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Biorthogonal wavelets for the direct integral formulation of the heat equation

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Abstract

We consider the direct integral formulation of the heat equation in a smooth domain of $\mathbf{R}^2$ with Neumann and Dirichlet boundary conditions. The unknown belongs to an anisotropic Sobolev space of positive order for the Neumann problem and of negative order for the Dirichlet one and is approximated by the Galerkin method using an appropriate biorthogonal wavelet basis. The use of such a basis allows to compress the stiffness matrix from $O(N^2)$ to $O(N)$, and to obtain a uniformly bounded condition number. Finally, we show that the compressed scheme converges as fast as the Galerkin.

1 Introduction

The boundary element method applied to the heat equation on a smooth domain Ω of $\mathbf{R}^2$ leads to the resolution of a two-dimensional problem [8, 16]. Unfortunately, the stiffness matrix is full, and in general ill-conditioned. Many authors (see [10, 18, 23]) have introduced new bases made of wavelets to overcome these difficulties, but only, to our knowledge, for elliptic problems. Therefore our goal is to adapt the strategy to a parabolic case.

The Dirichlet and Neumann problems will be solved using direct integral formulations (see [8, 16]). Their kernel involve an exponential function and the first step is then to check that this function satisfies a decay property in the sense of [23] to fit well in our compression procedure. The second step consists in the characterization of the anisotropic Sobolev spaces $\tilde{H}^{r,p}(\Sigma_T)$ with $r \neq p$ in term of biorthogonal wavelets, since the integral formulation of the heat equation is given in those spaces. This will be done using tensor products of one-dimensional wavelets. For the Neumann and Dirichlet problems, the associated bilinear form may be no more coercive and we overcome this difficulty by using compact perturbation arguments (see [3, 17]). As in elliptic problems, the use of wavelets allow to compress the stiffness matrix from $O(N^2)$ non-zero entries to $O(N)$. However, this procedure has to be done with precautions here. In fact, the wavelet coefficients of the stiffness matrix have a decreasing property involving a "pseudo-distance" instead of the classical euclidian one. Once this is proved, one can, as in elliptic problems, compress the matrix by replacing some small coefficients by zero. The order of convergence between the exact solution and the solution of the compressed Galerkin scheme is not affected by this procedure.

The schedule of the paper is the following one. In section 2 we recall the integral formulation of the problem and define the anisotropic Sobolev spaces which are characterized by a biorthogonal wavelet basis in section 3. The Galerkin method is presented in section 4. Due to the above characterization, we obtain a well-conditioned system. The next part is devoted to the decay of some coefficients of the stiffness matrix and a compression procedure is retailed to sparse it. Therefore in section 6 we give an error estimate between the exact solution and the solution of the compressed Galerkin scheme. We adapt our strategy to the study of the first kind formulation of the Neumann problem in section 7 and slights modifications are needed to treat the Dirichlet problem in the last section.

2 Integral formulation

In this section, we recall the integral formulation of the heat equation described in [8, 16]. We need the definition of the anisotropic Sobolev spaces $\tilde{H}^{r,p}(\Sigma_T)$. For $r, p \geq 0$, we set

$$H^{r,p}(\Sigma_T) = L^2((0,T); H^r(\Gamma)) \bigcap L^2(\Gamma; H^p(0,T)), \tag{1}$$

$$\tilde{H}^{r,p}(\Sigma_T) = \{u = U_{|\Sigma_T} : U \in H^{r,p}(\Sigma) \ , U(.,t) = 0, t < 0\}, \tag{2}$$

when $\Sigma = \Gamma \times \mathbf{R}$ and $H^r(\Gamma)$ is the classic Sobolev space because the boundary Γ is smooth.

The Neumann problem is the following one

$$\begin{cases} -\Delta\Phi + \partial_t \Phi = 0 & \text{in } Q_T = \Omega \times (0,T), \\ \partial_n \Phi_{|\Sigma_T} = g_1 & \text{on } \Sigma_T = \Gamma \times (0,T), \\ \Phi(x,0) = 0 & \quad \forall x \in \Omega. \end{cases} \tag{3}$$

Let V and W denote the classic single-layer and double-layer heat potentials :

$$(V\mu)(x,t) = \int_0^t \int_\Gamma \mu(y,\tau) E(x-y, t-\tau) d\Gamma_y d\tau, \tag{4}$$

$$(W\mu)(x,t) = \int_0^t \int_\Gamma \mu(y,\tau) \partial_{n_y} E(x-y, t-\tau) d\Gamma_y d\tau, \tag{5}$$

for all $(x,t) \in Q_T \cup Q_T^c$, with $Q_T^c = \Omega^c \times (0,T)$, where E is the fundamental solution of the heat equation :

$$E(x,t) = \frac{H(t)}{4\pi t} \exp\left(-\frac{|x|^2}{4t}\right), \tag{6}$$

$H(t)$ being the Heaviside's function.

We recall the definition of the single-layer operator S, the double-layer operator D, the spatial adjoint of the double-layer operator D' and the hypersingular heat operator H :

$$(S\sigma)(x,t) = \int_0^t \int_\Gamma \sigma(y,\tau) E(x-y, t-\tau) d\Gamma_y d\tau,$$

$$(D\mu)(x,t) = \int_0^t \int_\Gamma \mu(y,\tau) \partial_{n_y} E(x-y, t-\tau) d\Gamma_y d\tau,$$

$$(D'\sigma)(x,t) = \int_0^t \int_\Gamma \sigma(y,\tau) \partial_{n_x} E(x-y, t-\tau) d\Gamma_y d\tau,$$

$$(H\mu)(x,t) = -\partial_{n_x} \int_0^t \int_\Gamma \mu(y,\tau) \partial_{n_y} E(x-y, t-\tau) d\Gamma_y d\tau,$$

for $(x,t) \in \Sigma_T$.

The problem (3) admits the direct representation

$$\Phi = V g_1 - W\mu, \tag{7}$$

where μ is the solution of the following equation of the second kind :

$$\left(\frac{1}{2}I + D\right)\mu = S g_1, \tag{8}$$

or equivalently the equation of the first kind :

$$H\mu = \left(\frac{1}{2}I - D'\right) g_1. \tag{9}$$

In both equations (8) and (9), the unknown μ is given by

$$\mu = \Phi^- \in \tilde{H}^{1/2,1/4}(\Sigma_T). \tag{10}$$

In the next section, we construct explicitly a biorthogonal wavelet basis for the spaces $\tilde{H}^{r,r/2}(\Sigma_T)$ with $r \geq -1/2$.

3 Adapted wavelet basis

Using the construction of a biorthogonal wavelet basis on the interval satisfying complementary boundary conditions (see [15]) and some adapted tensor products, the anisotropic Sobolev spaces $\tilde{H}^{r,r/2}(\Sigma_T)$, with $r \geq -1/2$, are characterized in this section. In fact, we use tensor products of one-dimensional wavelets but with two different uniform meshes in space and time.

At first sight, we recall some notations and some results about the biorthogonal bases (see [12] for instance). Given a Hilbert space H, we suppose that we have a sequence of nested closed subspaces S_j of H, whose union is dense in H :

$$S_0 \subset S_1 \subset \cdots \subset H,$$

$$\mathrm{clos}_H\left(\bigcup_{j=0}^{\infty} S_j\right) = H.$$

The spaces S_j have the form

$$S_j = S(\Phi_j) = \mathrm{clos}_H(Span(\Phi_j)), \quad \Phi_j = \{\varphi_{j,k} : k \in \Delta_j\},$$

with Δ_j a countable set of indices and Φ_j are stable bases.

From the identity

$$S(\Phi_{j+1}) = S(\Phi_j) \bigoplus S(\Psi_j),$$

we introduce the set $W_j = S(\Psi_j)$, the complement of $S(\Phi_j)$ in $S(\Phi_{j+1})$. The set $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$ is the collection of the successive translates of the wavelet ψ_j, at a fixed level j.

For a function ψ, on $\mathbf{R}^n$ to fix the ideas, we will note

$$\Psi = \{\psi_\lambda : \lambda \in \nabla\},$$

where the indices $\lambda = (j, k) \in \nabla$ encode the level of resolution, which will be denoted by $|\lambda| = j$, the location of the function (k) and sometimes the type of wavelet which is used (e). ψ_λ can be written as

$$\psi_\lambda = 2^{jn/2}\psi_e(2^j \cdot -k), k \in \nabla_j. \tag{11}$$

Sometimes, as we will see later on, it is useful to use the condensed notation (11) instead of the original one $\psi_{j,k}(x) = 2^{jn/2}\psi(2^j x - k)$.

Now suppose that are given two sets of functions :

$$\begin{aligned} \Psi &= \{\psi_{j,k} : (j,k) \in \nabla\}, \\ \tilde{\Psi} &= \{\tilde{\psi}_{j,k} : (j,k) \in \nabla\}, \end{aligned}$$

where $\nabla = \{(j, k) : k \in \nabla_j, j = -1, 0, 1, 2, \ldots\}$ such that

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{(j,k),(j',k')}, \quad (j,k), (j',k') \in \nabla, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in H.

With this biorthogonal system, every $v \in H$ has a unique expansion written in these bases in the following form :

$$v = \sum_{(j,k)\in\nabla} \langle v, \tilde{\psi}_{j,k} \rangle \psi_{j,k} = \sum_{(j,k)\in\nabla} \langle v, \psi_{j,k} \rangle \tilde{\psi}_{j,k} \tag{13}$$

such that the systems are stable in the sense that

$$\|v\|_H^2 \sim \sum_{(j,k)\in\nabla} |\langle v, \tilde{\psi}_{j,k} \rangle|^2 = \sum_{(j,k)\in\nabla} |\langle v, \psi_{j,k} \rangle|^2. \tag{14}$$

Such a dual system is a good candidate for the characterization of the usual Sobolev spaces on $\mathbf{R}^n$ by means of the wavelet coefficients if the Bernstein and Jackson estimates hold. We recall the following general result ([10, 12]).

Theorem 3.1 *Let us assume that the Jackson estimate holds, namely*

$$\|v - Q_j v\|_\tau \lesssim 2^{j(\tau - t)} \|v\|_t, v \in H^t(\Gamma), \tag{15}$$

for $-\tilde{d} - 1 < \tau < \gamma, \tau \leq t, -\tilde{\gamma} < t \leq d + 1$, *with a similar inequality for* $(v - \tilde{Q}_j v)$, *by interchanging* d *and* $\tilde{d}$, γ *and* $\tilde{\gamma}$.

Moreover, if we have the following "inverse" property

$$\|v_j\|_t \lesssim 2^{j(t-\tau)} \|v_j\|_\tau, v_j \in S_j, \tag{16}$$

if $-\infty < \tau \leq t < \tilde{\gamma}$ *; and*

$$\|v_j\|_t \lesssim 2^{j(t-\tau)} \|v_j\|_\tau, v_j \in \tilde{S}_j, \tag{17}$$

for $-\infty < \tau \leq t < \gamma$, *the next eqivalences hold :*

$$\|v\|_t \sim \sum_{k \in \Delta_{j_0}} |\langle v, \varphi_{j_0,k} \rangle|^2 + \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} 2^{2jt} |\langle v, \tilde{\psi}_{j,k} \rangle|^2, \tag{18}$$

$$\|v\|_t \sim \sum_{k \in \Delta_{j_0}} |\langle v, \varphi_{j_0,k} \rangle|^2 + \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} 2^{2jt} |\langle v, \psi_{j,k} \rangle|^2, \tag{19}$$

for $-\tilde{\gamma} < t < \gamma, -\gamma < t < \tilde{\gamma}$.

Remark 3.2 *The choice* $d = \tilde{d} = 2$ *(i.e. the use of piecewise linear elements) allows to characterize* $H^s(\mathbf{R}^n)$ *for* $s \in [-\frac{1}{2}, \frac{3}{2})$ *and in this case,* $\gamma = \frac{3}{2}$.

We have now to adapt such a construction in order to give a characterization of anisotropic Sobolev spaces of the form $\tilde{H}^{r,r/2}(\Sigma_T)$ for $r \geq -1/2$.

First of all, there exists a smooth parametrisation of Σ_T by

$$\begin{aligned} \phi : [0,1]^2 &\rightarrow \Gamma \times (0,T) \\ (\theta, s) &\rightarrow (x(\theta), Ts). \end{aligned}$$

This mapping ϕ yields an isomorphism between $\tilde{H}^{r,r/2}(\Sigma_T)$ and $\tilde{H}^{r,r/2}([0,1]^2)$ and we construct a wavelet basis of this last space.

Therefore, we subdivise $[0,1]^2$ with two uniform meshes, dividing $[0,1]$ by 2^j subintervals (space-part) and 2^{2j} subintervals (time-part) with $j \geq 0$.

We take V_j as the following approximation space

$$V_j = \left\{ f(x,t) \in C([0,1]^2) \text{ s.t. } f(x,.) \text{ and } f(.,t) \text{ are piecewise linear on } [0,1] \right\}. \tag{20}$$

Remark 3.3 *If the characterization of spaces* $\tilde{H}^{r,r/2}([0,1]^2)$ *with* $r \geq 3/2$ *is needed, we have to use quadratic wavelets for the space part, i.e.* $d = 3$.

Because of the smoothness of Γ, the space-part basis is constructed by taking the periodized biorthogonal wavelet basis of $[0,1]$ and the resulting set will be denoted by

$$\Psi^X = \{\psi_\lambda^X \,:\, \lambda \in \nabla^X\}. \tag{21}$$

These wavelets are supposed to have $\tilde{d}^X$ vanishing moments.

For the time-part, we define the space :

$$\tilde{H}^s([0,1]) = \left\{u = U_{|[0,1]} \text{ s.t. } U \in H^s(\mathbf{R}) \text{ and } U(t) = 0, \forall t < 0\right\}. \tag{22}$$

We know (see [15]) the existence of a biorthogonal wavelet basis with complementary boundary conditions which characterizes such a Sobolev space. Typically here, the left end-point boundary wavelets must satisfy a Dirichlet condition. We recall the general strategy and the results of [15].

We introduce $\mathbf{Z} \subseteq \{0,1\}$ to specify where the Dirichlet boundary condition is located. Here $Z = \{0\}$ for our problem. The main results are the following ones (see Theorem 3.1 and 3.4 of [15]).

Theorem 3.4 *For any $\mathbf{Z} \subseteq \{0,1\}$ and any $\gamma, \tilde{\gamma} > 0$, there exist $d, \tilde{d} \in \mathbf{N}, d + \tilde{d} \in 2\mathbf{N}$ and a pair of wavelet bases :*

$$\Psi^Z = \{\psi_\lambda^Z \,:\, \lambda \in \nabla^Z\}, \quad \tilde{\Psi}^{\tilde{Z}} = \{\tilde{\psi}_\lambda^{\tilde{Z}} \,:\, \lambda \in \nabla^Z\}, \tag{23}$$

which are biorthogonal, satisfy complementary boundary conditions, are local and exact of degree d (respectively $\tilde{d}$).

From the direct and inverse estimates (see Theorem 3.2 of [15]) we obtain the next equivalence.

Theorem 3.5 *For $0 < \sigma < \gamma, 0 < \tilde{\sigma} < \tilde{\gamma}$, one has*

$$\left(\sum_{\lambda \in \nabla^Z} 2^{2\tau|\lambda|} \left|\langle v, \tilde{\psi}_\lambda^{\tilde{Z}} \rangle\right|^2\right)^{1/2} \sim \begin{cases} \|v\|_{H^\tau_{\mathbf{Z}}} & \text{for } 0 \leq \tau < \gamma, \\ \|v\|_{(H^\tau_{\mathbf{Z}})^*} & \text{for } -\tilde{\gamma} \leq \tau < 0, \end{cases}$$

where $H^\tau_{\mathbf{Z}}$ is the Sobolev space on $[0,1]$ whose functions satisfy the boundary conditions encoded by the set Z and $(H^\tau_{\mathbf{Z}})^*$ is its dual.

The construction of the wavelet bases on $[0,1]$ with boundary conditions is made in three steps. First of all, we recall the results of [11] concerning a multiresolution on the interval without any boundary conditions. We therefore expose the strategy for symmetric boundary conditions and finally, with few adaptations, we treat asymmetric boundary conditions (see [15]).

We start to recall the construction of a biorthogonal wavelet basis on $[0,1]$ without boundary conditions (see [11]).

As usual, we begin with a dual pair $(\theta, \tilde{\theta})$ of refinable functions, with respective masks $a_k, \tilde{a}_k$, satisfying $\langle \theta, \tilde{\theta}(\cdot - k)\rangle_{\mathbf{R}} = \delta_{0,k}, \forall k \in \mathbf{Z}$.

We take as θ the usual B-spline of order $d-1$, ${}_d\theta$, whose support is given by $Supp\ {}_d\theta = [l_1, l_2]$ and is centered at $\frac{\mu(d)}{2}$, with $\mu(d) := d \bmod 2$. One can show that for all $d, \tilde{d} \in \mathbf{N}$ such that $\tilde{d} \geq d$ and $d + \tilde{d}$ even there exists a function ${}_{d,\tilde{d}}\tilde{\theta}$ such that $({}_d\theta, {}_{d,\tilde{d}}\tilde{\theta})$ is a dual system. We write similarly

$$Supp\ {}_{d,\tilde{d}}\tilde{\theta} =: [\tilde{l}_1, \tilde{l}_2] = [l_1 - \tilde{d} + 1, l_2 + \tilde{d} - 1].$$

On the interval $[0,1]$, we have to construct a pair of generators $\Theta'_j, \tilde{\Theta}'_j$ of the spaces $S_j([0,1])$, $\tilde{S}_j([0,1])$, which are exact of orders $d, \tilde{d}$.

These sets of functions are defined in the following way :

$$\Theta'_j = \Theta^L_j \cup \Theta^I_j \cup \Theta^R_j, \tag{24}$$
$$\tilde{\Theta}'_j = \tilde{\Theta}^L_j \cup \tilde{\Theta}^I_j \cup \tilde{\Theta}^R_j, \tag{25}$$

respectively for the left-, interior- and right-parts of the generators.

The sets $\Theta^I_j, \tilde{\Theta}^I_j$ are made of the functions $\theta_{j,k}, \tilde{\theta}_{j,k}$, for $k \in \Delta^I_j, \tilde{\Delta}^I_j$. If we set

$$\alpha_{n,r} := \int_{\mathbf{R}} x^r \theta(x-n)dx, \quad \tilde{\alpha}_{n,r} := \int_{\mathbf{R}} x^r \tilde{\theta}(x-n)dx, \tag{26}$$

one can define the following left-boundary functions :

$$\theta^L_{j,l-d+r} := \sum_{m=-l_2+1}^{l-1} \tilde{\alpha}_{m,r} \theta_{j,m}|_{[0,1]}, \quad r = 0, \ldots, d-1, \tag{27}$$
$$\tilde{\theta}^L_{j,\tilde{l}-\tilde{d}+r} := \sum_{m=-\tilde{l}_2+1}^{\tilde{l}-1} \alpha_{m,r} \tilde{\theta}_{j,m}|_{[0,1]}, \quad r = 0, \ldots, \tilde{d}-1. \tag{28}$$

The right-end boundary functions are obtained by the following symmetry properties

$$\theta^R_{j,2^j-l+d-\mu(d)-r}(1-x) = \theta^L_{j,l-d+r}(x), \quad r = 0, \ldots, d-1, \tag{29}$$

$$\tilde{\theta}^R_{j,2^j-\tilde{l}+\tilde{d}-\mu(d)-r}(1-x) = \tilde{\theta}^L_{j,\tilde{l}-\tilde{d}+r}(x), \quad r = 0, \ldots, \tilde{d}-1. \tag{30}$$

We have to assume that $j \geq j_0$ if we do not want the boundary wavelets to interfere. Therefore, if we pose

$$S_j([0,1]) := S(\Theta'_j), \quad \tilde{S}_j([0,1]) := S(\tilde{\Theta}'_j), \tag{31}$$

these spaces are nested and are exact of order $d, \tilde{d}$.

The last point concerns the biorthogonalization of the system $(\Theta'_j, \tilde{\Theta}'_j)$ which is allways possible (see Theorem 4.1 of [15]).

We define the sets

$$Z \subseteq \{0,1\}, \quad \tilde{Z} := \{0,1\} \setminus Z. \tag{32}$$

Z precises where are the Dirichlet boundary conditions on $[0,1]$. In our concrete problem, $Z = \{0\}$ but we begin to suppose that $Z = \{0,1\}$ or $Z = \emptyset$.

First of all, we want to construct a dual system of generators $\Theta_{j,Z}, \tilde{\Theta}_{j,\tilde{Z}}$ for a set Z as in (32) which are biorthogonal :

$$\langle \Theta_{j,Z}, \tilde{\Theta}_{j,\tilde{Z}} \rangle_{[0,1]} = I, \tag{33}$$

and which satisfy, for $s \leq d-1$ (resp. $\tilde{s} \leq \tilde{d}-1$) the following boundary conditions

$$\frac{d^r}{dx^r}\Theta_{j,Z}(z) = 0, \quad r < s, z \in Z, \quad \frac{d^r}{dx^r}\tilde{\Theta}_{j,\tilde{Z}}(\tilde{z}) = 0, \quad r < \tilde{s}, \tilde{z} \in \tilde{Z}. \tag{34}$$

Let us remark that $s = 1$ for our concrete problem.

The obtention of such generators is made in two steps. First we construct collections satisfying the boundary conditions (34). Then we will show how to biorthogonalize the system.

For the interior generators, we take the usual splines, without any modification :

$$\begin{aligned} \theta_{j,k}^{I,Z} &:= 2^{j/2}{}_m\theta(2^j \cdot -k), \quad \forall k \in \Delta_{j,Z}^I, & (35) \\ \tilde{\theta}_{j,k}^{I,\tilde{Z}} &:= 2^{j/2}{}_{m,\tilde{m}}\tilde{\theta}(2^j \cdot -k), \quad \forall k \in \tilde{\Delta}_{j,\tilde{Z}}^I. & (36) \end{aligned}$$

In order to construct the generators Θ^L, Θ^R which satisfy the condition (34), we use both integration and differentiation. We especially remark that integrating a function raises the number of zero boundary conditions. Therefore, we adjust these integrations on the left by corresponding differentiations on the dual system, in order to preserve a certain stability which is necessary for the next step, that is the biorthogonalization of the resulting system. The different steps of the construction are summarized hereafter. The technical proofs are omitted and can be found in [15].

Now we write the basis obtained when there is no boundary constraints. We suppose that $d + \tilde{d}$ is even and we let $a', \tilde{a}' \in \mathbf{N}$ with $a' \geq d - s$, $\tilde{a}' \geq \tilde{d} + s$. The dual system is therefore given by

$$\begin{aligned} \Theta^{(+0)} &:= \left\{\theta^L_{j,a'-(d-s)+r}(d-s, \tilde{d}+s, a') : r = 0, \ldots, d-s-1\right\} \\ &\cup \left\{{}_{d-s}\theta_{[j,a']}, \ldots, {}_{d-s}\theta_{[j,a'+b']}\right\}, & (37) \\ \tilde{\Theta}^{(-0)} &:= \left\{\tilde{\theta}^L_{j,\tilde{a}'-(\tilde{d}+s)+r}(d-s, \tilde{d}+s, \tilde{a}') : r = 0, \ldots, \tilde{d}+s-1\right\} \\ &\cup \left\{{}_{d-s,\tilde{d}+s}\tilde{\theta}_{[j,\tilde{a}']}, \ldots, {}_{d-s,\tilde{d}+s}\tilde{\theta}_{[j,\tilde{a}'+\tilde{b}']}\right\}, & (38) \end{aligned}$$

where

$$a' - d + s = \tilde{a}' - \tilde{d} - s,$$

and we suppose that

$$a' + b' = \tilde{a}' + \tilde{b}'.$$

This system satisfies

$$\det \langle \Theta^{(+0)}, \tilde{\Theta}^{(-0)} \rangle_{[0,1]} \neq 0. \tag{39}$$

Integrating the primal system on the left and differentiation on the dual one s times lead to the new system

$$\begin{aligned}
\Theta^{(+s)} &:= \left\{ \theta^L_{j,l-s-d+r}(\cdot|d,\tilde{d},l-s) \,:\, r = s,\dots,d-1 \right\} \\
&\cup \ \left\{ {}_d\theta_{[j,l-s]}, \dots, {}_d\theta_{[j,l-s+b']} \right\}, \qquad (40)\\
\tilde{\Theta}^{(-s)} &:= \left\{ \tilde{\theta}^L_{j,\tilde{l}-\tilde{d}+r}(\cdot|d,\tilde{d},\tilde{l}) \,:\, r = 0,\dots,\tilde{d}-1 \right\} \\
&\cup \ \left\{ {}_{d,\tilde{d}}\tilde{\theta}_{[j,\tilde{l}]}, \dots, {}_{d-s+1,\tilde{d}+s-1}\tilde{\theta}_{[j,\tilde{l}+\tilde{b}'+s]} \right\}. \qquad (41)
\end{aligned}$$

The above sets allow to obtain the following Theorem (see Theorem 4.2 of [15] in the case $Z = \{0,1\}, \{0\}$). The other cases follow by symmetry arguments.

Theorem 3.6 *For every $Z \subseteq \{0,1\}$, $s, \tilde{s} \in \mathbf{N}$, $s < d$, $\tilde{s} < \tilde{d}$ and $j \geq j_0$, the pairs*

$$\Theta'_{j,Z} := \Theta^L_{j,Z} \cup \Theta^I_{j,Z} \cup \Theta^R_{j,Z}, \tag{42}$$

$$\tilde{\Theta}'_{j,\tilde{Z}} := \tilde{\Theta}^L_{j,\tilde{Z}} \cup \tilde{\Theta}^I_{j,\tilde{Z}} \cup \tilde{\Theta}^R_{j,\tilde{Z}}, \tag{43}$$

defined hereabove satisfy (34) and can be biorthogonalized.

We know define a multiresolution analysis corresponding to symmetric boundary conditions, namely $Z = \{0,1\}$ or $Z = \emptyset$.

For any integers $d', \tilde{d}', q, p, \tilde{p}$ such that $d' + \tilde{d}'$ is even and $d' + p = \tilde{d}' + \tilde{p}$, we introduce the notation

$$\Theta'_j(d',\tilde{d}',q,p) = \bigcup \left\{ \Theta^X_j(d',\tilde{d}',q,p) \,:\, X \in \{L,I,R\} \right\}. \tag{44}$$

Lemma 3.7 *If we define the ith integral of a function g on $[0,1]$ by $\left(\int^{(i)} g\right)(x) := \int_0^x \left(\int^{(i-1)} g\right)(t)dt$ when $i \in \mathbf{N}$ and $\left(\int^{(0)} g\right)(x) := g(x)$ and if we note $\partial_x^i S$ the set of the ith order derivatives of the elements of S, one has*

$$\partial_x^s \tilde{S}_j^{(-0)} =: \tilde{S}_j^{(-s)} = S(\tilde{\Theta}'_j(d,\tilde{d},0,\tilde{l})), \tag{45}$$

and

$$S(\Theta'_j(d,\tilde{d},s,l-s)) = S_j^{(+s)}, \tag{46}$$

with

$$S_j^{(+s)} = \left\{ w = \int^{(s)} v \,:\, v \in S_j^{(+0)}, \left(\int^{(i)} v \right)(1) = 0, i = 1, \ldots, s \right\}. \tag{47}$$

Now that the construction of the spaces $S(\Theta_{j,Z}), S(\tilde{\Theta}_{j,\tilde{Z}})$ is established, we are able to define the biorthogonal wavelet bases. Again, we start to study the case of symmetric boundary conditions.

For the case $Z = \emptyset$, the biorthogonal basis is well-known and can be expressed by

$$(\Psi_j^{(+0)})^T = (\Theta_{j+1}^{(+0)})^T \mathbf{M}_{j,1}^{(+0)}, \quad (\tilde{\Psi}_j^{(-0)})^T = (\tilde{\Theta}_{j+1}^{(-0)})^T \tilde{\mathbf{M}}_{j,1}^{(-0)}. \tag{48}$$

The wavelets $\Psi_j^{(+0)}, \tilde{\Psi}_j^{(-0)}$ are compactly supported, biorthogonal and the functions $\Psi_j^{(+0)}$ have $\tilde{d} + s$ vanishing moments.

As for the generators, we introduce succesively

$$\Psi_j^{(+i)} := 2^{ij} \left(\int^{(i)} \Psi_j^{(+0)} \right) \subset S_{j+1}^{(+i)}, \tag{49}$$

and

$$\tilde{\Psi}_j^{(-i)} = (-1)^i 2^{-ij} \frac{d^i}{dx^i} \tilde{\Psi}_j^{(-0)} \subset \tilde{S}_{j+1}^{(-i)}, \tag{50}$$

where

$$\tilde{S}_j^{(-i)} = \partial_x^i \tilde{S}_j^{(-0)}.$$

Therefore, one obtain the

Proposition 3.8 *The collections $\Psi_j^{(+i)}, \tilde{\Psi}_j^{(-i)}$ are biorthogonal bases with*

$$S_{j+1}^{(+i)} = S_j^{(+i)} \bigoplus S(\Psi_j^{(+i)}), \quad \tilde{S}_{j+1}^{(-i)} = \tilde{S}_j^{(-i)} \bigoplus S(\tilde{\Psi}_j^{(-i)}), \quad i = 0, \ldots, s. \tag{51}$$

Concerning symmetric boundary conditions, if the generators are given by

$$\Theta_{j,\{0,1\}} = \Theta_j^{(+s)}, \quad \tilde{\Theta}_{j,\emptyset} = \tilde{\Theta}_j^{(-s)}, \tag{52}$$

$$\Theta_{j,\emptyset} = \Theta_j^{(-\tilde{s})}, \quad \tilde{\Theta}_{j,\{0,1\}} = \tilde{\Theta}_j^{(+\tilde{s})}, \tag{53}$$

we pose:

$$\begin{array}{rclrcl} \Psi_j^{\{0,1\}} & := & \Psi_j^{(+s)}, & \tilde{\Psi}_j^{\emptyset} & := & \tilde{\Psi}_j^{(-s)} \\ \Psi_j^{\emptyset} & := & \Psi_j^{(-s)}, & \tilde{\Psi}_j^{\{0,1\}} & := & \tilde{\Psi}_j^{(+s)}. \end{array} \tag{54}$$

Remark 3.9 *The stationary interior wavelets of the sets $\Psi_j^{(+s)}$ and $\Psi_j^{(-\tilde{s})}$ coincide.*

Therefore, for our concrete problem, we have $Z = \{0\}$ and if we pose $\nabla_j^0 = \{p, p+1, \dots, 2^j - p - 1, 2^j - p\}$, the biorthogonal wavelet basis is defined by

$$\Psi_{j,L}^{\{0\}} := \{\psi_{j,k}^{\{0,1\}} : k = 1, \dots, p-1\}, \tag{55}$$
$$\Psi_{j,I}^{\{0\}} := \{\psi_{j,k}^{\emptyset} : k \in \nabla_j^0\}, \tag{56}$$
$$\Psi_{j,R}^{\{0\}} := \{\psi_{j,k}^{\emptyset} : k = 2^j - p + 1, \dots, 2^j\}, \tag{57}$$

which will be briefly denoted by $\{\Psi_j^Z\}$, and the dual system is given by

$$\tilde{\Psi}_{j,L}^{\{1\}} := \{\tilde{\psi}_{j,k}^{\emptyset} : k = 1, \dots, p-1\}, \tag{58}$$
$$\tilde{\Psi}_{j,I}^{\{1\}} := \{\tilde{\psi}_{j,k}^{\{0,1\}} : k \in \nabla_j^0\}, \tag{59}$$
$$\tilde{\Psi}_{j,R}^{\{1\}} := \{\tilde{\psi}_{j,k}^{\{0,1\}} : k = 2^j - p + 1, \dots, 2^j\}, \tag{60}$$

denoted by $\{\tilde{\Psi}_j^{\tilde{Z}}\}$.

Coming back to our problem, for each $j \geq 0$ and $Z = \{0\}$ we pose

$$\{\Psi_j^T\} = \{\Psi_{2j}^Z\} \bigcup \{\Psi_{2j-1}^Z\}. \tag{61}$$

We define a sequence of spaces W_j by $V_{j+1} = V_j \overset{\not\perp}{\bigoplus} W_j$, where

$$W_j = Span\left\{\Psi_{j,k_1}^X(x)\Psi_{j,k_2}^T(t)\right\}_{\substack{k_1 \in \nabla_j^X \\ k_2 \in \nabla_j^T}} \tag{62}$$
$$\overset{not}{=} Span\{\Xi_{j,K}(x,t)\}_{K \in \nabla^{XT}}, \tag{63}$$

for $j \geq j_0$ in order to avoid the overlapping of the left and right end-point wavelets and to ensure the presence of interior wavelets in $[0,1]$.

Now we can proove the following result :

Theorem 3.10 *Let $\{N_k\}_k$ be a basis of V_{j_0}. If*

$$u(x,t) = \sum_{k=0}^{k_0} c_k N_k(x,t) + \sum_{j \geq j_0} \sum_{K \in \nabla^{XT}} c_{j,K} \Xi_{j,K}(x,t)$$

belongs to $\tilde{H}^{r,r/2}([0,1]^2)$ for $r \in [0, 5/2)$ (with $d^X \geq 2$ if $r < 3/2$; $d^X \geq 3$ if $r \in [3/2, 5/2)$ and $d^Z \geq 2$), then we have

$$\|u\|^2_{\tilde{H}^{r,r/2}([0,1]^2)} \sim \sum_{k=0}^{k_0} |c_k|^2 + \sum_{j,K} 2^{2rj} |c_{j,K}|^2. \tag{64}$$

Proof : We recall that

$$\tilde{H}^{r,r/2}([0,1]^2) = L^2([0,1]; H^r([0,1])) \cap L^2([0,1]; \tilde{H}^{r/2}([0,1])) := H_1 \cap H_2.$$

For $r, p \geq 0$ given, we introduce the following bases, respectively for the space part and for the time part.

$$\mathcal{B}_X = \{_{d_X}\varphi_{j_0,k_1}(x)\}_{k_1} \cup \left\{ \bigcup_{\substack{j \geq j_0, \\ k_1 \in \nabla_j^X}} \{\Psi_{j,k_1}^X(x)\} \right\}_{j,k_1}, \tag{65}$$

$$\mathcal{B}_T = \{_{d_T}\varphi_{j_0,k_2}(t)\}_{k_2} \cup \left\{ \bigcup_{\substack{j \geq j_0, \\ k_2 \in \nabla_j^T}} \{\Psi_{j,k_2}^T(t)\} \right\}_{j,k_2}, \tag{66}$$

where the sets $\{\Psi^X\}$ and $\{\Psi^T\}$ are defined previously and characterize respectively $H^r([0,1])$ and $H^p([0,1])$.

We omit in the following the indices d_X, d_T. A wavelet basis on $\tilde{H}^{r,p}([0,1]^2)$ is defined by tensor products, namely $\mathcal{B} = \mathcal{B}_X \otimes \mathcal{B}_T$:

$$\begin{aligned} \mathcal{B} &= \{\varphi_{j_0,k_1}(x).\varphi_{j_0,k_2}(t)\}_K \\ &\bigcup \{\varphi_{j_0,k_1}(x).\Psi_{j,k_2}^T(t)\}_{j,K} \\ &\bigcup \{\Psi_{j,k_1}^X(x).\varphi_{j_0,k_2}(t)\}_{j,K} \\ &\bigcup \{\Psi_{j,k_1}^X(x).\Psi_{j,k_2}^T(t)\}_{j,K}, \end{aligned}$$

with $K = (k_1, k_2)$ for short.

Clearly, $\mathcal{B}$ is a Riesz basis of $L^2([0,1]^2)$. Moreover, we can show that it characterizes $\tilde{H}^{r,p}([0,1]^2)$. A function $u \in L^2([0,1]^2)$ can be written in this basis in the following form : for $j_0 > 0$ fixed, if we note $k_0 = 2^{j_0} - 1$, we write :

$$\begin{aligned} u(x,t) &= \sum_{k_1,k_2=0}^{k_0} d_{j_0;k_1,k_2}.\varphi_{j_0,k_1}(x).\varphi_{j_0,k_2}(t) \\ &+ \sum_{\substack{j \geq j_0, \\ k_1=0,\ldots,k_0, \\ k_2 \in \nabla_j^T}} e_{j;k_1,k_2}.\varphi_{j_0,k_1}(x).\Psi_{j_0,k_2}^T(t) \\ &+ \sum_{\substack{j \geq j_0, \\ k_1 \in \nabla_j^X, \\ k_2=0,\ldots,k_0}} \tilde{e}_{j;k_1,k_2}.\Psi_{j,k_1}^X(x).\varphi_{j_0,k_2}(t) \\ &+ \sum_{\substack{j \geq j_0, \\ k_1 \in \nabla_j^X, \\ k_2 \in \nabla_j^T}} c_{j;k_1,k_2}.\Psi_{j,k_1}^X(x).\Psi_{j,k_2}^T(t), \end{aligned}$$

which is noted shortly as :

$$u(x,t) = \sum_{i,j=0,1} u^{ij}(x,t). \tag{67}$$

We retail the characterization of the space H_1. We pose :

$$u(\cdot,t)=\sum u^{ij}(\cdot,t),\forall t\in(0,1).$$

Writing

$$u^{11}(\cdot,t)=\sum_{\substack{j\geq j_0,\\ k_1\in\nabla_j^X}} c_{j;k_1}(t).\Psi^X_{j,k_1}(\cdot), \tag{68}$$

with

$$c_{j,k_1}(t)=\sum_{k_2\in\nabla_j^T} c_{j;k_1,k_2}\Psi^T_{j,k_2}(t), \quad \text{for } j\geq j_0. \tag{69}$$

Therefore, due to the properties of Ψ^T, we get

$$\|u^{11}(\cdot,t)\|^2_{H^r([0,1])}\sim\sum_{j,k_1}2^{2rj}|c_{j,k_1}(t)|^2,$$

and

$$\begin{aligned}
\|u^{11}\|_{H_1} &= \int_0^1\|u^{11}(\cdot,t)\|^2_{H^r([0,1])}dt\\
&= \sum_{j,k_1}2^{2rj}\int_0^1|c_{j,k_1}(t)|^2dt\\
&\sim \sum_{j,k_1,k_2}2^{2rj}|c_{j,k_1,k_2}|^2.
\end{aligned}$$

By the same way, we show that :

$$\|u^{10}(\cdot,t)\|^2_{H^r}\sim\sum_{j,k_1}2^{2rj}|\tilde{e}_{j,k_1}(t)|^2,$$

with

$$\tilde{e}_{j,k_1}(t)=\sum_{k_2}\tilde{e}_{j,k_1,k_2}\varphi_{j_0,k_2}(t)\in V_{j_0},$$

and

$$\|\tilde{e}_{j,k_1}(t)\|^2_{L^2}\sim\sum_{k_2}|\tilde{e}_{j,k_1,k_2}|^2.$$

Consequently

$$\|u^{10}\|^2_{H_1}\sim\sum_{j;k_1,k_2}2^{2rj}|\tilde{e}_{j,k_1,k_2}|^2.$$

For the first term we have

$$\begin{aligned}
u^{01}(\cdot,t) &= \sum_{\substack{j\geq j_0,\\ k_1=0,\dots,k_0}} e_{j,k_1}(t)\varphi_{j_0,k_1}(x)\\
&= \sum_{k_1=0}^{k_0}\left[\sum_{j\geq j_0}e_{j,k_1}(t)\right]\varphi_{j_0,k_1}(x),
\end{aligned}$$

with

$$e_{j,k_1}(t) = \sum_{k_2 \in \nabla_j^T} e_{j,k_1,k_2} \Psi^T_{j,k_2}(t) \in W_j.$$

$$\|u^{01}\|^2_{H_1} \sim \sum_{j;k_1} |\tilde{e}_{j,k_1}(t)|^2 \sim \sum_{j;k_1,k_2} |\tilde{e}_{j,k_1,k_2}|^2.$$

We can therefore show the next two equivalences :

$$\begin{aligned}
\|u\|^2_{H_1} &\sim \sum_{k_1,k_2} |d_{j_0;k_1,k_2}|^2 + \sum_{j,k_1,k_2} |e_{j;k_1,k_2}|^2 \\
&+ \sum_{j,k_1,k_2} 2^{2rj} |\tilde{e}_{j;k_1,k_2}|^2 + \sum_{j,k_1,k_2} 2^{2rj} |c_{j;k_1,k_2}|^2, \\
\|u\|^2_{H_2} &\sim \sum_{k_1,k_2} |d_{j_0;k_1,k_2}|^2 + \sum_{j,k_1,k_2} 2^{4pj} |e_{j;k_1,k_2}|^2 \\
&+ \sum_{j,k_1,k_2} |\tilde{e}_{j;k_1,k_2}|^2 + \sum_{j,k_1,k_2} 2^{4pj} |c_{j;k_1,k_2}|^2.
\end{aligned}$$

In the following, we omit the indices of the wavelet coefficients for the sake of brevity. If $u \in H_1 \cap H_2 = \tilde{H}^{r,p}([0,1]^2)$, we write

$$\begin{aligned}
\|u\|^2_{r,p} &\overset{def}{=} \|u\|^2_{H_1} + \|u\|^2_{H_2} \\
&\sim 2\left(\sum |d|^2\right) + \left(\sum |e|^2 + \sum 2^{4pj}|e|^2\right) \\
&+ \left(\sum 2^{2rj}|\tilde{e}|^2 + \sum |\tilde{e}|^2\right) + \left(\sum (2^{2rj} + 2^{4pj})|c|^2\right).
\end{aligned}$$

For the general case $r, p \in \mathbf{R}$, we obtain :

$$\begin{aligned}
\|u\|^2_{r,p} &\sim \sum |d|^2 + \sum \max\left(1, 2^{4pj}\right) |e|^2 \\
&+ \sum \max\left(1, 2^{2rj}\right) |\tilde{e}|^2 + \sum \max\left(2^{2rj}, 2^{4pj}\right) |c|^2.
\end{aligned} \tag{70}$$

The next particular case is interesting for our problems.
Suppose now that $r = 2p$. We get :

$$\|u\|^2_{r,r/2} \sim \sum |d|^2 + \sum 2^{2rj}[|e|^2 + |\tilde{e}|^2 + |c|^2]. \tag{71}$$

■

4 Galerkin method

We begin to write the variational formulation of the problem : we want to find $\mu \in L^2(\Sigma_T)$ solution of

$$a(\mu, v) = \left\langle \left(\frac{1}{2}I + D\right)\mu, v \right\rangle_{\Sigma_T} = \langle Sg_1, v\rangle_{\Sigma_T}, \forall v \in L^2(\Sigma_T), \tag{72}$$

if we take the equation of the second kind, or

$$h(\mu, v) = \langle H\mu, v\rangle_{\Sigma_T} = \langle \left(\frac{1}{2}I - D'\right) g_1, v\rangle_{\Sigma_T}, \forall v \in \tilde{H}^{1/2,1/4}(\Sigma_T), \tag{73}$$

for the equation of the first kind.

The situation is not the same is these two cases. In (72), the bilinear form a is not coercive on $L^2(\Sigma_T)$. But we can overcome this difficulty by using the fact that the operator $\left(\frac{1}{2}I + D\right)$ is a compact perturbation of the identity and a result of [17]. The second case (73) is easier to treat because the operator H is strongly coercive. Therefore, we begin to study the second case.

We recall the properties of the hypersingular operator H (see [8, 16]).

Lemma 4.1 *The operator $H : \tilde{H}^{r,r/2}(\Sigma_T) \longrightarrow \tilde{H}^{r-1,(r-1)/2}(\Sigma_T)$ is an isomorphism, for all $r \geq \frac{1}{2}$.*

Furthermore, H is strongly coercive, i.e.

$$(u, Hu)_{\Sigma_T} \gtrsim \|u\|^2_{\frac{1}{2},\frac{1}{4}}, \forall u \in \tilde{H}^{1/2,1/4}(\Sigma_T). \tag{74}$$

The Galerkin approximation $\mu_L \in V_L$ of the solution μ of (9) is the unique solution of

$$(H\mu_L, v_L)_{\Sigma_T} = \left(\left(\frac{1}{2}I - D'\right) g_1, v_L\right)_{\Sigma_T}, \forall v_L \in V_L. \tag{75}$$

We will note H_L the corresponding stiffness matrix in the basis $\{\Xi_\lambda\}$.

Due to the coercivity on $\tilde{H}^{1/2,1/4}(\Sigma_T)$ and to the Theorem 3.10, we modify slightly the basis (by a diagonal preconditioning) in order to obtain a well-conditioned stiffness matrix.

Corollary 4.2 *We define the following set*

$$\hat{\Xi}_\lambda(x, t) = 2^{-|\lambda|/2}\Xi_\lambda(x, t), \tag{76}$$

as a new basis for $\tilde{H}^{1/2,1/4}(\Sigma_T)$ and we note $\hat{H}_L$ the stiffness matrix in this new basis $\{\tilde{\Xi}_\lambda\}$. We obtain therefore

$$\text{Cond } (\hat{H}_L) \lesssim 1. \tag{77}$$

A direct consequence of the Céa's Lemma and Therorem 3.10 is the obtention of an error estimate.

Theorem 4.3 *Let $r' \in \left(\frac{1}{2}, \frac{5}{2}\right)$ and $g_1 \in \tilde{H}^{r'-1,\frac{r'-1}{2}}(\Sigma_T)$. Then the solution μ of (9) belongs to the space $\tilde{H}^{r',\frac{r'}{2}}(\Sigma_T)$ and the error between μ and its Galerkin approximation μ_L is estimated by*

$$\|\mu - \mu_L\|_{\frac{1}{2},\frac{1}{4}} \lesssim 2^{(\frac{1}{2}-r')L}\|\mu\|_{r',\frac{r'}{2}}. \tag{78}$$

Now we study the equation of the second kind. The non-coercivity is overcomed by the use of the following Theorem (see [3, 17, 22]).

Theorem 4.4 *Let H be a separable Hilbert space with a norm denoted by $\|.\|$, let A be a one-to-one continuous operator on H and let H_n, $n \in \mathbf{N}$ be a family of finite-dimensional subspaces of H. Denote by P_n the orthogonal projection on H_n and assume that*

$$\|(I - P_n)w\| \to 0, \;\; as\; n \to \infty, \forall w \in H. \tag{79}$$

If there exists a positive constant α and an integer such that

$$\|P_n A x_n\| \geq \alpha \|x_n\|, \forall x_n \in H_n, \forall n > N, \tag{80}$$

then for all $f \in H$, the problem

$$Ay = f \tag{81}$$

has a unique solution $y \in H$. Moreover, for all $n > N$, there exists a unique solution $y_n \in H_n$ of the approximate problem

$$P_n A y_n = P_n f, \tag{82}$$

and satisfying

$$\|y - y_n\| \leq c\|y - P_n y\|. \tag{83}$$

Here the assumption (80) is satisfied thanks to the following result (see [3]) :

Lemma 4.5 *If $A = \frac{I}{2} - D$ is an isomorphism from H into itself with a compact operator D, then A satisfies (80).*

Proof : In fact, we take advantage of the compactness of D, which is shown in [8]. The proof is therefore a direct consequence of the Lemma in [17]. ■

The previous Theorem proves existence and uniqueness for the approximation problem: find $\mu_L \in V_L$ solution of

$$a(\mu_L, v_L) = \left\langle \left(\frac{1}{2}I + D\right) \mu_L, v_L \right\rangle_{\Sigma_T} = \langle S g_1, v_L \rangle_{\Sigma_T}, \forall v_L \in V_L. \tag{84}$$

The stiffness matrix of this system will be denoted by A_L. Consequently, the result hereafter is a direct consequence of the Theorems 3.10 and 4.4 :

Theorem 4.6 *Let $\mu \in \tilde{H}^{r,r/2}(\Sigma_T)$ be the solution of (8) with a datum $g_1 \in \tilde{H}^{r-1,\frac{r-1}{2}}(\Sigma_T)$, $r \in [0, 5/2)$. Then its Galerkin approximation $\mu_L \in V_L$ satisfies*

$$\|\mu - \mu_L\|_{L^2(\Sigma_T)} \lesssim 2^{-rL} \|u\|_{r,r/2}. \tag{85}$$

Moreover, the condition number of the stiffness matrix of (84) is uniformly bounded.

5 Compression of the stiffness matrix

We try in this section to adapt the ideas developped by [13, 23] among others to the case of the heat equation. One of the essential points is the decay property of the kernel. To our knowledge, in the above mentioned works, only elliptic problems were considered, involving logarithmic kernels or a power of distance function. Here the situation is quite different and the kernel is an exponential function. We give now a way to express correctly this property.

We introduce the abbreviated notation

$$\tilde{E}(x,t) = \frac{1}{t} e^{\frac{-x^2}{4t}}, \tag{86}$$

and using the Leibniz rule, we have an expression of the successive derivatives of $\tilde{E}$.

Lemma 5.1 *For all $\alpha, \gamma \geq 0$, there exist some real coefficients $a_{\alpha\beta\gamma}$ such that*

$$\frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial^\alpha \tilde{E}}{\partial x^\alpha} = \sum_{\beta=0}^{\alpha+2\gamma} a_{\alpha\beta\gamma} \frac{x^\beta}{t^{\alpha/2+\beta/2+\gamma+1}} e^{\frac{-x^2}{4t}}. \tag{87}$$

The second step consists of the following lemma.

Lemma 5.2 *Let $a_1 > 0, a_2 \in \mathbf{R}$ and set $b = a_2 - \frac{a_1}{2}$. Then it holds*

$$(x^2+t)^b \frac{x^{a_1}}{t^{a_2}} e^{\frac{-x^2}{4t}} \lesssim 1, \quad \forall x, t > 0. \tag{88}$$

Proof : Because $t > 0$, we have

$$(x^2+t)^b \frac{x^{a_1}}{t^{a_2}} = \left(\frac{x^2}{t} + 1\right)^b \left(\frac{x^2}{t}\right)^{\frac{a_1}{2}}.$$

Using the fact that

$$(x+1)^\delta x^\eta e^{\frac{-x}{4}} \lesssim 1, \forall x > 0,$$

when $\eta > 0$, we obtain the result. ■

We are now in a position to write the correct decay property of the kernel.

Lemma 5.3 *For all $\alpha, \gamma \geq 0$, we have*

$$\left| \frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial^\alpha \tilde{E}}{\partial x^\alpha} \right| \lesssim \frac{1}{(x^2+t)^{\alpha/2+\gamma+1}}, \quad \forall x, t > 0. \tag{89}$$

Proof : The identity (87) allows to write

$$\left|\frac{\partial^\gamma}{\partial t^\gamma}\frac{\partial^\alpha \tilde{E}}{\partial x^\alpha}\right| \lesssim \sum_{\beta=0}^{\alpha+2\gamma} \frac{x^\beta}{t^{\alpha/2+\beta/2+\gamma+1}} e^{\frac{-x^2}{4t}}.$$

We apply therefore the Lemma 5.2 with $a_1 = \beta$ and $a_2 = \alpha/2 + \beta/2 + \gamma + 1$, for all $\beta = 0, \ldots, \alpha + 2\gamma$ and get the result. ■

We exploit this property to show the decay of the stiffness matrix coefficients. We begin with the formulation of the second kind. We recall that we want to solve

$$a(\mu, v) = \left\langle \left(\frac{1}{2}I + D\right)\mu, v\right\rangle_{\Sigma_T} = (Sg_1, v)_{\Sigma_T}, \forall v \in L^2(\Sigma_T).$$

Because the operator I is diagonal, we are going to compress only the operator D. This suggests to introduce the following bilinear form

$$\begin{aligned} d(\mu, v) &:= \langle D\mu, v\rangle_{\Sigma_T} \\ &= \int_{\Gamma\times(0,T)} v(x,t)\left[\int_{\Gamma\times(0,t)} \mu(y,\tau)\partial n_y E(x-y, t-\tau) d\Gamma_y d\tau\right] d\Gamma_x dt. \end{aligned}$$

We use the notation

$$\begin{aligned} D_{\lambda,\lambda'} &= D_{j,j';k_1,k_2;k_1',k_2'} \\ &= d(\Psi^X_{j,k_1} \otimes \Psi^T_{j,k_2}; \Psi^X_{j',k_1'} \otimes \Psi^T_{j',k_2'}), \end{aligned}$$

for the coefficients of the matrix and we introduce the quantity

$$dist_{\lambda,\lambda'} = \delta\left(\Omega^X_{j,k_1}, \Omega^X_{j',k_1'}\right)^2 + \delta\left(\Omega^T_{j,k_2}, \Omega^T_{j',k_2'}\right), \tag{90}$$

where for two continuous functions χ, ψ with a compact support on Γ, $\Omega_{j,k}$ is the interior of the support of $\chi_{j,k}$ and $\delta(\Omega, \Omega')$ is the euclidian distance between Ω and Ω'.

After these necessary notations, we are able to formulate the decay property of the coefficients.

Proposition 5.4 *For all the coefficients of the stiffness matrix, one has*

$$|D_{\lambda,\lambda'}| \leq C\frac{2^{-b(j+j')}}{(dist_{\lambda,\lambda'})^b}, \tag{91}$$

with $b = \tilde{d}^X + 2\tilde{d}^T + \frac{3}{2}$.

Proof : By Fubini's theorem, we have

$$\begin{aligned} D_{\lambda,\lambda'} &= \int_{\Gamma^2\times\Delta} \Psi^T_{j',k'_2}(t)\Psi^X_{j',k'_1}(x)\Psi^T_{j,k_2}(\tau)\Psi^X_{j,k_1}(y) \\ &\times \qquad \partial_{n_y}E(x-y,t-\tau)d\Gamma_x d\Gamma_y dt d\tau, \end{aligned}$$

where $\Delta = \{(t,\tau)\in(0,T)^2 : \tau < t\}$.

The wavelets Ψ^X and Ψ^T have respectively $\tilde{d}^X$ and $\tilde{d}^T$ vanishing moments, which allows to make integration by parts in the previous estimate. Indeed, there exist $\theta_{2j,\hat{k}_2}$ and $\theta_{2j-1,\tilde{k}_2}$ such that

$$\Psi^T_{j,k_2} = {}_{d_T}\theta^{(\tilde{d}^T)}_{2j,\hat{k}_2}$$

or

$$\Psi^T_{j,k_2} = {}_{d_T}\theta^{(\tilde{d}^T)}_{2j-1,\tilde{k}_2}.$$

Similarly, we have

$$\Psi^X_{j,k_1} = {}_{d_X}\theta^{(\tilde{d}^X)}_{j,k_1}.$$

Furthermore, these functions satisfy

$$\|\theta_{j,k}\| \lesssim 2^{-(\tilde{d}+\frac{1}{2})j}. \tag{92}$$

Consequently, we obtain

$$\begin{aligned} |D_{\lambda,\lambda'}| &\lesssim \int_{[0,1]^2\times\Delta} \left| \frac{\partial^{2(\tilde{d}^X+\tilde{d}^T)}k(x,y,t,\tau)}{\partial x^{\tilde{d}^X}\partial y^{\tilde{d}^X}\partial t^{\tilde{d}^T}\partial \tau^{\tilde{d}^T}} \right| \\ &\times \left| \theta^T_{2j',\tilde{k}'_2}(t)\theta^X_{j',k'_1}(x)\theta^T_{2j,k_2}(\tau)\theta^X_{j,k_1}(y) \right| dxdydtd\tau, \end{aligned}$$

where k is, after a parametrisation, a first order partial derivative (by y) of E.

We get the conclusion by using the Lemma 5.3 with $\alpha = 2\tilde{d}^X + 1$ and $\gamma = 2\tilde{d}^T$. ∎

Now for $j,j' \in \{j_0,\ldots,L\}$, we can define the compressed subblocks

$$\tilde{D}_{j,j'} = \left(\tilde{D}_{\lambda,\lambda'}\right)_{\substack{k_1\in\nabla^X_j \;;\; k_2\in\nabla^T_j \\ k'_1\in\nabla^X_{j'} \;;\; k'_2\in\nabla^T_{j'}}} \tag{93}$$

where we pose

$$\tilde{D}_{\lambda,\lambda'} = \begin{cases} 0 & \text{if } dist_{\lambda,\lambda'} \geq \delta_{j,j'}, \\ D_{\lambda,\lambda'} & \text{else.} \end{cases} \tag{94}$$

We associate with D_L (resp. $\tilde{D}_L$, the global compressed matrix) the operator $\mathcal{D}_L$ (resp. $\tilde{\mathcal{D}}_L$) from V_L into its dual.

At this stage, we are in a position to estimate the number of non-zero elements $\mathcal{N}(\tilde{D}_{j,j'})$ of the subblock matrix $\tilde{D}_{j,j'}$.

Proposition 5.5 *If $\tilde{D}_{j,j'}$ is defined by (94) with a compression parameter $\delta_{j,j'} > 0$, then*

$$\mathcal{N}(\tilde{D}_{j,j'}) \lesssim 2^{3(j+j')} \quad \min \quad \Big\{ 2^{-3j} + 2^{-3j'} + 2^{-j-2j'} + 2^{-j'-2j} \tag{95}$$

$$+ \quad \delta_{j,j'}^{1/2}(2^{-2j} + 2^{-2j'}) + \delta_{j,j'}(2^{-j} + 2^{-j'}) + \delta_{j,j'}^{3/2}, 1 \Big\} . \tag{96}$$

Proof : The proof is adapted from the one-dimensional case (see for instance Lemma 5.2 of [2, 21]). We begin by fixing a wavelet Ξ_λ and we estimate the number of wavelets $\Xi_{\lambda'}$ satisfying $dist_{\lambda,\lambda'} < \delta_{j,j'}$. To do this, we separate the problem into two parts, because $dist_{\lambda,\lambda'} < \delta_{j,j'}$ implies that

$$\delta(\Omega^X_{j,k_1}, \Omega^X_{j',k'_1}) < \delta_{j,j'}^{1/2}$$

and

$$\delta(\Omega^T_{j,k_2}, \Omega^T_{j',k'_2}) < \delta_{j,j'}.$$

If we denote by $\mathrm{Card}\{k'_1\}$ and $\mathrm{Card}\{k'_2\}$ the respective numbers of wavelets such that the previous inequalities hold, we obtain (see [2, 21])

$$\mathrm{Card}\{k'_1\} \lesssim 1 + \frac{2^{-j} + \delta_{j,j'}^{1/2}}{2^{-j'}}, \quad \mathrm{Card}\{k'_2\} \lesssim 1 + \frac{2^{-2j} + \delta_{j,j'}}{2^{-2j'}}.$$

Because $\mathcal{N}(\tilde{D}_{j,j'}) \le \sum_{k'_1,k'_2} \mathrm{Card}\{k'_1\}\mathrm{Card}\{k'_2\}$, we conclude that

$$\mathcal{N}(\tilde{D}_{j,j'}) \lesssim 2^{3j} \left(1 + \frac{2^{-j} + \delta_{j,j'}^{1/2}}{2^{-j'}}\right) \left(1 + \frac{2^{-2j} + \delta_{j,j'}}{2^{-2j'}}\right).$$

■

We now estimate the difference $D_{j,j'} - \tilde{D}_{j,j'}$.

Proposition 5.6 *Let $\tilde{D}_{j,j'}$ be defined by (94) with $\delta_{j,j'} > 0$. Therefore, one has*

$$\|D_{j,j'} - \tilde{D}_{j,j'}\|_\infty \lesssim 2^{-bj} 2^{-(b-3)j'} \delta_{j,j'}^{-b} \max\{\delta_{j,j'}^{3/2}, 2^{-3j}, 2^{-3j'}\}, \tag{97}$$

with $b = \tilde{d}^X + 2\tilde{d}^T + \frac{3}{2}$.

Proof : We start to write the definition of the infinite norm thanks to the estimate (91):

$$\|D_{j,j'} - \tilde{D}_{j,j'}\|_\infty \lesssim \max_{\substack{k_1 \in \nabla^X_j \\ k_2 \in \nabla^T_j}} \sum_{k'_1 \in \nabla^X_{j'}} \sum_{\substack{k'_2 \in \nabla^T_{j'} : \\ dist_{\lambda,\lambda'} \ge \delta_{j,j'}}} 2^{-b(j+j')} (dist_{\lambda,\lambda'})^{-b}.$$

We estimate directly the terms in the sum corresponding to wavelets $\Xi_{\lambda'}$ whose support is closest to the support of Ξ_λ. Their number is bounded by $\max\{2^{3j'}(2^{-j} + \delta_{j,j'}^{1/2})(2^{-2j} +$

$\delta_{j,j'}), 1\}$ and if we denote by K_{far} the remainding terms, we obtain

$$\begin{aligned}\|D_{j,j'} - \tilde{D}_{j,j'}\|_\infty &\lesssim 2^{-b(j+j')}\delta_{j,j'}^{-b} \max\{2^{3j'}(2^{-j} + \delta_{j,j'}^{1/2})(2^{-2j} + \delta_{j,j'}), 1\} \\ &+ 2^{-b(j+j')}2^{3j'} \max_{\substack{k_1 \in \nabla_j^X \\ k_2 \in \nabla_j^T}} \sum_{(k_1', k_2') \in K_{far}} (dist_{\lambda,\lambda'})^{-b} \int_{\Omega_{\lambda'}} d\sigma(M) \\ &\lesssim 2^{-b(j+j')}\delta_{j,j'}^{-b} \max\{2^{3j'}(2^{-j} + \delta_{j,j'}^{1/2})(2^{-2j} + \delta_{j,j'}), 1\} \\ &+ 2^{-b(j+j')}2^{3j'} \max_{\substack{k_1 \in \nabla_j^X \\ k_2 \in \nabla_j^T}} \int_{x^2+t>\delta_{j,j'}} \{x^2 + t\}^{-b} dxdt,\end{aligned}$$

and this last integral is lower than

$$\delta_{j,j'}^{3/2-b} \int_{x'^2+t'>1} \{x'^2 + t'\}^{-b} dx'dt',$$

by the change of variables $x = \delta_{j,j'}^{1/2}x'$ and $t = \delta_{j,j'}t'$. We remark that this last integral is convergent because $b > 3/2$. ■

The next result gives an estimate for the consistancy part of the error.

Theorem 5.7 *Let $r, \tilde{r} \in [0, \frac{5}{2})$ with $\{\tilde{d}^X, \tilde{d}^T\} \in \{2, 3\}$. Assume that the parameters $\delta_{j,j'}$ satisfy*

$$\delta_{j,j'} \geq a \cdot \max\{2^{-2j}, 2^{-2j'}, 2^{\alpha(L-j)}2^{\tilde{\alpha}(L-j')}2^{-2L}\}, \tag{98}$$

for $\alpha, \tilde{\alpha}, a > 0$ such that

$$\alpha > \frac{r + b + \tau}{b - 3/2}, \quad \tilde{\alpha} > \frac{\tilde{r} + b - 3 + \tau}{b - 3/2}, \tag{99}$$

for some $\tau > 0$.

Then for any $\mu_L, \tilde{\mu}_L \in V_L$, we have

$$\left|\left\langle (\mathcal{D}_L - \tilde{\mathcal{D}}_L)\mu_L, \tilde{\mu}_L \right\rangle\right| \lesssim a^{3/2-b} 2^{-L(r+\tilde{r})} \|\mu_L\|_{r,r/2} \|\tilde{\mu}_L\|_{\tilde{r},\tilde{r}/2}, \tag{100}$$

with $b = \tilde{d}^X + 2\tilde{d}^T + \frac{3}{2}$.

Proof : By Theorem 3.10 and the definition of $\tilde{D}_L$, we have

$$\left|\left\langle (\mathcal{D}_L - \tilde{\mathcal{D}}_L)\mu_L, \tilde{\mu}_L \right\rangle\right| \lesssim 2^{-L(r+\tilde{r})} \|E_L\|_2 \|\mu_L\|_{r,r/2} \|\tilde{\mu}_L\|_{\tilde{r},\tilde{r}/2},$$

where we define the matrix E_L by

$$E_L = 2^{(L-j)r}2^{(L-j')\tilde{r}} \left(D_{\lambda,\lambda'} - \tilde{D}_{\lambda,\lambda'}\right)_{\lambda,\lambda'}.$$

We estimate $\|E_L\|_2$ by the Schur lemma (see [18]) with the sequence $\gamma_{j,k_1,k_2} = 2^{-\tau j}$.

$$\sum_{j',k'_1,k'_2} |E_{L;\lambda,\lambda'}|\gamma_{j',k'_1,k'_2} \lesssim a^{3/2-b}\gamma_{j,k_1,k_2} 2^{(L-j)[r+(3/2-b)(\alpha-\kappa_1)]} \times \sum_{j'=j_0}^{L-1} 2^{(L-j')[\tilde{r}+(3/2-b)(\tilde{\alpha}-\kappa'_1)]},$$

for two parameters κ_1, κ'_1 satisfying $\kappa_1 + \kappa'_1 = 2$, $\kappa_1(b-3/2) = b-\tau$ and $\kappa'_1(b-3/2) = b-3+\tau$.

Then we have

$$\sum_{j',k'_1,k'_2} |E_{L;\lambda,\lambda'}|\gamma_{j',k'_1,k'_2} \lesssim a^{3/2-b}\gamma_{j,k_1,k_2},$$

if $\alpha, \tilde{\alpha}$ satisfy

$$\alpha > \frac{r+b-\tau}{b-3/2}, \quad \tilde{\alpha} > \frac{\tilde{r}+b-3+\tau}{b-3/2}.$$

We similarly show that

$$\sum_{j,k_1,k_2} |E_{L;\lambda,\lambda'}|\gamma_{j,k_1,k_2} \lesssim a^{3/2-b}\gamma_{j',k'_1,k'_2},$$

for

$$\alpha > \frac{r+b+\tau}{b-3/2}, \quad \tilde{\alpha} > \frac{\tilde{r}+b-3-\tau}{b-3/2}.$$

■

Lemma 5.8 *Assume that $\alpha, \tilde{\alpha} < 2$ for all $\{\tilde{d}^X, \tilde{d}^T\} \in \{2,3\}$. Suppose that the parameters $\delta_{j,j'}$ satisfy*

$$\delta_{j,j'} = a \cdot \max\{2^{-2j}, 2^{-2j'}, 2^{\alpha(L-j)}2^{\tilde{\alpha}(L-j')}2^{-2L}\}. \tag{101}$$

Then the number of non-zero elements of $\tilde{D}_L$ is of order $N_L \log N_L$, when $N_L = 2^{3L}$ is the size of the matrix.

Proof : We distinguish the case $\delta_{j,j'} = a \cdot \max\{2^{-2j}, 2^{-2j'}\}$ from the case $\delta_{j,j'} = a2^{\alpha(L-j)}2^{\tilde{\alpha}(L-j')}2^{-2L}$ and one obtain the result for $\alpha, \tilde{\alpha} < 2$. ■

Remark 5.9 *We can also make a "second compression" between wavelets at different levels j and j', such that if $j' < j$ for instance, the support of the wavelet at the finest grid j is far from the singular support of the wavelet at the coarse grid j' (see [23]). To this end, we introduce the following notations :*

we write

$$\Omega^{S,X}_{j,k} := \textit{sing supp}\ \Psi^X_{j,k}, \tag{102}$$

as the singular support of the wavelet $\Psi^X_{j,k}$. By the same way, we define $\Omega^{S,T}_{j,k}$ for the wavelet $\Psi^T_{j,k}$. We finally introduce the "singular pseudo-distance"

$$dist^S_{\lambda,\lambda'} = \delta\left(\Omega^{S,X}_{j,k_1}, \Omega^{S,X}_{j',k'_1}\right)^2 + \delta\left(\Omega^{S,T}_{j,k_2}, \Omega^{S,T}_{j',k'_2}\right). \tag{103}$$

Therefore, if $j' < j$ and $dist_{\lambda,\lambda'} \lesssim 2^{-j'}$, we have the next estimate

$$|D_{\lambda,\lambda'}| \leq C \frac{2^{-bj} 2^{j'}}{(dist^S_{\lambda,\lambda'})^{\frac{b-2}{2}}}, \tag{104}$$

with $b = \tilde{d}^X + 2\tilde{d}^T + \frac{3}{2}$.

See [23] for further details.

6 Convergence

The compressed Galerkin scheme is defined by $\tilde{A}_L \vec{\tilde{u}}_L = b$. We recall a result corresponding to the first Strang lemma (see [5]) because here we do not have coercivity.

Theorem 6.1 *Under the assumption of the Theorem 4.4, assume that there exists a sequence of operators A_n from H_n to H_n such that A_n is injective. Then the error between the exact solution $y \in H$ of $Ay = f$ and the approximated one $z_n \in H_n$ of*

$$A_n z_n = P_n f,$$

is estimated as follows :

$$\|y - y_n\| \leq C\|y - P_n y\| + \|P_n A z_n - A_n z_n\|.$$

It remains to check that the compressed scheme is invertible.

Lemma 6.2 *Assume that (98) and (99) are satisfied, for all $\{d^X, d^T\} \in \{2, 3\}$ with $r = \tilde{r} = 0$. Then there exists $a_\star > 0$ (independent of L) such that for all $a \geq a_\star$, the operator $\tilde{\mathcal{D}}_L$ is injective.*

Proof : By Theorem 5.7 and assumption (80) satisfied for L large enough, we get the desired injectivity of $\tilde{D}_L$ with the help of the Neumann's series. ■

Corollary 6.3 *Under the assumptions of the previous Lemma, for all $a \geq a_\star$,*

$$\text{Cond }(\tilde{D}_L) \lesssim 1.$$

Finally,

Theorem 6.4 *Let the assumptions of the Theorem 4.6 be satisfied. Assume that (99) holds, for all* $\{d^X, d^T\} \in \{2,3\}, i = 1,2,\ r \in [0, 5/2)$ *and* $\tilde{r} = 0$. *Assume that the compression parameters satisfy (98). Then there exists* $a_\star > 0$ *(independent of L) such that for all* $a \geq a_\star$*, we may write*

$$\|\mu - \tilde{\mu}_L\|_{L^2(\Sigma_T)} \lesssim 2^{-rL} \|\mu\|_{\tilde{H}^{r,r/2}(\Sigma_T)}. \tag{105}$$

Proof : By Theorem 6.1, we get

$$\|\mu - \tilde{\mu}_L\|_{L^2(\Sigma_T)} \leq C \left(\|\mu - Q_L\mu\|_{L^2(\Sigma_T)} + \|P_L \mathcal{D}_L \tilde{\mu}_L - \tilde{\mathcal{D}}_L \tilde{\mu}_L\|_{L^2(\Sigma_T)} \right).$$

The first term on the right hand side is estimated by Theorem 4.6 while the second one is estimated by Theorem 5.7. ■

7 Results for the first kind formulation

The results of the two previous sections can be easily adapted to the first kind formulation of the Neumann problem. The difference is coming from the use of the hypersingular operator H instead of the double-layer potential D. It involves one more spatial derivative. Moreover, the strong-ellipticity of H on $\tilde{H}^{1/2,1/4}(\Sigma_T)$ leads to a change of the basis, as explained in section 4. We sum up hereafter the main results, the sketches of the prooves are the same as in the two previous sections.

We recall the variational formulation of the first kind (73).

Find $\mu \in L^2(\Sigma_T)$ such that

$$h(\mu, v) = \langle H\mu, v\rangle_{\Sigma_T} = \langle \left(\frac{1}{2}I - D'\right) g_1, v\rangle_{\Sigma_T}, \forall v \in \tilde{H}^{1/2,1/4}(\Sigma_T),$$

for the equation of the first kind.

The decay property of the coefficients reads as follow.

Proposition 7.1 *For all the coefficients of the stiffness matrix, one has*

$$|H_{\lambda,\lambda'}| \lesssim \frac{2^{-b_H(j+j')}}{(dist_{\lambda,\lambda'})^{b_H}}, \tag{106}$$

with $b_H = \tilde{d}^X + 2\tilde{d}^T + 2$.

Proof : By Fubini's theorem, we have

$$\begin{aligned} |H_{\lambda,\lambda'}| \ &\lesssim \ 2^{-\frac{j+j'}{2}} \int \left| \frac{\partial^{2(\tilde{d}^X + \tilde{d}^T)} k(x,y,t,\tau)}{\partial x^{\tilde{d}^X} \partial y^{\tilde{d}^X} \partial t^{\tilde{d}^T} \partial \tau^{\tilde{d}^T}} \right| \\ &\times \ \left| \theta^T_{2j',k'_2}(t) \theta^X_{j',k'_1}(x) \theta^T_{2j,k_2}(\tau) \theta^X_{j,k_1}(y) \right| dx dy dt d\tau, \end{aligned}$$

where k is, after a parametrisation, a second order partial derivative of E. The conclusion follows by using the Lemma 5.3 and the estimate (92). ■

Remark 7.2 *The difference between the second kind and the first kind integral formulations is only visible on the barameter b. Here the new parameter b_H is simply related to the parameter b by*

$$b_H = \tilde{d}^X + 2\tilde{d}^T + 2 = b + \frac{1}{2}.$$

A consequence is that the Propositions 5.5 and 5.6 are still valid, replacing the parameter b by the new parameter b_H.

This property allows to define a compress stiffness matrix $\tilde{H}_L$ from the initial Galerkin one H_L and we have the following estimate.

Theorem 7.3 *Let $r, \tilde{r} \in [0, \frac{5}{2})$ and $\{d^X, d^T\} \in \{2, 3\}$. Assume that the truncation parameter $\delta_{j,j'}$ satisfies*

$$\delta_{j,j'} \geq a \cdot \max\{2^{-2j}, 2^{-2j'}, 2^{\alpha(L-j)} 2^{\tilde{\alpha}(L-j')} 2^{-2L}\}, \tag{107}$$

for some $\alpha, \tilde{\alpha}, a > 0$ such that

$$\alpha > \frac{r + b_H + \tau}{b_H - 2}, \quad \tilde{\alpha} > \frac{\tilde{r} + b_H - 4 + \tau}{b_H - 2}, \tag{108}$$

for some $\tau > 0$.

Then for any $\mu_L, \tilde{\mu}_L \in V_L$, we have

$$\left| \left\langle (\mathcal{H}_L - \tilde{\mathcal{H}}_L)\mu_L, \tilde{\mu}_L \right\rangle \right| \lesssim a^{3/2 - b_H} 2^{-L(r + \tilde{r} - 1)} \|\mu_L\|_{r, r/2} \|\tilde{\mu}_L\|_{\tilde{r}, \tilde{r}/2}, \tag{109}$$

with $b_H = \tilde{d}^X + 2\tilde{d}^T + 2$.

Theorem 7.4 *Let the assumptions of the Theorem 4.3 be satisfied. Assume that (107) and (108) hold, for all $\{d^X, d^T\} \in \{2, 3\}$ and for $r \in (1/2, 5/2)$ and $\tilde{r} = \frac{1}{2}$. Then there exists $a_\star > 0$ (independent of L) such that for all $a \geq a_\star$, there exists a unique solution $\tilde{\mu}_L \in V_L$ of*

$$\left(\tilde{H}_L \tilde{\mu}_L, v_L \right)_{\Sigma_T} = \left(\left(\frac{1}{2} I - D' \right) g_1, v_L \right)_{\Sigma_T}, \forall v_L \in V_L, \tag{110}$$

and the next error estimate holds :

$$\|\mu - \tilde{\mu}_L\|_{\frac{1}{2}, \frac{1}{4}} \lesssim 2^{-(r' - \frac{1}{2})L} \|\mu\|_{r', \frac{r'}{2}}. \tag{111}$$

Proof : We simply write the first Strang Lemma and use Theorems 4.3 and 7.3 with $\tilde{r} = \frac{1}{2}$. ■

8 The Dirichlet problem

The Dirichlet problem is the following one

$$\begin{cases} -\Delta\Phi + \partial_t\Phi = 0 & \text{in } Q_T = \Omega \times (0,T), \\ \Phi_{|\Sigma_T} = g_0 & \text{on } \Sigma_T = \Gamma \times (0,T), \\ \Phi(x,0) = 0 & \quad \forall x \in \Omega. \end{cases} \tag{112}$$

The problem (112) admits the direct representation

$$\Phi = V\sigma - Wg_0, \tag{113}$$

where σ is the solution of the following equation of the first kind :

$$S\sigma = \left(\frac{1}{2}I + D\right) g_0, \tag{114}$$

or the "adjoint" equation :

$$\left(\frac{1}{2}I - D'\right) \sigma = Hg_0, \tag{115}$$

with

$$\sigma = \partial_n \Phi^- \in \tilde{H}^{-1/2,-1/4}(\Sigma_T). \tag{116}$$

The single-layer operator S is defined by (see [8, 16]) :

$$(S\sigma)(x,t) = \int_0^t \int_\Gamma \sigma(y,\tau)E(x-y,t-\tau)d\Gamma_y d\tau, \tag{117}$$

for $(x,t) \in \Sigma_T$ and has the following properties (see [8, 16]) :

Lemma 8.1 *The operator $S : \tilde{H}^{r,r/2}(\Sigma_T) \longrightarrow \tilde{H}^{r+1,(r+1)/2}(\Sigma_T)$ is an isomorphism, for all $r \geq -\frac{1}{2}$.*

Furthermore, S is strongly coercive, i.e.

$$(u, Su)_{\Sigma_T} \gtrsim \|u\|^2_{-\frac{1}{2},-\frac{1}{4}}, \forall u \in \tilde{H}^{-1/2,-1/4}(\Sigma_T). \tag{118}$$

The Galerkin method is defined by : find $\sigma_L \in V_L$ solution of

$$(S\sigma_L, v_L)_{\Sigma_T} = \left((\frac{1}{2}I + D)g_0, v_L\right)_{\Sigma_T}, \forall v_L \in V_L. \tag{119}$$

Again, we make use of the Theorem 3.10 and modify slightly the basis in order to obtain a good condition number.

Corollary 8.2 *Take the functions*

$$\hat{\Xi}_\lambda(x,t) = 2^{|\lambda|/2}\Xi_\lambda(x,t), \tag{120}$$

as a new basis for $\tilde{H}^{-1/2,-1/4}(\Sigma_T)$.

Then the condition number of the stiffness matrix is uniformly bounded.

The strong coercivity of the operator S allows to apply the Céa's Lemma, combined with the Theorem of characterization 3.10 to obtain the following error estimate.

Theorem 8.3 *Let* $r' \in (-\frac{1}{2}, \frac{3}{2}]$ *and* $g_0 \in \tilde{H}^{r'+1,(r'+1)/2}(\Sigma_T)$*. Then the solution of (114) satisfies* $\sigma \in \tilde{H}^{r',\frac{r'}{2}}(\Sigma_T)$ *and*

$$\|\sigma - \sigma_L\|_{-\frac{1}{2},-\frac{1}{4}} \lesssim 2^{-(r'+\frac{1}{2})L}\|\sigma\|_{r',\frac{r'}{2}}.$$

We briefly give the results of the compression procedure.

The weak formulation of the problem is defined by

$$s(\sigma, v) = \langle S\sigma, v\rangle = \left((\frac{1}{2}I + D)g_0, v\right)_{\Sigma_T}, \forall v \in \tilde{H}^{-1/2,-1/4}(\Sigma_T). \tag{121}$$

Then if we note

$$S_{\lambda,\lambda'} = s(\hat{\Xi}_\lambda; \hat{\Xi}_{\lambda'}), \tag{122}$$

one obtains

$$|S_{\lambda,\lambda'}| \lesssim \frac{2^{-b_S(j+j')}}{(dist_{\lambda,\lambda'})^{b_S}}, \tag{123}$$

with $b_S = \tilde{d}^X + 2\tilde{d}^T + 1$.

We are now able to estimate the difference between the initial Galerkin scheme and the compressed one.

Theorem 8.4 *Let* $r, \tilde{r} \in (-\frac{1}{2}, \frac{3}{2}]$*. Assume that the truncation parameter* $\delta_{j,j'}$ *satisfies (107) for some* $\alpha, \tilde{\alpha}, a > 0$ *such that*

$$\alpha > \frac{r + b_S + \tau}{b_S - 1}, \tag{124}$$

$$\tilde{\alpha} > \frac{\tilde{r} + b_S - 2 + \tau}{b_S - 1}, \tag{125}$$

for some $\tau > 0$*. Then for any* $\sigma_L, \tilde{\sigma}_L \in V_L$*, we have*

$$\left|\left\langle(\mathcal{S}_L - \tilde{\mathcal{S}}_L)\sigma_L, \tilde{\sigma}_L\right\rangle\right| \lesssim a^{3/2-b_S}2^{-L(r+\tilde{r}+1)}\|\sigma_L\|_{r,r/2}\|\tilde{\sigma}_L\|_{\tilde{r},\tilde{r}/2}, \tag{126}$$

with $b_S = \tilde{d}^X + 2\tilde{d}^T + 1$*.*

Moreover, the compressed matrix $\tilde{S}_L$ *has* $O(N_L)$ *non-zero elements after the second compression and its condition number is uniformly bounded.*

Finally, we obtain the following error between the exact solution σ and the approximated one of the compressed scheme.

Theorem 8.5 *Under the same assumptions as in Theorem 8.3, assume that (124), (125) and (107) are satisfied for $r' \in (-\frac{1}{2}, \frac{3}{2})$ and $\tilde{r} = -\frac{1}{2}$. Then there exists $a_\star > 0$ such that for all $a > a_\star$, there exists a unique solution $\tilde{\sigma}_L \in V_L$ of*

$$(\tilde{\mathcal{S}}_L \tilde{\sigma}_L, v_L)_{\Sigma_T} = \left((\frac{1}{2} I + D) g_0, v_L \right)_{\Sigma_T}, \forall v_L \in V_L,$$

and it holds

$$\|\sigma - \tilde{\sigma}_L\|_{-\frac{1}{2},-\frac{1}{4}} \lesssim 2^{-(r'+\frac{1}{2})L} \|\sigma\|_{r',\frac{r'}{2}}. \tag{127}$$

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