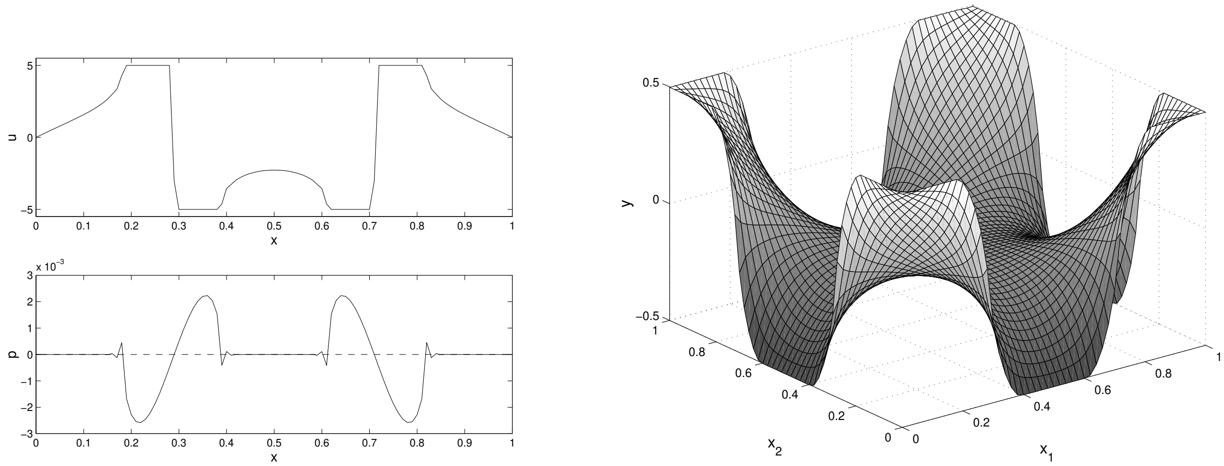

Figure 8: Optimal control, adjoint state, and state function for Example 4

6.4 Comparison with minimization of the L^2-norm

Example 5: Regard the desired state

$$y_d(x_1,x_2) = \begin{cases} 1 - 10 \cdot |(x_1,x_2) - (0.5,0.5)| & \text{if } |(x_1,x_2) - (0.5,0.5)| < 0.1 \\ 0 & \text{otherwise,} \end{cases}$$

which is shown in Figure 9, and

$$-5 \le u(s) \le 5 \quad \text{for all } s \in \Gamma. \tag{30}$$

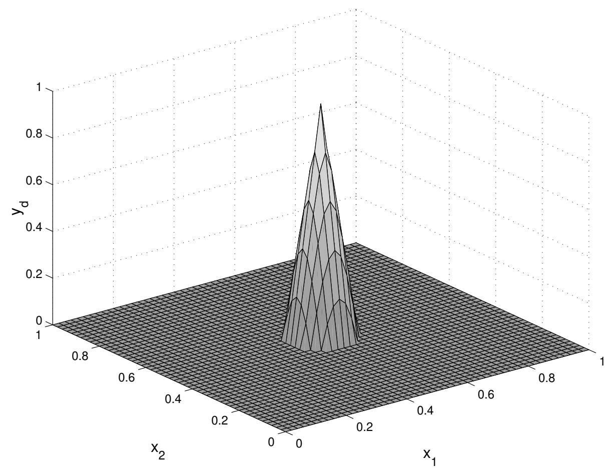

Figure 9: Function y_d for Example 5

For this example we want to compare the numerical solution $(y_{\overline{u}_C}, \overline{u}_C)$ of problem

$$(\mathcal{P}_C) \qquad \text{Minimize} \quad J_C(y,u) = \|y - y_d\|_{C(\overline{\Omega})}$$

subject to the state equation (1) and (30),

with the solution $(y_{\overline{u}_{L^2}}, \overline{u}_{L^2})$ of problem

$$(\mathcal{P}_{L^2}) \qquad \text{Minimize} \quad J_{L^2}(y,u) = \|y - y_d\|_{L^2(\overline{\Omega})}$$

subject to the state equation (1) and (30).

The results are shown in Figure 10 and Figure 11, the functional values are

$J_C(y_{\overline{u}_C}, \overline{u}_C) = 0.518,$

$J_C(y_{\overline{u}_{L^2}}, \overline{u}_{L^2}) = 0.989,$

$J_{L^2}(y_{\overline{u}_C}, \overline{u}_C) = 0.257,$

$J_{L^2}(y_{\overline{u}_{L^2}}, \overline{u}_{L^2}) = 0.00516.$

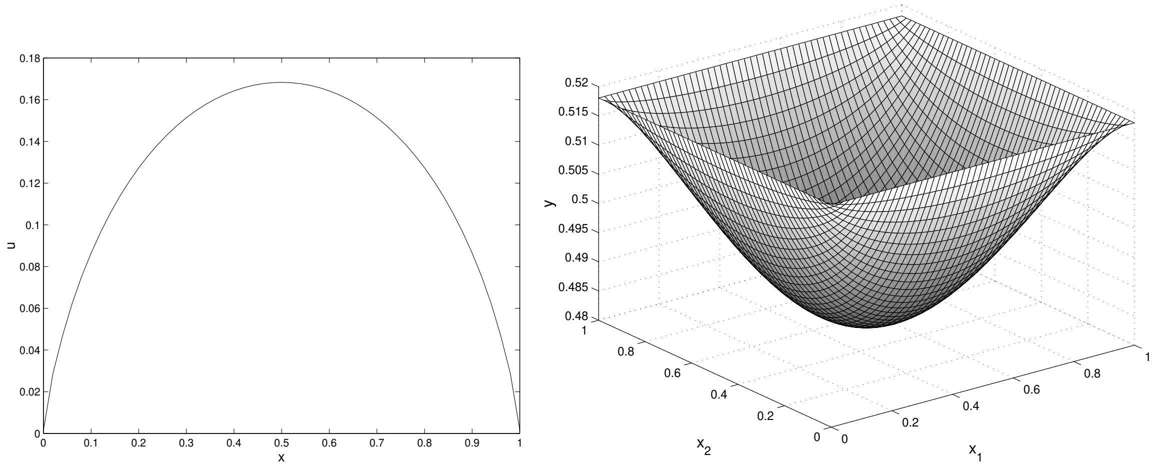

Figure 10: Optimal control $\overline{u}_C$ and state $y_{\overline{u}_C}$ for Example 5

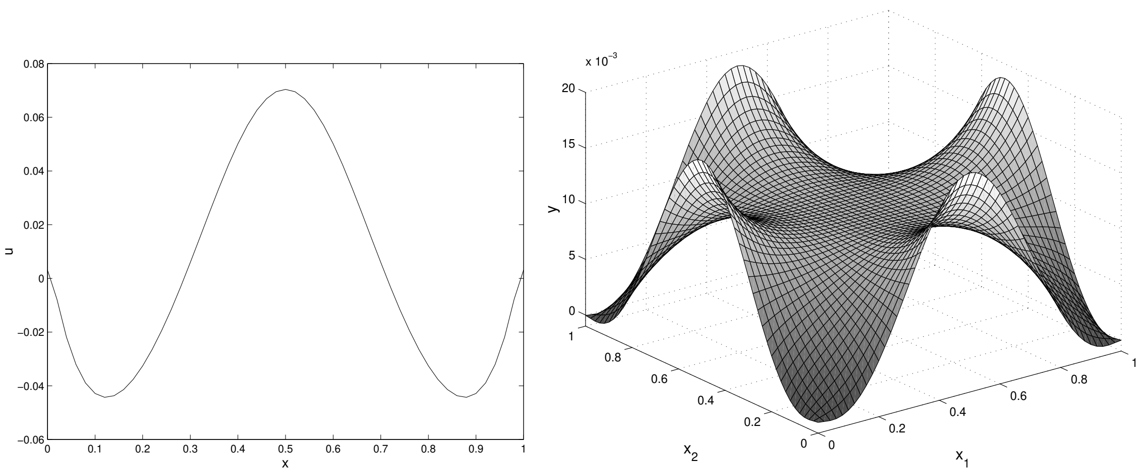

Figure 11: Optimal control $\overline{u}_{L^2}$ and state $y_{\overline{u}_{L^2}}$ for Example 5

A Discretization of control, state function, and state equation

Choose a discretization parameter $N \in \mathbb{N}$. Following Maurer and Mittelmann [10], we introduce index sets

$$I(\Omega) = \{(i,j) \in \mathbb{N}^2 : 1 \leq i,j \leq N-1\},$$
$$I(\overline{\Omega}) = \{(i,j) \in \mathbb{N}^2 : 0 \leq i,j \leq N\},$$
$$I(\Gamma) = I(\overline{\Omega}) - I(\Omega),$$

the mesh size $h = 1/N$, and grid points

$$x_{i,j} = (i \cdot h, j \cdot h), \quad (i,j) \in I(\overline{\Omega}).$$

Discretization of control functions u

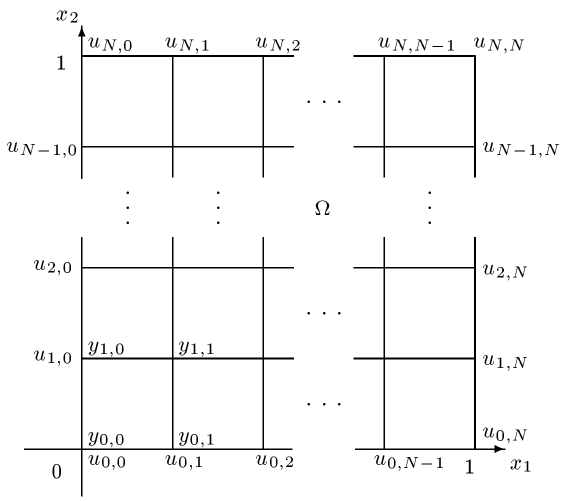

Figure 12: Discretization of u and y

The controls $u \in L^t(\Gamma)$ are approximated by continuous, piecewise linear functions $u_h : \Gamma \to \mathbb{R}$,

$$u_h = \sum_{(i,j)\in I(\Gamma)} u_{i,j} e^{i,j},$$

with real numbers $u_{i,j}$, $(i,j) \in I(\Gamma)$, see Figure 12, and continuous, piecewise linear functions $e^{i,j} : \Gamma \to \mathbb{R}$, satisfying

$$e^{i,j}(s) = \begin{cases} 1 & \text{if } s = x_{i,j} \\ 0 & \text{if } s = x_{k,l} \text{ for some } (k,l) \in I(\Gamma), (i,j) \neq (k,l). \end{cases}$$

Furthermore, each function $e^{i,j}$, $(i,j) \in I(\Gamma)$, is differentiable in all non-grid-points of Γ.

Discretization of state functions y

We approximate the state functions $y \in H^1(\Omega) \cap C(\overline{\Omega})$ by continuous functions $y_h : \overline{\Omega} \to \mathbb{R}$,

$$y_h = \sum_{(i,j) \in I(\overline{\Omega})} y_{i,j} \eta^{i,j},$$

with real numbers $y_{i,j}$, $(i,j) \in I(\overline{\Omega})$, and continuous functions $\eta^{i,j} : \overline{\Omega} \to \mathbb{R}$, satisfying

$$\eta^{i,j}(x) = \begin{cases} 1 & \text{if } x = x_{i,j} \\ 0 & \text{if } x = x_{k,l} \text{ for some } (k,l) \in I(\overline{\Omega}), (i,j) \neq (k,l). \end{cases}$$

Restricted to one of the squares of size h (see Figure 12), each function $\eta^{i,j}$ admits the form

$$\eta^{i,j}(x_1, x_2) = a x_1 x_2 + b x_1 + c x_2 + d,$$

with real numbers a, b, c, d.

Discretization of the constraints

These are the inequality constraints (3), (4), and (5). The discrete counterparts of these constraints are given by

$$\begin{aligned} y_{i,j} - \delta &\leq y_d(x_{i,j}) && (i,j) \in I(\overline{\Omega}), \\ -y_{i,j} - \delta &\leq -y_d(x_{i,j}) && (i,j) \in I(\overline{\Omega}), \\ u_{i,j} &\leq u_2 && (i,j) \in I(\Gamma), \\ -u_{i,j} &\leq -u_1 && (i,j) \in I(\Gamma), \\ y_{i,j} &\leq y_2 && (i,j) \in I(\overline{\Omega}), \\ -y_{i,j} &\leq -y_1 && (i,j) \in I(\overline{\Omega}). \end{aligned}$$

Discretization of the state equation

We use a finite element method. Let V_h be the real vector space $V_h = \text{span}\{\eta^{i,j}\}_{(i,j) \in I(\overline{\Omega})}$, where the $\eta^{i,j}$ have been defined above. Instead of solving (1) we solve the discretized variational equation

Find $y_h \in V_h$ such that

$$\int_\Omega (\nabla\varphi \cdot \nabla y_h + \varphi y_h)\, dx = \int_\Gamma \varphi u_h \, ds \tag{31}$$

holds for every $\varphi \in V_h$. Introducing the vectors

$$\mathbf{u} = (u_{0,0}, u_{0,1}, \ldots, u_{0,N}, \ldots, u_{N,N}, \ldots, u_{N,0}, \ldots, u_{1,0})^T,$$

$$\mathbf{y} = (y_{0,0}, y_{0,1}, \dots, y_{0,N}, y_{1,0}, \dots, y_{N,N})^T,$$

and evaluating equation (31) for $\varphi = \eta^{i,j}$, $(i,j) \in I(\overline{\Omega})$, we get a system of linear equations,

$$\mathbf{E}\mathbf{y} = \mathbf{B}\mathbf{u}. \tag{32}$$

Defining constants $c_1 = h^2/9 + 2/3,\ c_2 = h^2/36 - 1/3,\ c_3 = h^2/18 - 1/6$, this system reads as

$4c_1 y_{i,j} + 2c_3(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1}) + c_2(y_{i-1,j-1} + y_{i+1,j-1} + y_{i-1,j+1} + y_{i+1,j+1}) = 0 \qquad (i,j) \in I(\Omega)$

$$\begin{aligned}
&2c_1 y_{i,0} + c_3(y_{i-1,0} + y_{i+1,0} + 2y_{i,1}) + c_2(y_{i-1,1} + y_{i+1,1}) \\
&\quad = h/6(u_{i-1,0} + 4u_{i,0} + u_{i+1,0}) \qquad 1 \le i \le N-1 \\
&2c_1 y_{0,j} + c_3(y_{0,j-1} + y_{0,j+1} + 2y_{1,j}) + c_2(y_{1,j-1} + y_{1,j+1}) \\
&\quad = h/6(u_{0,j-1} + 4u_{0,j} + u_{0,j+1}) \qquad 1 \le j \le N-1 \\
&2c_1 y_{i,N} + c_3(y_{i-1,N} + y_{i+1,N} + 2y_{i,N-1}) + c_2(y_{i-1,N-1} + y_{i+1,N-1}) \\
&\quad = h/6(u_{i-1,N} + 4u_{i,N} + u_{i+1,N}) \qquad 1 \le i \le N-1 \\
&2c_1 y_{N,j} + c_3(y_{N,j-1} + y_{N,j+1} + 2y_{N-1,j}) + c_2(y_{N-1,j-1} + y_{N-1,j+1}) \\
&\quad = h/6(u_{N,j-1} + 4u_{N,j} + u_{N,j+1}) \qquad 1 \le j \le N-1
\end{aligned}$$

$$\begin{aligned}
&c_1 y_{0,0} + c_3(y_{1,0} + y_{0,1}) + c_2 y_{1,1} = h/6(u_{1,0} + 4u_{0,0} + u_{0,1}) \\
&c_1 y_{N,0} + c_3(y_{N,1} + y_{N-1,0}) + c_2 y_{N-1,1} = h/6(u_{N,1} + 4u_{N,0} + u_{N-1,0}) \\
&c_1 y_{0,N} + c_3(y_{1,N} + y_{0,N-1}) + c_2 y_{1,N-1} = h/6(u_{1,N} + 4u_{0,N} + u_{0,N-1}) \\
&c_1 y_{N,N} + c_3(y_{N-1,N} + y_{N,N-1}) + c_2 y_{N-1,N-1} = h/6(u_{N-1,N} + 4u_{N,N} + u_{N,N-1}).
\end{aligned}$$

The matrices $\mathbf{E}$ and $\mathbf{B}$ are not written down explicitly but can be extracted.

Discretization of the functional

Evaluating the integral expression in the functional of $(\mathcal{P}_d)$ at $u = u_h$, we get $\int_\Gamma u_h^2 \, ds = \mathbf{u}^T \mathbf{A} \mathbf{u}$, with the matrix $\mathbf{A}$,

$$\mathbf{A} = \frac{h}{6} \begin{pmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 1 & & & 1 & 4 \end{pmatrix}.$$

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