Numerical study of eigenvector statistics for random banded matrices

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Abstract

The statistics of eigenvector amplitudes near the band center in random banded matrix ensembles is studied numerically. The nonlinear σ model provides a rigorous description of the statistics in these ensembles. We are interested in the extension of the predictions of the σ model approach to complex quantum systems. We study the validity range of the perturbation theory beginning from the well-known formulas in random matrix theory.

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1. Introduction

Disordered quantum systems exhibit irregular fluctuations of eigenfunctions. These determine the fluctuations of the conductance through quantum dots and wires. They can be studied by looking at the statistics of the local amplitudes of the eigenfunctions [1].

In the metallic regime, characterized by a large conductance, the eigenfunctions spread uniformly throughout the whole system. Their statistics is described by random matrix theory (RMT) [2]. RMT deals with ensembles of Hermitian matrices, corresponding to physical operators, whose matrix elements are randomly distributed. In Dyson's ensembles of random matrices the matrix elements are Gaussian distributed. These ensembles are mathematically well-understood, and expressions for the statistical distribution functions can be derived exactly [1].

Random banded matrix (RBM) ensembles [3] can be regarded as generalization of the Dyson ensembles in RMT. In the former, the matrix elements are still Gaussian distributed, but their variance decays outside a certain range from the main diagonal. These matrix ensembles can be said to model systems which are classically strongly chaotic but subject to quantum localization [4]. The kicked rotator is an example for such systems [3].

Due to localization [4] the statistics of eigenfunctions in RBM ensembles show deviations from RMT, depending on how much they differ from the Dyson ensembles. Here, the supersymmetric σ model approach can be applied [5]. Particularly, in the weakly localized regime, where the matrices are almost full, the deviations are small, and can be treated within perturbation theory.

In this paper, we present numerical results for the eigenfunction statistics in RBM ensembles, and compare these to the expressions obtained for the σ model. We are particularly interested in the extension of the σ model approach to complex quantum systems, which can be no longer labelled as classically chaotic, but are still known to follow the predictions of RMT in the metallic regime. A well-known example is the Anderson model of localization [6], for which we shall present numerical results elsewhere [7].

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2. Description of the model

In this paper, we consider the simplest version of the RBM ensembles [3], where the matrix elements H_{ij} of the Hamiltonians are taken randomly, if $|i-j| \leq B$, where B is the band width. Other matrix elements are zero. In the ensemble of orthogonal matrices all the nonzero nondiagonal matrix elements are chosen to be real and Gaussian distributed with variance 1 and mean 0, whereas in the ensemble of unitary matrices these elements additionally have equally distributed imaginary parts. The diagonal elements in both orthogonal and unitary ensembles are real and Gaussian distributed with variance 2 and mean 0.

Physically, the RBM ensembles can be related to one-dimensional (1D) disordered systems with long-range hopping, or, alternatively, to quasi-1D systems with B associated to the number of transverse channels for electron propagation. The orthogonal ensemble describes systems, which are time-reversal symmetric, whereas in the unitary ensemble this symmetry is broken. They correspond to the Dyson ensembles in RMT, if B equals the rank N of the matrices.

The RBM ensembles can be reduced to the nonlinear σ model, if $B \gg 1$ [5]. The main properties of spectra and eigenfunctions in RBM ensembles have been shown to depend only on the scaling parameter $\lambda = B^2/N$ [3,8]. The strength of localization of the eigenstates can be tuned by adjusting λ. For $\lambda \ll 1$ the states are highly localized and for $\lambda \gg 1$ they are delocalized. Here, we are mainly interested in the delocalized regime, where the perturbation theory provides formulas which can be generalized to other disordered quantum systems [9].

3. Eigenfunction statistics

Let E_α and $\psi_\alpha = (\psi_\alpha^1, \ldots \psi_\alpha^N)$ be the eigenvalues and eigenvectors of a Hamiltonian, belonging to an orthogonal or unitary ensemble described above. In this paper, we look at the statistics of the eigenvector amplitudes $\{N|\psi_\alpha^i|^2\,;\, i = 1, \ldots, N\}$, weighted by the dimension N of the eigenvectors. We define [10]

$$f(t) = \frac{\Delta}{N} \sum_\alpha \sum_i \delta[t - |\psi_\alpha^i|^2 N]\, \delta(E - E_\alpha)\,. \quad (1)$$

Here Δ is the mean level spacing and E is the energy. In the delocalized regime, where B is large and the predictions of RMT are valid, $f(t)$ is given by [1]

$$f^{(\mathrm{O})}_{\mathrm{RMT}}(t) = \frac{1}{\sqrt{2\pi t}} \exp(-t/2), \quad (2)$$

$$f^{(\mathrm{U})}_{\mathrm{RMT}}(t) = \exp(-t)\,. \quad (3)$$

The superscripts O and U refer to Dyson's orthogonal and unitary ensembles of random matrices. Eq. (2) is usually referred to as the Porter-Thomas distribution [11].

Small deviations from Eqs. 2 and 3 due to localization can be written as perturbation expansions [9]

$$f^{(\mathrm{O})}(t) \simeq f^{(\mathrm{O})}_{\mathrm{RMT}}(t) \left[1 + a_d\,(3 - 6t + t^2)/2\right], \quad (4)$$

$$f^{(\mathrm{U})}(t) \simeq f^{(\mathrm{U})}_{\mathrm{RMT}}(t) \left[1 + a_d\,(2 - 4t + t^2)\right]. \quad (5)$$

These expansions are valid for small amplitudes $t < a_d^{-1/2}$. They are parametrized by

$$a_d = \frac{\Delta}{2\pi} \sum_{\mathbf{q}} \frac{1}{D\mathbf{q}^2}. \quad (6)$$

which is a sum over the eigenmodes of the diffusion propagator in the system. D is the diffusion constant.

As indicated, a_d generally depends on the dimension d of the system. It also depends on the boundary conditions. The random banded Hamiltonians correspond to the quasi-1D case with open boundary conditions, where $a_1 = X/6$ [9]. The parameter $X = (2/\beta)L/\xi$, where $\beta = 1$ in the orthogonal ensemble, and $\beta = 2$ in the unitary one. L is the length of the quasi-one dimensional wire, and ξ is the localization length. Without direct connection to a physical system, L and ξ are not well defined in the RBM ensembles. Therefore we only consider the ratio L/ξ here.

A non-perturbative solution for $f(t)$ in the nonlinear σ model was given in Ref. [12]. The formulas in the orthogonal and unitary ensemble consist of rather long integral and differential expressions, parametrized by X. Because of the lack of space these formulas are not repeated here. They are valid only in the quasi-1D case, and cannot be generalized to higher dimensional systems, in contrary to the perturbational expressions (4) and (5) [12].

4. Numerical results

We studied ensembles of RBMs with rank $N = 1000$. The distributions of eigenvector amplitudes are in good agreement with Eqs. (4) and (5) for large B, as can be seen in Fig. 1. Here, we plot the function $\delta f(t) = f(t)/f_{\mathrm{RMT}}(t) - 1$ to see the deviations from RMT. For $B < 70$ or $B^2/N < 5$ the agreement rapidly gets poorer, indicating that the first order perturbation theory no longer applies.

The values obtained for the parameter X in the orthogonal and unitary ensembles for different values of B are shown in Fig. 2. The values are the same in the orthogonal and unitary ensembles, and they are inversely proportional to B^2. This fact lets us extrapolate values of X even for small B.

The formulas from Ref. [12], compared to our results, are shown in Fig. 3. We get a very good agreement for all the values of B we have used. It is somewhat surprising that the theory applies even for rather small B. For large B, the agreement is less good for tails, i.e. for large t. This disagreement may point towards a fallacy of the formulas in that regime. The tails are, however, also numerically difficult to compute, because they represent rarely occurring high amplitudes of weakly localized states.

5. Conclusions

We have analyzed the statistics of eigenvector amplitudes in RBM ensembles and their deviations from RMT as a function of the band width B. We found good agreement with the perturbational expressions, based on the nonlinear σ model for large B. Our numerical results can be fitted by the expressions derived in Ref. [12],

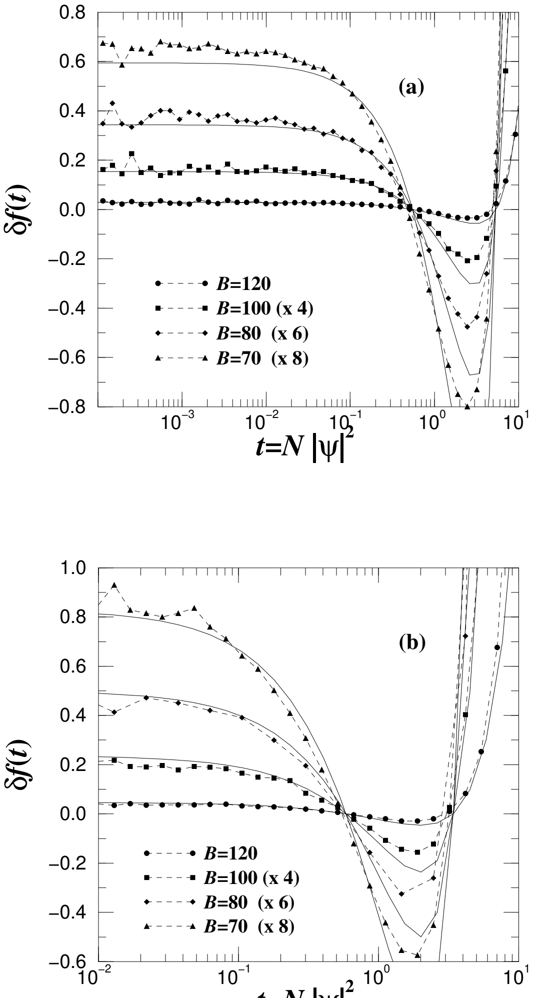

Figure 1. Deviations of $f(t)$ from RMT for the (a) orthogonal and (b) unitary RBMs of order $N = 1000$ (symbols). The averages are taken over the energy interval $[-0.1, 0.1]$ and 1000 RBM realizations. The roughness of the curves show the degree of inaccuracy due to the finite number of realizations. The solid lines show the first order correction term in Eqs. (4) and (5), respectively. The values of a_1 for different band widths B have been fitted to get the best agreement for small t. The curves have been multiplied by a constant for clarity as indicated.

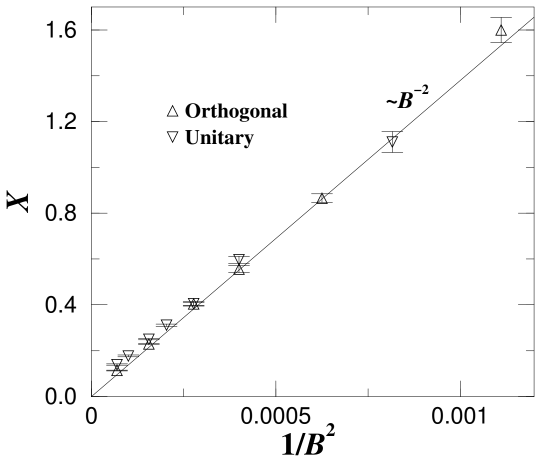

Figure 2. Parameter $X = (2/\beta)L/\xi$ for RBMs in the orthogonal and unitary ensemble. The errorbars show the standard errors of the average over RBM realizations.

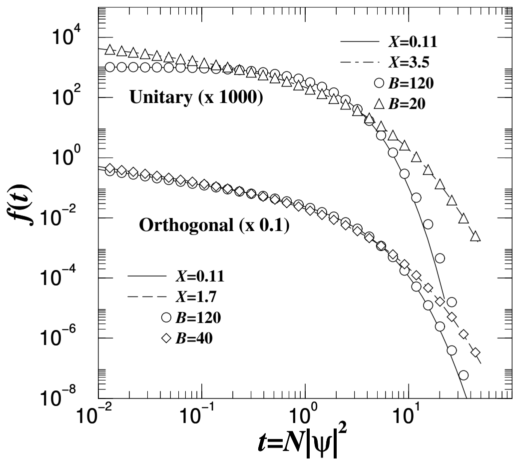

Figure 3. Distribution of eigenvector amplitudes for the orthogonal and unitary RBM ensembles with different values of the band width B (symbols). The lines show predictions of Ref. [12]. The curves have been multiplied by a constant for clarity as indicated.

which are exact in the quasi-1D case, but numerically somewhat difficult to use. The agreement is less good in the tails of distribution functions, which may indicate the failure of the σ model approach in this regime. The tails are of specific interest in two- and three-dimensional disordered systems, because there are contradictory predictions for them in the literature, based on the σ model [13] or direct optimal fluctuations [14] approaches. Our numerical studies for such systems are in progress.

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