Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

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On Fractional Exponents in Perturbed Matrix Spectra of Defective Matrices

Preprint SFB393/00-17

Preprint-Reihe des Chemnitzer SFB 393

SFB393/00-17 **April 2000**

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ON FRACTIONAL EXPONENTS IN PERTURBED MATRIX SPECTRA OF DEFECTIVE MATRICES

Abstract

In this paper we discuss the perturbation theory for defective matrices. We consider the asymptotic expansions of the perturbed spectrum, when the matrix A is perturbed to $A + tE$ and in particular we analyse the rational exponents that may occur and discuss the possible exponents of the leading coefficients when we vary the matrix E over the unit sphere $\|E\| = 1$. We partially characterize the possible leading coefficients but the description of the set of all leading exponents remains a challenging open problem.

Keywords: Perturbation of eigenvalues, defective matrices, matrix computations

AMS Subject classification: 15A18, 65F15

1 Introduction and Preliminaries

We consider perturbation theory for eigenvalues of real or complex matrices. Obtaining perturbation bounds (including asymptotic estimates) for the eigenvalues of A under perturbations is one of the most studied problems in matrix perturbation theory [10]. In particular, the eigenvalue problem for defective matrices is of great interest in view of some theoretical and significant numerical difficulties accompanying its solution, see e.g. [5, 1]. We recall that an $n \times n$ matrix A is defective if it has less than n linearly independent eigenvectors, or equivalently if it is not diagonalizable, i.e., it has at least one Jordan block of second or higher order in its Jordan canonical form.

In this paper we analyse the following eigenvalue perturbation problem.

Let A be a real or complex matrix and let it be perturbed to $A + tE$, where $\|E\| = 1 > 0$ and $t > 0$. Suppose we know the partial algebraic multiplicites of the eigenvalues of A, then characterize (and possibly compute) all the possible (for varying E) leading fractional exponents p/q in the asymptotically small eigenvalue perturbations of order $O(t^{p/q})$ for $t \to 0$.

We propose a simple procedure to determine these leading fractional exponents based on a correspondence between the lattice of integer partitions and certain sets of fractional exponents called fractional intervals.

Note that we vary E so as to successively determine the possible fractional exponents in the asymptotic expansion of the eigenvalues of $A + tE$. Thus our problem is different from the classical problem, where the matrix E is fixed.

The main construction is based on specific perturbations, which 'fill in' the zeros in the superdiagonal of the Jordan form of A to create larger and larger Jordan blocks.
We characterize subsets of the set of all possible leading exponents and give the complete set for special cases. The general problem remains a challenging open problem.
We use the following notation: We denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Q}_1 = \{p/q : p,q \in \mathbb{N},\ p < q\}$ the set of positive rational numbers less than 1 and by $\mathcal{R}_n = \{p/q : p,q \in \mathbb{N},\ p < q \leq n\} \subset \mathbb{Q}_1$ the set of proper rational fractions with denominator not exceeding $n \in \mathbb{N}$.
The entire part of a real number $x > 0$ is denoted by $\mathrm{Ent}(x) \in \mathbb{N} \cup \{0\}$ and $[n]$ is the least common multiple of the integers $1, \dots, n \in \mathbb{N}$.
We denote the set of unordered integer partitions $\nu = (\nu_1, \dots, \nu_n)$ of $n \in \mathbb{N}$ by $\mathcal{P}_n$, i.e., $\nu_1 \geq \cdots \geq \nu_n \geq 0$ and $n = \nu_1 + \cdots + \nu_n$, and the partial sums by $\sigma_i(\nu) = \nu_1 + \cdots + \nu_i$. On $\mathcal{P}_n$ we consider the partial oder $\preceq$, such that $\mu \preceq \nu$ if $\sigma_i(\mu) \leq \sigma_i(\nu)$ for $i = 1, \dots, n$.
In $\mathbf{M}_n(\mathbb{F})$, the space of $n \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we denote by $\|\cdot\|$ the spectral norm, by $\mathbf{GL}_n(\mathbb{F}) \subset \mathbf{M}_n(\mathbb{F})$ the group of invertible matrices and by $\mathbf{S}(\mathbb{F}) = \{X \in \mathbf{M}_n(\mathbb{F}) : \|X\| = 1\}$ – the unit sphere in $\mathbf{M}_n(\mathbb{F})$.
Single nilpotent Jordan blocks are denoted by $N_n = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} \in \mathbf{M}_n(\mathbb{R})$, where for $n = 1$ we set $N_1 = 0$. The unit basis elements of $\mathbf{M}_n(\mathbb{R})$ of matrices with a single non-zero element 1 in position (i,j) are denoted by $E_n(i,j)$ and $J(\lambda, n) = \lambda I_n + N_n$ is an $n \times n$ Jordan block with eigenvalue λ. The m pairwise distinct eigenvalues of a matrix A with algebraic multiplicities $k_1, \dots, k_m$ are denoted by $\lambda_1(A), \dots, \lambda_k(A)$, where $k_1 + \cdots + k_m = n$.
Finally we introduce fractional intervals. For $p, q, s \in \mathbb{N}$ introduce the set $\mathcal{B}_p(q,s)$ as follows:

1. If $p < s \leq q$ then
$$\mathcal{B}_p(q,s) := \left\{ \frac{p}{q}, \frac{p}{q-1}, \dots, \frac{p}{s} \right\}. \tag{1}$$
The set $\mathcal{B}_p(q,s)$ is referred to as a *fractional interval of first kind* with endpoints p/q and p/s. It has $q - s + 1$ elements.

2. If $s \leq p < q$ then
$$\mathcal{B}_p(q,s) := \left\{ \frac{p}{q}, \frac{p}{q-1}, \dots, \frac{p}{p+1} \right\}. \tag{2}$$
The set $\mathcal{B}_p(q,s)$ is referred to as a *fractional interval of second kind* with endpoints p/q and $p/(p+1)$. It has $q - p$ elements.

3. If $p \geq q$ and/or $s > q$ then
$$\mathcal{B}_p(q,s) := \emptyset. \tag{3}$$

Hence the fractional interval $\mathcal{B}_p(q,s)$ either has elements less than 1 (if it is of first or second kind) or is empty. Also, if the interval $\mathcal{B}_p(q,s)$ is of second kind, as a set it is equal to the interval $\mathcal{B}_p(q, p+1)$ of first kind.

Note that if the fractional intervals $\mathcal{B}_p(q_1, s_1)$ and $\mathcal{B}_p(q_2, s_2)$ are of first kind, then

$$\mathcal{B}_p(q_1, s_1) \cup \mathcal{B}_p(q_2, s_2) = \mathcal{B}_p(q_1, s_2)$$

if and only if $s_2 \leq s_1 \leq q_2 + 1$ and $q_2 \leq q_1$. In particular

$$\mathcal{B}_p(q, s_1) \cup \mathcal{B}_p(s_1 - 1, s) = \mathcal{B}_p(q, s)$$

provided the intervals in the left-hand side are of first kind. Similar results are valid for intervals of second kind as well.
The following Lemma will be used in the construction of specific perturbations.

Lemma 1 *Let $n-1$ real numbers $t_1, \ldots, t_{n-1}$ be given. Then*

$$\det\left(\lambda I_n - \sum_{i=1}^{n-1} t_i E_n(i, i+1) - t_n E_n(n, j)\right) = \lambda^{j-1}\left(\lambda^{n+1-j} - \prod_{i=j}^{n} t_i\right) \tag{4}$$

for all $j = 1, \ldots, n$. In particular, taking $t_1 = \cdots = t_{n-1} = 1$ and $t_n = t$ we have

$$\det\left(\lambda I_n - N_n - tE_n(n, j)\right) = \lambda^{j-1}\left(\lambda^{n+1-j} - t\right). \tag{5}$$

Proof. Relations (4) and (5) follow immediately from the fact that the left-hand side of (4) is the determinant of the matrix

$$\begin{bmatrix} \lambda & -t_1 & 0 & \ldots & 0 \\ 0 & \lambda & -t_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -t_{n-1} \\ -t_n & 0 & 0 & \ldots & \lambda \end{bmatrix}$$

for $j = 1$, or of the block upper-triangular matrix

$$\left[\begin{array}{ccccc|ccccc} \lambda & -t_1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda & -t_2 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -t_{j-2} & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \lambda & -t_{j-1} & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & \lambda & -t_j & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \lambda & -t_{j+1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & -t_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & -t_n & 0 & 0 & \ldots & \lambda \end{array}\right]$$

if $j \geq 2$. $\square$

2 Fractional Exponents in Asymptotic Spectral Perturbations

In this section we consider the problem of determining the possible and the leading fractional exponents in asymptotically small spectral perturbations.

2.1 Problem Statement

Let a given matrix $A \in \mathbf{M}_n(\mathbb{F})$ of unit norm be perturbed to $A + tE$, where $t > 0$ and $E \in \mathbf{S}_n(\mathbb{F})$. The aim of *asymptotic spectral perturbation analysis* is to find asymptotic bounds for the quantities

$$|\lambda_s(A + tE) - \lambda_s(A)|$$

as functions of the real parameter $t \to 0$, where $\lambda_i(B)$ are the eigenvalues of B and E varies over the unit sphere $\mathbf{S}_n(\mathbb{F})$.
Under the (small) perturbation $A \to A+tE$ each eigenvalue $\lambda_i(A)$ of algebraic multiplicity k_i perturbes into k_i (not necesserily different) eigenvalues of $A + tE$, say

$$\lambda_{k_1+\cdots+k_{i-1}+1}(A + tE), \ldots, \lambda_{k_1+\cdots+k_i}(A + tE)$$

(for $i = 1$ the sum from k_1 to k_0 is considered void). The corresponding functions

$$t \mapsto l_s(t) := \lambda_{k_1+\cdots+k_{i-1}+s}(A + tE) \in \mathbb{C}$$

are *algebraic*. Indeed, their values are the roots of the characteristic polynomial

$$\det(\lambda I_n - A - tE) = \lambda^n - c_1(t)\lambda^{n-1} + \cdots + (-1)^n c_n(t) \tag{6}$$

of $A+tE$ with coefficients $c_j(t)$, which, being sums of principal j-th order minors of $A+tE$, are polynomials of degree up to j in t. In particular the functions l_s are continuous on $\mathbb{R}$. In fact, l_s are branches of algebraic functions which are piece-wise differentiable and hence piece-wise analytic, see e.g., [5]. At the isolated set $\Theta := \{t_1, \ldots, t_\sigma\}$ of *exceptional points* in an open interval $T \subset \mathbb{R}$, where the functions l_s are not differentiable, there exists $\varepsilon = \varepsilon(\tau, \Theta) > 0$ such that differentiability holds on the *pierced neighbourhood*

$$N(\tau, \varepsilon) := (\tau - \varepsilon, \tau) \cup (\tau, \tau + \varepsilon) = (\tau - \varepsilon, \tau + \varepsilon)\backslash\{\tau\} \subset T.$$

Indeed, given $\tau \in T$ there are two alternatives. If $\tau \notin \Theta$, then l_s is differentiable on each open subinterval $T_0 \ni \tau$ of T which does not contain exceptional points, and in particular on each $N(\tau, \varepsilon) \subset T_0$. On the other hand if $\tau \in \Theta$, then l_s is not differentiable at τ but on $T_0\backslash\{\tau\}$, where $T_0 \subset T$ is an open interval such that $T_0 \cap \Theta = \{\tau\}$.
We are interested in the local behaviour of l_s in a small neighbourhood of $t = 0$. This behaviour is completely determined by the matrices A and E if E is fixed. In particular, not only the fractional powers in the asymptotics of the eigenvalues of $A+tE$, but also the corresponding coefficients are uniquely determined. But often the matrix E is not known,

e.g., the perturbation tE may be due to rounding errors during the computation of the eigenvalues of A in finite arithmetic. If the matrix A is scaled to have unit norm, which we assume in the following, then we may consider $t = \text{eps}$, where eps is the rounding unit. Our goal is to determine the fractional exponents in the asymptotic eigenvalue expansions. When E varies over $\mathbf{S}(\mathbb{F})$ then the fractional exponents describe the behaviour of l_s modulo $\mathbf{S}(\mathbb{F})$ and this is determined by A only. In this case the whole information is coded in the Jordan form of A and in particular in the arithmetic invariant of A under the similarity action

$$(X, A) \to X^{-1}AX, \ \ X \in \mathbf{GL}_n(\mathbb{F})$$

of the general linear group $\mathbf{GL}_n(\mathbb{F})$. The arithmetic invariant consists of the partial algebraic multiplicities of the eigenvalues of A which are exactly the orders of the diagonal blocks of the complex Jordan form

$$J_A = R^{-1}AR$$

of A relative to $\mathbf{GL}_n(\mathbb{C})$. In order to treat in an uniform way the cases of real ($\mathbb{F} = \mathbb{R}$) and complex ($\mathbb{F} = \mathbb{C}$) matrices we shall always use the complex Jordan form. Of course, using the real Jordan form for real matrices would give analogous results.

If A is non-defective, i.e., J_A is diagonal, then all functions l_s are analytic in a sufficiently small neighbourhood of $t = 0$. This case is not interesting from the point of view of asymptotic analysis, since then

$$|l_s(t) - l_s(0)| = O(t), \ \ t \to 0,$$

see, e.g., [10].

If A is defective, then the functions l_s are analytic in a small pierced neighbourhood $N(0, \varepsilon)$ of $t = 0$, see [5]. This means that for some $\varepsilon > 0$ and some $t_0 \in N_\varepsilon$ all derivatives $l_s^{(j)}(t_0)$ of l_s at $t = t_0$ exist, and

$$l_s(t) = \sum_{j=0}^{\infty} \frac{l_s^{(j)}(t_0)}{j!}(t - t_0)^j$$

for all $t \in N_\varepsilon$. For $t = 0$, however, some of the functions l_s may not be differentiable, since the limit

$$\lim_{t \to 0} \left| l_s^{(j)}(t) \right| = \infty$$

is possible. In this case l_s has the asymptotic expansion

$$l_s(t) = l_s(0) + O\left(t^{p_s/q_s}\right) + O\left(t^{p_s/q_s + r_s}\right), \ t \to 0$$

where $p_s/q_s \in \mathbb{Q}_1$ and $r_s > 0$.

Consider for example the case $i = 1$ and $s = 1$ and set

$$\delta_E(t) := |l_1(t) - l_1(0)| = |\lambda_1(A + tE) - \lambda_1(A)|.$$

Note that here l_1 may be *any* of the continuous functions l_s satisfying $l_s(0) = \lambda_1(A)$ and that l_1 is continuous on $\mathbb{R}$ and analytic on some $N(0, \varepsilon)$.
If the elementary divisors of $\lambda I_n - A$ corresponding to $\lambda_1 = \lambda_1(A)$ are linear we have

$$\delta_E(t) = at + O(t^2),\ t \to 0,$$

were $a > 0$ may be taken as

$$a = \min\left\{ \|X\|\,\|X^{-1}\| : X \in \mathbf{GL}_n(\mathbb{C}),\ X^{-1}AX = \begin{bmatrix} \lambda_1 I_{k_1} & 0 \\ 0 & * \end{bmatrix} \right\}$$

where $*$ denotes an unspecified matrix block.
If the matrix A is defective and there are non-linear elementary divisors corresponding to λ_1, then the asymptotics of $\delta_E(t)$ is more involved and may include fractional exponents, namely [5, 6]

$$\delta_E(t) = \sum_i a_i t^{p_i/q_i} + O(t),\ t \to 0. \tag{7}$$

Here a_i are non-negative constants and $p_i/q_i \in \mathbb{Q}_1$, (the set of positive rational numbers less than 1) are the fractional exponents, where p_i and q_i are coprime integers with $p_i < q_i \leq n$. Note that if we perturb A to $A + t^\alpha E$ with $\alpha = [n]$ then the fractional powers of t^α will become integer powers of t. Also, the fractions p_i/q_i are necessarily elements of $\mathcal{R}_n$, the set of proper rational fractions with denominators not exceeding n, e.g.

$$\mathcal{R}_2 = \left\{\frac{1}{2}\right\},\ \mathcal{R}_3 = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\},\ \mathcal{R}_4 = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}, \ldots$$

This follows from (6) and a simple application of the technique of Newton diagrams.
Let $\mathcal{P} \subset \mathcal{R}_n$ be the set of *possible fractional exponents* p_i/q_i in (7) when E varies over $\mathbf{S}(\mathbb{F})$. A fraction $p/q \in \mathcal{R}_n$ is an element of $\mathcal{P}$ if and only if there are $E \in \mathbf{S}(\mathbb{F})$ and $j \in \mathbb{N}$ such that $p = p_j$, $q = q_j$ and $a_j > 0$ in expression (7).
We also introduce the subset $\mathcal{L} \subset \mathcal{P}$ of *leading fractional exponents* as follows. A fraction $p/q \in \mathcal{R}_n$ is an element of $\mathcal{L}$ if and only if there are $E \in \mathbf{S}_\rho(\mathbb{F})$, $a > 0$ and $r > 0$ such that the expression (7) may be written as

$$\delta_E(t) = at^{p/q} + O(t^{p/q+r}),\ t \to 0. \tag{8}$$

We recall that the sets $\mathcal{P}$ and $\mathcal{L}$ correspond to a given eigenvalue of A, in this case the eigenvalue $\lambda_1(A)$ and we could denote them by $\mathcal{P}_{\lambda_1}$ and $\mathcal{L}_{\lambda_1}$, respectively. For the other eigenvalues of A there are other (possibly) different sets $\mathcal{P}_{\lambda_i}$ and $\mathcal{L}_{\lambda_i}$, $i = 2, \ldots, m$, of possible and leading exponents.
It is obvious that we have

$$\mathcal{L} \subset \mathcal{P} \subset \mathcal{R}_n. \tag{9}$$

Whether one or both of these inclusions are proper depends on the Jordan structure of A. In general to determine the sets $\mathcal{L}$ and $\mathcal{P}$ on the basis of the arithmetic invariant of A is an open problem. We will present partial results for this problem below.

Example 1 For the matrix $A = N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have

$$\mathcal{L} = \mathcal{P} = \mathcal{R}_2.$$

Indeed, for $E = E_2(2,1)$ the characteristic polynomial of $A+tE$ is $\lambda^2 - t$ with roots $\pm t^{1/2}$. Thus the only element $1/2$ of $\mathcal{R}_2$ is an element of $\mathcal{L}$ as well.

Example 2 Consider the matrix $A = N_3$. For $E = E_3(3,1)$ the characteristic polynomial of $A + tE$ is $\lambda^3 - t$ and the eigenvalues are the cube roots of t, e.g.,

$$\lambda_1(A + tE) = t^{1/3}. \tag{10}$$

For $E = E_3(3,2)$ the characteristic polynomial of $A + tE$ is $\lambda(\lambda^2 - t)$ and has roots $\pm t^{1/2}$ and 0. Hence

$$\lambda_1(A + tE) = t^{1/2}. \tag{11}$$

For $E = E_3(3,1) + E_3(3,2)$ the characteristic polynomial is $\lambda^3 - t\lambda - t$ and has a root with Pisseaux series [5]

$$\lambda_1(A + tE) = t^{1/3} + \frac{t^{2/3}}{3} + 0t^{3/3} - \frac{t^{4/3}}{81} + O(t^{5/3}),\ t \to 0$$

in $t^{1/3}$. Hence $1/3, 1/2, 2/3 \in \mathcal{P}$ and, since these are all elements of $\mathcal{R}_3$, we see that $\mathcal{P} = \mathcal{R}_3$. Relations (10) and (11) show that $1/3$ and $1/2$ are also elements of $\mathcal{L}$. To show that $2/3$ is also an element of $\mathcal{L}$ is more complicated. With

$$E = E_3(2,1) - E_3(2,2) - E_3(3,2) + E_3(3,3),$$

the characteristic polynomial of the matrix

$$A + tE = \begin{bmatrix} 0 & 1 & 0 \\ t & -t & 1 \\ 0 & -t & t \end{bmatrix}$$

is $\lambda^3 - t^2\lambda + t^2$ and has a root

$$\lambda_1(A + tE) = -t^{2/3} - \frac{t^{4/3}}{3} + O(t^2),\ t \to 0.$$

Hence $2/3$ is also an element of $\mathcal{L}$ which gives $\mathcal{L} = \mathcal{R}_3$. Hence in this particular example we also have

$$\mathcal{L} = \mathcal{P} = \mathcal{R}_3.$$

Example 3 For the matrix $A = \operatorname{diag}(N_2, N_1)$, taking $E = E_3(2,1)$ we obtain the characteristic polynomial $\lambda(\lambda^2 - t)$, i.e., $1/2$ is an element of $\mathcal{L}$. Taking $E = E_3(2,3) + E_3(3,1)$

we get the characteristic polynomial $\lambda^3 - t^2$. Hence $2/3$ is also an element of $\mathcal{L}$. But $1/3$ is not an element of $\mathcal{L}$. Indeed, in this case the characteristic polynomial of $A + tE$ must be

$$\lambda^3 - \cdots - \left(c_{30}t^3 + c_{31}t^2 + c_{32}t\right)$$

with $c_{32} \neq 0$. But $\det(A + tE)$ is a polynomial in t of degree not less than 2, so $c_{32} = 0$ and the exponents $1/3$ cannot be leading. Thus

$$\mathcal{L} = \mathcal{P} = \left\{\frac{1}{2}, \frac{2}{3}\right\}$$

are proper subsets of $\mathcal{R}_3 = \{1/3, 1/2, 2/3\}$.

Note that, in general, if $p/q \in \mathcal{L}$ then we can find E so that

$$\delta_E(t) = \sum_{i=1}^{\infty} b_i t^{ip/q}, t \to 0.$$

So the numbers ip/q are elements of $\mathcal{P}$ for $i = 1, \ldots, \mathrm{Ent}(q/p)$.

Example 4 Let $A \in \mathbf{M}_n(\mathbb{R})$ be zero except for the first superdiagonal, where $a_{1,2} = 1$ and $a_{i,i+1}$ is equal to 1 or 0 in such a way that there are $j - 1$ zero elements $(1 \leq j \leq n - 1)$. Take a matrix $E = F + G$ so that F has all its elements equal to zero with the exception of $j - 1$ elements on its superdiagonal equal to 1 in the positions of the zero elements $a_{i,i+1}$, and $G = E_n(n, 1) + E_n(n, 2)$. Then the matrix $\lambda I_n - (A + tE)$ has the form

$$\begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & -t_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -t_{n-1} \\ -t & -t & 0 & \ldots & \lambda \end{bmatrix}$$

where $j - 1$ among the numbers $t_2, \ldots, t_{n-1}$ are equal to t and the other $n - j - 1$ are equal to 1. Then by (4) we see that

$$\det(\lambda I_n - (A + tE)) = \lambda^n - t^j \lambda - t^j.$$

The roots of this polynomial have Pisseaux series [5] in $t^{j/n}$, e.g.

$$\lambda_1(A + tE) = t^{j/n} + \frac{t^{2j/n}}{n} - \frac{n-3}{2n^2} t^{3j/n} + O(t^{4j/n}),\ t \to 0.$$

This example shows that the set $\mathcal{L}$ of leading fractional exponents is of major interest. The computation (and even the estimation) of a_i and p_i, q_i in (7) is usually a very difficult task. However, if the orders of the blocks in the complex Jordan form J_A of A are known,

then some direct calculations, together with an implementation of the technique of Newton diagrams, allow to determine a subset of the set of leading fractional exponents $\mathcal{L}$ for a general defective matrix. In [7, 8] the problem is solved for some of the leading exponents of a defective matrix with only one eigenvalue. Here we extend these results, but still the complete analysis is an open problem.
Of course, we have the fractional exponents

$$\frac{1}{k_{1,1}},\ \frac{1}{k_{1,1}-1}, \ldots,\ \frac{1}{2} \in \mathcal{L} \tag{12}$$

corresponding to the largest block $J(\lambda_1, k_{1,1})$ of J of order $k_{1,1}$ with eigenvalue λ_1. Similarly, we also have the exponents

$$\frac{1}{k_{1,j}},\ \frac{1}{k_{1,j}-1}, \ldots,\ \frac{1}{2} \in \mathcal{L}$$

corresponding to the smaller blocks $J(\lambda_1, k_{1,j})$ of J_A, $k_{1,j} < k_{1,1}$, but these exponents are already in the list (12). We also have that for each $1/\alpha$ from the list (12) the integer multiples i/α with $i < \alpha$ belong to $\mathcal{P}$. At the same time there may be other fractional exponents with denominators up to n and nominators up to $n-1$ as in the following example of Wilkinson [12].

Example 5 Consider the nilpotent matrix $A = \operatorname{diag}(J(0,3), J(0,2)) \in \mathbf{M}_5(\mathbb{R})$ with Jordan blocks of order $k_{1,1} = 3$, $k_{1,2} = 2$, which is already in Jordan form. Choosing the matrix E as $E_5(3,1)$ we get the fractional exponent $1/3$ while choosing E as $E_5(3,2)$ or $E_5(5,4)$ we get the exponent $1/2$, both of which are in the list (12) for $k_{1,1} = 3$. However, we have also the exponents $2/5$, corresponding to $E = E_5(3,4) + E_5(5,1)$ and $2/3$, corresponding to $E = E_5(3,4) + E_5(5,3)$, which are not in the list (12). Choosing

$$E = E_5(2,4) + E_5(3,5) - E_5(4,1) - E_5(4,4) + E_5(5,3) + E_5(5,5) \tag{13}$$

we also find the leading exponent $3/4$, since the characteristic polynomial of $A + tE$ in this case is

$$\lambda(\lambda^4 - t^3\lambda - t^3).$$

On the other hand $1/5$ is not an element of $\mathcal{L}$, since the lowest power of t in $\det(A + tE)$ is 2. Whether some of the other candidates $1/4$, $3/5$ or $4/5$ is indeed a member of $\mathcal{L}$ is not known. Thus the list of leading exponents for this case satisfies

$$\left\{\frac{1}{3},\ \frac{1}{2},\ \frac{2}{5},\ \frac{2}{3},\ \frac{3}{4}\right\} \subset \mathcal{L}.$$

The set $\mathcal{P}$ contains in addition the exponent $4/5$ as $2/5$ multiplied by 2. In particular, the characteristic polynomial of the matrix $A + tE$ with

$$E = E_5(3,4) + E_5(5,1) + E_5(5,2)$$

is $\lambda^5 - t^2\lambda - t^2$ and has roots

$$\lambda_1(A + tE) = t^{2/5} + \frac{t^{4/5}}{5} + O(t^{6/5}),\ t \to 0.$$

A key factor in obtaining the fractions 2/5 and 2/3 from $\mathcal{L}$ is that for $t > 0$ the matrix $J_A + tE_5(3,4)$ has non-zero superdiagonal and its Jordan form consists of a single Jordan block $J(0,5)$. Thus an additional perturbation $tE_5(5,j)$ gives the fractions $2/(6-j)$ with $j = 1,2,3$ of which 2/5 and 2/3 are 'new', i.e., not in the list (12).
In turn, the exponent $3/4 \in \mathcal{L}$ was found after a number of trials, which finally gave (13) having in mind the desired form of the characteristic polynomial of $A + tE$ capable of producing this particular exponent as leading.

2.2 Calculation of fractional exponents

In this subsection we present a simple algorithm to compute a subset $\mathcal{F}$ of $\mathcal{L}$. This method is based on the idea to perturb J_A to $J_A + t(F+G)$ so that the Jordan form of the matrix $J_A + tF$ consists of larger blocks than those of J_A. This is done using perturbations tF which contain matrices of the form $tE_n(i,i+1)$ whenever the i-th row of J_A is zero. After that the perturbation tG leads to a matrix $J_A + t(F+G)$ with a simple characteristic polynomial with roots which are the desired leading fractional powers of t.
Let A have m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities

$$k_1 \geq \ldots \geq k_m, \ k_1 + \cdots + k_m = n,$$

which means that the characteristic polynomial of A is

$$\det(\lambda I_n - A) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{k_i}.$$

Suppose that each λ_i takes part in n_i Jordan blocks of orders

$$k_{i,1} \geq \cdots \geq k_{i,n_i}, \ k_{i,1} + \cdots + k_{i,n_i} = k_i$$

i.e., that the minimal polynomial of A is

$$\prod_{i=1}^{m} (\lambda - \lambda_i)^{k_{i,1}}.$$

The numbers $k_{i,j}$ are the *partial algebraic multiplicities* of λ_i. The number n_i is the *geometric multiplicity* of λ_i, i.e., the number of linearly independent eigenvectors of A, corresponding to λ_i. If at least one geometrical multiplicity is larger than 1, then the matrix A is *derogatory.* We recall that $k_i - n_i$ is the *defect* of the eigenvalue λ_i, while $n - n_1 - \cdots - n_m$ is the *defect* of the matrix A and is denoted by $\operatorname{def}(A)$. With this notation the Jordan form $J_A = R^{-1}AR$ may be written as

$$J_A = \operatorname{diag}\left(J\left(\lambda_1, k_{1,1}\right), \ldots, J\left(\lambda_1, k_{1,n_1}\right), \ldots, J\left(\lambda_m, k_{m,1}\right), \ldots, J\left(\lambda_m, k_{m,n_m}\right)\right).$$

The matrices $A + tE$ and $J_A + tR^{-1}ER$ have the same fractional powers of t in their asymptotic eigenvalue expansions. Since we are interested in the fractional exponents p_i/q_i

and not in the coefficients a_i in (7), we assume in the following that the matrix A is already transformed to Jordan form, i.e., that $A = J_A$. We shall also assume that $k_{1,1} > 1$, which is necessary and sufficient for existence of fractional exponents in the asymptotic expansion of $\delta_E(t)$, $t \to 0$, for some E.
To simplify the notation further, we set

$$k := k_1,\ \ r := n_1;\ \ \kappa_i := k_{1,i},\ \ i = 1, \ldots, r.$$

Then we have the (unordered) partition

$$\kappa := (\kappa_1, \ldots, \kappa_r),\ \ \kappa_1 \geq \cdots \geq \kappa_r,\ \ k = \kappa_1 + \cdots + \kappa_r$$

of k with $\kappa_1 > 1$ and $r < k$. If necessary, we set $\kappa_i = 0$ for $i > r$ and write the partition as

$$\kappa = (\kappa_1, \ldots, \kappa_k) = (\kappa_1, \ldots, \kappa_r, 0, \ldots, 0).$$

Denoting the partial sums as

$$\sigma_i(\kappa) := \kappa_1 + \cdots + \kappa_i,$$

the set $\mathcal{P}_k$ of unordered partitions of k is a lattice with a partial order $\alpha \preceq \beta$ if and only if

$$\sigma_i(\alpha) \leq \sigma_i(\beta),\ \ i = 1, \ldots, k.$$

Note that this order relation is not linear, since for $k \geq 6$ there are uncomparable partitions such as $6 = 4 + 1 + 1$ and $6 = 3 + 3$.
The set $\mathcal{F} \subset \mathcal{L}$ that we determine depends not only on the partition κ of k but also on $\mathrm{def}(A)$. As a set $\mathcal{F} \subset \mathcal{Q}$ will consist of pair-wise disjoint fractions p/q in which p and q are coprime. However, in the description of $\mathcal{F}$ given below, in order to simplify the notations, we allow that p_i and q_i have common divisors, e.g. sometimes we write $2/6$ instead of $1/3$, etc. For instance we use sets of rational fractions as

$$\left\{ \frac{2}{q},\ \frac{2}{q-1}, \ldots,\ \frac{2}{3} \right\}$$

with $q \geq 3$ in which some of the fractions may not be coprime.
Finally, introduce the number

$$\pi_i := \min\{\sigma_i(\kappa) - i - 1, n - r - \mathrm{def}(A)\}$$

(we recall that $n - r = n_2 + \cdots + n_m$). Thus π_i depends not only on κ but also on the defect of A.
Now we can state our main result.

Theorem 1 *Consider the fractional intervals*

$$\begin{aligned} \mathcal{F}_{ij} &:= \mathcal{B}_{i+j}(\sigma_i(\kappa), 1 + k - \sigma_{r+1-i}(\kappa)) \\ &= \mathcal{B}_{i+j}(\sigma_i(\kappa), 1 + \kappa_{r+2-i} + \cdots + \kappa_r) \end{aligned} \tag{14}$$

for $i = 1, \ldots, r$ and $j = 0, \ldots, \pi_i$. (Here the sum from $r+1$ to r for $i = 1$ above is considered void).
Then for the set of leading fractional exponents $\mathcal{L}$ corresponding to the eigenvalue λ_1 of A we have

$$\mathcal{F} = \bigcup_{i=1,\, j=0}^{r,\, \pi_i} \mathcal{F}_{ij} \subset \mathcal{L}. \tag{15}$$

Furthermore, for $n = 2$ we have $\mathcal{F} = \mathcal{L}$, while for $n > 2$ the inclusion in (15) may be proper.

Proof. When studying the absolute perturbations in λ_1 we may always assume without loss of generality that $\lambda_1 = 0$ (otherwise we may consider the matrix $A - \lambda_1 I_n$ with the same Jordan structure and a zero eigenvalue of multiplicity k). For the proof we consider two cases. First we consider the case when A has a single zero eigenvalue and second the case when A has two or more pairwise distinct eigenvalues.
Case 1. If A has a single zero eigenvalue, i.e., $m = 1$, then $k = k_1 = n$ and $\pi_1 = \cdots = \pi_r = 0$. In this case we have to show that $\mathcal{L}$ contains the union of the sets

$$\begin{aligned}
\mathcal{F}_{10} &= \mathcal{B}_1(\sigma_1(\kappa), 2) \\
&= \left\{ \frac{1}{\kappa_1}, \frac{1}{\kappa_1 - 1}, \ldots, \frac{1}{2} \right\}; \\
\mathcal{F}_{20} &= \mathcal{B}_2(\sigma_2(\kappa), 1 + \kappa_r) \\
&= \left\{ \frac{2}{\kappa_1 + \kappa_2}, \frac{2}{\kappa_1 + \kappa_2 - 1}, \ldots, \frac{2}{1 + \kappa_r} \right\} \setminus \{1\}; \\
&\vdots \\
\mathcal{F}_{i0} &= \mathcal{B}_i(\sigma_i(\kappa), 1 + k - \sigma_{r+1-i}(\kappa)) \\
&= \left\{ \frac{i}{\sigma_i(\kappa)}, \frac{i}{\sigma_i(\kappa) - 1}, \ldots, \frac{i}{1 + \kappa_{r+2-i} + \cdots + \kappa_r} \right\} \setminus \{1\}; \\
&\vdots \\
\mathcal{F}_{r0} &= \mathcal{B}_r(k, 1 + k - \sigma_1(\kappa)) \\
&= \left\{ \frac{r}{k}, \frac{r}{k-1}, \ldots, \frac{r}{1 + \kappa_2 + \cdots + \kappa_r} \right\} \setminus \{1\}.
\end{aligned}$$

Here we have used the fact that

$$\kappa_{r+2-i} + \cdots + \kappa_r = k - \sigma_{r+1-i}(\kappa), \ i \geq 2.$$

Recall that for $m = 1$ the Jordan form of A is

$$J_A = \operatorname{diag}(J(0, \kappa_1), \ldots, J(0, \kappa_r)). \tag{16}$$

We show successively how each member p/q of $\mathcal{F}_{10}, \ldots, \mathcal{F}_{r0}$ appears as an actual fractional exponent in a term $O(t^{p/q})$ in the asymptotic expansion of $\delta_E(t)$, $t \to 0$, for an appropriate choice of E. Note that to have an exponent p/q with p and q coprime, one must have at least p non-zero elements $e_{i_1,j_1},\ldots,e_{i_p,j_p}$ of E in different rows and columns, i.e., $i_1<\cdots<i_p$, $j_1<\cdots<j_p$. Only in this case a term t^p may appear in the characteristic polynomial of J_A+tE.

The set $\mathcal{F}_{10}$. Consider perturbations which affect only one block $J(0,\kappa_j)$ of J_A from (16) with $\kappa_j\geq 2$. Since $\kappa_1\geq 2$ at least one such block exists. Taking the matrix E as

$$E=E_n(\sigma_j(\kappa),\sigma_{j-1}(\kappa)+s)$$

for $s=1,\ldots,\kappa_j$ and using relation (5) we see that

$$\det(\lambda I_n-J_A-tE)=\lambda^{n-s}(\lambda^s-t).$$

Hence the fractions

$$\frac{1}{\kappa_j},\ \frac{1}{\kappa_j-1},\ldots,\ \frac{1}{2}$$

are among the leading exponents. All these exponents are in the fractional interval $\mathcal{F}_{10}=\mathcal{B}_1(\kappa_1,2)$, while for $j=1$ they are exactly the elements of $\mathcal{F}_{10}$.

The set $\mathcal{F}_{i0}$, $1<i<r$ (this case exists for $r\geq 3$). Take first a perturbation $E=E_1$ which affects the superdiagonal over the first i blocks $J(0,\kappa_1),\ldots,J(0,\kappa_i)$ of the matrix J. Let $E_1=F_1+G_1$, where

$$\begin{aligned} F_1 &:= \sum_{j=1}^{i-1} E_n(\sigma_j(\kappa),\sigma_j(\kappa)+1)\\ G_1 &:= E_n(\sigma_i(\kappa),s),\ 1\leq s\leq \kappa_1. \end{aligned}$$

According to relation (4) we have

$$\det(\lambda I_n-J_A-tE_1)=\lambda^{n-1+s-\sigma_i(\kappa)}\left(\lambda^{1-s+\sigma_i(\kappa)}-t^i\right).$$

This gives us exponents from the set

$$\begin{aligned} &\left\{\frac{i}{\sigma_i(\kappa)},\ \frac{i}{\sigma_i(\kappa)-1},\ldots,\ \frac{i}{1+\kappa_2+\cdots+\kappa_i}\right\}\setminus\{1\} \qquad (17)\\ &\quad=\mathcal{B}_i(\sigma_i(\kappa),1+\kappa_2+\cdots+\kappa_i)\\ &\quad=\mathcal{B}_i(\sigma_i(\kappa),1+\sigma_i(\kappa)-\sigma_1(\kappa)), \end{aligned}$$

some of which are in the fractional interval $\mathcal{F}_{i0}$. The set $\mathcal{F}_{i0}$ may contain larger elements than the elements of the set (17). We now show that these larger elements are also leading exponents by the construction of suitable perturbations E_2, E_3, etc.

Determine the perturbation $E=E_2$ which affects the superdiagonal over the next i blocks $J(0,\kappa_2),\ldots,J(0,\kappa_{i+1})$ of J_A as $E_2=F_2+G_2$, where

$$\begin{aligned} F_2 &:= \sum_{j=2}^{i} E_n(\sigma_j(\kappa),\sigma_j(\kappa)+1)\\ G_2 &:= E_n(\sigma_{i+1}(\kappa),s),\ \kappa_1+1\leq s\leq \sigma_2(\kappa). \end{aligned}$$

We have

$$\det(\lambda I_n - J_A - tE_2) = \lambda^{n-1+s-\sigma_{i+1}(\kappa)+\kappa_1}\left(\lambda^{1-s+\sigma_{i+1}(\kappa)-\kappa_1} - t^i\right)$$

which gives exponents from the set

$$\left\{\frac{i}{\kappa_2+\cdots+\kappa_{i+1}},\ \frac{i}{\kappa_2+\cdots+\kappa_{i+1}-1},\ldots,\ \frac{i}{1+\kappa_3+\cdots+\kappa_{i+1}}\right\}\setminus\{1\} \tag{18}$$
$$= \mathcal{B}_i(\sigma_{i+1}(\kappa)-\sigma_1(\kappa), 1+\sigma_{i+1}(\kappa)-\sigma_2(\kappa)).$$

Since $\kappa_{i+1} \geq 1$ we have

$$\frac{i}{\kappa_2+\cdots+\kappa_{i+1}} \leq \frac{i}{\kappa_2+\cdots+\kappa_i}.$$

Hence the first (smallest) element of (18) is not strictly larger than the last (largest) element of (17). Therefore the union of the fractional intervals (17) and (18) is

$$\left\{\frac{i}{\sigma_i(\kappa)},\ \frac{i}{\sigma_i(\kappa)-1},\ldots,\ \frac{i}{1+\kappa_3+\cdots+\kappa_{i+1}}\right\}\setminus\{1\}$$
$$= \mathcal{B}_i(\sigma_i(\kappa), 1+\sigma_{i+1}(\kappa)-\sigma_2(\kappa)).$$

Continuing this process with a matrix E_3 affecting the superdiagonal over the blocks $J(0,\kappa_3),\ldots,J(0,\kappa_{i+2})$ of J_A, etc., we see that all exponents from the set

$$\left\{\frac{i}{\sigma_i(\kappa)},\ \frac{i}{\sigma_i(\kappa)-1},\ldots,\ \frac{i}{1+\kappa_{r+1-i}+\cdots+\kappa_r}\right\}\setminus\{1\} \tag{19}$$
$$= \mathcal{B}_i(\sigma_i(\kappa), 1+\sigma_r(\kappa)-\sigma_{r+1-i}(\kappa))$$

are leading. But the set (19) is exactly the fractional interval $\mathcal{F}_{i0}$.

The set $\mathcal{F}_{r0}$. Here we may take $E = F + G$ with

$$\begin{aligned} F &:= \sum_{j=1}^{r-1} E_n(\sigma_j(\kappa), \sigma_j(\kappa)+1) \\ G &:= E_n(n,s),\ 1 \leq s \leq \kappa_1. \end{aligned}$$

The characteristic polynomial of $J_A + tE$ is

$$\lambda^{n-k-1+s}(\lambda^{k+1-s} - t^r)$$

and it follows that all exponents from $\mathcal{F}_{r0}$ are leading.

Hence we have proved the inclusions

$$\mathcal{F}_{10},\ldots,\mathcal{F}_{r0} \subset \mathcal{L}.$$

Case 2. If A has more than one eigenvalue, or $m \geq 2$, then we take a perturbation tF that puts a quantity t in the places of the zero elements of the superdiagonal of J_A over i blocks

$$J(0,\kappa_{\alpha_1}),\ldots,J(0,\kappa_{\alpha_i})$$

of J_A with zero eigenvalue and over j blocks

$$J(\lambda_{\beta_1}, \kappa_{\beta_1}), \ldots, J(\lambda_{\beta_j}, \kappa_{\beta_j})$$

with eigenvalues $\lambda_{\beta_1}, \ldots \lambda_{\beta_j} \neq 0$ (we recall that in J_A the first blocks have eigenvalue zero). Add a perturbation tG with

$$G := E_n(k_{\alpha_1} + \cdots + k_{\alpha_i} + k_{\beta_1} + \cdots + k_{\beta_j}, \gamma),\ 1 \leq \gamma \leq \alpha_1.$$

Then the characteristic polynomial of $J_A + tE$ is

$$\lambda^{n-1+\gamma-k_{\alpha_1}-\cdots-k_{\alpha_i}-k_{\beta_1}-\cdots-k_{\beta_j}} \left(\lambda^{k_{\alpha_1}+\cdots+k_{\alpha_i}+1-\gamma} \prod_{\rho=1}^{j} (\lambda - \lambda_{\beta_\rho})^{\kappa_{\beta_\rho}} - t^{i+j} \right).$$

This polynomial has

$$q := k_{\alpha_1} + \cdots + k_{\alpha_i} + 1 - \gamma$$

zeros of the form

$$e_\omega b\, t^{(i+j)/q} \left(1 + O(t^{1/q})\right),\ t \to 0,\ \omega = 0, \ldots, q-1,$$

where $e_\omega := \exp(2\pi\omega\imath/q)$ are the q roots of 1, and

$$b := \left(\prod_{\rho=1}^{j} \left(-\lambda_{\beta_\rho}\right)^{\kappa_{\beta_\rho}} \right)^{-1/q}.$$

Thus we have the leading exponents $(i+j)/q$ for $j < q - i$. Also, j cannot exceed $n - r - \mathrm{def}(A)$, which is the number of eigenvalues of A, which are different from zero. Hence we have proved that the fractional intervals

$$\mathcal{B}_{i+j}(k_{\alpha_1} + \cdots + k_{\alpha_i} + 1 - \gamma, 1 + k_{\alpha_2} + \cdots + k_{\alpha_i}),\ 1 \leq \gamma \leq \alpha_1 \tag{20}$$

are subsets of $\mathcal{L}$. Taking all combinations of $\alpha_1, \ldots, \alpha_i$ from the partition κ and repeating the argument from Case 1., we obtain that the union of the intervals (20) is $\mathcal{F}_{ij}$, i.e., $\mathcal{F}_{ij} \subset \mathcal{L}$.
The case $n = 2$ is trivial, see Example 1 which gives

$$\mathcal{F} = \mathcal{L} = \mathcal{R}_2 = \left\{ \frac{1}{2} \right\}.$$

That the inclusion may be proper for $n > 2$ is demonstrated below with some examples.
□

Remark 1 *Note that in view of (1) - (14) the sets $\mathcal{F}_{ij}$ and their union $\mathcal{F}$ are in fact easily computable despite of their not very pleasant appearance. Thus Theorem 1 gives a simple algorithm to compute some of the leading fractional exponents in the eigenvalues of the matrix $A+tE$, $t \to 0$, given the partial algebraic multiplicities $\kappa_{i,j}$ of the eigenvalues $\lambda_i(A)$ of A.*

Remark 2 *The key step in the constructive proof of Theorem 1 is that we have split the matrix E as $E = F + G$. The perturbation tF puts a quantity t in positions $(i, i+1)$ whenever the i-th row of J_A is zero, i.e., it fills in the superdiagonal of J_A. Thus the partition of n containing the partial multiplicities of the eigenvalues of $J_A + tF$ is larger (relative to the partial order $\preceq$) than the partition containing the partial multiplicities of the eigenvalues of J_A. Finally a perturbation tG gives the necessary fractional exponents from the set $\mathcal{F}$*

To see that $\mathcal{F}$ may be a proper subset of $\mathcal{L}$, consider the following examples.

Example 6 For $n = 3$ the first possible case is $A = \mathrm{diag}(N_2, N_1)$. Here we also have

$$\mathcal{F} = \mathcal{L} = \left\{\frac{1}{2}, \frac{2}{3}\right\}.$$

Indeed, if $\mathcal{F} \neq \mathcal{L}$ then $\mathcal{L}$ must contain the fraction $1/3$. The only possibility for $1/3$ to appear as a leading exponents is when the constant term $\det(A+tE)$ in the characteristic polynomial of $A+tE$ has t in first power (see also the Appendix). But this is impossible, since the lowest possible degree of t in the polynomial $\det(A+tE)$ is 2. Hence $1/3$ is not an element of $\mathcal{L}$ and hence $\mathcal{F} = \mathcal{L}$. The second possible case is $A = N_3$. But Example 2 shows that here we have $\mathcal{F} = \{1/3, 1/2\}$ and $\mathcal{L} = \mathcal{R}_3 = \{1/3, 1/2, 2/3\}$, see also [7]. Thus $\mathcal{F}$ is a proper subset of $\mathcal{L}$.

Example 7 For 4×4 matrices we have 4 subcases.
1. For $A = \mathrm{diag}(N_2, N_1, N_1)$ we have

$$\mathcal{F} = \mathcal{L} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}.$$

Indeed, the leading fractions $1/2$, $2/3$ and $3/4$ from $\mathcal{F}$ are achieved for E equal to $E_4(2,1)$, $E_4(2,3) + E_4(3,1)$ and $E_4(2,3) + E_4(3,4) + E_4(4,1)$, respectively. We shall show that $\mathcal{L}$ has no other elements than these of $\mathcal{F}$. The other elements from $\mathcal{R}_4$ are $1/4$ and $1/3$. The exponent $1/4$ cannot be leading, since it requires the polynomial $\det(A+tE)$ to contain a multiple of t. But the lowest possible degree of t in this polynomial is 3. The fraction $1/3$ also cannot be a leading exponent because there are no principal minors of $A+tE$ of order 3 containing t in first power (see also the next subcase).
2. For $A = \mathrm{diag}(N_2, N_2)$ we have

$$\mathcal{F} = \left\{\frac{1}{2}, \frac{2}{3}\right\}$$

but

$$\mathcal{L} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$$

and hence $\mathcal{F}$ is a proper subset of $\mathcal{L}$. Indeed, $\mathcal{F}$ does not contain the elements $1/4$, $1/3$ and $3/4$ from $\mathcal{R}_4$ which are candidates for members of $\mathcal{L}$. We prove first that $1/4$ and $1/3$ cannot be elements of $\mathcal{L}$ and then we show that $3/4 \in \mathcal{L}$ by a special choice of E.

Note that for each E the lowest power of t in the polynomial $c_4(t) = \det(A+tE)$ is 2, i.e.,

$$c_4(t) = \gamma_{40}t^4 + \gamma_{41}t^3 + \gamma_{42}t^2. \tag{21}$$

If $1/4 \in \mathcal{L}$ then the characteristic polynomial of $A+tE$ must have the form

$$\lambda^4 + \cdots + c_4(t),$$

with

$$c_4(t) = \gamma_{40}t^4 + \gamma_{41}t^3 + \gamma_{42}t^2 + \gamma_{43}t$$

and $\gamma_{43} \neq 0$, but this is impossible in view of (21).
Suppose that $1/3 \in \mathcal{L}$. Then the characteristic polynomial $A+tE$ must have the form

$$(\lambda - \tau t)\left(\lambda^3 + b_1(t)\lambda^2 + b_2(t)\lambda + b_3(t)\right),$$

where

$$b_3(t) = \beta_{30}t^3 + \beta_{31}t^2 + \beta_{32}t$$

and $\beta_{32} \neq 0$. Hence

$$b_3(t) - \tau t b_2(t) = -c_3(t)$$

and at least one principal minor of order 3 of the matrix $A+tE$ must contain t in first degree. But all such minors contain t in second or higher degree.
To show that 3/4 is an element of $\mathcal{L}$ take E as

$$E = E_4(1,3) + E_4(2,3) - E_4(3,1) - E_4(3,3) + E_4(4,2) + E_4(4,4).$$

Then the characteristic polynomial of the matrix

$$A + tE = \begin{bmatrix} 0 & 1 & t & 0 \\ 0 & 0 & t & 0 \\ -t & 0 & -t & 1 \\ 0 & t & 0 & t \end{bmatrix}$$

is $\lambda^4 - t^3\lambda - t^3$ and has roots of the form

$$\lambda = t^{3/4} + O(t^{3/2}),\ t \to 0.$$

3. For $A = \mathrm{diag}(N_3, N_1)$ we have

$$\mathcal{F} = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}.$$

That 1/4 is not element of $\mathcal{L}$ is clear from the fact that the lowest power of t in the polynomial $\det(A+tE)$ is 2 (see the previous subcase). Whether 3/4 is a member of $\mathcal{L}$, and hence $\mathcal{F}$ is a proper subset of $\mathcal{L}$, is an open question.

4. For $A = N_4$ we have

$$\mathcal{F} = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}.$$

The set $\mathcal{L}$ contains the fraction 2/3. Indeed, according to Example 2 we can take

$$E = E_4(2,1) - E_4(2,2) - E_4(3,2) + E_4(3,3),$$

which gives the characteristic polynomial

$$\lambda(\lambda^3 - t^2\lambda + t^2)$$

of $A + tE$ with one zero root and three roots of order $t^{2/3}$. Thus $\mathcal{F}$ is a proper subset of $\mathcal{L}$. Whether $\mathcal{L}$ contains 3/4 is an open question.

The set $\mathcal{L}$ can be studied via the characteristic polynomial of $A + tE$ and the technique of Newton diagrams, see e.g. [8]. The coefficients $c_j(t)$ of this polynomial are polynomials in t of degrees not exceeding j,

$$c_j(t) = c_{j0}t^j + c_{j1}t^{j-1} + \cdots + c_{j,j-1}t, \ c_{\alpha\beta} \in \mathbb{F}.$$

Define the numbers d_j, $j = 1, \ldots, n$, as follows. If $c_j(t)$ is the zero polynomial set $d_j = 0$. If $c_j(t) \neq 0$ set $d_j = c_{j,\alpha_j}$, where $\alpha_j := \max\{s : c_{js} \neq 0\}$. Thus d_j is the coefficient corresponding to the lowest degree α_j of t in $c_j(t)$. Keeping only the lowest powers of t in each $c_j(t)$ we get the polynomial

$$\lambda^n - d_1 t^{\alpha_1}\lambda^{n-1} + \cdots + (-1)^n d_n t^{\alpha_n}$$

which is in general different from the characteristic polynomial of $A + tE$ but has roots of the same low asymptotic order for $t \to 0$. We call two such polynomials *asymptotically equivalent*. Such asymptotically equivalent polynomials are discussed in the Appendix. For low order examples they allow to calculate directly some of the leading exponents. This was demonstrated in the examples presented above.

3 Fractional Exponents for Low Order Matrices

In this section we present a list of the fractional exponents from the set $\mathcal{F}$ according to Theorem 1 for matrices of order up to 7.

3.1 Matrices with a single eigenvalue

In this case $m = 1$, $k = k_1 = n$ and κ is a partition of n,

$$n = \kappa_1 + \cdots + \kappa_r.$$

Table 1 below gives the fractional exponents for different partitions of n. (Wilkinson's example corresponds to $5 = 3 + 2$.) Here and in the following we partition the different exponents with ; to indicate that the exponents come form different Jordan blocks.
The smallest partition $n = 1 + \cdots + 1$ does not produce fractional exponents. The largest partition $n = n$ gives exponents $1/p$, $p = 2, \ldots, n$, which are exactly the elements of $\mathcal{B}_1(n, 2)$. The partition $n = 2 + 1 + \cdots + 1$ gives the exponents $p/(p+1)$ which are exactly the elements of the union of one-element sets $\mathcal{B}_p(p+1, p+1)$ for $p = 1, \ldots, n-1$.

3.2 Matrices with two or more eigenvalues

Table 1 from the previous subsection describes the union of the sets $\mathcal{F}_{i0}$. If the matrix A has more than one eigenvalue, then the sets $\mathcal{F}_{ij}$ with $j \geq 1$ also 'contribute' to $\mathcal{F}$.
If the number m of different eigenvalues of A is close to n the set $\mathcal{F}$ is small as shown below.
For $m = n$, $n \geq 2$, we have $k = 1$ and there are no fractional exponents. We can formally set

$$\mathcal{F} = \emptyset.$$

For $m = n - 1$ and $n \geq 3$ we have $k = 2$ and hence

$$\mathcal{F} = \left\{ \frac{1}{2} \right\}.$$

For $m = n - 2$ and $n \geq 4$ we have $k = 3$ and there are two possible subcases: $\kappa = (3)$ and $\kappa = (2, 1)$. For both of them

$$\mathcal{F} = \left\{ \frac{1}{3}, \frac{1}{2}; \frac{2}{3} \right\}.$$

For $m = n - 3$, $n \geq 5$ there are 7 subcases as follows. If $k = 4$, $\kappa = (4)$ and $n = 5$ then

$$\mathcal{F} = \left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}; \frac{2}{3} \right\}.$$

If $k = 4$ and $n \geq 6$ then

$$\mathcal{F} = \left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}; \frac{2}{3}; \frac{3}{4} \right\}.$$

If $k = 4$ and $\kappa = (3, 1)$ then

$$\mathcal{F} = \left\{ \frac{1}{3}, \frac{1}{2}; \frac{2}{3}; \frac{3}{4} \right\}.$$

If $k = 4$ and $\kappa = (2, 2)$ or $\kappa = (2, 2, 1)$ then

$$\mathcal{F} = \left\{ \frac{1}{2}; \frac{2}{3}; \frac{3}{4} \right\}.$$

Table 1

Partition	Fractional Exponents
$2 = 2$	$1/2$
$3 = 3$	$1/3,\ 1/2$
$3 = 2 + 1$	$1/2;\ \ 2/3$
$4 = 4$	$1/4,\ 1/3,\ 1/2$
$4 = 3 + 1$	$1/3,\ 1/2;\ \ 2/3$
$4 = 2 + 2$	$1/2;\ \ 2/3$
$4 = 2 + 1 + 1$	$1/2;\ \ 2/3;\ \ 3/4$
$5 = 5$	$1/5,\ 1/4,\ 1/3,\ 1/2$
$5 = 4 + 1$	$1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3$
$5 = 3 + 2$	$1/3,\ 1/2;\ \ 2/5,\ 2/3$
$5 = 3 + 1 + 1$	$1/3,\ 1/2;\ \ 2/3;\ \ 3/5,\ 3/4$
$5 = 2 + 2 + 1$	$1/2;\ \ 2/3;\ \ 3/5,\ 3/4$
$5 = 2 + 1 + 1 + 1$	$1/2;\ \ 2/3;\ 3/4;\ \ 4/5$
$6 = 6$	$1/6,\ 1/5,\ 1/4,\ 1/3,\ 1/2$
$6 = 5 + 1$	$1/5,\ 1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3$
$6 = 4 + 2$	$1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3;$
$6 = 4 + 1 + 1$	$1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3;\ 3/5,\ 3/4$
$6 = 3 + 3$	$1/3,\ 1/2;\ \ 2/5$
$6 = 3 + 2 + 1$	$1/3,\ 1/2;\ \ 2/5,\ 2/3;\ \ 3/5,\ 3/4$
$6 = 3 + 1 + 1 + 1$	$1/3,\ 1/2;\ \ 2/3;\ \ 3/5,\ 3/4;\ \ 4/5$
$6 = 2 + 1 + 1 + 1 + 1$	$1/2;\ \ 2/3;\ \ 3/4;\ \ 4/5;\ \ 5/6$
$7 = 7$	$1/7,\ 1/6,\ 1/5,\ 1/4,\ 1/3,\ 1/2$
$7 = 6 + 1$	$1/6,\ 1/5,\ 1/4,\ 1/3,\ 1/2;\ \ 2/7,\ 2/5,\ 2/3$
$7 = 5 + 2$	$1/5,\ 1/4,\ 1/3,\ 1/2;\ \ 2/7,\ 2/5,\ 2/3$
$7 = 5 + 1 + 1$	$1/5,\ 1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3;\ \ 3/7,\ 3/5,\ 3/4$
$7 = 4 + 3$	$1/4,\ 1/3,\ 1/2;\ \ 2/7,\ 2/5$
$7 = 3 + 3 + 1$	$1/3,\ 1/2;\ \ 2/5,\ 2/3;\ \ 3/7,\ 3/5$
$7 = 4 + 2 + 1$	$1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3;\ 3/7,\ 3/5,\ 3/4$
$7 = 4 + 1 + 1 + 1$	$1/4,\ 1/3,\ 1/2;\ \ 2/5,\ 2/3;\ 3/5,\ 3/4;\ 4/7,\ 4/5$
$7 = 3 + 2 + 2$	$1/3,\ 1/2;\ 2/5,\ 2/3;\ \ 3/7,\ 3/5$
$7 = 3 + 2 + 1 + 1$	$1/3,\ 1/2;\ \ 2/5,\ 2/3;\ \ 3/5,\ 3/4;\ \ 4/7,\ 4/5$
$7 = 2 + 2 + 1 + 1 + 1$	$1/2;\ \ 2/3;\ \ 3/5,\ 3/4;\ \ 4/5;\ \ 5/7,\ 5/6$
$7 = 2 + 1 + 1 + 1 + 1 + 1$	$1/2;\ \ 2/3;\ \ 3/4;\ \ 4/5;\ \ 5/6;\ \ 6/7$

If $k=3$ and $\kappa=(3)$ then

$$\mathcal{F}=\left\{\frac{1}{3},\ \frac{1}{2};\ \frac{2}{3}\right\}.$$

If $k=3$ and $\kappa=(2,1)$ then

$$\mathcal{F}=\left\{\frac{1}{2};\ \frac{2}{3}\right\}.$$

If $k=2$ then

$$\mathcal{F}=\left\{\frac{1}{2}\right\}.$$

Let us now consider the possible fractional exponents for $n\leq 7$ and $m\leq n-4$, since the cases with $m\geq n-3$ have been already analysed. Here we have additional exponents to those given in Table 1. We consider the cases $n=6$ with $m=2$, $n=7$ with $m=2$ and $n=7$ with $m=3$. For each of these case there are subcases depending on the multiplicity k of λ_1 and its partitions κ. We do not consider the partition $k=2+1+\cdots+1$, since it gives no additional exponents other than $1/2, 2/3, \ldots, (k-1)/k$.
1) Case $n=6$, $m=2$ and $k=5$. Here we have 5 subcases according to the partitions of 5 in Table 1 except the last one. In addition to the exponents from the second column of Table 1 we have new exponents as follows:

Partition	Additional Fractional Exponents
$5=5$	$2/5$, $2/3$
$5=4+1$	$3/5$, $3/4$; $4/5$
$5=3+2$	$3/5$, $3/4$
$5=3+1+1$	$4/5$
$5=2+2+1$	$4/5$

2) Case $n=6$, $m=2$ and $k=4$. There are 3 subcases according to the partitions of 4 in Table 1. In addition to the exponents from the second column of Table 1 we have new exponents as follows:

Partition	Additional Fractional Exponents
$4=4$	$2/3$
$4=3+1$	$3/4$
$4=2+2$	$3/4$

3) Case $n=6$, $m=2$ and $k=3$ or $k=2$. No additional exponents are added to the exponents already mentioned in the second column of Table 1 for the partitions of 3 and 2.
4) Case $n=7$, $m=2$ and $k=6$. We have 7 subcases according to the partitions of 6 in Table 1. In addition to the exponents from the second column of Table 1 the new

exponents are:

Partition	Additional Fractional Exponents
$6 = 6$	$2/5$, $2/3$
$6 = 5 + 1$	$3/5$, $3/4$
$6 = 4 + 2$	$3/5$, $3/4$
$6 = 4 + 1 + 1$	$4/5$
$6 = 3 + 3$	$2/3$; $3/5$, $3/4$
$6 = 3 + 2 + 1$	$4/5$
$6 = 3 + 1 + 1 + 1$	$5/6$

5) Case $n = 7$*,* $m = 2$ and $k = 5$. There are 5 subcases according to the partitions of 5 in Table 1. In addition to the exponents from the second column of Table 1 there are new exponents as follows:

Partition	Additional Fractional Exponents
$5 = 5$	$2/5$, $2/3$
$5 = 4 + 1$	$3/5$, $3/4$
$5 = 3 + 2$	$3/5$, $3/4$
$5 = 3 + 1 + 1$	$4/5$
$5 = 2 + 2 + 1$	$4/5$

6) Case $n = 7$*,* $m = 2$ and $k = 4$. There are 3 subcases according to the partitions of 4 in Table 1. In addition to the exponents from the second column of Table 1 we have new exponents as follows:

Partition	Additional Fractional Exponents
$4 = 4$	$2/3$
$4 = 3 + 1$	$3/4$
$4 = 2 + 2$	$3/4$

7) Case $n = 7$*,* $m = 2$ and $k = 3$ or $k = 2$. No additional exponents are added to the exponents already listed in Table 1 for the partitions of 3 and 2.
8) Case $n = 7$*,* $m = 3$ and $k = 5$. Here we have 5 subcases according to the partitions of 5 in Table 1. In addition to the exponents from the second column of Table 1 the new exponents are:

Partition	Additional Fractional Exponents
$5 = 5$	$2/5$, $2/3$; $3/5$, $3/4$
$5 = 4 + 1$	$3/5$, $3/4$; $4/5$
$5 = 3 + 2$	$3/5$, $3/4$; $4/5$
$5 = 3 + 1 + 1$	$4/5$
$5 = 2 + 2 + 1$	$4/5$

9) Case $n = 7$, $m = 3$ and $k = 4$. Here we have 3 subcases according to the partitions of 4 in Table 1. In addition to the exponents from the second column of Table 1 there are new exponents as follows:

Partition	Additional Fractional Exponents
$4 = 4$	$2/3$; $3/4$
$4 = 3 + 1$	$3/4$
$4 = 2 + 2$	$3/4$

10) Case $n = 7$, $m = 3$ and $k = 3$ or $k = 2$. No additional exponents are added.

Consider finally a particular example of larger order. Let $n = 19$ and the matrix A have three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ $(m = 3)$ with partial multiplicities 7,4,2 for the first, 3,1 for the second and 1,1 for the third. Then $k = 13 = 7 + 4 + 2$, $r = 3$, the defect of A is 12 and $n - r - \operatorname{def}(A) = 4$ (the number of Jordan blocks with eigenvalues λ_2 or λ_3). The sets $\mathcal{F}_{i0}$ are

$$\begin{aligned}
\mathcal{F}_{10} &= \left\{\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\} \\
\mathcal{F}_{20} &= \left\{\frac{2}{11}, \frac{2}{10} = \frac{1}{5}, \frac{2}{9}, \frac{2}{8} = \frac{1}{4}, \frac{2}{7}, \frac{2}{6} = \frac{1}{3}, \frac{2}{5}, \frac{2}{4} = \frac{1}{2}, \frac{2}{3}\right\} \\
\mathcal{F}_{30} &= \left\{\frac{3}{11}, \frac{3}{10}, \frac{3}{9} = \frac{1}{3}, \frac{3}{8}, \frac{3}{7}\right\}
\end{aligned}$$

and their union is the 15-element set

$$\left\{\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}; \frac{2}{11}, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{2}{3}, \frac{3}{11}, \frac{3}{10}, \frac{3}{8}, \frac{3}{7}\right\}.$$

The consideration of $\mathcal{F}_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ gives 22 new exponents, listed below

$$\frac{3}{5}, \frac{3}{4}; \frac{4}{13}, \frac{4}{11}, \frac{4}{9}, \frac{4}{7}, \frac{4}{5}; \frac{5}{13}, \frac{5}{12}, \frac{5}{9}, \frac{5}{8}, \frac{5}{7}, \frac{5}{6}$$
$$\frac{6}{13}, \frac{6}{11}, \frac{6}{7}; \frac{7}{13}, \frac{7}{12}, \frac{7}{11}, \frac{7}{10}, \frac{7}{9}, \frac{7}{8}.$$

Thus $\mathcal{F}$ has 37 elements in this case.

4 Conclusions

In this paper we have presented a simple description of a subset of the set of leading fractional exponents in the asymptotic expansion of the eigenvalue perturbations of a defective matrix. The set of leading fractional exponents corresponding to a given eigenvalue depends on its partial algebraic multiplicities and on the defect of the whole matrix. For each case we have explicitely constructed the perturbation which produces the corresponding fractional exponent.

Computationally the problem of determining the Jordan structure of a general matrix is very difficult, see [3, 2, 4, 9]. The standard codes for eigenvalue analysis based on the QR algorithm in general do not produce reliable results in this case. Only the codes based on regularisation, techniques for root localization and clustering and other sophisticated tools [3, 2, 9] may give satisfactory results for defective matrices. A preliminary knowledge of possible and leading fractional exponents based on theoretical or experimental considerations (e.g. computation of the pseudo-spectra of the matrix [11]) may be very helpful in this area.

As a whole, the numerical solution of the eigenvalue problem for general dense (and even very low order) matrices remains a challenging problem in numerical linear algebra scientific computing.

Another challenging problem is the determination of the whole set of leading fractional exponents as discussed above. Even the completion of the list of leading exponents for low order matrices (of order say up to 7) is still an open problem.

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5 Appendix

Here we present conditions for the characteristic polynomial of the matrix $A + tE$ to have a root λ_0 with an asymptotic expansion

$$\lambda_0 = at^{p/q} + O(t^{2p/q}),\ t \to 0 \tag{22}$$

where $p < q$ and p, q are coprime.
Keeping only the leading terms $d_j t^{\alpha_j}$ in the coefficients $c_j(t)$ of the characteristic polynomial of $A + tE$ we obtain the so called *asymptotically equivalent* polynomial. Now λ_0 is a root of this latter polynomial satisfying (22) only if it has a multiplier of the form

$$\lambda^{rq} + \sum_{j=1}^{rq-1} \gamma_j t^{\alpha_j} \lambda^j + \gamma_0 t^{rp},\ \gamma_0 \neq 0 \tag{23}$$

where $r \in \mathbb{N}$ and $rq \leq n$. Substituting (22) in (23) we obtain

$$a^{rq} + \gamma_0 + \sum_{j=1}^{rq-1} \gamma_j t^{\alpha_j + jp/q - rp} + O(t^{p/q}),\ t \to 0.$$

Hence for each $j = 1, \ldots, rq - 1$ we have either $\gamma_j = 0$ or

$$\gamma_j \neq 0 \quad \text{and} \quad jp \geq q(rp - \alpha_j).$$

Suppose that $\gamma_{j_1}, \ldots, \gamma_{j_s}$ are the non-zero coefficients in the sum from (23) for which $jp = q(rp - \alpha_j)$. Then the coefficient a in (22) may be determined from

$$a = (-\gamma_0 - \gamma_{j_1} - \cdots - \gamma_{j_s})^{1/(rq)}.$$

Let us demonstrate this with an example.

Example 8 Let $A = N_3$ and suppose we are interested in the leading exponent $2/3$. The characteristic polynomial of $N_3 + tE$ may be written as

$$\lambda^3 - c_{10} t \lambda^2 + (c_{20} t^2 + c_{21} t)\lambda - (c_{30} t^3 + c_{31} t^2 + c_{32} t).$$

We obtain that

$$c_{10} = c_{21} = c_{32} = 0,\ c_{31} \neq 0.$$

If we are interested only in the first two coefficients in the asymptotic expansion

$$\lambda = a_1 t^{2/3} + a_2 t^{4/3} + O(t^2),\ t \to 0,$$

then we may omit the term containing t^3 as well. Hence we have the asymptotically equivalent equation

$$\lambda^3 + c_{20} t^2 \lambda - c_{31} t^2 = 0$$

which gives

$$a_1 = c_{31}^{1/3},\ a_2 = -\frac{c_{20}}{3c_{31}^{1/3}}.$$