Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Helmut Harbrecht Reinhold Schneider

Wavelet Galerkin Schemes for 2D-BEM

Preprint SFB393/00-20

Preprint-Reihe des Chemnitzer SFB 393

SFB393/00-20 **April 2000**

Contents

This paper is concerned with the implementation of the wavelet Galerkin scheme for the Laplacian in two dimensions. We utilize biorthogonal wavelets constructed by A. Cohen, I. Daubechies and J.-C. Feauveau in [3] for the discretization leading to quasi-sparse system matrices which can be compressed without loss of accuracy. We develop algorithms for the computation of the compressed system matrices whose complexity is optimal, i.e., the complexity for assembling the system matrices in the wavelet basis is $\mathcal{O}(N_J)$, where N_J denotes the number of unknowns.

1. Introduction

Various problems in science and engineering can be formulated by integral equations. One of the most prominent type of those integral equations are boundary integral equations. Usually, these equations are solved numerically by the boundary element method (BEM). The boundary element method has been considered as an appropriate tool to solve certain boundary value problems. For example, the boundary element method is a favourable approach for the treatment of exterior boundary value problems. Nevertheless, traditional discretizations of integral equations suffer from a major disadvantage. The corresponding system matrices are densely populated. Therefore, the complexity for solving such equations is at least $\mathcal{O}(N_J^2)$, where N_J denotes the number of equations. This fact restricts the maximal size of the linear equations seriously.

Modern methods for the fast solution of BEM reduce the complexity to a suboptimal rate, i.e., $\mathcal{O}(N_J \log^\alpha N_J)$, or even an optimal rate, i.e., $\mathcal{O}(N_J)$. Prominent examples for such methods are the *fast multipole method* [19], the *panel clustering* [22] or *hierarchical matrices* [21, 36]. As introduced by [1] wavelet bases offer another tool for the fast solution of integral equations. In fact, a Galerkin discretization based on wavelet bases results in numerically sparse matrices, i.e., many matrix entries are negligible and can be treated as zero. Discarding these nonrelevant matrix entries is called matrix compression. Therefore, the full matrix is replaced by a sparse matrix. Of course, the compression procedure induces a perturbation of the original Galerkin discretization. Consequently, the resulting solution differs from the solution of the uncompressed scheme. In [1] this error has been estimated in L_2. It has been shown that, for any $\varepsilon > 0$, a sparse matrix exists such that the compression introduces an error $\leq \varepsilon$.

The article [1] has initiated the investigation of wavelet methods for integral equations, pseudodifferential equations and boundary integral equations [8, 9, 10]. These papers written by Siegfried Prößdorf and coauthors considered also operators of nonzero order by an appropriate preconditioning. Based on norm equivalences [7, 17, 27] it has been shown that for strongly elliptic operators after a diagonal preconditioning the wavelet Galerkin matrices are well conditioned. A new strategy has been introduced to retain the convergence behaviour of the corresponding Galerkin scheme without compromising the complexity of the compression. These early results have been improved by several authors [11, 12, 35] and [30, 31, 33, 34]. Concerning boundary integral equations a strong effort has been spent on the construction of appropriate wavelet bases on surfaces [6, 13, 14, 30, 35]. Furthermore, the efficient computation of the relevant matrix coefficients turned out to be an important task for the successful application of the wavelet Galerkin method [12, 32, 33, 35].

The purpose of the present paper is to describe a fully discrete wavelet Galerkin scheme for boundary integral equations in 2D since this has been omitted in the previous papers. Although the three dimensional boundary value problems are of higher interest, the development and practical realization of 2D wavelet Galerkin methods is of importance by its own. Two dimensional or axial symmetric boundary value problems play an important role in practical applications. Wavelets offer a highly accurate tool to solve these equations. This fact is mainly retained in the presence of piecewise smooth boundaries. In particular, the combination of the finite element method with the boundary element method, or equivalently, the exact use of artificial boundary conditions, is a practically and highly interesting approach for which wavelets seems to be advantageous [24, 25]. For all kind of those problems, the two dimensional case is an excellent object for numerical studies and experiments. One reason is that, in connection with wavelet methods, on curves more scales can be realized. Therefore, one can achieve a relatively high accuracy of the discretization. This allows the study of the asymptotic behaviour of the solution together with the complexity of the algorithm. Furthermore, the implementation is much easier than it is in the three dimensional case. For this reason, testing or modifying algorithms and ideas can be realized quickly. Further developments and improvements of higher dimensional methods will benefit from the experiences in two dimensions. Of course, there are particularities in the two dimensional case which have no counterpart in three dimensions. A proper realization has also to exploit the special properties of the two dimensional case. One purpose of the present paper is to focus on these particular properties, whereas for the theoretical foundations we refer to [5, 11, 12, 24, 31, 35].

The present paper is organized as follows. As typical examples for boundary integral equations, we consider in section 2. the indirect formulations for the Dirichlet problem. Then, only a single function appears on the right hand side of the integral equation. We employ all kind of integral operators resulting from second order boundary value problems and derive Fredholm integral equations of the first kind and of the second kind. For the sake of completeness we mention also Neumann problems. In this respect, we indicate the treatment of the hypersingular operator. It is worth to mention that the present approach can also be applied to direct formulations, see e.g. [24, 25] for more details. For the sake of brevity, in the present paper we treat only the Laplacian explicitly. For further literature on boundary integral equations we refer for example to [4, 20, 28]. The present variational formulations are chosen such that only globally smooth kernels must be handled by numerical integration while the singular parts can be computed analytically. This trick simplifies and accelerates the matrix generation.

In section 3. we define the multiresolution analysis and the wavelet bases on curves through a regular parametrization. Since the coarse scale bases are defined on rather coarse grids, we need a global representation of the boundary curve. From this perspective it becomes obligatory that a wavelet Galerkin scheme has to be a fully discrete one [12, 32, 35], i.e., the computation of the relevant matrix entries requires numerical integration. There are two approximating steps: The first one is the matrix compression, the other one results from numerical integration.

The proposed matrix compression strategy has been developed in [8, 12, 31, 35]. In order to avoid logarithmic terms in complexity a second compression step, introduced in [35], is applied to the matrix entries corresponding to wavelets with overlapping supports. Moreover, we briefly recall the wavelet preconditioning.

We explain in section 4. concretely how to establish the matrix pattern of the compressed matrix and how to compute the required matrix entries. It turns out that a naive calculation of the required entries leads to a repeated computation of integrals. To avoid this multiple computation an improved strategy is proposed. We mention that the described algorithms are also practicable in higher dimensions.

Section 5. is devoted to the analysis of the numerical integration. Similarly to the matrix compression, the perturbation resulting from the numerical integration is studied in [12, 32, 35]. It turns out that one has to achieve a certain accuracy $\varepsilon_{j,j'}$ which depends strongly on the corresponding scales (j,j'). In order to achieve the desired accuracies, we utilize exponentially convergent quadratures rules [12, 32, 35], e.g. Gauß-Legendre formulas. For this reason we require (piecewise) analytic curves and parametric representations. The present quadrature algorithm differs from those proposed in [12, 32, 35] since global analyticity of the underlying kernels is exploited. Moreover, we describe a strategy improving the stability and accuracy of the quadrature while the efficiency is not compromised. We want to remark, that this strategy can also be used in 3D, but the present analysis of the convergence and the complexity is valid only for two dimensional problems.

At the end of the paper, in section 6., we present various numerical experiments. These experiments confirm the theoretical results quite well. The accuracy of the Galerkin scheme has never been deteriorated by the matrix compression. In fact, in many cases the solution of the compressed scheme is even slightly better than the solution of the uncompressed scheme. Though it has not been considered in the present paper, the fast wavelet Galerkin scheme can be applied to nonsmooth boundaries as well.

Finally, let us remark, that we have left open several problems. We have not fully exploited the operator splitting $A = A_{\text{sing}} + A_\infty$ into the operator A_{sing} with a singular kernel and the operator A_∞ with an analytical kernel. In fact, A_∞ can be compressed in a more appropriate way, cf. [29]. Whereas, this strategy fails for nonsmooth boundaries, e.g. on boundaries with corners. Perhaps, the treatment of nonsmooth boundaries requires a more careful analysis combined with some minor modifications. The development will be deferred to a forthcoming paper. However, this would be performed in conjunction with an adaptive strategy [2]. Throughout the present paper we only consider asymptotical behaviour. There are situations where this might be far from realistic computations. For instance, if highly oscillating solutions occur, like for the Helmholtz equation with a large Helmholtz number, or homogenization phenomena, the realization of asymptotic convergence and high accuracy is far beyond hardware facilities.

Incorporating the adaptive approximation of the solution or the multiscale approximation of the geometry fit completely into the multiscale concept [2, 16]. From this perspective the fast wavelet method offers a highly powerful tool for the numerical solution of two dimensional problems. However, for three dimensional problems the situation is still quite different. Due to geometric and topological subtleties, the realization of the present concept of wavelet Galerkin schemes is much more difficult in 3D. A strong effort has been spent into this direction during the past five years. Nevertheless, at the present stage of development, the methods based on a hp-approximation of the potentials, like the fast multipole method or the panel clustering, are more flexible and robust to handle complex three dimensional geometries.

Throughout this paper $a \lesssim b$ expresses that a can be bounded by a constant multiple of b uniformly in any parameter on which a and b may depend. Likewise $a \sim b$ means that $a \lesssim b$ and $a \gtrsim b$.

2. The boundary element method

2.1. Dirichlet problems

Let $\Omega^- \in \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary $\Gamma := \partial\Omega^-$. We set $\Omega^+ = \mathbb{R} \setminus \Omega^-$ and choose $\Omega = \Omega^+$ or $\Omega = \Omega^-$. We denote by $L^2(\Gamma)$ the function space of all square integrable functions on Γ with respect to the canonical inner product

$$(u, v)_{L^2(\Gamma)} = \int_\Gamma u(x)v(x)ds_x \tag{2.1}$$

and by $H^q(\Gamma)$ ($q \in \mathbb{R}$) the corresponding Sobolev spaces. Moreover, $L^2(\Omega)$ indicates the function space of all square integrable functions on Ω with respect to the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx.$$

For $q \in \mathbb{R}$ the space $H^q(\Omega)$ denotes the corresponding Sobolev space.

For a given $f \in H^{1/2}(\Gamma)$ we consider a Dirichlet problem, i.e., we seek $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\triangle u &= 0 \qquad \text{in } \Omega, \\ u &= f \qquad \text{on } \Gamma. \end{aligned} \tag{2.2}$$

For $\Omega = \Omega^-$ the problem is called interior Dirichlet problem. On the other hand, for $\Omega = \Omega^+$ we obtain an exterior Dirichlet problem. In the latter case one additionally demands that

$$u(x) = \mathcal{O}(1) \quad \text{as } |x| \to \infty$$

uniformly for all directions $x/|x|$.

1. Fredholm's integral equation of the first kind for Dirichlet problems: For solving an interior or exterior Dirichlet problem (2.2) by a Fredholm's integral equation of the first kind we introduce the *single layer operator* $\mathcal{V}$

$$(\mathcal{V}\rho)(x) := \int_\Gamma E(x,y)\rho(y)ds_y, \quad x \in \Gamma, \tag{2.3}$$

where the fundamental solution $E(x,y)$ is given by

$$E(x,y) = -\frac{1}{2\pi}\log|x-y|. \tag{2.4}$$

Then, one finds the boundary integral equation

$$\mathcal{V}\rho = f \quad \text{on } \Gamma \tag{2.5}$$

for the unknown density ρ. Knowing ρ the solution u of the Dirichlet problem (2.2) is given by

$$u(x) = \int_\Gamma E(x,y)\rho(y)ds_y, \quad x \in \Omega.$$

In the context of the boundary integral equation (2.5) the single layer operator $\mathcal{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defines an operator of order -1, which is symmetric and positive definite if $\operatorname{diam}\Omega^- < 1$, cf. [26].

2. Fredholm's integral equation of the second kind for Dirichlet problems: For solving the Dirichlet problem (2.2) by a Fredholm's integral equation of the second kind we define the *double layer operator*

$$(\mathcal{K}\rho)(x) := \int_\Gamma \frac{\partial}{\partial n_y}E(x,y)\rho(y)ds_y, \quad x \in \Gamma, \tag{2.6}$$

with $E(x,y)$ from (2.4). Note that, here and in the sequel, n_y denotes the unit normal at $y \in \Gamma$ which is oriented to the outside of Ω^-. One finds the equation

$$\big(\mathcal{K} \pm \tfrac{1}{2}I\big)\rho = f \quad \text{on } \Gamma, \tag{2.7}$$

where one chooses "+" for an exterior and "−" for an interior problem. In both cases the solution u is represented by

$$u(x) = \int_\Gamma \frac{\partial}{\partial n_y}E(x,y)\rho(y)ds_y, \quad x \in \Omega.$$

The operator on the left hand side of (2.7) defines an operator of order zero, i.e., $\mathcal{K} \pm \frac{1}{2}I : L^2(\Gamma) \to L^2(\Gamma)$. Let us remark that sometimes it is convenient to consider $\mathcal{K} \pm \frac{1}{2}I : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, see e.g. [24]. The equation for the interior problem is uniquely solvable while the equation for the exterior problem has no unique solution since $-\frac{1}{2}$ is an eigenvalue of $\mathcal{K}$. Hence, to ensure uniqueness of the solution ρ in the case of the exterior problem we have to suppose that $\int_\Gamma f(x)ds_x = 0$ and $\int_\Gamma \rho(x)ds_x = 0$.

It is well known [20, 28], that these formulations yield a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem (2.2). Since the solution u is not obtained directly by the boundary integral equations (2.5) and (2.7), respectively, these formulations are called *indirect methods*.

2.2. Variational formulations

The smooth boundary Γ can be parametrized by a 1-periodic function $\gamma : [0,1] \to \Gamma$ such that

$$\alpha(t) := |\gamma'(t)| > 0 \tag{2.8}$$

for all $t \in [0,1]$. Indeed, we suppose that γ is analytic in $[0,1]$. In addition to the spaces $L^2(\Gamma)$ and $H^q(\Gamma)$ we introduce 1-periodic spaces $L^2(0,1)$ and $H^q(0,1)$, respectively. Precisely, let $L^2(0,1)$ be the space of all 1-periodic square integrable functions. Its inner product is denoted by

$$(v,w)_{L^2(0,1)} := \int_0^1 v(t)w(t)dt. \tag{2.9}$$

Then, for any real number q the 1-periodic Sobolev space $H^q(0,1)$ is defined as the closure with respect to the norm

$$\|v\|^2_{H^q(0,1)} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2q} |\hat{v}(n)|^2$$

of the space of all 1-periodic C^∞-functions. Here, $\hat{v}(n)$ indicate the Fourier coefficients

$$\hat{v}(n) = \int_0^1 e^{-2\pi i n s} v(s)ds, \quad n \in \mathbb{Z}.$$

Invoking the definition (2.8) the comparison of (2.1) and (2.9) implies

$$(v,w)_{L^2(\Gamma)} = \big(v\circ\gamma, (w\circ\gamma)\alpha\big)_{L^2(0,1)} \tag{2.10}$$

for all $v \in H^q(\Gamma)$ and $w \in H^{-q}(\Gamma)$.

1. Variational formulation for the equation of the first kind: The variational formulation of (2.5) in the $L^2(\Gamma)$ inner product (2.1) is given by

$$\text{seek } \rho \in H^{-1/2}(\Gamma): \quad (\mathcal{V}\rho, \delta)_{L^2(\Gamma)} = (f, \delta)_{L^2(\Gamma)} \quad \forall\ \delta \in H^{-1/2}(\Gamma).$$

We define the integral operator

$$V : H^{-1/2}(0,1) \to H^{1/2}(0,1), \quad (V\mu)(s) := \int_0^1 E\big(\gamma(s),\gamma(t)\big)\mu(t)dt. \tag{2.11}$$

Using (2.10) we find for $\rho, \delta \in H^{-1/2}(\Gamma)$ the identity $(\mathcal{V}\rho, \delta)_{L^2(\Gamma)} = \big(V\big((\rho \circ \gamma)\alpha\big), (\delta \circ \gamma)\alpha\big)_{L^2(0,1)}$. Thereby, V inherits symmetry and positive definiteness from the operator $\mathcal{V}$. Setting $\mu := (\rho \circ \gamma)\alpha \in H^{-1/2}(0,1)$ and $\nu := (\delta \circ \gamma)\alpha \in H^{-1/2}(0,1)$ leads to the variational formulation in $L^2(0,1)$

$$\begin{aligned} &\text{seek } \mu \in H^{-1/2}(0,1) : \\ &\qquad (V\mu, \nu)_{L^2(0,1)} = (f \circ \gamma, \nu)_{L^2(0,1)} \quad \forall\, \nu \in H^{-1/2}(0,1). \end{aligned} \tag{2.12}$$

Knowing the density μ the solution u is obtained from

$$u(x) = \int_0^1 E\big(x, \gamma(t)\big)\mu(t)dt, \qquad x \in \Omega. \tag{2.13}$$

2. Variational formulation for the equation of the second kind: The variational formulation of (2.7) in $L^2(\Gamma)$ is given by

$$\text{seek } \rho \in L^2(\Gamma) : \quad (\mathcal{K}\rho, \delta)_{L^2(\Gamma)} \pm \frac{1}{2}(\rho, \delta)_{L^2(\Gamma)} = (f, \delta)_{L^2(\Gamma)} \quad \forall\, \delta \in L^2(\Gamma).$$

For $t \in [0,1]$ it is convenient to abbreviate here and in the sequel $n_t := n_{\gamma(t)}$. We introduce

$$K : L^2(0,1) \to L^2(0,1), \quad (K\mu)(s) := \int_0^1 \frac{\partial}{\partial n_t} E\big(\gamma(s), \gamma(t)\big)\mu(t)\alpha(t)dt, \tag{2.14}$$

with $E(x,y)$ from (2.4). Comparing the definitions of $\mathcal{K}$ (2.6) and K (2.14) one finds for $\rho, \delta \in L^2(\Gamma)$ the identity $(\mathcal{K}\rho, \delta)_{L^2(\Gamma)} = \big(K(\rho \circ \gamma), (\delta \circ \gamma)\alpha\big)_{L^2(0,1)}$. The substitutions $\mu := \rho \circ \gamma \in L^2(0,1)$ and $\nu := (\delta \circ \gamma)\alpha \in L^2(0,1)$ yield the formulation

$$\begin{aligned} &\text{seek } \mu \in L^2(0,1) : \\ &\qquad (K\mu, \nu)_{L^2(0,1)} \pm \frac{1}{2}(\mu, \nu)_{L^2(0,1)} = (f \circ \gamma, \nu)_{L^2(0,1)} \quad \forall\, \nu \in L^2(0,1). \end{aligned} \tag{2.15}$$

For the exterior problem we suppose $(f \circ \gamma, \alpha)_{L^2(0,1)} = 0$. Further, the density μ has to satisfy the side condition

$$(\mu, \alpha)_{L^2(0,1)} = 0. \tag{2.16}$$

The solution u is represented by the potential evaluation

$$u(x) = \int_0^1 \frac{\partial}{\partial n_t} E\big(x, \gamma(t)\big)\mu(t)\alpha(t)dt, \quad x \in \Omega. \tag{2.17}$$

2.3. Discretization

For a given $j \in \mathbb{N}$, $j \geq j_0$, choose $N_j = 2^j$ and $\triangle_j := \{0, 1, \ldots, N_j - 1\}$. We subdivide the (periodic) interval $[0,1]$ by $t_k^{(j)} = k/N_j$, $k \in \triangle_j$, to obtain an equidistant partition with step width $h_j := 1/N_j = 2^{-j}$. For the sake of simplicity we identify here and in the sequel the points $t_{k+lN_j}^{(j)}$, $l \in \mathbb{Z}$, with the points $t_k^{(j)}$. For the discretization we employ L^2-normalized piecewise constant functions

$$\phi_{j,k}^{(1)} := 2^{j/2} \chi_{[t_k^{(j)}, t_{k+1}^{(j)}]}, \quad k \in \triangle_j, \tag{2.18}$$

or piecewise linear functions

$$\phi_{j,k}^{(2)}(x) := 2^{3j/2} \begin{cases} x - t_{k-1}^{(j)}, & \text{for } x \in [t_{k-1}^{(j)}, t_k^{(j)}], \\ t_{k+1}^{(j)} - x, & \text{for } x \in [t_k^{(j)}, t_{k+1}^{(j)}], \\ 0, & \text{elsewhere}, \end{cases} \qquad k \in \triangle_j. \tag{2.19}$$

The spaces $V_j := \operatorname{span}\{\phi_{j,k}^{(d)} : k \in \triangle_j\}$, $j \geq j_0$, $d = 1, 2$, form a nested sequence of subspaces in $L^2(0,1)$

$$V_{j_0} \subset V_{j_0+1} \subset \ldots, \qquad \overline{\bigcup_{j \geq j_0} V_j} = L^2(0,1), \qquad \bigcap_{j \geq j_0} V_j = V_{j_0}.$$

For the Galerkin scheme we replace the energy spaces $H^{-1/2}(0,1)$ and $L^2(0,1)$ in the variational formulations (2.12) and (2.15) by the finite dimensional spaces V_j, respectively. Then, for the equation of the first kind (2.12), the Ansatz $\mu = \sum \mu_k \phi_{j,k}^{(d)}$ together with

$$\begin{aligned} [\mathbf{V}_\phi]_{k,k'} &= \big(V\phi_{j,k'}^{(d)}, \phi_{j,k}^{(d)}\big)_{L^2(0,1)}, \\ [\boldsymbol{\mu}_\phi]_k = \mu_k, \qquad [\mathbf{f}_\phi]_k &= \big(f \circ \gamma, \phi_{j,k}^{(d)}\big)_{L^2(0,1)}, \end{aligned} \tag{2.20}$$

leads to the linear system of equations

$$\mathbf{V}_\phi \boldsymbol{\mu}_\phi = \mathbf{f}_\phi. \tag{2.21}$$

Likewise, with $\boldsymbol{\mu}_\phi$, $\mathbf{f}_\phi$ as in (2.20) and

$$[\mathbf{K}_\phi]_{k,k'} = \big(K\phi_{j,k'}^{(d)}, \phi_{j,k}^{(d)}\big)_{L^2(0,1)}, \qquad [\mathbf{B}_\phi]_{k,k'} = \big(\phi_{j,k'}^{(d)}, \phi_{j,k}^{(d)}\big)_{L^2(0,1)}. \tag{2.22}$$

we obtain for the equation of the second kind (2.15) the linear system of equations

$$\big(\mathbf{K}_\phi \pm \tfrac{1}{2}\mathbf{B}_\phi\big)\boldsymbol{\mu}_\phi = \mathbf{f}_\phi. \tag{2.23}$$

For an exterior Dirichlet problem the system matrix $\mathbf{K}_\phi + \frac{1}{2}\mathbf{B}_\phi$ is singular. Hence, we discard in this case one of the unknowns in (2.23) which yields a reduced but regular system matrix with a unique solution $\tilde{\boldsymbol{\mu}}$. Without loss of generality we may assume that the skipped unknown is the last one. Then, the vector $\left[\begin{smallmatrix}\tilde{\boldsymbol{\mu}}\\ 0\end{smallmatrix}\right]$ solves the original singular linear system of equations. To satisfy the side condition (2.16) one computes the vector $[\mathbf{a}]_k := \big(\phi_{j,k}^{(d)}, \alpha\big)_{L^2(0,1)}$ and sets $\boldsymbol{\mu}_\phi := \left[\begin{smallmatrix}\tilde{\boldsymbol{\mu}}\\ 0\end{smallmatrix}\right] - \big(\mathbf{a}^T\left[\begin{smallmatrix}\tilde{\boldsymbol{\mu}}\\ 0\end{smallmatrix}\right]\big)\mathbf{a}$.

The most expensive work for solving a given Dirichlet problem is the computation of the system matrices $\mathbf{V}_\phi$ (2.20) and $\mathbf{K}_\phi$ (2.22). For assembling these matrices one has to evaluate for all $k, k' \in \triangle_j$ the integrals

$$\alpha_{(j,k),(j,k')}^{(d)} := \int_0^1 \int_0^1 k(s,t)\phi_{j,k'}^{(d)}(t)\phi_{j,k}^{(d)}(s)dtds \tag{2.24}$$

with some kernel function $k(s,t)$. More precisely, for piecewise constant functions, that is $d = 1$, one has to compute for all $k, k' \in \triangle_j$ the values

$$\alpha_{(j,k),(j,k')}^{(1)} = h_j \int_0^1 \int_0^1 k\big(t_k^{(j)} + h_j s, t_{k'}^{(j)} + h_j t\big)dtds. \tag{2.25}$$

Since the piecewise linear functions ($d = 2$) are not smooth in their support we compute in this case for all $k, k' \in \triangle_j$ the required integrals on the domain $[t_k^{(j)} + t_{k+1}^{(j)}] \times [t_{k'}^{(j)} + t_{k'+1}^{(j)}]$

$$\alpha_{(j,k),(j,k')}^{(2,i)} := h_j \int_0^1 \int_0^1 k\big(t_k^{(j)} + h_j s, t_{k'}^{(j)} + h_j t\big)\, p_i(s,t)\, dtds, \tag{2.26}$$

where $p_i(s,t)$, $i = 1,2,3,4$, are the polynomials

$$\begin{array}{ll} p_1(s,t) := st, & p_3(s,t) := (1-s)t, \\ p_2(s,t) := s(1-t), & p_4(s,t) := (1-s)(1-t). \end{array} \tag{2.27}$$

Then, we add them to the corresponding matrix entries according to

$$\alpha_{(j,k),(j,k')}^{(2)} = \alpha_{(j,k-1),(j,k'-1)}^{(2,1)} + \alpha_{(j,k-1),(j,k')}^{(2,2)} + \alpha_{(j,k),(j,k'-1)}^{(2,3)} + \alpha_{(j,k),(j,k')}^{(2,4)}. \tag{2.28}$$

A full quadrature scheme for the calculation of the integrals (2.25) and (2.26) by employing Gauß-Legendre quadrature rules is given in section 5..

2.4. Solving Neumann problems

For a given $g \in H^{-1/2}(\Gamma)$ with

$$\int_\Gamma g(x)ds_x = 0, \tag{2.29}$$

we consider a Neumann problem on the domain Ω, that is, we seek $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\triangle u &= 0 \qquad \text{in } \Omega, \\ \tfrac{\partial u}{\partial n} &= g \qquad \text{on } \Gamma. \end{aligned} \tag{2.30}$$

Analogously to the Dirichlet problem, depending on the choice of Ω the problem indicates an interior Neumann problem and an exterior Neumann problem, respectively. For an exterior Neumann problem it is additionally required that

$$u(x) = o(1) \quad \text{as } |x| \to \infty$$

uniformly for all directions $x/|x|$.

1. Fredholm's integral equation of the first kind for Neumann problems: The *hypersingular operator* $\mathcal{W}$ is given by

$$(\mathcal{W}\rho)(x) := -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} E(x,y)\rho(y)ds_y, \quad x \in \Gamma, \tag{2.31}$$

and defines an operator of order $+1$, i.e., $\mathcal{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. We seek the density ρ satisfying the Fredholm integral equation of the first kind

$$\mathcal{W}\rho = g \quad \text{on } \Gamma. \tag{2.32}$$

Since $\mathcal{W}$ is symmetric and positive semidefinite, cf. [20, 28], one restricts ρ by $\int_\Gamma \rho(x)ds_x = 0$. The variational formulation of (2.32) reads

$$\text{seek } \rho \in H^{1/2}(\Gamma): \quad (\mathcal{W}\rho, \delta)_{L^2(\Gamma)} = (g, \delta)_{L^2(\Gamma)} \quad \forall\, \delta \in H^{1/2}(\Gamma).$$

With $W : H^{1/2}(0,1) \to H^{-1/2}(0,1)$,

$$(W\mu)(s) := -\frac{\partial}{\partial n_s} \int_0^1 \frac{\partial}{\partial n_t} E\big(\gamma(s), \gamma(t)\big)\mu(t)\alpha(s)\alpha(t)dt, \tag{2.33}$$

it follows for all $\rho, \delta \in H^{1/2}(\Gamma)$ the equation $(\mathcal{W}\rho, \delta)_{L^2(\Gamma)} = \big(W(\rho\circ\gamma), \delta\circ\gamma\big)_{L^2(0,1)}$. Hence, replacing $\rho\circ\gamma \in H^{1/2}(0,1)$ and $\delta\circ\gamma \in H^{1/2}(0,1)$ by μ and ν, respectively, gives the variational formulation

$$\text{seek } \mu \in H^{1/2}(0,1): \quad (W\mu, \nu)_{L^2(0,1)} = (g\circ\gamma, \nu\alpha)_{L^2(0,1)} \quad \forall\, \nu \in H^{1/2}(0,1).$$

Herein, the restriction on the density takes the form (2.16). Since the energy space is $H^{1/2}(0,1)$ we have to discretize the variational formulation by piecewise linear functions, that is $d = 2$, which yields the Galerkin system

$$\mathbf{W}_\phi \boldsymbol{\mu}_\phi = \mathbf{g}_\phi \tag{2.34}$$

with $\boldsymbol{\mu}_\phi$ as in (2.20) and

$$[\mathbf{W}_\phi]_{k,k'} = \big(W\phi^{(2)}_{j,k'}, \phi^{(2)}_{j,k}\big)_{L^2(0,1)}, \qquad [\mathbf{g}_\phi]_k = \big(g\circ\gamma, \phi^{(2)}_{j,k}\alpha\big)_{L^2(0,1)}.$$

Since the piecewise linear functions lie in $H^1(0,1)$ we may employ the identity $(W\mu,\nu)_{L^2(0,1)} = (V\mu',\nu')_{L^2(0,1)}$, which holds for all $\mu,\nu\in H^1(0,1)$, see [18, 24] for details. Hence, observing that the derivatives of the Ansatz functions are piecewise constant functions, we find for the system matrix $\mathbf{W}_\phi$ the equation

$$\mathbf{W}_\phi = \mathbf{H}\mathbf{V}_\phi\mathbf{H}^T, \tag{2.35}$$

where $\mathbf{H}\in\mathbb{R}^{N_j\times N_j}$ is a band matrix defined by

$$\mathbf{H} := \frac{1}{h_j}\begin{bmatrix} -1 & & & & 1 \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{bmatrix},$$

and $\mathbf{V}_\phi$ given by (2.20) corresponds to the single layer operator discretized via piecewise constant functions ($d=1$). Since $\mathbf{W}_\phi$ inherits the symmetry and positive semidefiniteness from $\mathcal{W}$ the linear system of equations (2.34) is singular. To solve this system under the given side condition, one proceeds exactly as for the equation of the second kind for exterior Dirichlet problems. The density $\mu = \sum \mu_k\phi^{(2)}_{j,k}$ approximates the solution u of the Neumann problem (modulo some constant in the case of an interior problem) by the potential evaluation (2.17).

2. Fredholm's integral equation of the second kind for Neumann problems: We introduce the adjoint $\mathcal{K}^\star$ of the double layer operator

$$(\mathcal{K}^\star u)(x) := \int_\Gamma \frac{\partial}{\partial n_x}E(x,y)v(y)ds_y, \quad x\in\Gamma, \tag{2.36}$$

which indicates an operator of order zero, i.e., $\mathcal{K}^\star : L^2(\Gamma)\to L^2(\Gamma)$. Then, for the Neumann problem (2.30) we formulate the Fredholm integral equation of the second kind

$$\big(\pm\tfrac{1}{2}I - \mathcal{K}^\star\big)\rho = g \quad \text{on } \Gamma, \tag{2.37}$$

where we have to choose "+" for an exterior and "−" for an interior problem. Since $\big(-\frac{1}{2}I - \mathcal{K}^\star\big) = -\big(\frac{1}{2}I + \mathcal{K}\big)^\star$ this equation is not uniquely solvable in the case of the interior problem. Hence, again the side condition $\int_\Gamma \rho(x)ds_x = 0$ is required for the unknown density ρ. We employ $(\mathcal{K}^\star\rho,\delta)_{L^2(\Gamma)} = (\rho,\mathcal{K}\delta)_{L^2(\Gamma)}$ and substitute $\mu = (\rho\circ\gamma)\alpha$, $\nu = \delta\circ\gamma$ to find the variational formulation

$$\begin{aligned} &\text{seek } \mu\in L^2(0,1): \\ &\quad \pm\frac{1}{2}(\mu,\nu)_{L^2(0,1)} - (\mu,K\nu)_{L^2(0,1)} = (g\circ\gamma,\nu\alpha)_{L^2(0,1)} \quad \forall\,\nu\in L^2(0,1). \end{aligned}$$

Herein, the required side condition for the interior problem reads $(\mu, 1)_{L^2(0,1)} = 0$. With $\boldsymbol{\mu}_\phi$ from (2.20), $\mathbf{K}_\phi$, $\mathbf{B}_\phi$ from (2.22) and $\mathbf{g}_\phi$ similarly as above the Galerkin system is given by

$$\big(\pm \tfrac{1}{2}\mathbf{B}_\phi - \mathbf{K}_\phi^\star \big)\boldsymbol{\mu}_\phi = \mathbf{g}_\phi. \tag{2.38}$$

Note, the system corresponding to the exterior Neumann problem is uniquely solvable while for the interior problem it is singular. To solve the singular system one proceeds likewise to the equation of the second kind for the Dirichlet problem. Note that, this time, we get $[\mathbf{a}]_k := (\phi_{j,k}^{(d)}, 1)_{L^2(0,1)} = 1/\sqrt{h_j}$. The density $\mu = \sum \mu_k \phi_{j,k}^{(d)}$ defines an approximated solution to the Neumann problem (modulo some constant in the case of an interior problem) by (2.13).

3. Wavelet approximation for BEM

3.1. Motivation

We have introduced in subsection 2.3. an equidistant partition of the interval $[0, 1]$. On this partition the unknown density is discretized via (periodic) piecewise constant and linear functions, respectively. Instead of using this *single-scale basis* we want to apply wavelets with vanishing moments (more precisely: biorthogonal wavelet bases) yielding numerically sparse system matrices.

The outline is as follows. We first introduce biorthogonal wavelet bases on $\mathbb{R}$. Next, we obtain the wavelet bases on the interval $[0, 1]$ by periodization. According to [35] we briefly recall in the third subsection the compression strategy of matrices arising from BEM. Since the system matrices resulting from boundary integral operators of order $\neq 0$ are ill conditioned, we give a simple diagonal preconditioner based on the wavelet expansion in the last subsection.

3.2. Biorthogonal multiresolution on $\mathbb{R}$

On $\mathbb{R}$ the piecewise polynomial splines of degree $d-1$ can be defined as follows. Denoting by $[x_0, \ldots, x_d]f$ the d-th order *divided difference* at the points $x_0, \ldots, x_d \in \mathbb{R}$ (see e.g. [15]) the (centered) cardinal B-spline of order d is given by

$$\phi^{(d)}(x) = d[0, 1, \ldots, d]\Big(\cdot - x - \Big\lfloor \frac{d}{2} \Big\rfloor \Big)_+^{d-1}.$$

where $x_+^l := (\max\{0, x\})^l$ and $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer less (greater) than or equal to x. This *scaling function* is normalized

$$\big\| \phi^{(d)} \big\|_{L^1(\mathbb{R})} = 1,$$

compactly supported

$$\operatorname{supp}\phi^{(d)} = \Big[-\Big\lfloor\frac{d}{2}\Big\rfloor, \Big\lceil\frac{d}{2}\Big\rceil\Big] \tag{3.1}$$

and refinable

$$\phi^{(d)}(x) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} a_k \phi^{(d)}(2x-k) \tag{3.2}$$

with *mask coefficients*

$$a_k = \begin{cases} 2^{1-d}\binom{d}{k}, & -\lfloor\frac{d}{2}\rfloor \le k \le \lceil\frac{d}{2}\rceil, \\ 0, & \text{elsewhere.}\end{cases} \tag{3.3}$$

Introducing for $j,k\in\mathbb{Z}$ the translates and dilates of the scaling function $\phi^{(d)}_{j,k} := 2^{j/2}\phi^{(d)}(2^j\cdot -k)$, the sets $\Phi^{(d)}_j := \big\{\phi^{(d)}_{j,k} : k\in\mathbb{Z}\big\}$ generate a sequence of spaces $V_j := \overline{\operatorname{span}\Phi^{(d)}_j}$, which is nested

$$\ldots \subset V_j \subset V_{j+1} \subset \ldots$$

and dense in $L^2(\mathbb{R})$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

The spaces V_j are exact of order d, i.e., denoting the space of all polynomials of degree $< d$ by $\Pi_d(\mathbb{R})$, there holds

$$\Pi_d(\mathbb{R}) \subset V_j.$$

Furthermore, $\Phi^{(d)}_j$ forms a Riesz basis in V_j

$$\big\|\Phi^{(d)}_j \mathbf{c}\big\|_{L^2(\mathbb{R})} \sim \|\mathbf{c}\|_{l^2(\mathbb{Z})} \quad \forall\, \mathbf{c}\in l^2(\mathbb{Z}).$$

Due to [3] there exists for every integer $\tilde{d}\ge d$ with $\tilde{d}+d$ even a *dual scaling function* $\widetilde{\phi}^{(d,\tilde{d})}\in L^2(\mathbb{R})$ which is biorthogonal to the first scaling function

$$\big(\phi^{(d)}, \widetilde{\phi}^{(d,\tilde{d})}(\cdot - k)\big)_{L^2(\mathbb{R})} = \delta_{0,k}, \quad k\in\mathbb{Z}.$$

Moreover, similarly to the primal scaling function, this function is normalized, compactly supported

$$\operatorname{supp}\widetilde{\phi}^{d,\tilde{d}} = \Big[-\Big\lfloor\frac{d}{2}\Big\rfloor + 1 - \tilde{d}, \Big\lceil\frac{d}{2}\Big\rceil - 1 + \tilde{d}\Big] \tag{3.4}$$

and refinable

$$\widetilde{\phi}^{(d,\tilde{d})}(x) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \tilde{a}_k \widetilde{\phi}^{(d,\tilde{d})}(2x-k). \tag{3.5}$$

The mask coefficients in (3.5) can be derived by the *z-notation*: The coefficient $\tilde{a}_k$ of the sequence $\sum_k \tilde{a}_k z^k = p(z)q(z)$,

$$p(z) = 2^{1-\tilde{d}} \sum_{k=0}^{\tilde{d}} \binom{\tilde{d}}{k} z^{k-\lfloor \frac{\tilde{d}}{2} \rfloor},$$

$$q(z) = \sum_{k=0}^{\frac{d+\tilde{d}}{2}-1} 2^{-k} \binom{\frac{d+\tilde{d}}{2}-1+k}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-z)^{l-k},$$

coincides with the mask coefficient $\tilde{a}_k$, see [3] for details. Exactly like the primal side, the translates and dilates of the dual scaling function $\widetilde{\phi}_{j,k}^{(d,\tilde{d})} := 2^{j/2}\widetilde{\phi}^{(d,\tilde{d})}(2^j \cdot -k)$, $j,k \in \mathbb{Z}$, generate collections of Riesz bases $\widetilde{\Phi}_j^{(d,\tilde{d})} := \{\widetilde{\phi}_{j,k}^{(d,\tilde{d})} : k \in \mathbb{Z}\}$ in the spaces $\widetilde{V}_j := \overline{\operatorname{span} \widetilde{\Phi}_j^{(d,\tilde{d})}}$ which are nested, dense in $L^2(\mathbb{R})$ and exact of order $\tilde{d}$.

According to [3] a dual pair of *wavelets* $\psi^{(d,\tilde{d})}$, $\widetilde{\psi}^{(d,\tilde{d})} \in L^2(\mathbb{R})$ satisfying

$$\left(\psi^{(d,\tilde{d})}, \widetilde{\psi}^{(d,\tilde{d})}(\cdot - k)\right)_{L^2(\mathbb{R})} = \delta_{0,k}, \quad k \in \mathbb{Z},$$

is defined by

$$\begin{aligned} \psi^{(d,\tilde{d})}(x) &:= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} b_k \phi^{(d)}(2x-k), \\ \widetilde{\psi}^{(d,\tilde{d})}(x) &:= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \tilde{b}_k \widetilde{\phi}^{(d,\tilde{d})}(2x-k), \end{aligned} \tag{3.6}$$

where the mask coefficients b_k and $\tilde{b}_k$ are given by

$$b_k = (-1)^k \tilde{a}_{1-k}, \qquad \tilde{b}_k = (-1)^k a_{1-k}, \qquad k \in \mathbb{Z}, \tag{3.7}$$

with a_k from (3.3) and $\tilde{a}_k$ from (3.5). As a consequence of the finite masks and the compact supports of the scaling functions both wavelets are compactly supported

$$\operatorname{supp} \psi^{(d,\tilde{d})} = \operatorname{supp} \widetilde{\psi}^{(d,\tilde{d})} = \left[1 - \frac{d+\tilde{d}}{2}, \frac{d+\tilde{d}}{2}\right]. \tag{3.8}$$

Setting analogously to the scaling functions

$$\psi_{j,k}^{(d,\tilde{d})} := 2^{j/2}\psi^{(d,\tilde{d})}(2^j \cdot -k), \qquad \widetilde{\psi}_{j,k}^{(d,\tilde{d})} := 2^{j/2}\widetilde{\psi}^{(d,\tilde{d})}(2^j \cdot -k),$$

the sets

$$\Psi_j^{(d,\tilde{d})} := \big\{\psi_{j,k}^{(d,\tilde{d})} : k \in \mathbb{Z}\big\}, \qquad \widetilde{\Psi}_j^{(d,\tilde{d})} := \big\{\widetilde{\psi}_{j,k}^{(d,\tilde{d})} : k \in \mathbb{Z}\big\},$$

generate complement spaces $W_j := \overline{\operatorname{span} \Psi_j^{(d,\tilde{d})}}$ and $\widetilde{W}_j := \overline{\operatorname{span} \widetilde{\Psi}_j^{(d,\tilde{d})}}$ with

$$V_j \oplus W_j = V_{j+1}, \qquad \widetilde{V}_j \oplus \widetilde{W}_j = \widetilde{V}_{j+1},$$

where $\oplus$ denotes the direct sum. Thus, recursively one obtains

$$\overline{\bigoplus_{j\in\mathbb{Z}} W_j} = \overline{\bigoplus_{j\in\mathbb{Z}} \widetilde{W}_j} = L^2(\mathbb{R}).$$

Since biorthogonality implies $W_j \perp \widetilde{V}_j$ the primal wavelets have vanishing moments of order $\tilde{d}$, i.e.,

$$\big((\cdot)^\alpha, \psi_{j,k}^{(d,\tilde{d})}\big)_{L^2(\mathbb{R})} = 0, \quad 0 \le \alpha < \tilde{d}.$$

Furthermore, the wavelets

$$\Psi^{(d,\tilde{d})} := \bigcup_{j\in\mathbb{Z}} \Psi_j^{(d,\tilde{d})}, \qquad \widetilde{\Psi}^{(d,\tilde{d})} := \bigcup_{j\in\mathbb{Z}} \widetilde{\Psi}_j^{(d,\tilde{d})},$$

form Riesz bases in $L^2(\mathbb{R})$

$$\|\mathbf{c}\|^2_{l^2(\mathbb{Z}\times\mathbb{Z})} \sim \big\|\Psi^{(d,\tilde{d})}\mathbf{c}\big\|^2_{L^2(\mathbb{R})} \sim \big\|\widetilde{\Psi}^{(d,\tilde{d})}\mathbf{c}\big\|^2_{L^2(\mathbb{R})} \quad \forall\, \mathbf{c} \in l^2(\mathbb{Z}\times\mathbb{Z}). \tag{3.9}$$

3.3. Periodization

The above setting is clearly not suitable for the treatment of equations which are defined on bounded domains. In the sequel we will utilize a periodic version of a multiscale resolution. It essentially retains all the structural and computational advantages of the stationary and shift-invariant case considered in the previous subsection.

To this end, the simple trick is to replace the meaning of $u_{j,k} := 2^{j/2}u(2^j \cdot -k)$, $k \in \mathbb{Z}$, for compactly supported $u \in L^2(\mathbb{R})$ by its periodized counterpart

$$u_{j,k} := 2^{j/2} \sum_{n\in\mathbb{Z}} u(2^j(\cdot + n) - k).$$

In this way, given any dual pair $\phi^{(d)}$ and $\widetilde{\phi}^{(d,\tilde{d})}$ on $\mathbb{R}$ of compactly supported scaling functions, and setting $\triangle_j := \mathbb{Z} \setminus 2^j\mathbb{Z}$, the corresponding sets

$$\Phi_j^{(d)} := \big\{\phi_{j,k} : k \in \triangle_j\big\}, \qquad \Psi_j^{(d,\tilde{d})} := \big\{\psi_{j,k}^{(d,\tilde{d})} : k \in \triangle_j\big\}, \qquad j \ge j_0,$$

and likewise $\widetilde{\Phi}_j^{(d,\tilde{d})}$ and $\widetilde{\Psi}_j^{(d,\tilde{d})}$, have finite cardinality 2^j and consist of functions which are 1-periodic. Note that this definition preserves the biorthogonality relations. One easily checks that the scaling functions are biorthogonal

$$\big(\Phi_j^{(d)}, \widetilde{\Phi}_j^{(d,\tilde{d})}\big)_{L^2(0,1)} = \mathbf{I},$$

where $(\cdot,\cdot)$ denotes the inner product on $L^2(0,1)$ defined by (2.9). Moreover, the wavelet bases

$$\Psi^{(d,\tilde{d})} := \Phi_{j_0}^{(d)} \bigcup_{j\geq j_0} \Psi_j^{(d,\tilde{d})}, \qquad \widetilde{\Psi}^{(d,\tilde{d})} := \widetilde{\Phi}_{j_0}^{(d,\tilde{d})} \bigcup_{j\geq j_0} \widetilde{\Psi}_j^{(d,\tilde{d})}$$

are biorthogonal, i.e.,

$$\big(\Psi^{(d,\tilde{d})}, \widetilde{\Psi}^{(d,\tilde{d})}\big)_{L^2(0,1)} = \mathbf{I}.$$

For the sake of simplicity in representation, we will indicate in the sequel the scaling functions on the coarsest level of the wavelet bases by $\Psi_{j_0-1}^{(d,\tilde{d})} := \Phi_{j_0}^{(d)}$ and $\widetilde{\Psi}_{j_0-1}^{(d,\tilde{d})} := \widetilde{\Phi}_{j_0}^{(d,\tilde{d})}$. Moreover, we set $\triangle_{j_0-1} := \triangle_{j_0}$. Clearly, the coarsest level j_0 has to be chosen large enough to ensure that the diameter of the supports of the scaling functions and the wavelets is smaller that 1. Comparing (3.1), (3.4) and (3.8) implies

$$j_0 \geq \log_2\big[\operatorname{diam}\big(\operatorname{supp}\widetilde{\phi}^{(d,\tilde{d})}\big)\big] = \log_2(d + 2\tilde{d} - 2).$$

3.4. Matrix compression

Discretizing an integral operator $A : H^q(0,1) \to H^{-q}(0,1)$ by biorthogonal wavelet bases leads to quasi-sparse matrices. In a first compression step all matrix entries, for which the distances of the supports (on the given boundary Γ) of the corresponding Ansatz and test functions are bigger than a level depending cut-off parameter $\mathcal{B}_{j,j'}$, are set to zero. In the second compression step some of those matrix entries are set to zero, for which the corresponding Ansatz and test functions have overlapping supports.

More precisely, for a given $J > j_0$ we discretize the integral operator A by the wavelet basis $\Psi_J^{(d,\tilde{d})} := \bigcup_{j=j_0-1}^{J-1} \Psi_j^{(d,\tilde{d})}$ instead by the single-scale basis $\Phi_J^{(d)}$. We introduce the abbreviations

$$\begin{aligned}
\Theta_{j,k}^{(d,\tilde{d})} &:= \big\{x = \gamma(s) \in \mathbb{R}^2 : s \in \operatorname{supp}\psi_{j,k}^{(d,\tilde{d})}\big\},\\
\Xi_{j,k}^{(d,\tilde{d})} &:= \big\{x = \gamma(s) \in \mathbb{R}^2 : \nexists\, \varepsilon > 0 \;\big|\; \psi_{j,k}^{(d,\tilde{d})} \in C^\infty(s-\varepsilon, s+\varepsilon)\big\}.
\end{aligned}$$

Note that $\Theta_{j,k}^{(d,\tilde{d})}$ denotes the support of $\psi_{j,k}^{(d,\tilde{d})} \circ \gamma^{-1}$ while $\Xi_{j,k}^{(d,\tilde{d})}$ denotes the so-called *singular support* of $\psi_{j,k}^{(d,\tilde{d})} \circ \gamma^{-1}$, i.e., those points where $\psi_{j,k}^{(d,\tilde{d})} \circ \gamma^{-1}$ is not smooth.

The compressed system matrix $\mathbf{A}_\psi$ corresponding to A is given by

$$[\mathbf{A}_\psi]_{(j,k),(j',k')} := \begin{cases} 0, & \operatorname{dist}\left(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j,j'},\ \ j,j' \geq j_0, \\ 0, & \operatorname{dist}\left(\Xi_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}'_{j,j'},\ \ j' > j, \\ 0, & \operatorname{dist}\left(\Theta_{j,k}^{(d,\tilde{d})}, \Xi_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}'_{j,j'},\ \ j > j', \\ \left(A\psi_{j',k'}^{(d,\tilde{d})}, \psi_{j,k}^{(d,\tilde{d})}\right)_{L^2(0,1)}, & \text{otherwise}. \end{cases} \tag{3.10}$$

Herein, choosing

$$a, a' > 1, \qquad d < \delta, \delta' < \tilde{d} + 2q, \tag{3.11}$$

the cut-off parameters $\mathcal{B}_{j,j'}$ and $\mathcal{B}'_{j,j'}$ are set as follows

$$\begin{aligned} \mathcal{B}_{j,j'} &= a \ \max\left\{2^{-\min\{j,j'\}}, 2^{\frac{2J(\delta-q)-(j+j')(\delta+\tilde{d})}{2(\tilde{d}+q)}}\right\}, \\ \mathcal{B}'_{j,j'} &= a' \max\left\{2^{-\max\{j,j'\}}, 2^{\frac{2J(\delta'-q)-(j+j')\delta'-\max\{j,j'\}\tilde{d}}{\tilde{d}+2q}}\right\}. \end{aligned} \tag{3.12}$$

It is shown in [35] that this compression strategy reduces the number of nonzero entries to $\mathcal{O}(N_J)$ without any loss of accuracy and stability of the underlying Galerkin scheme. The resulting structure of the compressed matrix is figuratively called *finger structure*, cf. figure 1.

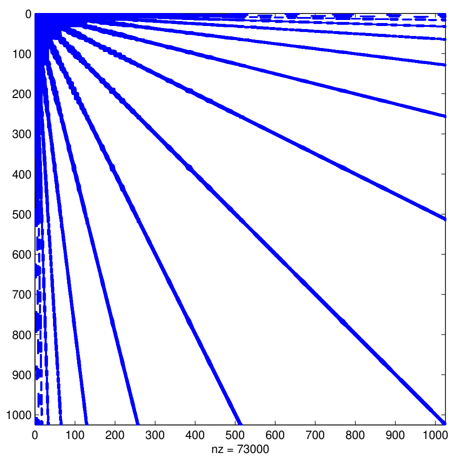

Figure 1: The structure of the compressed system matrix.

3.5. Wavelet preconditioning

The single layer operator and the hypersingular operator are operators of order different from zero. Hence, the system matrix $\mathbf{A}_\psi$ is ill conditioned. According to [5, 35], for example, the wavelet approach offers an simple diagonal preconditioner.

Let us define the number $\overline{q} := \sup\big\{q \in \mathbb{R} : \widetilde{\psi}^{(d,\tilde{d})} \in H^q(0,1)\big\}$ which characterizes the regularity of the dual wavelet basis. Moreover, we introduce the diagonal matrix $\mathbf{D}^r$ by

$$\big[\mathbf{D}^r\big]_{(j,k),(j',k')} = 2^{rj}\delta_{j,j'}\delta_{k,k'}, \quad k \in \triangle_j, \quad k' \in \triangle_{j'}, \quad j_0 - 1 \leq j, j' < J. \tag{3.13}$$

Then, if $A : H^q(0,1) \to H^{-q}(0,1)$ denotes an integral operator of order $2q$ with $\overline{>}-q$, the corresponding system matrix $\mathbf{A}_\psi$ is spectrally equivalent to $\mathbf{D}^{2q}$. Therefore, the system matrix $\mathbf{D}^{-q}\mathbf{A}_\psi\mathbf{D}^{-q}$ is well conditioned, i.e.,

$$\operatorname{cond}_{l^2}(\mathbf{D}^{-q}\mathbf{A}_\psi\mathbf{D}^{-q}) \sim 1.$$

Note that the coefficients on the main diagonal of $\mathbf{A}_\psi$ satisfy

$$\big(A\psi_{j,k}^{(d,\tilde{d})}, \psi_{j,k}^{(d,\tilde{d})}\big)_{L^2(0,1)} \sim 2^{2qj}, \quad k \in \triangle_j, \quad j_0 - 1 \leq j < J.$$

Therefore, the above preconditioning can be replaced by a diagonal scaling. In fact, the diagonal scaling improves and simplifies the wavelet preconditioning.

4. The discrete wavelet Galerkin scheme

4.1. Changing bases

Generally written, in the single-scale basis we obtain the Galerkin system

$$(\mathbf{A}_\phi + \lambda\mathbf{B}_\phi)\boldsymbol{\mu}_\phi = \mathbf{f}_\phi \tag{4.1}$$

with the unknown density vector $\boldsymbol{\mu}_\phi$. Hereby, $\mathbf{A}_\phi$ is the system matrix corresponding to a boundary integral operator $A : H^q(0,1) \to H^{-q}(0,1)$, $\mathbf{B}_\phi$ is the mass matrix belonging to the identity operator, $\mathbf{f}_\phi$ is the vector of the right hand side and λ is some real constant. We denote by $\mathbf{T}$ the transposed fast wavelet transform, i.e., the basis transform for the dual bases given by

$$\sum_{k\in\triangle_J} a_k\widetilde{\phi}_{k,J}^{(d,\tilde{d})} = \sum_{j=j_0-1}^{J-1}\sum_{k\in\triangle_j} b_{j,k}\widetilde{\psi}_{j,k}^{(d,\tilde{d})}, \qquad [b_{j,k}]_{k\in\triangle_j, j_0-1\leq j<J} = \mathbf{T}[a_k]_{k\in\triangle_J},$$

see [3] for details. Then, the change from the single-scale basis into the biorthogonal wavelet basis corresponds to the system

$$\mathbf{T}(\mathbf{A}_\phi + \lambda \mathbf{B}_\phi)\mathbf{T}^\star \boldsymbol{\mu}_\psi = \mathbf{T}\mathbf{f}_\phi, \quad \boldsymbol{\mu}_\phi = \mathbf{T}^\star \boldsymbol{\mu}_\psi.$$

Clearly, assembling $\mathbf{B}_\phi$ and $\mathbf{f}_\phi$ has a complexity $\mathcal{O}(N_J)$. Moreover, it is well known that the application of $\mathbf{T}$ and $\mathbf{T}^\star$ to a vector requires only $\mathcal{O}(N_J)$ operations, cf. [3]. Hence, utilizing iterative methods for the solution of the linear system of equations we may concentrate in the sequel on the efficient computation of the compressed matrix $\mathbf{A}_\psi \approx \mathbf{T}\mathbf{A}_\phi\mathbf{T}^\star$ solving the system

$$(\mathbf{A}_\psi + \lambda \mathbf{T}\mathbf{B}_\phi\mathbf{T}^\star)\boldsymbol{\mu}_\psi = \mathbf{T}\mathbf{f}_\phi, \quad \boldsymbol{\mu}_\phi = \mathbf{T}^\star \boldsymbol{\mu}_\psi. \tag{4.2}$$

Remark 4.1. The equation of the second kind for the exterior Dirichlet problem (2.23), the equation of the second kind for the interior Neumann problem (2.38) and the equation of the first kind for the Neumann problem (2.34), lead to a singular linear system (4.2). Therefore, we discarded one unknown of the single-scale basis solving a reduced system. Analogously in the multiscale basis we omit one of the scaling functions on the coarsest grid. Without loss of generality we discard the first scaling function $\phi_{j_0,0}^{(d)} = \psi_{j_0-1,0}^{(d,\tilde{d})}$. Then, the application of $\mathbf{T}\mathbf{B}_\phi\mathbf{T}^\star$ to the reduced vector $\tilde{\boldsymbol{\mu}}$ in the iterative solver is performed as follows: We compute the vector $\left[\begin{smallmatrix} a \\ \mathbf{b} \end{smallmatrix}\right] := \mathbf{T}\mathbf{B}_\phi\mathbf{T}^\star\left[\begin{smallmatrix} 0 \\ \tilde{\boldsymbol{\mu}} \end{smallmatrix}\right]$ and continue the iteration with $\mathbf{b}$. At the end we get the solution $\boldsymbol{\mu}_\phi = \mathbf{T}^\star\left[\begin{smallmatrix} 0 \\ \tilde{\boldsymbol{\mu}} \end{smallmatrix}\right]$, for which the postulated side conditions are computed as seen in section 2..

4.2. Computing distances between wavelets

To set up the compression pattern according to (3.10), we have to compute the distances $\operatorname{dist}\big(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big)$ and $\operatorname{dist}\big(\Xi_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big)$. The first distance is required if $j, j' \geq j_0$ while the second distance is required if $j' > j \geq j_0 - 1$.

The crucial idea for the evaluation of the distance $\operatorname{dist}\big(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big)$ is to find disks $B(m_{j,k}, r_{j,k}) := \{x \in \mathbb{R}^2 : |x - m_{j,k}| \leq r_{j,k}\}$ with $\Theta_{j,k}^{(d,\tilde{d})} \subset B(m_{j,k}, r_{j,k})$. Then, consequently, one has

$$\operatorname{dist}\big(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big) \leq \max\{0, |m_{j,k} - m_{j',k'}| - r_{j,k} - r_{j',k'}\}.$$

We start on the finest level $J-1$ to compute these disks. For this, note that we obtain for all $k \in \triangle_j$, $j \geq j_0$, from (3.8)

$$\operatorname{supp} \psi_{j,k}^{(d,\tilde{d})} = 2^{-j}\Big[k + 1 - \frac{d+\tilde{d}}{2}, k + \frac{d+\tilde{d}}{2}\Big] =: 2^{-j}[k + \underline{l}, k + \overline{l}]. \tag{4.3}$$

Invoking (4.3) we calculate

$$\underline{x}_k := \gamma\big(2^{1-J}(k + \underline{l})\big), \qquad \overline{x}_k := \gamma\big(2^{1-J}(k + \overline{l})\big), \qquad k \in \triangle_{J-1}.$$

Then, assuming that J is large enough, the disk determined by $m_{J-1,k} := (\underline{x}_k + \overline{x}_k)/2$ and $r_{J-1,k} := |\underline{x}_k - \overline{x}_k|/2$ satisfies $\Theta_{J-1,k}^{(d,\tilde{d})} \subset B(m_{J-1,k}, r_{J-1,k})$. Next, from (4.3) we deduce

$$\operatorname{supp} \psi_{j,k}^{(d,\tilde{d})} = \operatorname{supp} \psi_{j+1,2k+\underline{l}}^{(d,\tilde{d})} \cup \operatorname{supp} \psi_{j+1,2k+\overline{l}}^{(d,\tilde{d})}$$

for all $k \in \triangle_j$, $j \geq j_0$. Hence, computing recursively $B(m_{j,k}, r_{j,k})$ with

$$B(m_{j,k}, r_{j,k}) \supset B(m_{j+1,2k+\underline{l}}, r_{j+1,2k+\underline{l}}) \cup B(m_{j+1,2k+\overline{l}}, r_{j+1,2k+\overline{l}})$$

we obtain for all $k \in \triangle_j$, $J-1 > j \geq j_0$, the disks on the coarser levels. Note, the complexity for determining all the disks $B(m_{j,k}, r_{j,k})$ is $\mathcal{O}(N_J) + \mathcal{O}(N_J/2) + \mathcal{O}(N_J/4) + \ldots + \mathcal{O}(1) = \mathcal{O}(N_J)$, i.e., the complexity is linear. Moreover, with these disks we also can determine the distance $\operatorname{dist}\big(\Xi_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big)$ in the following way. For $j \geq j_0$ the singular support $\Xi_{j,k}^{(d,\tilde{d})}$ consists only of points $\gamma(s)$, where $s \in [0,1]$ is a point of the grid on the level $j+1$. Consequently, observing (4.3), we define

$$S_{j,k} := \{\gamma(2^{-(j+1)}l) : l \in \triangle_{j+1} \text{ and } 2(k+\underline{l}) \leq l \leq 2(k+\overline{l})\}$$

to obtain a set of $2(d+\tilde{d})-1$ points with $\Xi_{j,k}^{(d,\tilde{d})} \subset S_{j,k}$. Furthermore, (3.1) implies that the singular support of the scaling functions $\Phi_{j_0}^{(d)} = \Psi_{j_0-1}^{(d,\tilde{d})}$ consists of the $d+1$ points contained in

$$S_{j_0-1,k} := \{\gamma(2^{-j_0}l) : l \in \triangle_{j_0} \text{ and } k - \lfloor d/2 \rfloor \leq l \leq k + \lceil d/2 \rceil\}.$$

By these sets we obtain

$$\operatorname{dist}\big(\Xi_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\big) \leq \max\{0, \min_{x \in S_{j,k}} |x - m_{j',k'}| - r_{j',k'}\}.$$

4.3. Setting up the compression pattern

The compression strategy introduced in subsection 3.4. leads to a sparse system matrix $\mathbf{A}_\psi$. We first collect some important properties concerning the structure of this matrix.

- As one easily checks by (3.12), the cut-off parameters $\mathcal{B}_{j,j'}$ and $\mathcal{B}'_{j,j'}$, respectively, are symmetric with respect to the levels j and j', i.e.,

  $$\mathcal{B}_{j,j'} = \mathcal{B}_{j',j}, \qquad \mathcal{B}'_{j,j'} = \mathcal{B}'_{j',j}.$$

  Therefore, due to the symmetry of the distance function, one deduces that the structure of the compressed system matrix is symmetric.

- Naturally, the wavelet basis has a father-son relation with respect to the supports. More precisely, there holds

$$\Theta_{j+1,2k}^{(d,\tilde{d})}, \Theta_{j+1,2k+1}^{(d,\tilde{d})} \subset \Theta_{j,k}^{(d,\tilde{d})}, \quad k \in \triangle_j, \quad j \geq j_0, \tag{4.4}$$

cf. (4.3). In addition, (3.12) implies

$$\mathcal{B}_{j+1,j'+1} \leq \mathcal{B}_{j+1,j'} \leq \mathcal{B}_{j,j'}, \qquad j, j' \geq j_0.$$

Hence, combining the father-son relation (4.4) with the latter relation, it follows for $j, j' \geq j_0$

$$\begin{aligned} &\operatorname{dist}\left(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j,j'} \\ &\qquad \Longrightarrow \quad \begin{cases} \operatorname{dist}\left(\Theta_{j+1,2k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'}, \\ \operatorname{dist}\left(\Theta_{j+1,2k+1}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'}, \end{cases} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\operatorname{dist}\left(\Theta_{j,k}^{(d,\tilde{d})}, \Theta_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j,j'} \\ &\qquad \Longrightarrow \quad \begin{cases} \operatorname{dist}\left(\Theta_{j+1,2k}^{(d,\tilde{d})}, \Theta_{j'+1,2k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'+1}, \\ \operatorname{dist}\left(\Theta_{j+1,2k}^{(d,\tilde{d})}, \Theta_{j'+1,2k'+1}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'+1}, \\ \operatorname{dist}\left(\Theta_{j+1,2k+1}^{(d,\tilde{d})}, \Theta_{j'+1,2k'}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'+1}, \\ \operatorname{dist}\left(\Theta_{j+1,2k+1}^{(d,\tilde{d})}, \Theta_{j'+1,2k'+1}^{(d,\tilde{d})}\right) > \mathcal{B}_{j+1,j'+1}. \end{cases} \end{aligned} \tag{4.6}$$

- On the other hand, for the cut-off parameter $\mathcal{B}'_{j,j'}$ we find by (3.12)

$$\mathcal{B}'_{j+1,j'} \leq \mathcal{B}_{j,j'}, \qquad j > j' \geq j_0 - 1.$$

Invoking (4.4) this inequality yields

$$\begin{aligned} &\operatorname{dist}\left(\Theta_{j,k}^{(d,\tilde{d})}, \Xi_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}'_{j,j'} \\ &\qquad \Longrightarrow \quad \begin{cases} \operatorname{dist}\left(\Theta_{j+1,2k}^{(d,\tilde{d})}, \Xi_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}'_{j+1,j'}, \\ \operatorname{dist}\left(\Theta_{j+1,2k+1}^{(d,\tilde{d})}, \Xi_{j',k'}^{(d,\tilde{d})}\right) > \mathcal{B}'_{j+1,j'}, \end{cases} \end{aligned} \tag{4.7}$$

for $j > j' \geq j_0 - 1$.

Performing the compression we are interested in the index sets $\mathcal{I}_{j,j'}$, $j_0 - 1 \leq j, j' < J$, consisting of all pairs (k, k') of indices corresponding to matrix entries which have to be evaluated. Combining (4.5), (4.6) and (4.7) with the symmetric compression structure, one may formulate the following algorithm, given in pseudo code language. Herein, the result of the function **compute**$\big(\psi_{j,k}^{(d,\tilde{d})}, \psi_{j',k'}^{(d,\tilde{d})}\big)$ is supposed to be **true**, if the matrix entry $\big(A\psi_{j',k'}^{(d,\tilde{d})}, \psi_{j,k}^{(d,\tilde{d})}\big)_{L^2(0,1)}$ has to be computed according to (3.10). Else it is **false**.

Algorithm 4.2.

> **initialisation:** $\mathcal{I}_{j_0-1,j_0-1} := \mathcal{I}_{j_0-1,j_0} := \mathcal{I}_{j_0,j_0-1} := \mathcal{I}_{j_0,j_0} := \{\triangle_{j_0} \times \triangle_{j_0}\}$
> **for** $j := j_0 + 1$ **to** $J - 1$ **do begin**
>   **for** $j' := j_0 - 1$ **to** $j - 1$ **do begin**
> $\mathcal{I}_{j,j'} := \{\}$ *C: compute* $\mathcal{I}_{j,j'}$ *from* $\mathcal{I}_{j-1,j'}$ (4.5), (4.7)
>   **for all** $(k,k') \in \mathcal{I}_{j-1,j'}$ **do begin**
>   **if** $\big($**compute**$\big(\psi_{j,2k}^{(d,\tilde{d})}, \psi_{j',k'}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j'} := \mathcal{I}_{j,j'} \cup \{(2k,k')\}$
>   **if** $\big($**compute**$\big(\psi_{j,2k+1}^{(d,\tilde{d})}, \psi_{j',k'}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j'} := \mathcal{I}_{j,j'} \cup \{(2k+1,k')\}$
>   **end**
> $\mathcal{I}_{j',j} := \mathcal{I}_{j,j'}$ *C: according to symmetry*
>   **end**
> $\mathcal{I}_{j,j} := \{\}$ *C: compute* $\mathcal{I}_{j,j}$ *from* $\mathcal{I}_{j-1,j-1}$ (4.6)
>   **for all** $(k,k') \in \mathcal{I}_{j-1,j-1}$ **do begin**
>   **if** $\big($**compute**$\big(\psi_{j,2k}^{(d,\tilde{d})}, \psi_{j,2k'}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j} := \mathcal{I}_{j,j} \cup \{(2k,2k')\}$
>   **if** $\big($**compute**$\big(\psi_{j,2k+1}^{(d,\tilde{d})}, \psi_{j,2k'}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j} := \mathcal{I}_{j,j} \cup \{(2k+1,2k')\}$
>   **if** $\big($**compute**$\big(\psi_{j,2k}^{(d,\tilde{d})}, \psi_{j,2k'+1}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j} := \mathcal{I}_{j,j} \cup \{(2k,2k'+1)\}$
>   **if** $\big($**compute**$\big(\psi_{j,2k+1}^{(d,\tilde{d})}, \psi_{j,2k'+1}^{(d,\tilde{d})}\big) =$ **true**$\big)$ $\mathcal{I}_{j,j} := \mathcal{I}_{j,j} \cup \{(2k+1,2k'+1)\}$
>   **end**
> **end**

Obviously, the complexity of this algorithm can be estimated by the number of nonzero elements of the compressed system matrix $\mathbf{A}_\psi$, that is by $\mathcal{O}(N_J)$.

4.4. Computing the system matrix

For the computation of the system matrix $\mathbf{A}_\psi$ one has to calculate for $j_0 - 1 \le j, j' < J$ all coefficients $\big(A\psi_{j',k'}^{(d,\tilde{d})}, \psi_{j,k}^{(d,\tilde{d})}\big)_{L^2(0,1)}$, for which the pair (k,k') is found in the set $\mathcal{I}_{j,j'}$. For the sake of simplicity we assume that $j, j' \ge j_0$, i.e., we omit the scaling functions on the coarsest level $j_0 - 1$. The described strategy can also used in the case $j = j_0 - 1$ or $j' = j_0 - 1$ but the formulas must be modified slightly.

We abbreviate in the sequel analogously to (2.24)

$$\alpha^{(d)}_{(j,k),(j',k')} := \big(A\phi^{(d)}_{j',k'}, \phi^{(d)}_{j,k}\big)_{L^2(0,1)}. \tag{4.8}$$

Then, utilizing the refinement relation (3.6) each matrix entry splits into the finite sum

$$\big(A\psi^{(d,\tilde{d})}_{j',k'}, \psi^{(d,\tilde{d})}_{j,k}\big)_{L^2(0,1)} = \sum_{l}\sum_{l'} b_l b_{l'} \alpha^{(d)}_{(j+1,2k+l),(j'+1,2k'+l')} \tag{4.9}$$

with b_k from (3.7). Since the intersection of the supports of different wavelets might not be empty, a naive calculation of (4.9) leads to repeated computation of the values $\alpha^{(d)}_{(j,k),(j',k')}$. Hence, we develop a strategy to avoid multiple calculation.

In the case of piecewise constant functions one finds analogously to (2.25)

$$\alpha^{(1)}_{(j,k),(j',k')} = \sqrt{h_j h_{j'}} \int_0^1 \int_0^1 k\big(t^{(j)}_k + h_j s, t^{(j')}_{k'} + h_{j'} t\big) dt ds, \tag{4.10}$$

where $k(s,t)$ is the given kernel function. We introduce the sets $\mathcal{Q}_{j,j'}$, $j_0 < j, j' \leq J$, where every calculated triple $\big((k,k'), \alpha^{(1)}_{(j,k),(j',k')}\big)$ is stored. Then, if a special value $\alpha^{(1)}_{(j,k),(j',k')}$ is required, one first checks if the corresponding triple $\big((k,k'), \alpha^{(1)}_{(j,k),(j',k')}\big)$ is contained in $\mathcal{Q}_{j,j'}$, else one computes and stores it.

For piecewise linear function we perform a related strategy. We define analogously to (2.26)

$$\alpha^{(2,i)}_{(j,k),(j',k')} := \sqrt{h_j h_{j'}} \int_0^1 \int_0^1 k\big(t^{(j)}_k + h_j s, t^{(j')}_{k'} + h_{j'} t\big)\, p_i(s,t)\, dt ds, \tag{4.11}$$

with $p_i(s,t)$, $i = 1,2,3,4$, given by (2.27). Then, similar to (2.28), one finds

$$\alpha^{(2)}_{(j,k),(j',k')} = \alpha^{(2,1)}_{(j,k-1),(j',k'-1)} + \alpha^{(2,2)}_{(j,k-1),(j',k')} + \alpha^{(2,3)}_{(j,k),(j',k'-1)} + \alpha^{(2,4)}_{(j,k),(j',k')}. \tag{4.12}$$

Inserting this equation into (4.9) yields

$$\begin{aligned}\big(A\psi^{(2,\tilde{d})}_{j',k'}, \psi^{(2,\tilde{d})}_{j,k}\big)_{L^2(0,1)} = \sum_{l}\sum_{l'} b_l b_{l'} \Big[&\alpha^{(2,1)}_{(j+1,2k+l-1),(j'+1,2k'+l'-1)} \\ &+ \alpha^{(2,2)}_{(j+1,2k+l-1),(j'+1,2k'+l')} + \alpha^{(2,3)}_{(j+1,2k+l),(j'+1,2k'+l'-1)} \\ &+ \alpha^{(2,4)}_{(j+1,2k+l),(j'+1,2k'+l')}\Big].\end{aligned}$$

Resorting the sums leads to

$$\begin{aligned}\big(A\psi_{j',k'}^{(2,\tilde{d})}, \psi_{j,k}^{(2,\tilde{d})}\big)_{L^2(0,1)} = \sum_l \sum_{l'} \Big[& b_{l+1} b_{l'+1} \alpha^{(2,1)}_{(j+1,2k+l),(j'+1,2k'+l')} \\ & + b_{l+1} b_{l'} \alpha^{(2,2)}_{(j+1,2k+l),(j'+1,2k'+l')} + b_l b_{l'+1} \alpha^{(2,3)}_{(j+1,2k+l),(j'+1,2k'+l')} \\ & + b_l b_{l'} \alpha^{(2,4)}_{(j+1,2k+l),(j'+1,2k'+l')}\Big]. \end{aligned} \tag{4.13}$$

Hence, to avoid multiple calculations we define sets $\mathcal{Q}_{j,j'}$, $j_0 < j, j' \leq J$, where we store the required tuples $\big((k,k'), \alpha^{(2,1)}_{(j,k),(j',k')}, \alpha^{(2,2)}_{(j,k),(j',k')}, \alpha^{(2,3)}_{(j,k),(j',k')}, \alpha^{(2,4)}_{(j,k),(j',k')}\big)$. Then, the strategy is similar to above, i.e., before calculating values we search for them in the corresponding set $\mathcal{Q}_{j,j'}$. If these values are not contained in this set, we calculate and store them.

Remark 4.3. As we have seen in subsection 2.4., in the single-scale basis the system matrix corresponding to the hypersingular can be derived from the system matrix corresponding to the single layer operator. A similar approach exists for the wavelet bases: According to [3] we conclude $\big(\psi^{(d,\tilde{d})}\big)' = \psi^{(d-1,\tilde{d}+1)}$ which implies

$$\big(W\psi_{j',k'}^{(2,\tilde{d})}, \psi_{j,k}^{(2,\tilde{d})}\big)_{L^2(0,1)} = \big(V\psi_{j',k'}^{(1,\tilde{d}+1)}, \psi_{j,k}^{(1,\tilde{d}+1)}\big)_{L^2(0,1)}.$$

In other words, one computes an entry of the system matrix of the hypersingular operator with respect to piecewise linear wavelets $\Psi^{(2,\tilde{d})}$ like an entry of the system matrix of the single layer operator with respect to piecewise constant wavelets $\Psi^{(1,\tilde{d}+1)}$.

Since we have to compute $\mathcal{O}(N_J)$ entries in $\mathbf{A}_\psi$, the required memory for storing the sets $\mathcal{Q}_{j,j'}$, $j_0 < j, j' \leq J$, is $\mathcal{O}(N_J)$. Moreover, note that, if one has evaluated all pairs $(k,k') \in \mathcal{I}_{j,j'}$, one may delete the set $\mathcal{Q}_{j+1,j'+1}$ to reduce the required main memory.

5. Numerical quadrature

5.1. Error estimations on the reference domain

As we have seen in the last sections it is sufficient to develop quadrature schemes on $\square := [0,1]^2$ for the single layer and the double layer kernel, cf. (2.25), (2.26), (4.10) and (4.11). For the quadrature we utilize tensor product Gauß-Legendre rules on $\square$, for which we state first a general error estimation. For this, let us set

$$I^{[0,1]}f := \int_0^1 f(s)ds.$$

The g-point Gauß-Legendre formula on $[0,1]$

$$Q_g^{[0,1]} f := \sum_{i=1}^{g} \omega_{g,i} f\big(\xi_{g,i}\big)$$

applied to $f \in C^{2g}(0,1)$ can be estimated as

$$R_g^{[0,1]} f := \big|I^{[0,1]} f - Q_g^{[0,1]} f\big| \lesssim \frac{2^{-4g}}{(2g)!} \max_{s\in[0,1]} \big|f^{(2g)}(s)\big|, \tag{5.1}$$

cf. [23]. Further, we define for a given $f \in L^2(\Box)$

$$I^{\Box} f := \big(I^{[0,1]} \times I^{[0,1]}\big) f = \int_{\Box} f(s,t) d(s,t)$$

and the product Gauß-Legendre quadrature formula

$$Q_{g,g'}^{\Box}(f) := \big(Q_g^{[0,1]} \times Q_{g'}^{[0,1]}\big) f = \sum_{i=1}^{g} \sum_{i'=1}^{g'} \omega_{g,i}\omega_{g',i'} f\big(\xi_{g,i}, \xi_{g',i'}\big).$$

Since

$$\begin{aligned} R_{g,g'}^{\Box} f &:= I^{\Box} f - Q_{g,g'}^{\Box} f \\ &= \Big[\big(I^{[0,1]} \times I^{[0,1]}\big) - \big(Q_g^{[0,1]} \times Q_{g'}^{[0,1]}\big)\Big] f \\ &= \Big[\big(I^{[0,1]} \times I^{[0,1]}\big) - \big(I^{[0,1]} \times Q_{g'}^{[0,1]}\big) + \big(I^{[0,1]} \times Q_{g'}^{[0,1]}\big) - \big(Q_g^{[0,1]} \times Q_{g'}^{[0,1]}\big)\Big] f \\ &= \Big[I^{[0,1]} \times \big(I^{[0,1]} - Q_{g'}^{[0,1]}\big)\Big] f + \Big[\big(I^{[0,1]} - Q_g^{[0,1]}\big) \times Q_{g'}^{[0,1]}\Big] f \end{aligned}$$

we obtain the estimate

$$\big|R_{g,g'}^{\Box} f\big| \leq I^{[0,1]} \max_{t\in[0,1]} \big|R_{g'}^{[0,1]} f(\cdot,t)\big| + Q_{g'}^{[0,1]} \max_{s\in[0,1]} \big|R_g^{[0,1]} f(s,\cdot)\big|,$$

Hence, if $f \in C^{2g}(0,1) \times C^{2g'}(0,1)$, we estimate according to (5.1) the quadrature error $R_{g,g'}^{\Box} f$ by

$$R_{g,g'}^{\Box} f \lesssim \frac{2^{-4g}}{(2g)!} \max_{(s,t)\in\Box} \left|\frac{\partial^{2g} f(s,t)}{\partial s^{2g}}\right| + \frac{2^{-4g'}}{(2g')!} \max_{(s,t)\in\Box} \left|\frac{\partial^{2g'} f(s,t)}{\partial t^{2g'}}\right|. \tag{5.2}$$

5.2. Quadrature of the double layer operator

We abbreviate

$$k_K(s,t) := \frac{\partial}{\partial n_t} E\big(\gamma(s), \gamma(t)\big)\alpha(t) = \frac{\alpha(t)}{2\pi} \cdot \frac{n_t\big(\gamma(s) - \gamma(t)\big)}{|\gamma(s) - \gamma(t)|^2}. \tag{5.3}$$

Since the parametrization γ is smooth, we may utilize a Taylor expansion

$$\gamma(s) = \gamma(t) + \gamma'(t)(s-t) + \frac{1}{2}\gamma''(t)(s-t)^2 + \mathcal{O}(|s-t|^3).$$

Observing that $n_t\gamma'(t) = 0$ this implies

$$k_K(s,t) = \frac{1}{2\pi}\begin{cases} \frac{n_t(\gamma(s)-\gamma(t))}{|\gamma(s)-\gamma(t)|^2}\alpha(t), & s \neq t, \\ \frac{n_t\gamma''(t)}{2\alpha(t)}, & s = t. \end{cases}$$

Moreover, employing the Taylor expansion of γ, it is easy to verify that the kernel (5.3) is analytic in $\square$ if γ is analytic in $[0,1]$. Hence, there exists a constant $r > 0$ such that

$$\frac{\partial^\alpha k_K(s,t)}{\partial s^\alpha} \lesssim \frac{\alpha!}{r^\alpha}, \qquad \frac{\partial^\alpha k_K(s,t)}{\partial t^\alpha} \lesssim \frac{\alpha!}{r^\alpha}, \tag{5.4}$$

uniformly for $\alpha \in \mathbb{N}$. According to (2.25) and (4.10), we have to calculate for piecewise constant functions

$$\alpha^{(1)}_{(j,k),(j',k')} = \sqrt{h_j h_{j'}} \int_0^1 \int_0^1 k_K\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big)dtds.$$

Setting $f(s,t) := \sqrt{h_j h_{j'}} k_K\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big)$ the estimate (5.4) implies

$$\frac{\partial^\alpha f(s,t)}{\partial s^\alpha} \lesssim \sqrt{h_j h_{j'}} \frac{h_j^\alpha}{r^\alpha}\alpha!, \qquad \frac{\partial^\alpha f(s,t)}{\partial t^\alpha} \lesssim \sqrt{h_j h_{j'}} \frac{h_{j'}^\alpha}{r^\alpha}\alpha!.$$

Combining this with (5.2) yields

$$R^\square_{g,g'} f \lesssim \sqrt{h_j h_{j'}}\big(2^{-4g} h_j^{2g} r^{-2g} + 2^{-4g'} h_{j'}^{2g'} r^{-2g'}\big). \tag{5.5}$$

Observing $h_j = 2^{-j}$ and $h_{j'} = 2^{-j'}$, a quadrature error $\lesssim \varepsilon_{j,j'}$ is ensured by the choice

$$g_{j,j'} = \left\lceil -\frac{\log_2 \varepsilon_{j,j'} + \frac{j+j'}{2}}{2(j + 2 + \log_2 r)} \right\rceil, \qquad g'_{j,j'} = \left\lceil -\frac{\log_2 \varepsilon_{j,j'} + \frac{j+j'}{2}}{2(j' + 2 + \log_2 r)} \right\rceil, \tag{5.6}$$

For piecewise linear functions we are interested in the degree of the quadrature required for computing

$$\alpha^{(2,i)}_{(j,k),(j',k')} = \sqrt{h_j h_{j'}} \int_0^1 \int_0^1 k_K\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big) p_i(s,t) dtds$$

with a given precision $\varepsilon_{j,j'}$. Herein, $p_i(s,t)$ are polynomials of degree one defined by (2.27). One easily checks

$$|p_i(s,t)| \leq 1, \qquad \Big|\frac{\partial p_i(s,t)}{\partial s}\Big| \leq 1, \qquad \Big|\frac{\partial p_i(s,t)}{\partial t}\Big| \leq 1, \qquad i = 1,2,3,4. \tag{5.7}$$

Substituting $f(s,t) := \sqrt{h_j h_{j'}} k_K\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big) p_i(s,t)$ yields by (5.4) and (5.7) the estimate

$$\frac{\partial^\alpha f(s,t)}{\partial s^\alpha} \lesssim \sqrt{h_j h_{j'}} \frac{h_j^{\alpha-1}}{r^{\alpha-1}} (\alpha-1)!, \qquad \frac{\partial^\alpha f(s,t)}{\partial t^\alpha} \lesssim \sqrt{h_j h_{j'}} \frac{h_{j'}^{\alpha-1}}{r^{\alpha-1}} (\alpha-1)!.$$

Consequently, from (5.2) and the latter estimate we deduce

$$R_{g,g'}^{\square} f \lesssim \sqrt{h_j h_{j'}} \big(2^{-4g} h_j^{2g-1} r^{1-2g} + 2^{-4g'} h_{j'}^{2g'-1} r^{1-2g'}\big). \tag{5.8}$$

Therefore, for getting $R_{g,g'}^{\square} f \lesssim \varepsilon_{j,j'}$ we set

$$g_{j,j'} = \left\lceil -\frac{\log_2 \varepsilon_{j,j'} + \frac{j'-j}{2}}{2(j+2+\log_2 r)} \right\rceil, \qquad g_{j,j'}' = \left\lceil -\frac{\log_2 \varepsilon_{j,j'} + \frac{j-j'}{2}}{2(j'+2+\log_2 r)} \right\rceil. \tag{5.9}$$

5.3. Quadrature of the single layer operator

It is sufficient to develop a quadrature scheme for evaluating double integrals of the kind

$$\beta_{(j,k),(j',k')} := \int_0^1 \int_0^1 k_V\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big) p(s,t) dt ds,$$

where $k_V(s,t)$ denotes the weakly singular kernel function

$$k_V(s,t) := E\big(\gamma(s), \gamma(t)\big) = -\frac{1}{2\pi} \log|\gamma(s) - \gamma(t)|$$

and $p(s,t)$ indicates a polynomial in s and t of degree ≤ 1. The goal is to split the kernel function

$$k_V(s,t) = k_V^{(1)}(s,t) + k_V^{(2)}(s,t)$$

into a smooth kernel $k_V^{(1)}(s,t)$ and a weakly singular, but analytically integrable, kernel $k_V^{(2)}(s,t)$. Then, the second integral on the right hand side of

$$\begin{aligned}\beta_{(j,k),(j',k')} = &\int_0^1 \int_0^1 k_V^{(1)}\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big) p(s,t) dt ds \\ &+ \int_0^1 \int_0^1 k_V^{(2)}\big(t_k^{(j)} + h_j s, t_{k'}^{(j')} + h_{j'} t\big) p(s,t) dt ds\end{aligned} \tag{5.10}$$

can be computed exactly. In the sequel we assume that $j, j' \geq 2$. Consequently, $h_j = \operatorname{diam}[t_k^{(j)}, t_{k+1}^{(j)}] \leq \frac{1}{4}$ and $h_{j'} = \operatorname{diam}[t_{k'}^{(j')}, t_{k'+1}^{(j')}] \leq \frac{1}{4}$. This is no restriction since we may subdivide one or both integrals to ensure this supposition. Denoting by m, m' the midpoints of the integration intervals,

$$m := \frac{1}{2}\Big(t_k^{(j)} + t_{k+1}^{(j)}\Big), \qquad m' := \frac{1}{2}\Big(t_{k'}^{(j')} + t_{k'+1}^{(j')}\Big),$$

there occur three cases:

1. Case: $|m - m'| \le \frac{1}{2}$
   We split the kernel by
   $$\begin{aligned} k_V(s,t) &= -\frac{1}{4\pi}\log|\gamma(s)-\gamma(t)|^2 \\ &= -\frac{1}{4\pi}\log\frac{\big|\gamma(s)-\gamma(t)\big|^2}{(s-t)^2} - \frac{1}{4\pi}\log(s-t)^2 =: k_V^{(1)}(s,t) + k_V^{(2)}(s,t). \end{aligned}$$
   Due to the smoothness of γ we find
   $$k_V^{(1)}(s,t) = -\frac{1}{4\pi}\begin{cases} \log\frac{|\gamma(s)-\gamma(t)|^2}{(s-t)^2}, & s \neq t, \\ \log\big(\alpha^2(t)\big), & s = t, \end{cases}$$
   hence, the kernel $k_V^{(1)}(s,t)$ is analytic in the domain $Q := \{(s,t) \in \square : |s-t| < 1\}$. From $|m - m'| < \frac{1}{2}$ and $h_j, h_{j'} \le \frac{1}{4}$ we deduce that
   $$|s-t| < \frac{3}{4} \qquad \forall\ (s,t) \in [t_k^{(j)}, t_{k+1}^{(j)}] \times [t_{k'}^{(j')}, t_{k'+1}^{(j')}],$$
   i.e., the distance to the singularity point (s,t) with $|s-t| = 1$ is $\ge \frac{1}{4}$.

2. Case: $1 + m - m' < \frac{1}{2}$
   We choose now
   $$k_V^{(1)}(s,t) := -\frac{1}{4\pi}\log\frac{\big|\gamma(s)-\gamma(t)\big|^2}{(1+s-t)^2}, \qquad k_V^{(2)}(s,t) := -\frac{1}{4\pi}\log(1+s-t)^2.$$
   It follows
   $$k_V^{(1)}(s,t) = -\frac{1}{4\pi}\begin{cases} \log\frac{|\gamma(s)-\gamma(t)|^2}{(1+s-t)^2}, & (s,t) \neq (0,1), \\ \log\big(\alpha^2(t)\big), & (s,t) = (0,1), \end{cases}$$
   which implies that the kernel $k_V^{(1)}(s,t)$ is analytic in the domain $Q := \{(s,t) \in \square \setminus (0,1)\}$. From $1 + m - m' < \frac{1}{2}$ and $h_j, h_{j'} \le \frac{1}{4}$, we deduce
   $$|1+s-t| < \frac{3}{4} \quad \forall\ (s,t) \in [t_k^{(j)}, t_{k+1}^{(j)}] \times [t_{k'}^{(j')}, t_{k'+1}^{(j')}],$$
   i.e., the distance to the singularity $(s,t) = (0,1)$ is $\ge \frac{1}{4}$.

3. Case: $1 + m' - m < \frac{1}{2}$

   Similarly to the second case we set

$$k_V^{(1)}(s,t) := -\frac{1}{4\pi} \log \frac{\left|\gamma(s) - \gamma(t)\right|^2}{(1+t-s)^2}, \qquad k_V^{(2)}(s,t) := -\frac{1}{4\pi} \log(1+t-s)^2.$$

   For the kernel $k_V^{(1)}(s,t)$ one finds

$$k_V^{(1)}(s,t) = -\frac{1}{4\pi} \begin{cases} \log \frac{|\gamma(s)-\gamma(t)|^2}{(1+t-s)^2}, & (s,t) \neq (1,0), \\ \log \bigl(\alpha^2(t)\bigr), & (s,t) = (1,0). \end{cases}$$

   The kernel $k_V^{(1)}(s,t)$ is analytic in the domain $Q := \{(s,t) \in \Box \setminus (1,0)\}$. Furthermore, analogously to the second case, the distance to the singularity $(s,t) = (1,0)$ is $\geq \frac{1}{4}$.

In all three cases the second term on the left hand side of (5.10) can be derived by the following primitives

$$\int\int \log(s-t)^2 dtds = \frac{1}{2}(s-t)^2\bigl(3 - \log(s-t)^2\bigr),$$
$$\int\int \log(s-t)^2\, s\, dtds = \frac{1}{18}(s-t)^2\bigl(16s + 11t - (6s+3t)\log(s-t)^2\bigr),$$
$$\int\int \log(s-t)^2\, t\, dtds = \frac{1}{18}(s-t)^2\bigl(16t + 11s - (6t+3s)\log(s-t)^2\bigr),$$
$$\int\int \log(s-t)^2\, s\, t\, dtds = -\frac{1}{16}(s+t)^2\bigl((s+t)^2 + 2(s-t)^2\log(s-t)^2\bigr).$$

We utilize numerical quadrature for the first term on the left hand side of (5.10). As we have seen, in all cases the distance to the weak singularity of $k_V^{(1)}(s,t)$ is $\geq \frac{1}{4}$ independently of (j,k) and (j',k'). Since the kernel $k_V^{(1)}(s,t)$ is analytic except in the singularity, we deduce that there exists a constant $0 < r \leq 1/4$ such that

$$\frac{\partial^\alpha k_V^{(1)}(s,t)}{\partial s^\alpha} \lesssim \frac{\alpha!}{r^\alpha}, \qquad \frac{\partial^\alpha k_V^{(1)}(s,t)}{\partial t^\alpha} \lesssim \frac{\alpha!}{r^\alpha}, \tag{5.11}$$

uniformly for $\alpha \in \mathbb{N}$. Since (5.11) agrees with (5.4) we may adopt the argumentation of the previous subsection. Therefore, to ensure a quadrature error $\lesssim \varepsilon_{j,j'}$ we have to choose the degrees $g_{j,j'}$ and $g'_{j,j'}$ from (5.6) if $d = 1$ and from (5.9) if $d = 2$.

5.4. The recycling scheme

Using the analysis of the wavelet Galerkin scheme as a guideline, it turns out that the required accuracy for the computation of the integrals $\alpha^{(1)}_{(j,k),(j',k')}$ and $\alpha^{(2,i)}_{(j,k),(j',k')}$, respectively, depends strongly on the involved scales (j, j'), cf. [12, 32, 35]. In particular, the highest accuracy is required on the coarsest levels, that is $(j, j') = (j_0, j_0)$, while for the integrals on the finest levels, that is $(j, j') = (J, J)$, the required accuracy is low. More precisely, it is sufficient to choose

$$\varepsilon_{j,j'} \lesssim 2^{-2J(d-q)} 2^{(j+j')d},$$

cf. [5, 12, 32]. According to the formulas (5.6) and (5.9) one has to be very careful if the parameter r is small. In this case, one has to avoid denominators close to zero. This requires that $h_j, h_{j'} < r$. In order to satisfy this condition, one has to subdivide the large domains of integration. On the other hand, on finer scales we already have to compute such integrals over small domains. Therefore, we propose another approach. We compute integrals on the fine scales with the accuracy required on the coarse scales and reuse these results also for the computation of the coarse scale coefficients. This reduces the number of functions significantly.

In particular, by the formulas (4.9) and (4.13) the computation of an matrix entry $\big(A\psi^{(d,\tilde{d})}_{j',k'}, \psi^{(d,\tilde{d})}_{j,k}\big)_{L^2(0,1)}$ of the compressed system matrix $\mathbf{A}_\psi$ reduces to the computation of the integrals $\alpha^{(1)}_{(j+1,2k+l),(j'+1,2k'+l')}$ and $\alpha^{(2,i)}_{(j+1,2k+l),(j'+1,2k'+l')}$, respectively. The crucial idea of the *recycling scheme* is to invoke the refinement relation (3.2) of the scaling functions. Then, for $d = 1$ one finds for the integrals $\alpha^{(1)}_{(j,k),(j',k')}$ the relations

$$\begin{aligned}
\alpha^{(1)}_{(j,k),(j',k')} &= \frac{1}{\sqrt{2}}\Big(\alpha^{(1)}_{(j+1,2k),(j',k')} + \alpha^{(1)}_{(j+1,2k+1),(j',k')}\Big),\\
\alpha^{(1)}_{(j,k),(j',k')} &= \frac{1}{\sqrt{2}}\Big(\alpha^{(1)}_{(j,k),(j'+1,2k')} + \alpha^{(1)}_{(j,k),(j'+1,2k'+1)}\Big).
\end{aligned}$$

Since for $d = 2$ we compute the integrals over the smooth parts of the scaling functions, we additionally invoke (4.11) which gives for the integrals $\alpha^{(2,i)}_{(j,k),(j',k')}$ the relations

$$\begin{aligned}
\alpha^{(2,1)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,1)}_{(j+1,2k),(j',k')} + 2\alpha^{(2,1)}_{(j+1,2k+1),(j',k')} + \alpha^{(2,3)}_{(j+1,2k+1),(j',k')}\Big),\\
\alpha^{(2,2)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,2)}_{(j+1,2k),(j',k')} + 2\alpha^{(2,2)}_{(j+1,2k+1),(j',k')} + \alpha^{(2,4)}_{(j+1,2k+1),(j',k')}\Big),\\
\alpha^{(2,3)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,1)}_{(j+1,2k),(j',k')} + 2\alpha^{(2,3)}_{(j+1,2k),(j',k')} + \alpha^{(2,3)}_{(j+1,2k+1),(j',k')}\Big),\\
\alpha^{(2,4)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,4)}_{(j+1,2k),(j',k')} + 2\alpha^{(2,4)}_{(j+1,2k),(j',k')} + \alpha^{(2,2)}_{(j+1,2k+1),(j',k')}\Big),
\end{aligned}$$

and

$$
\begin{aligned}
\alpha^{(2,1)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,1)}_{(j,k),(j'+1,2k')} + 2\alpha^{(2,1)}_{(j,k),(j'+1,2k'+1)} + \alpha^{(2,2)}_{(j,k),(j'+1,2k'+1)}\Big),\\
\alpha^{(2,2)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,1)}_{(j,k),(j'+1,2k')} + 2\alpha^{(2,2)}_{(j,k),(j'+1,2k')} \quad + \alpha^{(2,2)}_{(j,k),(j'+1,2k'+1)}\Big),\\
\alpha^{(2,3)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,3)}_{(j,k),(j'+1,2k')} + 2\alpha^{(2,3)}_{(j,k),(j'+1,2k'+1)} + \alpha^{(2,4)}_{(j,k),(j'+1,2k'+1)}\Big),\\
\alpha^{(2,4)}_{(j,k),(j',k')} &= \frac{1}{2\sqrt{2}}\Big(\alpha^{(2,3)}_{(j,k),(j'+1,2k')} + 2\alpha^{(2,4)}_{(j,k),(j'+1,2k')} \quad + \alpha^{(2,4)}_{(j,k),(j'+1,2k'+1)}\Big).
\end{aligned}
$$

Hence, exploiting these *recycling formulas* one may use already computed values for the computation of $\alpha^{(1)}_{(j,k),(j',k')}$ and $\alpha^{(2,i)}_{(j,k),(j',k')}$ instead of applying a direct quadrature. Starting the computation of the required values $\big(A\psi^{(d,\tilde{d})}_{j',k'}, \psi^{(d,\tilde{d})}_{j,k}\big)_{L^2(0,1)}$ of the compressed matrix $\mathbf{A}_\psi$ on the finest levels, i.e., calculating first the entries for which $(j,j') = (J-1, J-1)$, a systematically exploitation of the recycling formulas leads to a natural subdivision of the coarser scales. We mention that the structure of the compressed matrix implies that almost all computed $\alpha^{(1)}_{(j,k),(j',k')}$ can be used for the computation of the required $\alpha^{(1)}_{(j-1,k),(j',k')}$ and $\alpha^{(1)}_{(j,k),(j'-1,k')}$, and likewise the $\alpha^{(2,i)}_{(j,k),(j',k')}$. Moreover, this strategy reduces the number of those values, for which a direct quadrature is required.

By using the above recycling techniques the computed integrals on the finer scales are part of the integrals on the coarser scales. Assuming the worst case, $\alpha^{(1)}_{(j_0,k),(j_0,k')}$ and $\alpha^{(2,i)}_{(j_0,k),(j_0,k')}$ consist of $2^{(j-j_0)+(j'-j_0)}$ values on the scale (j,j'). On the other hand, we have to take into consideration the L^2-normalization of the scaling functions. It turns out that we attain a quadrature error $\lesssim \varepsilon_{j_0,j_0}$ on the coarsest level (j_0,j_0) for both, piecewise constant and piecewise linear functions, by choosing

$$
g_{j,j'} = \left\lceil -\frac{\log_2 \varepsilon_{j_0,j_0}}{2(j+2+\log_2 r)} \right\rceil, \qquad g'_{j,j'} = \left\lceil -\frac{\log_2 \varepsilon_{j_0,j_0}}{2(j'+2+\log_2 r)} \right\rceil, \tag{5.12}
$$

Herein, $\varepsilon_{j_0,j_0} \lesssim 2^{-2J(d-q)}$ is required to attain the optimal order of convergence $2(d-q)$ of the Galerkin scheme. As one easily checks, for $\varepsilon_{j_0,j_0} = 2^{-2J(d-q)}$ the quadrature degree on the finest scale tends to the (fixed) degree $g_{J,J} = g'_{J,J} = \lceil d-q \rceil$ if $J \to \infty$. We mention that this degree of quadrature is also necessary to obtain the optimal order of convergence in the traditional Galerkin scheme.

The main fact to reduce the number of the nonzero coefficients of $\mathbf{A}_\psi$ from $\mathcal{O}(N_J \log^3 N_J)$ to an optimal rate $\mathcal{O}(N_J)$ relies on the second compression. This compression is active if

$$
\operatorname{dist}\big(\Theta^{(d,\tilde{d})}_{j,k}, \Theta^{(d,\tilde{d})}_{j',k'}\big) \lesssim 2^{-\min\{j,j'\}}.
$$

As shown in [35, p. 177] the block matrix $\big[\mathbf{A}_\psi\big]_{(j,\triangle_j),(j',\triangle_{j'})}$, $j > j'$, consists in this case of $\mathcal{O}\big(\dim(W_j)^\alpha\big)$ instead of $\mathcal{O}\big(\dim(W_j)\big)$ entries. Here, $\alpha = \alpha(j, j')$ denotes some paramter with $0 < \alpha < 1$. Choosing the degrees of quadrature as in (5.12), each matrix element can be computed with $\mathcal{O}\big(J^2/((j+c)(j'+c))\big)$ function calls. From an estimate of the form

$$2^{-\alpha j} j^2 \lesssim 2^{-\alpha' j}$$

with some $0 < \alpha' < \alpha$, the argumentation in [35] can be repeated to prove that the over-all number of function calls is $\mathcal{O}(2^J) = \mathcal{O}(N_J)$. Since the detailed proof of this result is rather technical, we refer to [12] for further details. Let us remark, that the proof in [12] is also valid in the three dimensional case where singular kernel functions appear. Indeed, the present case is much easier since only smooth kernels are involved. Employing the recycling technique gives a further reduction of the complexity. However, the asymptotic behaviour cannot exceed the optimal complexity rate.

6. Numerical results

6.1. The model problem

For testing the algorithms we introduce a numerical example for which the solution is known analytically. We choose Γ given by

$$\gamma : [0,1] \to \Gamma, \qquad \gamma(t) = \tfrac{1}{20}\big(4 + \cos(10\pi t) + \cos(2\pi t)\big)\begin{bmatrix}\cos(2\pi t)\\ \sin(2\pi t)\end{bmatrix},$$

and Ω as the interior domain Ω^-, see figure 2 for this constellation.

One easily checks that the function

$$U(x) = \frac{x_1 - 2x_2 + 0.2}{(x_1 - 0.2)^2 + (x_2 - 0.2)^2} \tag{6.13}$$

is harmonic in $\mathbb{R}^2 \setminus \{\left[\begin{smallmatrix}0.2\\0.2\end{smallmatrix}\right]\}$. Hence, the interior Dirichlet problem

$$\begin{aligned} -\triangle u &= 0 \qquad && \text{in } \Omega,\\ u &= U\big|_\Gamma && \text{on } \Gamma, \end{aligned} \tag{6.14}$$

has the unique solution $u = U$. The function U is plotted in figure 2. Since U is harmonic in $\overline{\Omega}$, it follows that

$$\int_\Gamma \frac{\partial U(x)}{\partial n_x} ds_x = \int_\Gamma n_x^T \nabla U(x) ds_x = 0.$$

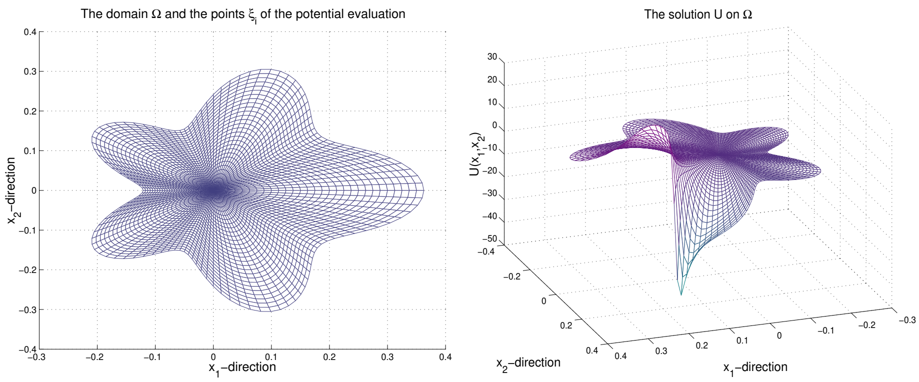

Figure 2: The test domain Ω with the points ξ_i of the potential evaluation (left) and the function $U(x) = \frac{x_1-2x_2+0.2}{(x_1-0.2)^2+(x_2-0.2)^2}$ on Ω (right).

Therefore, the interior Neumann problem

$$\begin{aligned} -\triangle u &= 0 \qquad \text{in } \Omega, \\ u &= \frac{\partial U}{\partial n} \qquad \text{on } \Gamma, \end{aligned} \tag{6.15}$$

has the unique solution $u = U + c$, where c denotes some real constant.

In the sequel we solve both, the Dirichlet and the Neumann problem, by the boundary integral equations described in this paper. We compute the approximated density vector $\boldsymbol{\mu}_\phi$ corresponding to the single-scale basis by the traditionally single-scale Galerkin scheme (4.1) and by the wavelet Galerkin scheme (4.2). For assembling the Galerkin matrix with respect to the single-scale basis, we choose the fixed degree of quadrature $\lceil d-q \rceil$. In the wavelet Galerkin scheme the degree of quadrature is computed by (5.12) with $\varepsilon_{j_0,j_0} = 2^{-2J(d-q)}$ and $r = 1/4$. We determine the numerical solution u_ϕ and u_ψ, respectively, by the potential evaluations (2.13) and (2.17) in fixed points ξ_i lying in Ω, cf. figure 2. For the Dirichlet problem (6.14) we measure the absolute errors $\max_i |(u-u_\phi)(\xi_i)|$ and $\max_i |(u-u_\psi)(\xi_i)|$, respectively. Since for the Neumann problem (6.15) the solution is given modulo a constant, we calculate in this case the absolute errors $\min_{c\in\mathbb{R}} \max_i |(u-u_\phi+c)(\xi_i)|$ and $\min_{c\in\mathbb{R}}\{\max_i |(u-u_\psi+c)(\xi_i)|\}$, respectively.

6.2. The choice of the compression parameters

First we study the influence of the parameters a, a', δ and δ' to the compression and accuracy of the wavelet Galerkin scheme. For this, we solve the interior Dirichlet problem (6.14) by the equation of the first kind (2.5) and piecewise linear wavelets $\Psi_J^{(2,4)}$. Since the single layer operator $\mathcal{V}$ is an operator of order -1, condition (3.11) implies that $a, a' > 1$ and $2 < \delta, \delta' < 3$. For the sake of simplicity we set $a = a'$ and $\delta = \delta'$ and choose a fixed $N_J := 2048$. We measure the density of $\mathbf{V}_\psi$ and the absolute error of u_ψ for the parameters a and δ lying in the range $1 \le a \le 2$ and $2 \le \delta \le 3$. Herein, the density is the ratio of the number of nonzero elements divided by N_J^2. The obtained results are shown in figure 3. From the plot on the left hand side one deduces that the absolute error of the numerical solution u_ψ does not drop under a fixed value which is independent of a and δ. Obviously, this value is the discretization error. On the other hand, if one chooses $a + \delta$ too small the error of compression is larger than the discretization error. Hence, the error of the numerical solution u_ψ increases. The plot on the right hand side shows the density of the system matrix $\mathbf{V}_\psi$ while changing the parameters. One observes that the density seems to grow linearly with $a+\delta$. Summarizing it turns out, that $a + \delta$ has to be big enough to ensure that the compression error is not higher than the discretization error. But $a + \delta$ should be as small as possible to obtain the best compression rate.

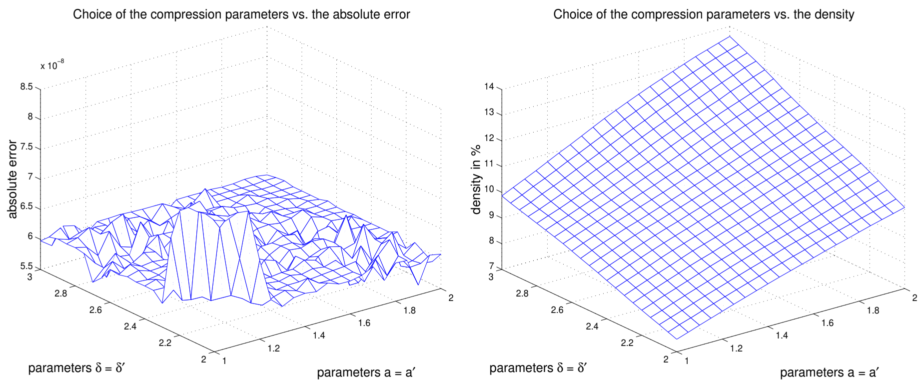

Figure 3: The choice of the parameters a, a', δ and δ' from (3.11) vs. the absolute error (left) and vs. the density of the compressed matrix (right).

6.3. The asymptotic behaviour of the compression

Next, we count the number of nonzero elements in the compressed system matrices while increasing J. To satisfy (3.11) we have to choose wavelets with a sufficient numer of vanishing moments $\tilde{d}$. Since, however, the supports of the wavelets increase proportionally with the number of vanishing moments (which reduces in our experience the compression rates), one has to apply wavelets with minimal number of vanishing moments and minimal supports, respectively. Hence, we choose $\Psi_J^{(1,3)}$ or $\Psi_J^{(2,4)}$ for the discretization of the single and double layer operator and $\Psi_J^{(2,2)}$ for the discretization of the hypersingular operator, respectively. For the computation of the compression we choose the parameters from (3.11) for the double and single layer operator as $a = a' = \delta = \delta' = 1.5$ if $d = 1$ and $a = a' = 1.25$, $\delta = \delta' = 2.5$ if $d = 2$. For the hypersingular operator we choose $a = a' = 1.5$ and $\delta = \delta' = 2.5$. As the numerical results in the next subsections confirm these settings are sufficient to attain approximatively the accuracy of the traditional single-scale Galerkin scheme.

The diagrams in figure 4 show the number of nonzero elements of the compressed system matrices corresponding to the double layer operator $\mathcal{K}$, the single layer operator $\mathcal{V}$ and the hypersingular operator $\mathcal{W}$. On the left hand side one finds the number of nonzero elements of the system matrices $\mathbf{K}_\psi$ and $\mathbf{V}_\psi$ computed with respect to the piecewise constant wavelets $\Psi_J^{(1,3)}$. On the right hand side one sees the number of nonzero elements of the system matrices $\mathbf{K}_\psi$, $\mathbf{V}_\psi$ and $\mathbf{W}_\psi$ computed with respect to the piecewise linear wavelets $\Psi_J^{(2,4)}$ and $\Psi_J^{(2,2)}$, respectively. Both diagrams confirm that we obtain asymptotically only $\mathcal{O}(N_J)$ nonzero matrix elements, whereby the constant for the piecewise linear wavelets is larger than for the piecewise constant wavelets.

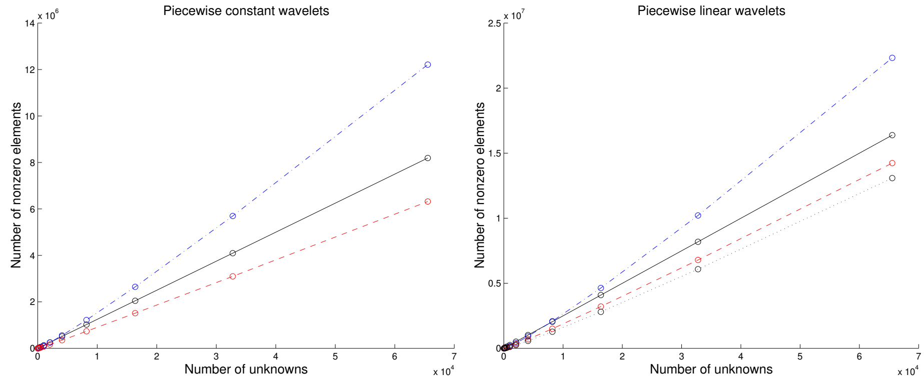

Figure 4: The number of nonzero elements of the compressed system matrices: the dashed line belongs to $\mathbf{V}_\psi$, the dashdotted line belongs to $\mathbf{K}_\psi$, the dotted line belongs to $\mathbf{W}_\psi$ and the solid line indicates linear behaviour ($125N_J$ on the left hand side and $250N_J$ on the right hand side).

6.4. The wavelet preconditioner

Since the single layer operator $\mathcal{V}$ and the hypersingular operator $\mathcal{W}$ are operators of order different from zero, the resulting Galerkin matrices $\mathbf{V}_\phi$, $\mathbf{V}_\psi$, $\mathbf{W}_\phi$ and $\mathbf{W}_\psi$ are ill conditioned. More precisely, without any preconditioning the condition numbers in l^2 behave like 2^J, where J indicates the level of discretization. According to subsection 3.5. the wavelet approach provides a simple diagonal preconditioner.

In table 1 we compare the l_2-condition of the matrices $\mathbf{V}_\phi$ and $\mathbf{W}_\phi$ and the preconditioned matrices $\mathbf{V}_\psi$ and $\mathbf{W}_\psi$. The given numbers result from a diagonal scaling of the computed system matrices, which additionally improves the wavelet preconditioner (3.13). As one figures out of the given results, the behaviour of the condition numbers of the single-scale matrices is $\sim 2^J$ while the condition numbers of the system matrices with respect to the (preconditioned) wavelet bases seem to

Level J		5	6	7	8	9	10	11
Unknowns N_J		32	64	128	256	512	1024	2048
single layer operator	$\Phi_J^{(1)}$	54.37	107.3	219.8	447.2	900.2	1800	3588
	$\Psi_J^{(1,3)}$	8.887	8.967	8.992	9.005	9.010	9.013	9.014
	$\Phi_J^{(2)}$	131.2	268.8	552.8	1137	2308	4640	9291
	$\Psi_J^{(2,4)}$	58.87	65.69	72.71	77.30	81.17	84.01	86.29
hypersingular operator	$\Phi_J^{(2)}$	57.03	130.1	290.6	641.9	1409	3070	6645
	$\Psi_J^{(2,2)}$	4.889	4.946	4.950	4.950	4.951	4.951	4.951

Table 1: The condition numbers of the diagonally scaled system matrices $\mathbf{V}_\phi$, $\mathbf{V}_\psi$, $\mathbf{W}_\phi$ and $\mathbf{W}_\psi$, respectively.

be bounded. Note, since the matrices $\mathbf{W}_\phi$ and $\mathbf{W}_\psi$, respectively, are only positive semidefinite, we discard one unknown to compute the condition number.

6.5. Numerical results for the Dirichlet problem

We consider the numerical solution of the given Dirichlet problem (6.14). In the tables 2 and 3 we list the over-all computing time and the accuracy obtained by the traditional (single-scale) Galerkin scheme and the compressed wavelet Galerkin scheme. Precisely, table 2 shows the values attained by piecewise constant functions for the equation of the first kind (2.21) while in table 3 we give the results for piecewise linear functions for the equation of the second kind (2.23). We determine for the single-scale scheme and the multiscale scheme the absolute errors of the numerical solutions u_ϕ and u_ψ as described above. The columns titled by “contr.” (contraction) give the ratio of the absolute error divided by the absolute error of the previous level. Since the optimal convergence rate is given by $2(d-r)$, these numbers should be close to $2^{2(d-q)}$. Hence, the optimal ratios 8 in table 2 and 16 in table 3 are achieved approximately. The data listed in brackets are determined via extrapolation, since these problems do not fit into the available main memory of the computer. As one figures out of the tables 2 and 3, the accuracy is not deteriorated by the compression strategy. The break even point of the compressed wavelet Galerkin scheme is about 256 unknowns. Moreover, as the columns titled by “density” (in %) confirm, the compression yields an enormous save of memory.

6.6. Numerical results for the Neumann problem

For the Neumann problem (6.15) we tabulate similarly to the Dirichlet problem the over-all computing time and the accuracy attained by the traditional (single-scale) Galerkin scheme and the compressed wavelet Galerkin scheme. Table 4 gives the results for the equation of the second kind (2.38) obtained with piecewise constant functions and table 5 shows the values for the equation of the first kind (2.34) measured for piecewise linear functions. Likewise to the Dirichlet problem the optimal convergence rate is $2(d-q)$. Therefore, the optimal contraction is $2^{2(d-q)}$.

N_J	single-scale scheme: $d=1$			multiscale scheme: $(d,\tilde{d})=(1,3)$			
	abs. error	contr.	time	abs. error	contr.	time	density
128	8.8e-01	—	0.17	8.7e-01	—	0.21	48.4
256	8.4e-02	11	0.71	8.4e-02	10	0.56	30.5
512	8.7e-03	9.6	2.91	8.7e-03	9.6	1.51	18.0
1024	1.0e-03	8.3	13.2	1.0e-03	8.3	3.75	10.5
2048	1.3e-04	8.1	54.7	1.3e-04	8.2	9.58	5.94
4096	1.6e-05	8.1	225	1.6e-05	8.1	21.4	3.31
8192	(2.0e-06)	(8.0)	(902)	2.0e-06	8.0	50.3	1.82
16384	(2.5e-07)	(8.0)	(3609)	2.5e-07	7.8	116	0.99
32768	(3.1e-08)	(8.0)	(14438)	3.3e-08	7.8	267	0.53
65536	(3.9e-09)	(8.0)	(57751)	4.2e-09	7.8	597	0.28
131072	(4.9e-10)	(8.0)	(231000)	5.4e-10	7.7	1439	0.15

Table 2: Over-all computing times (in seconds) and accuracy of the single-scale and the multiscale scheme obtained for the single layer operator and piecewise constant functions.

N_J	single-scale scheme: $d=2$			multiscale scheme: $(d,\tilde{d})=(2,4)$			
	abs. error	contr.	time	abs. error	contr.	time	density
128	6.0e-01	—	0.13	6.0e-01	—	0.22	62.9
256	2.3e-02	26.5	0.51	2.3e-02	26.5	0.62	39.5
512	1.1e-03	20.3	2.18	1.1e-03	20.3	1.62	23.7
1024	4.0e-05	28.0	10.0	4.0e-05	28.0	4.18	13.6
2048	2.4e-06	16.3	40.6	2.4e-06	16.3	10.5	7.61
4096	1.5e-07	16.1	166	1.5e-07	16.1	23.7	4.18
8192	(9.5e-09)	(16.0)	(663)	9.5e-09	16.0	55.6	2.25
16384	(5.9e-10)	(16.0)	(2653)	5.9e-10	16.0	125	1.20
32768	(3.7e-11)	(16.0)	(10612)	3.7e-11	16.1	268	0.63
65536	(2.3e-12)	(16.0)	(42447)	2.3e-12	16.2	700	0.33

Table 3: Over-all computing times (in seconds) and accuracy of the single-scale and the multiscale scheme obtained for the double layer operator and piecewise linear functions.

This is 4 in table 4 and 8 in table 5. The data listed in brackets signify again extrapolated results.

N_J	single-scale scheme: $d=1$			multiscale scheme: $(d,\tilde{d})=(1,3)$			
	abs. error	contr.	time	abs. error	contr.	time	density
128	1.35	—	0.02	1.35	—	0.09	42.2
256	2.0e-01	6.7	0.12	2.0e-01	6.6	0.19	24.9
512	6.1e-02	3.3	0.65	6.1e-02	3.3	0.50	13.9
1024	1.5e-02	4.0	4.03	1.5e-02	4.0	1.09	7.54
2048	3.8e-03	4.0	16.4	3.8e-03	4.0	2.69	4.02
4096	9.5e-04	4.0	66.5	9.5e-04	4.0	6.03	2.10
8192	(2.4e-04)	(4.0)	(266)	2.4e-04	4.0	13.4	1.09
16384	(5.9e-05)	(4.0)	(1064)	5.9e-05	4.0	28.3	0.56
32768	(1.5e-05)	(4.0)	(4255)	1.5e-05	4.0	61.6	0.29
65536	(3.7e-06)	(4.0)	(17019)	3.7e-06	4.0	137	0.15
131072	(9.3e-07)	(4.0)	(68076)	9.4e-07	4.0	326	0.08
262144	(2.3e-07)	(4.0)	(272304)	2.4e-07	4.0	856	0.04

Table 4: Over-all computing times (in seconds) and accuracy of the single-scale and the multiscale scheme obtained for the double layer operator and piecewise constant functions.

N_J	single-scale scheme: $d=2$			multiscale scheme: $(d,\tilde{d})=(2,2)$			
	abs. error	contr.	time	abs. error	contr.	time	density
128	9.6e-01	—	0.18	1.09	—	0.21	48.4
256	7.1e-02	14	0.73	1.0e-01	11	0.56	30.5
512	8.3e-03	8.6	3.10	7.7e-03	13	1.52	18.2
1024	1.0e-03	8.1	17.5	8.1e-04	9.5	3.79	10.7
2048	1.3e-04	8.0	81.8	1.0e-04	8.0	9.68	6.11
4096	1.6e-05	8.0	399	1.3e-05	8.0	21.9	3.44
8192	(2.0e-06)	(8.0)	(1595)	1.6e-06	8.0	51.8	1.91
16384	(2.5e-07)	(8.0)	(6378)	2.0e-07	7.9	121	1.04
32768	(3.1e-08)	(8.0)	(25513)	2.6e-08	7.8	281	0.57
65536	(3.9e-09)	(8.0)	(102050)	3.3e-09	7.8	641	0.31
131072	(4.9e-10)	(8.0)	(408210)	4.3e-10	7.7	1544	0.16

Table 5: Over-all computing times (in seconds) and accuracy of the single-scale and the multiscale scheme obtained for the hypersingular operator and piecewise linear functions.

References

[1] G. Beylkin, R. Coifman, and V. Rokhlin. The fast wavelet transform and numerical algorithms. *Comm. Pure and Appl. Math.*, 44:141–183, 1991.

[2] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations – convergence rates. *IGPM-Report, RWTH Aachen*, 1998. to appear in Math. Comp.

[3] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Pure Appl. Math.*, 45:485–560, 1992.

[4] M. Costabel. Principles of boundary element methods. *Comp. Phys. Reports*, 6:243–274, 1987.

[5] W. Dahmen. Wavelet and multiscale methods for operator equations. *Acta Numerica*, 6:55–228, 1997.

[6] W. Dahmen, B. Kleemann, S. Prößdorf, and R. Schneider. A multiscale method for the double layer potential equation on a polyhedron. In H.P. Dikshit and C.A. Micchelli, editors, *Advances in Computational Mathematics*, pages 15–57, World Scientific Publ., Singapore, 1994.

[7] W. Dahmen and A. Kunoth. Multilevel preconditioning. *Numer. Math.*, 63:315–344, 1992.

[8] W. Dahmen, S. Prößdorf, and R. Schneider. Multiscale methods for pseudodifferential equations. In L.L. Schumaker and G. Webb, editors, *Wavelet Analysis and its Applications*, volume 3, pages 191–235, 1993.

[9] W. Dahmen, S. Prößdorf, and R. Schneider. Wavelet approximation methods for periodic pseudodifferential equations. Part II – Fast solution and matrix compression. *Advances in Computational Mathematics*, 1:259–335, 1993.

[10] W. Dahmen, S. Prößdorf, and R. Schneider. Wavelet approximation methods for periodic pseudodifferential equations. Part I – Convergence analysis. *Mathematische Zeitschrift*, 215:583–620, 1994.

[11] W. Dahmen, S. Prößdorf, and R. Schneider. Multiscale methods for pseudodifferential equations on smooth manifolds. In C.K. Chui, L. Montefusco, and L. Puccio, editors, *Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications*, pages 385–424, 1995.

[12] W. Dahmen and R. Schneider. Wavelets on manifolds II: Application to boundary element methods and pseudodifferential equations. Manuscript.

[13] W. Dahmen and R. Schneider. Composite wavelet basis for operator equations. *Math. Comp.*, 68:1533–1567, 1999.

[14] W. Dahmen and R. Schneider. Wavelets on manifolds I. Construction and domain decomposition. *Math. Anal.*, 31:184–230, 1999.

[15] C. de Boor. *A practical guide to splines*. Springer, 1978.

[16] R. DeVore. Nonlinear approximation. *Acta Numerica*, 7:51–150, 1998.

[17] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Functional Anal.*, 93:34–170, 1990.

[18] G. Gatica and G. Hsiao. *Boundary-field equation methods for a class of nonlinear problems*, volume 331 of *Pitman research notes in mathematics series*. Longman, Essex, 1995.

[19] L. Greengard and V. Rokhlin. A fast algorithm for particle simulation. *J. Comput. Phys.*, 73:325–348, 1987.

[20] W. Hackbusch. *Integralgleichungen*. B.G. Teubner, Stuttgart, 1989.

[21] W. Hackbusch. A sparse matrix arithmetic based on $\mathcal{H}$-matrices. Part I: Introduction to $\mathcal{H}$-matrices. *Computing*, 64:89–108, 1999.

[22] W. Hackbusch and Z.P. Nowak. On the fast matrix multiplication in the boundary element method by panel clustering. *Numer. Math.*, 54:463–491, 1989.

[23] G. Hämmerlin and K.-H. Hoffmann. *Numerische Mathematik*. Springer-Verlag, Berlin-Heidelberg, 1992. 3. Auflage.

[24] H. Harbrecht, F. Paiva, C. Pérez, and R. Schneider. Biorthogonal wavelet approximation for the coupling of FEM-BEM. *Preprint SFB 393/99-32, TU Chemnitz*, 1999. submitted to Numer. Math.

[25] H. Harbrecht, F. Paiva, C. Pérez, and R. Schneider. Wavelet preconditioning for the coupling of FEM-BEM. *Preprint SFB 393/00-07, TU Chemnitz*, 2000. submitted to J. Numer. Lin. Alg. with Appl.

[26] G. Hsiao and W. Wendland. A finite element method for some equations of first kind. *J. Math. Anal. Appl.*, 58:449–481, 1977.

[27] S. Jaffard. Wavelet methods for fast resolution of elliptic equations. *SIAM J. Numer. Anal.*, 29:965–986, 1992.

[28] R. Kress. *Linear Integral Equations*. Springer-Verlag, Berlin-Heidelberg, 1989.

[29] S.V. Pereverzev. Hyperbolic cross and the complexity of the approximate solution of Fredholm integral equations of the second kind with differentiable kernels. *Sib. Math. J.*, 32(1):85–92, 1991. English, Russian original, translation from Sib. Mat. Zh. 32, 1(185):107–115, 1991.

[30] T. von Petersdorff, R. Schneider, and C. Schwab. Multiwavelets for second kind integral equations. *SIAM J. Num. Anal.*, 34:2212–2227, 1997.

[31] T. von Petersdorff and C. Schwab. Wavelet approximation for first kind integral equations on polygons. *Numer. Anal.*, 74:479–516, 1996.

[32] T. von Petersdorff and C. Schwab. Fully discretized multiscale Galerkin BEM. In W. Dahmen, A. Kurdila, and P. Oswald, editors, *Multiscale wavelet methods for PDEs*, pages 287–346, Academic Press, San Diego, 1997.

[33] A. Rathsfeld. On a hierarchical three point basis of piecewise linear functions over smooth boundaries. to appear in the proceedings of the *11. Tagung math. Physik*, Birkhäuser.

[34] A. Rathsfeld. A wavelet algorithm for the boundary element solution of a geodetic boundary value problem. *Comput. Methods Appl. Mech. Engrg.*, 157:267–287, 1998.

[35] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung großer vollbesetzter Gleichungssysteme*. B.G. Teubner, Stuttgart, 1998.

[36] V.M. Tyrtyshnikov. Mosaic sceleton approximation. *Calcolo*, 33:47–57, 1996.

Helmut Harbrecht and Reinhold Schneider
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz, Germany

Other titles in the SFB393 series:

00-01 G. Kunert. Anisotropic mesh construction and error estimation in the finite element method. January 2000.

00-02 V. Mehrmann, D. Watkins. Structure-preserving methods for computing eigenpairs of large sparse skew-Hamiltonian/Hamiltonian pencils. January 2000.

00-03 X. W. Guan, U. Grimm, R. A. Römer, M. Schreiber. Integrable impurities for an open fermion chain. January 2000.

00-04 R. A. Römer, M. Schreiber, T. Vojta. Disorder and two-particle interaction in low-dimensional quantum systems. January 2000.

00-05 P. Benner, R. Byers, V. Mehrmann, H. Xu. A unified deflating subspace approach for classes of polynomial and rational matrix equations. January 2000.

00-06 M. Jung, S. Nicaise, J. Tabka. Some multilevel methods on graded meshes. February 2000.

00-07 H. Harbrecht, F. Paiva, C. Perez, R. Schneider. Multiscale Preconditioning for the Coupling of FEM-BEM. February 2000.

00-08 P. Kunkel, V. Mehrmann. Analysis of over- and underdetermined nonlinear differential-algebraic systems with application to nonlinear control problems. February 2000.

00-09 U.-J. Görke, A. Bucher, R. Kreißig, D. Michael. Ein Beitrag zur Lösung von Anfangs-Randwert-Problemen einschließlich der Materialmodellierung bei finiten elastisch-plastischen Verzerrungen mit Hilfe der FEM. März 2000.

00-10 M. J. Martins, X.-W. Guan. Integrability of the D_n^2 vertex models with open boundary. March 2000.

00-11 T. Apel, S. Nicaise, J. Schöberl. A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges. March 2000.

00-12 B. Lins, P. Meade, C. Mehl, L. Rodman. Normal Matrices and Polar Decompositions in Indefinite Inner Products. March 2000.

00-13 C. Bourgeois. Two boundary element methods for the clamped plate. March 2000.

00-14 C. Bourgeois, R. Schneider. Biorthogonal wavelets for the direct integral formulation of the heat equation. March 2000.

00-15 A. Rathsfeld, R. Schneider. On a quadrature algorithm for the piecewise linear collocation applied to boundary integral equations. March 2000.

00-16 S. Meinel. Untersuchungen zu Druckiterationsverfahren für dichteveränderliche Strömungen mit niedriger Machzahl. März 2000.

00-17 M. Konstantinov, V. Mehrmann, P. Petkov. On Fractional Exponents in Perturbed Matrix Spectra of Defective Matrices. April 2000.

00-18 J. Xue. On the blockwise perturbation of nearly uncoupled Markov chains. April 2000.

00-19 N. Arada, J.-P. Raymond, F. Tröltzsch. On an Augmented Lagrangian SQP Method for a Class of Optimal Control Problems in Banach Spaces. April 2000.

00-20 H. Harbrecht, R. Schneider. Wavelet Galerkin Schemes for 2D-BEM. April 2000.

The complete list of current and former preprints is available via
`http://www.tu-chemnitz.de/sfb393/preprints.html`.