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Integrability of the D_n^2 vertex models with open boundary

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Abstract

We investigate various aspects of the integrability of the vertex models associated to the D_n^2 affine Lie algebra with open boundaries. We first study the solutions of the corresponding reflection equation compatible with the minimal symmetry of this system. We find three classes of general solutions, one diagonal solution and two non-diagonal families with a free parameter. Next we perform the Bethe ansatz analysis for some of the associated open D_2^2 spin chains and we identify the boundary having quantum group invariance. We also discuss a new D_2^2 R-matrix.

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1 Introduction

Much work has been done in integrable lattice statistical mechanics models with open boundary conditions, since Sklyanin [1] generalized the quantum inverse scattering method to tackle the boundary problem. The bulk Boltzmann weights of an exactly solvable lattice system are usually the non-null matrix elements of a R-matrix $R(\lambda)$ which satisfies the Yang-Baxter equation. The integrability at boundary, for a given bulk theory, is governed by the reflection equation, which reads

$$R_{12}(\lambda-\mu) \stackrel{1}{K}_{-}(\lambda) R_{21}(\lambda+\mu) \stackrel{2}{K}_{-}(\mu) = \stackrel{2}{K}_{-}(\mu) R_{12}(\lambda+\mu) \stackrel{1}{K}_{-}(\lambda) R_{21}(\lambda-\mu) \tag{1}$$

where the matrix $K_{-}(\lambda)$ describes the reflection at one of the ends of an open chain. Similar equation should also hold for the reflection $K_{+}(\lambda)$ at the opposite boundary. However, for several relevant lattice models $K_{+}(\lambda)$ can be directly obtained from $K_{-}(\lambda)$. For example, this is the case of models whose $R(\lambda)$ matrix satisfies extra properties such as unitarity, P and T invariances and crossing symmetry [1, 2].

Therefore, the first step toward constructing integrable models with open boundaries is to search for solutions of the reflection equation. To date, solutions of this equation have been found for a number of lattice models ranging from vertex systems based on Lie algebras [3, 4, 6, 5] to solid-on-solid models and their restriction [7]. Classification of such solutions for particular systems [8] as well as extensions to include supersymmetric models [9] can also be found in the literature.

In spite of all these works, there is an interesting vertex model based on the non-exceptional D_n^2 Lie algebra for which little is known about the solution of the corresponding reflection equation. This is probably related to the fact that the D_n^2 R-matrix does not commute for different values of the rapidity [10], consequently the trivial diagonal solution $K_{-}(\lambda) = I$ does not hold for this system [3]. The purpose of this paper is to bridge this gap, by presenting what we hope to be the minimal solution of the reflection equation for D_n^2 vertex models. This result offers us the possibility to understand a relevant open problem which is the integrability of the D_n^2 vertex model with quantum algebra symmetry. In fact, this symmetry has been found for all vertex models based on non-exceptional Lie algebras [3] except for the D_n^2 model. It turns out that, by carring out a Bethe ansatz analysis, we are able to identify this symmetry for the simplest D_2^2 model and conjecture it for arbitrary values of n.

We have organized this paper as follows. We start next section by considering the reflection equation for the D_2^2 vertex model. We find one diagonal solution without free parameters and two non-diagonal families which depend on a free parameter. We also derive the corresponding integrable one-dimensional open spin chains. In section 3 we present the Bethe ansatz solutions of the open D_2^2 spin chain associated to the diagonal K-matrix and to a special manifold of the first non-diagonal family. This allows us to identify the quantum group symmetry for the D_2^2 model. In section 4 we generalize the K-matrices results of section 2 for arbitrary values of $n > 2$. Section 5 is reserved for our conclusions as well as a discussion on possible new D_n^2 R-matrices. In Appendix A we collect some useful relations and Appendix B contains a new D_2^2 R-matrix as well as its boundary behaviour.

2 The D_2^2 K-matrices

The D_2^2 vertex model has four independent degrees of freedom per bond and its Boltzmann weights preserve only one $U(1)$ symmetry out of two possible ones. Here we are interested in looking at solutions of the reflection equation that commute with this symmetry. We find that the most general K-matrix having this property is

$$K_-(\lambda) = \begin{pmatrix} Y_1(\lambda) & 0 & 0 & 0 \\ 0 & Y_2(\lambda) & Y_5(\lambda) & 0 \\ 0 & Y_6(\lambda) & Y_3(\lambda) & 0 \\ 0 & 0 & 0 & Y_4(\lambda) \end{pmatrix} \tag{2}$$

Our next step is to substitute this ansatz in equation (1) and look for relations that constraint the unknown elements $Y_j(\lambda), j = 1, \dots, 6$. Although we have many functional equations, a few of them are actually independent, and the most suitable ones have been collected in Appendix A. The basic idea is to try to solve such equations algebraically, which hopefully will produce a general ansatz for functions $Y_j(\lambda)$ containing several arbitrary parameters. The general strategy we use is to separate these equations in terms of ratio of functions depending either on λ or on μ. From the relations (A.5-A.7) one easly concludes that the simplest possible solution is to take $Y_5(\lambda) = Y_6(\lambda) = 0$. This is the diagonal solution, and by employing the "separation variable method" described above for the relations (A.5-A.7) we are able to fix the following ratios

$$\frac{Y_2(\lambda)}{Y_1(\lambda)} = \frac{e^{\lambda} - \beta_1}{e^{-\lambda} - \beta_1}, \quad \frac{Y_3(\lambda)}{Y_1(\lambda)} = \frac{e^{\lambda} - \beta_2}{e^{-\lambda} - \beta_2}, \quad \frac{Y_4(\lambda)}{Y_3(\lambda)} = \frac{e^{\lambda} - \beta_3}{e^{-\lambda} - \beta_3} \tag{3}$$

where $\beta_j, \ j = 1, 2, 3$ are arbitrary constants.

These relations enable us to write an ansatz for three unknown functions in terms of a normalizing factor, say $Y_1(\lambda)$. Substituting the relations (3) back to the reflection equation (1), we conclude that all the parameters β_j are fixed by

$$\beta_1 = -\beta_2 = -1/\beta_3 = \frac{I}{\sqrt{q}} \tag{4}$$

where q is the deformation parameter of the D_2^2 R-matrix [10]. This leads us to our first solutions with no free parameter,

$$Y_1^{(1)}(\lambda) = 1, \ \ Y_2^{(1)}(\lambda) = \frac{e^{\lambda} - \frac{I}{\sqrt{q}}}{e^{-\lambda} - \frac{I}{\sqrt{q}}}, \ \ Y_3^{(1)}(\lambda) = \frac{e^{\lambda} + \frac{I}{\sqrt{q}}}{e^{-\lambda} + \frac{I}{\sqrt{q}}}, \ \ Y_4^{(1)}(\lambda) = \frac{e^{\lambda} + \frac{I}{\sqrt{q}}}{e^{-\lambda} + \frac{I}{\sqrt{q}}} \frac{e^{\lambda} - I\sqrt{q}}{e^{-\lambda} - I\sqrt{q}} \tag{5}$$

Next we turn our search for non-diagonal solutions now with both $Y_5(\lambda)$ and $Y_6(\lambda)$ non null. From the equations (A.8-A.11), we notice that it is possible to solve $Y_2(\lambda), Y_3(\lambda)$ and $Y_6(\lambda)$ in terms of $Y_5(\lambda)$. At this point we should keep in mind that we are looking for regular K-matrices, i.e. $K_-(0) \sim$ identity. After some simplifications, we find the following general solutions

$$\epsilon_1(1 + e^{2\lambda})[Y_5(\lambda) + Y_6(\lambda)] = \epsilon_2 e^{\lambda}[Y_5(\lambda) - Y_6(\lambda)] \tag{6}$$

$$(e^{2\lambda} - 1)[Y_2(\lambda) + Y_3(\lambda)] = \epsilon_3 e^{\lambda}[Y_6(\lambda) - Y_5(\lambda)] \tag{7}$$

$$Y_2(\lambda) - Y_3(\lambda) = \epsilon_4[Y_5(\lambda) - Y_6(\lambda)] \tag{8}$$

where ϵ_j are four arbitrary parameters. These are linear equations which can be easily solved for the ratios $Y_2(\lambda)/Y_5(\lambda), Y_3(\lambda)/Y_5(\lambda)$ and $Y_6(\lambda)/Y_5(\lambda)$. Taking this into account as well as equations (A.5) and (A.7), we end up with the following ansatz for functions $Y_j(\lambda)$

$$Y_1(\lambda) = (\epsilon_5 e^{2\lambda} + \epsilon_6 e^{\lambda} + \epsilon_7)/e^{\lambda}, \quad Y_2(\lambda) = (1 + e^{2\lambda})\left[\epsilon_4(e^{2\lambda} - 1) - \epsilon_3 e^{\lambda}\right] \tag{9}$$

$$Y_3(\lambda) = (1 + e^{2\lambda})\left[-\epsilon_4(e^{2\lambda} - 1) - \epsilon_3 e^{\lambda}\right], \quad Y_4(\lambda) = (\epsilon_8 e^{2\lambda} + \epsilon_9 e^{\lambda} + \epsilon_{10})e^{\lambda} \tag{10}$$

$$Y_5(\lambda) = (e^{2\lambda} - 1)\left[\epsilon_2 e^{\lambda} + \epsilon_1(1 + e^{2\lambda})\right], \quad Y_6(\lambda) = (e^{2\lambda} - 1)\left[\epsilon_2 e^{\lambda} - \epsilon_1(1 + e^{2\lambda})\right] \tag{11}$$

having altogether ten free parameters. Substituting this ansatz back to the reflection equation and after involving algebraic manipulations, we find that nine parameters are in fact fixed, leading us to two classes of non-diagonal solution with a free parameter. The first class is given by

$$Y_1^{(2)}(\lambda, \xi_-) = (e^{2\lambda} + q)(\xi_-^2 q e^{2\lambda} - 1)e^{-\lambda}, \quad Y_4^{(2)}(\lambda, \xi_-) = (e^{2\lambda} + q)(\xi_-^2 q - e^{2\lambda})e^{\lambda} \tag{12}$$

$$Y_2^{(2)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}\left[2(e^{2\lambda} - 1)\xi_- q - e^{\lambda}(1 + q)(1 - \xi_-^2 q)\right] \tag{13}$$

$$Y_3^{(2)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}\left[-2(e^{2\lambda} - 1)\xi_- q - e^{\lambda}(1 + q)(1 - \xi_-^2 q)\right] \tag{14}$$

$$Y_5^{(2)}(\lambda, \xi_-) = Y_6^{(2)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}(1 - q)(\xi_-^2 q + 1)e^{\lambda} \tag{15}$$

while the second family is

$$Y_1^{(3)}(\lambda, \xi_-) = (e^{2\lambda} - q)(\xi_- e^{2\lambda} - 1)e^{-\lambda}, \quad Y_4^{(3)}(\lambda, \xi_-) = (e^{2\lambda} - q)(\xi_- - e^{2\lambda})e^{\lambda} \tag{16}$$

$$Y_2^{(3)}(\lambda, \xi_-) = Y_3^{(3)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}(1 - q)(\xi_- - 1)e^{\lambda} \tag{17}$$

$$Y_5^{(3)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}\left[2(e^{2\lambda} + 1)\sqrt{\xi_- q} + (1 + q)(1 + \xi_-)e^{\lambda}\right] \tag{18}$$

$$Y_6^{(3)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}\left[-2(e^{2\lambda} + 1)\sqrt{\xi_- q} + (1 + q)(1 + \xi_-)e^{\lambda}\right] \tag{19}$$

where ξ_- is an arbitrary parameter.

Since the D_2^2 R-matrix is PT invariant and crossing symmetric, the $K_+(\lambda)$ matrices at the opposite boundary are easily derived from the above solutions [2, 3]. More precisely, we have

$$K_+(\lambda, \xi_+) = K_-^t(\ln[q] - \lambda, \xi_+)M \tag{20}$$

where M is a matrix related to the crossing matrix V by $M = V^t V$ [3]. From the results of Appendix A, we have that for the D_2^2 model M is given by

$$M = \mathrm{diag}(q, 1, 1, q^{-1}) \tag{21}$$

Having found the $K_\pm(\lambda)$ matrices, one can construct the corresponding commuting transfer matrix $\tau(\lambda)$. Following Sklyanin [1], we have

$$t^{(l,m)}(\lambda) = \mathrm{Tr}_a \left[\overset{a\ (m)}{K_+} (\lambda) T(\lambda) \overset{a\ (l)}{K_-} (\lambda) T^{-1}(-\lambda) \right] \tag{22}$$

where $T(\lambda) = R_{aL}(\lambda) \cdots R_{a1}(\lambda)$ is the monodromy matrix of the associated closed chain with L sites. This means that the three families of $K_-(\lambda)$ matrices we found will produce nine possible types of open boundary conditions. The corresponding Hamiltonian of the spin chains with open boundaries are obtained by expanding the transfer matrix $t^{(l,m)}(\lambda)$ in powers of λ. When $\mathrm{Tr}[K_+^{(m)}(0)]$ is non-null, the Hamiltonian $H^{(l,m)}$ is proportional to the first-order expansion [1]

$$H^{(l,m)} = \sum_{k=1}^{L-1} H_{k,k+1} + \frac{1}{2\zeta} \frac{d \overset{a\ (l)}{K_-} (\lambda)}{d\lambda}\Big|_{\lambda=0} + \frac{\mathrm{Tr}_a \left[\overset{a\ (m)}{K_-} (0) H_{La} \right]}{\mathrm{Tr}\left[K_+^{(m)}(0) \right]} \tag{23}$$

where $H_{k,k+1} = P_{k,k+1} \frac{d}{d\lambda} R_{k,k+1}(\lambda)|_{\lambda=0}$ is the two-body bulk Hamiltonian and ζ is the normalization $R_{12}(0) = \zeta P_{12}$ [1]

For the first two solutions we indeed have $\mathrm{Tr}[K_+(0)] \neq 0$ while for the third one $\mathrm{Tr}[K_+(0)] = 0$. In this last case one has to consider the second order expansion in the spectral parameter λ [11]. We find convenient to write the expression for the Hamiltonians in terms of Pauli matrices $\sigma_{\alpha,i}^{\pm}$ and $\sigma_{\alpha,i}^{z}$ with components $\alpha =\uparrow, \downarrow$ acting on the site i of a lattice of size L. In terms of

[1]The normalization we use for $R(\lambda)$ (see Appendix A) produces $\zeta = (q - 1/q)^2$ for D_2^2 model.

these operators and up to irrelevant additive constants[2], we have

$$
\begin{aligned}
H^{(l,m)} \;=\; & -\frac{I(q-1/q)}{2}\sum_{k=1}^{L-1}\tilde{H}_{k,k+1} + I\frac{(q-1/q)^2}{2}\Big\{ \\
& \sum_{\alpha=\uparrow,\downarrow}\mu_\alpha^{(l)}(\xi_-)\sigma^z_{\alpha,1} + \delta^{(l)}\sigma^z_{\uparrow,1}\sigma^z_{\downarrow,1} + J_\uparrow^{(l)}(\xi_-)\sigma^+_{\uparrow,1}\sigma^-_{\downarrow,1} + J_\downarrow^{(l)}(\xi_-)\sigma^+_{\downarrow,1}\sigma^-_{\uparrow,1} \\
& - \sum_{\alpha=\uparrow,\downarrow}\mu_\alpha^{(m)}(\xi_+)\sigma^z_{\alpha,L} + \delta^{(m)}\sigma^z_{\uparrow,L}\sigma^z_{\downarrow,L} + J_\downarrow^{(m)}(\xi_+)\sigma^+_{\uparrow,L}\sigma^-_{\downarrow,L} + J_\uparrow^{(m)}(\xi_+)\sigma^+_{\downarrow,L}\sigma^-_{\uparrow,L}\Big\}
\end{aligned}
\tag{24}
$$

where the expression of the bulk part $\tilde{H}_{k,k+1}$ is

$$
\begin{aligned}
\tilde{H}_{k,k+1} \;=\; & \frac{(q-1/q)}{2}\Big[(\sigma^z_{\uparrow,k}+\sigma^z_{\downarrow,k})(\sigma^+_{\uparrow,k+1}\sigma^-_{\downarrow,k+1}+\sigma^-_{\uparrow,k+1}\sigma^+_{\downarrow,k+1}) - (\sigma^+_{\uparrow,k}\sigma^-_{\downarrow,k}+\sigma^-_{\uparrow,k}\sigma^+_{\downarrow,k})(\sigma^z_{\uparrow,k+1}+\sigma^z_{\downarrow,k+1})\Big] \\
& +(\sqrt{q}-\frac{1}{\sqrt{q}})^2\Big[\sigma^+_{\uparrow,k}\sigma^-_{\uparrow,k+1}\sigma^-_{\downarrow,k}\sigma^+_{\downarrow,k+1} + \sigma^-_{\uparrow,k}\sigma^+_{\uparrow,k+1}\sigma^+_{\downarrow,k}\sigma^-_{\downarrow,k+1} \\
& +\sigma^+_{\uparrow,k}\sigma^+_{\uparrow,k+1}\sigma^-_{\downarrow,k}\sigma^-_{\downarrow,k+1} + \sigma^-_{\uparrow,k}\sigma^-_{\uparrow,k+1}\sigma^+_{\downarrow,k}\sigma^+_{\downarrow,k+1}\Big] \\
& -2\Big[(\sigma^+_{\uparrow,k}\sigma^-_{\uparrow,k+1}+\sigma^-_{\uparrow,k}\sigma^+_{\uparrow,k+1})(1+\sigma^z_{\downarrow,k}\sigma^z_{\downarrow,k+1}) + (\sigma^+_{\downarrow,k}\sigma^-_{\downarrow,k+1}+\sigma^-_{\downarrow,k}\sigma^+_{\downarrow,k+1})(1+\sigma^z_{\uparrow,k}\sigma^z_{\uparrow,k+1})\Big] \\
& +(\sqrt{q}+\frac{1}{\sqrt{q}})\Big[(\sigma^+_{\uparrow,k}\sigma^-_{\uparrow,k+1}+\sigma^-_{\uparrow,k}\sigma^+_{\uparrow,k+1})(1-\sigma^z_{\downarrow,k}\sigma^z_{\downarrow,k+1}) \\
& +(\sigma^+_{\downarrow,k}\sigma^-_{\downarrow,k+1}+\sigma^-_{\downarrow,k}\sigma^+_{\downarrow,k+1})(1-\sigma^z_{\uparrow,k}\sigma^z_{\uparrow,k+1})\Big] \\
& -(\sqrt{q}-\frac{1}{\sqrt{q}})\Big[(\sigma^+_{\uparrow,k}\sigma^-_{\downarrow,k+1}+\sigma^-_{\uparrow,k}\sigma^+_{\downarrow,k+1})(\sigma^z_{\downarrow,k}-\sigma^z_{\uparrow,k+1}) + (\sigma^+_{\downarrow,k}\sigma^-_{\uparrow,k+1}+\sigma^-_{\downarrow,k}\sigma^+_{\uparrow,k+1})(\sigma^z_{\uparrow,k}-\sigma^z_{\downarrow,k+1})\Big] \\
& +[1-\frac{(q+1/q)}{2}](\sigma^z_{\uparrow,k}\sigma^z_{\downarrow,k}+\sigma^z_{\uparrow,k+1}\sigma^z_{\downarrow,k+1}) - \frac{(\sqrt{q}-\frac{1}{\sqrt{q}})^2}{4}(\sigma^z_{\uparrow,k}\sigma^z_{\downarrow,k+1}+\sigma^z_{\downarrow,k}\sigma^z_{\uparrow,k+1}) \\
& -[\frac{(q+1/q)}{4}+\frac{3}{2}](\sigma^z_{\uparrow,k}\sigma^z_{\uparrow,k+1}+\sigma^z_{\downarrow,k}\sigma^z_{\downarrow,k+1}) + \frac{(q-1/q)}{2}\sum_{\alpha=\uparrow,\downarrow}(\sigma^z_{\alpha,k}-\sigma^z_{\alpha,k+1}) - 2(q+\frac{1}{q})I_{k,k+1}
\end{aligned}
$$

Turning to the boundary interactions we found that the chemical potentials are given by

$$
\mu_\alpha^{(l)}(\xi) = \begin{cases} \mu_\uparrow^{(1)}(\xi) = 1/2 - I\frac{\sqrt{q}}{1+q}, \quad \mu_\downarrow^{(1)}(\xi) = 1/2 + I\frac{\sqrt{q}}{1+q} \\ \mu_\uparrow^{(2)}(\xi) = -\frac{(1+q+2\xi q)}{(1+q)(\xi^2q^2-1)}, \quad \mu_\downarrow^{(2)}(\xi) = -\frac{(1+q-2\xi q)}{(1+q)(\xi^2q^2-1)} \\ \mu_\uparrow^{(3)}(\xi) = \mu_\downarrow^{(3)} = \frac{1}{1-\xi} \end{cases}
\tag{25}
$$

[2]We also note that we have normalized the Hamiltonian by the pure imaginary number.

while the on-site parameters $\delta^{(l)}$ and $J_{\alpha}^{(l)}(\xi)$ are

$$\delta^{(l)} = \begin{cases} \delta^{(1)} = \frac{(q-1)}{2(1+q)} \\ \delta^{(2)} = \frac{(q-1)}{2(1+q)} \\ \delta^{(3)} = \frac{(1+q)}{2(q-1)} \end{cases} \tag{26}$$

and

$$J_{\alpha}^{(l)}(\xi) = \begin{cases} J_{\uparrow}^{(1)}(\xi) = J_{\downarrow}^{(1)} = 0 \\ J_{\uparrow}^{(2)}(\xi) = J_{\downarrow}^{(2)}(\xi) = \frac{(q-1)(1+\xi^2 q)}{(1+q)(\xi^2 q-1)} \\ J_{\uparrow}^{(3)}(\xi) = \frac{(1+q)(1+\xi)+4\sqrt{q\xi}}{(q-1)(\xi-1)}, \quad J_{\downarrow}^{(3)}(\xi) = \frac{(1+q)(1+\xi)-4\sqrt{q\xi}}{(q-1)(\xi-1)} \end{cases} \tag{27}$$

A natural question to be asked is which (if any) of these solutions would lead us to an integrable D_2^2 model with quantum algebra symmetry. One way to investigate that is by applying the Bethe ansatz method to diagonalize the above open spin chains. This allows us to extract information about the eigenspectrum, which in the case of quantum algebra invariance, should be highly degenerated (see e.g. [12]). In next section we will discuss this problem in details.

3 Bethe ansatz analysis

The purpose of this section is to study the spectrum of some of the open spin chains presented in section 2 by the coordinate Bethe ansatz formalism. One of our motivations is to identify the boundary that leads us to the quantum group symmetry. We begin by noticing that the total number of spins $\hat{N}_s = \sum_{i=1}^{L} \sum_{\alpha=\uparrow,\downarrow} \sigma_{\alpha,i}^{z}$ is a conserved quantity and its eigenvalues ns labels the many possible disjoint sectors of the Hilbert space. Therefore, the wave function solving the eigenvalue problem $H\left|\Psi_{ns}\right\rangle = E^{(l,m)}(L)\left|\Psi_{ns}\right\rangle$ can be written as follows

$$\left|\Psi_{ns}\right\rangle = \sum_{\alpha_j} \sum_{x_{Q_j}} f^{(\alpha_1,\cdots,\alpha_n)}(x_{Q_1},\cdots,x_{Q_{ns}}) \sigma^{+}_{\alpha_1,x_{Q_1}} \cdots \sigma^{+}_{\alpha_{ns},x_{Q_{ns}}} \left|0\right\rangle \tag{28}$$

where $|0\rangle$ denotes the ferromagnetic state (all spins up) and $1 \leq x_{Q_1} \leq x_{Q_2} \leq \cdots \leq x_{Q_{ns}} \leq L$ indicate the positions of the spins.

We will start our study by first considering the open spin chain $H^{(1,1)}$ corresponding to the diagonal K-matrix solution. As it is customary we begin our discussion of the eigenvalue problem in the sector of one down spin, $ns = 1$. In this sector, we find that for $1 < x < L$

$$\frac{I}{2(1/q-q)} E^{(1,1)}(L) f^{(\alpha)}(x) = (L-2)\Delta f^{(\alpha)}(x) + f^{(\alpha)}(x+1) + f^{(\alpha)}(x-1) + \frac{(1-q)^2}{4q} f^{(\alpha)}(x), \;\; \alpha = \uparrow, \downarrow \tag{29}$$

where we have defined $\Delta = q + 1/q$. The matching condition at the left and right boundaries gives us the following constraints

$$\begin{pmatrix} f^{(\uparrow)}(0) \\ f^{(\downarrow)}(0) \end{pmatrix} = \begin{pmatrix} \Delta - p_{1\uparrow} & d_{1\uparrow} \\ d_{1\downarrow} & \Delta - p_{1\downarrow} \end{pmatrix} \begin{pmatrix} f^{(\uparrow)}(1) \\ f^{(\downarrow)}(1) \end{pmatrix} \tag{30}$$

and

$$\begin{pmatrix} f^{(\uparrow)}(L+1) \\ f^{(\downarrow)}(L+1) \end{pmatrix} = \begin{pmatrix} \Delta - p_{L\uparrow} & d_{L\uparrow} \\ d_{L\downarrow} & \Delta - p_{L\downarrow} \end{pmatrix} \begin{pmatrix} f^{(\uparrow)}(L) \\ f^{(\downarrow)}(L) \end{pmatrix} \tag{31}$$

where the matrices parameters are given by

$$p_{1\uparrow} = \frac{3I + \sqrt{q} + q^2(I + 3\sqrt{q})}{4q(I + \sqrt{q})}, \quad p_{1\downarrow} = \frac{-3I + \sqrt{q} + q^2(-I + 3\sqrt{q})}{4q(-I + \sqrt{q})} \tag{32}$$

$$p_{L\uparrow} = \frac{3 - 2q + 3q^2 - 2I\sqrt{q} + 2Iq\sqrt{q}}{4q}, \quad p_{L\downarrow} = \frac{3 - 2q + 3q^2 + 2I\sqrt{q} - 2Iq\sqrt{q}}{4q} \tag{33}$$

$$d_{1\uparrow} = d_{1\downarrow} = -d_{L\uparrow} = -d_{L\downarrow} = \frac{q - 1/q}{4} \tag{34}$$

In order to go ahead it is crucial to notice that both boundary constraints (30) and (31) can be diagonalized by the *same* unitary transformation U. After performing this transformation the new components $\tilde{f}^{\alpha}(x) = U f^{\alpha}(x)$ satisfy

$$\begin{pmatrix} \tilde{f}^{(\uparrow)}(0) \\ \tilde{f}^{(\downarrow)}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\Delta}{2} \end{pmatrix} \begin{pmatrix} \tilde{f}^{(\uparrow)}(1) \\ \tilde{f}^{(\downarrow)}(1) \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} \tilde{f}^{(\uparrow)}(L+1) \\ \tilde{f}^{(\downarrow)}(L+1) \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{f}^{(\uparrow)}(L) \\ \tilde{f}^{(\downarrow)}(L) \end{pmatrix} \tag{36}$$

Clearly, equation (29) for $1 < x < L$ remains the same but now for the transformed amplitudes $\tilde{f}^{(\alpha)}(x)$. Now we reached a point in which one can try the usual Bethe ansatz (e.g. see ref. [13, 14]), namely

$$\tilde{f}^{(\alpha)}(x) = A_\alpha(k)e^{ikx} - A_\alpha(-k)e^{-ikx} \tag{37}$$

and by substituting this ansatz in (29) we obtain the following eigenvalue

$$\frac{I}{2(1/q-q)}E^{(1,1)}(L) = (L-2)\Delta + 2\cos(k) + \frac{(1-q)^2}{4q} \tag{38}$$

The fact that this ansatz should be also valid for the ends $x = 1$ and $x = L$ provides us constraints for the amplitudes $A(k)$ and $A(-k)$, which reads

$$A(-k) = -e^{ik}A(k) \quad \text{and} \quad A(-k) = \frac{(1-\frac{\Delta}{2}e^{-ik})}{(1-\frac{\Delta}{2}e^{ik})}e^{2i(L+1)k}A(k) \tag{39}$$

whose compatibility gives a restriction on the momentum k, namely

$$e^{2ikL}\frac{(e^{ik}-\frac{\Delta}{2})}{(\frac{\Delta}{2}e^{ik}-1)} = 1 \tag{40}$$

The next task is to generalize these results for arbitrary numbers of down spins. For a general multiparticle state, we assume the Bethe ansatz wave function

$$\tilde{f}^{(\alpha_1,\cdots,\alpha_n)}(x_{Q_1},\cdots,x_{Q_{ns}}) = \sum_P \mathrm{sgn}(P)\prod_{j=1}^{ns} e^{[ik_{p_j}x_{Q_j}]}A(k_{PQ_1},\cdots,k_{PQ_{N_e}})_{\alpha_{Q_1},\cdots,\alpha_{Q_{ns}}} \tag{41}$$

where P is the sum over all the permutations of the momenta, including the negations $k_j \to -k_j$, and the symbol sgn accounts for the sign of the permutations and negations. It turns out that for configurations such that $|x_{Q_i} - x_{Q_j}| \geq 2$ the open spin chain $H^{(1,1)}$ behaves as a free theory and the corresponding eigenvalues are

$$\frac{I}{2(1/q-q)}E^{(1,1)}(L) = (L-1)\Delta + \sum_{j=1}^{ns}[2\cos(k_j) - \Delta] + \frac{(1-q)^2}{4q} \tag{42}$$

The new ingredient for $ns \geq 2$ is that the nearest neighbor spin configurations enforce constraints on the amplitude of the wave function. This condition enhances a relation between the exchange of two states such as $\{(k_i, \alpha_i); (k_j, \alpha_j)\}$ and $\{(k_j, \alpha_j); (k_i, \alpha_i)\}$ which ultimately is represented by the two-body scattering

$$A_{\cdots\alpha_j,\alpha_i\cdots}(\cdots,k_j,k_i,\cdots) = S_{i,j}(k_i,k_j)A_{\cdots\alpha_i,\alpha_j\cdots}(\cdots,k_i,k_j,\cdots) \tag{43}$$

while the reflection at the left and right ends generalizes equation (29), which now reads

$$A_{\alpha_i,\cdots}(-k_j,\cdots) = -e^{ik_j}A_{\alpha_i,\cdots}(k_j,\cdots) \tag{44}$$

$$A_{\cdots,\alpha_i}(\cdots,-k_j) = \frac{(1-\frac{\Delta}{2}e^{-ik_j})}{(1-\frac{\Delta}{2}e^{ik_j})}e^{2i(L+1)k_j}A_{\cdots\alpha_i,}(\cdots,k_j) \tag{45}$$

Fortunately, the bulk two-body scattering amplitude $S_{i,j}(k_i,k_j)$ has been recently identified in ref.[15] for the periodic chain. This result is of enormous help here since it allows us to choose the suitable parametrization for the momenta k_j in terms of the S-matrix rapidities λ_j, which is

$$e^{ik_j} = \frac{\sinh(\lambda_j - i\gamma/2)}{\sinh(\lambda_j + i\gamma/2)} \tag{46}$$

where we have conveniently defined $q = e^{i\gamma}$. For explicit expression of the non-null S-matrix elements see ref.[15].

In this general case, the compatibility between the bulk and boundary scattering constraints (43-45) leads us to the Bethe ansatz equation for the momenta k_j

$$e^{2ik_jL}\frac{(e^{ik_j}-\frac{\Delta}{2})}{(\frac{\Delta}{2}e^{ik_j}-1)} = \Lambda_j(k_1,\cdots,k_{ns}) \tag{47}$$

where $\Lambda_j(k_1,\cdots,k_{ns})$ are the eigenvalues of the auxiliary inhomogeneous transfer matrix $t_j = S_{jns}(k_j,k_{ns})\cdots S_{j1}(k_j,k_1)S_{1j}(k1,-k_j)\cdots S_{nsj}(k_{ns},-k_j)$. The integrability of this latter inhomogenous problem follows from the fact that the 2×2 identity K-matrix is a solution of the reflection equation associated to the two-body scattering S_{ij}. As was shown in ref. [15] there is no need of a second Bethe ansatz to solve this auxiliary eigenvalue problem. By adapting the results of ref.[15] to our case and by relating the momenta k_j and the rapidities λ_j by equation (46) we find that the Bethe ansatz equations are given by

$$\left[\frac{\sinh(\lambda_j - i\gamma/2)}{\sinh(\lambda_j + i\gamma/2)}\right]^{2L}\frac{\cosh(\lambda_j + i\gamma/2)}{\cosh(\lambda_j - i\gamma/2)} = \prod_{k=1}^{ns}\frac{\sinh(\lambda_j/2-\lambda_k/2-i\gamma/2)}{\sinh(\lambda_j/2-\lambda_k/2+i\gamma/2)}\frac{\sinh(\lambda_j/2+\lambda_k/2-i\gamma/2)}{\sinh(\lambda_j/2+\lambda_k/2+i\gamma/2)}$$
$$j = 1,\cdots,ns \tag{48}$$

and the eigenvalues (42) in terms of the rapidities λ_j are

$$E^{(1,1)}(L) = -8\sin^3(\gamma)\sum_{j=1}^{ns}\frac{1}{\cos(\gamma)-\cosh(2\lambda_j)} - 4(L-1)\sin(2\gamma) + 4\sin(\gamma)\sin^2(\gamma/2) \tag{49}$$

Clearly, the Bethe ansatz equations for the open spin chain $H^{(1,1)}$ are not just the "doubling" of the corresponding results of the closed chain with periodic boundary conditions [16, 15] due to an additional boundary left hand factor. We recall here that the "doubling" property has been argued [17] to be one of the main features of a quantum algebra invariant open spin chain at least for standard forms of comultiplication. Looking at the spectrum of $H^{(1,1)}$, however, we notice a certain pattern of degeneracies which suggests an underlying hidden symmetry. It could be that the diagonal boundary solution corresponds to an asymmetric form of coproduct since this, in principle, is allowed too [18].

Next we turn our attention to the first non-diagonal solution and its corresponding open spin chain. In this case, at least for generic values of $\xi_{\pm}$, the Bethe ansatz construction we just explained above needs further generalizations. This can be seen even at the level of one down spin state, since there is not a unique transformation that diagonalizes both left and right boundary matrix problems. However, there is a particular manifold, $\xi_+ = q\xi_-$, in which our previous Bethe ansatz formulation is still valid. Fortunately, as we shall see below, this special manifold will be sufficient to single out the boundary leading us to the quantum algebra symmetry. Since for $\xi_+ = q\xi_-$, the Bethe ansatz analysis is very similar to the one just described above, we restrict ourselves to present only the final results. We found that the Bethe ansatz equations for the Hamiltonian $H^{(2,2)}$ at $\xi_+ = q\xi_-$ are

$$\left[\frac{\sinh(\lambda_j - i\gamma/2)}{\sinh(\lambda_j + i\gamma/2)}\right]^{2L} = \prod_{k=1}^{ns} \frac{\sinh(\lambda_j/2 - \lambda_k/2 - i\gamma/2)}{\sinh(\lambda_j/2 - \lambda_k/2 + i\gamma/2)} \frac{\sinh(\lambda_j/2 + \lambda_k/2 - i\gamma/2)}{\sinh(\lambda_j/2 + \lambda_k/2 + i\gamma/2)}, \quad j = 1, \cdots, ns \tag{50}$$

while the corresponding eigenvalues are given by

$$E^{(1,1)}(L) = -8\sin^3(\gamma) \sum_{j=1}^{ns} \frac{1}{\cos(\gamma) - \cosh(2\lambda_j)} - 4(L-1)\sin(2\gamma) - 4\sin(\gamma)\left[\sum_{\alpha=\uparrow,\downarrow} (\mu_{\alpha}^{(2)}(\xi_-) - \mu_{\alpha}^{(2)}(\xi_+)) + 2\delta^{(2)} \right. \tag{51}$$

Now the Bethe ansatz equations do have the "doubling" property at $\xi_+ = q\xi_-$ and this is an extra motivation to investigate the eigenspectrum of $H^{(2,2)}$. It turns out that at the value $\xi_- = 0$ and therefore $\xi_+ = 0$ we discover that the spectrum of the open chain $H^{(2,2)}$ is specially highly degenerated. In fact, after some algebraic manipulations, we check that for $\xi_{\pm} = 0$

the Hamiltonian $H^{(2,2)}$ has the appropriate boundary coefficients to ensure commutation with $U_q(D_2^2)$. Therefore, we finally managed to identify the quantum algebra symmetry for the D_2^2 vertex model.

Finally, it seems desirable to solve the open spins chains associated to the non-diagonal solutions for arbitrary values of the parameters $\xi_{\pm}$. The coordinate Bethe ansatz method, however, leads us to cumbersome calculations even for the first excitation over the reference state. In such general case it seems wise to tackle this problem by using a more unifying technique such as the algebraic Bethe ansatz approach. Since the basics of this method has been recently developed for the D_n^2 vertex models [15] we hope to return to this problem elsewhere.

4 The D_n^2 K-matrices

Here we shall consider the generalizations of the K-matrices solutions of section 2 for the general D_n^2 model. This system has $n-1$ distinct $U(1)$ conserved charges, and the K-matrix ansatz compatible with these symmetries can be represented by the following block diagonal matrix

$$K_{-}(\lambda) = \mathrm{diag}(Y_1(\lambda), \cdots, Y_{n-1}(\lambda), \hat{A}(\lambda), Y_{n+2}(\lambda), \cdots, Y_{2n}(\lambda)) \tag{52}$$

where $\hat{A}(\lambda)$ is a 2×2 matrix

$$\hat{A}(\lambda) = \begin{pmatrix} Y_n(\lambda) & Y_{2n+1}(\lambda) \\ Y_{2n+2}(\lambda) & Y_{n+1}(\lambda) \end{pmatrix} \tag{53}$$

where $Y_j(\lambda),\ j = 1, \cdots, 2n+2$ are functions we have determined by solving the reflection equation. Notice that for $n = 2$ we recover our starting ansatz of section 2.

Substituting this ansatz into the reflection equation, we realize that the simplest possible solution is the symmetric one, namely

$$Y_1(\lambda) = Y_2(\lambda) = \cdots = Y_{n-1}(\lambda) \quad \text{and} \quad Y_{n+2}(\lambda) = Y_{n+1}(\lambda) = \cdots = Y_{2n}(\lambda) \tag{54}$$

It turns out that the remaining functional equations for the functions $Y_1(\lambda)$, $Y_n(\lambda)$, $Y_{n+1}(\lambda)$, $Y_{2n}(\lambda)$, $Y_{2n+1}(\lambda)$ and $Y_{2n+2}(\lambda)$ are very similar to those presented in the appendix A. Therefore, they can be solved by the same procedure described in section 2 and in what follows we only quote our final results. As before we find three general families of K-matrices, and the diagonal one is given by

$$Y_1^{(1)}(\lambda) = 1, \quad Y_n^{(1)}(\lambda) = \frac{e^{\lambda} - Iq^{-(n-1)/2}}{e^{-\lambda} - Iq^{-(n-1)/2}} \tag{55}$$

$$Y_{n+1}^{(1)}(\lambda) = \frac{e^{\lambda} + Iq^{-(n-1)/2}}{e^{-\lambda} + Iq^{-(n-1)/2}}, \quad Y_{2n}^{(1)}(\lambda) = \frac{e^{\lambda} + Iq^{-(n-1)/2}}{e^{-\lambda} + Iq^{-(n-1)/2}} \frac{e^{\lambda} - Iq^{(n-1)/2}}{e^{-\lambda} - Iq^{(n-1)/2}} \tag{56}$$

The one-parameter families of non-diagonal K-matrices are given by

$$Y_1^{(2)}(\lambda, \xi_-) = (e^{2\lambda} + q^{n-1})(\xi_-^2 q^{n-1} e^{2\lambda} - 1)e^{-\lambda}, \quad Y_{2n}^{(2)}(\lambda, \xi_-) = (e^{2\lambda} + q^{n-1})(\xi_-^2 q^{n-1} - e^{2\lambda})e^{\lambda} \tag{57}$$

$$Y_n^{(2)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}\left[2(e^{2\lambda} - 1)\xi_- q^{n-1} - e^{\lambda}(1 + q^{n-1})(1 - \xi_-^2 q^{n-1})\right] \tag{58}$$

$$Y_{n+1}^{(2)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}\left[-2(e^{2\lambda} - 1)\xi_- q^{n-1} - e^{\lambda}(1 + q^{n-1})(1 - \xi_-^2 q^{n-1})\right] \tag{59}$$

$$Y_{2n+1}^{(2)}(\lambda, \xi_-) = Y_{2n+2}^{(2)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}(1 - q^{n-1})(\xi_-^2 q^{n-1} + 1)e^{\lambda} \tag{60}$$

and

$$Y_1^{(3)}(\lambda, \xi_-) = (e^{2\lambda} - q^{n-1})(\xi_- e^{2\lambda} - 1)e^{-\lambda}, \quad Y_{2n}^{(3)}(\lambda, \xi_-) = (e^{2\lambda} - q^{n-1})(\xi_- - e^{2\lambda})e^{\lambda} \tag{61}$$

$$Y_n^{(3)}(\lambda, \xi_-) = Y_{n+1}^{(3)}(\lambda, \xi_-) = \frac{(1 + e^{2\lambda})}{2}(1 - q^{n-1})(\xi_- - 1)e^{\lambda} \tag{62}$$

$$Y_{2n+1}^{(3)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}\left[2(e^{2\lambda} + 1)\sqrt{\xi_-}q^{(n-1)/2} + (1 + q^{n-1})(1 + \xi_-)e^{\lambda}\right] \tag{63}$$

$$Y_{2n+2}^{(3)}(\lambda, \xi_-) = \frac{(e^{2\lambda} - 1)}{2}\left[-2(e^{2\lambda} + 1)\sqrt{\xi_-}q^{(n-1)/2} + (1 + q^{n-1})(1 + \xi_-)e^{\lambda}\right] \tag{64}$$

The next natural step is to search for asymmetric K-matrices for $n \geq 3$, i.e. those having $Y_1(\lambda) \neq Y_2(\lambda) \neq \cdots Y_{n-1}(\lambda)$ and $Y_{n+2}(\lambda) \neq Y_{n+3}(\lambda) \neq \cdots Y_{2n}(\lambda)$. In this case the number of free parameters grows rapidly with n and the solution of the reflection equation becomes more involving. To illustrate that, we consider the D_3^2 model and for sake of simplicity we look first for diagonal solutions. There are six functions $Y_j(\lambda)$ to be determined and their ratios are fixed by choosing some easy looking relations coming from the reflection relation. More precisely, we have found the following equations

$$\frac{Y_2(\lambda)}{Y_1(\lambda)} = \frac{e^{2\lambda} - c_1}{e^{-2\lambda} - c_1}, \quad \frac{Y_3(\lambda)}{Y_1(\lambda)} = \frac{e^{\lambda} - c_2}{e^{-\lambda} - c_2}, \quad \frac{Y_4(\lambda)}{Y_1(\lambda)} = \frac{e^{\lambda} - c_3}{e^{-\lambda} - c_3} \tag{65}$$

$$\frac{Y_6(\lambda)}{Y_4(\lambda)} = \frac{e^{\lambda} - c_4}{e^{-\lambda} - c_4}, \quad \frac{Y_5(\lambda)}{Y_6(\lambda)} = \frac{e^{-2\lambda} - c_5}{e^{2\lambda} - c_5} \tag{66}$$

where c_j are once again constants yet to be determined. Substituting these relations back to the reflection equation we find only one possible manifold for the parameters c_j, which reads

$$c_1 = c_5 = -1 \quad \text{and} \quad c_2 = c_3 = c_4 = 0 \tag{67}$$

After an appropriate normalization, this solution leads us to a new diagonal K-matrix for the D_3^2 model

$$K_-^{D_3^2}(\lambda) = \mathrm{diag}(e^{-2\lambda}, 1, 1, 1, 1, e^{2\lambda}) \tag{68}$$

It is plausible that this "almost unity" solution and its extensions generalizes for arbitrary values of $n \geq 4$. Next we have looked at the possibility of asymmetric non-diagonal solutions for the D_3^2 model. It turns out that, within our algebraic approach, we did not found any of such solutions. However, this possibility should not be completely rule out, at least for general n, since we have so many free parameters that the chance to miss a particular integrable manifold is high. In general, classification of the solutions of the reflection equation seems to be an intricated problem even for simpler models [8]. We hope, however, that our K-matrices results prompt further investigation concerning this problem for the D_n^2 vertex models.

We would like to conclude this section with the following remarks. The $K_+(\lambda)$ matrices can be obtained from $K_-(\lambda)$ by the isomorphism

$$K_+(\lambda) = K_-^t[(n-1)\ln[q] - \lambda]M \tag{69}$$

where M is a diagonal matrix given by

$$M = \mathrm{diag}(q^{(2n-3)}, q^{2n-5}, \cdots, 1, 1, \cdots, q^{-(2n-5)}, q^{-(2n-3)}) \tag{70}$$

Once we are equipped with $K_\pm(\lambda)$ matrices, the construction of the corresponding open spin chains is possible along the lines of section 2. Similarly, at least for the diagonal solution, one can also repeat our Bethe ansatz construction without further technical difficulties. In particular, we conjecture that the open spin chain associated to the first non-diagonal solution at $\xi_\pm = 0$ is the one having the underlying quantum group symmetry.

5 Concluding Remarks

In this paper we have made a great deal of progress towards the understanding of the integrability of the D_n^2 vertex model with open boundaries. We have investigated the solutions of the associated reflection equation and found three general families of K-matrices which respect the minimal $U(1)$ symmetries of this system. We have carried out a Bethe ansatz analysis for the simplest case, D_2^2 model, revealing to us that the first non-diagonal solution at $\xi_{\pm} = 0$ possesses the special quantum algebra symmetry. In fact, the structure of the K-matrices at this particular point leads us to conjecture that this will be the case for arbitrary values of n.

We believe that our results open an enormous avenue for further investigations. One clear possibility is to use the Bethe ansatz results of section 3 to compute the thermodynamic behaviour, the bulk and the surface critical exponents. It would be also interesting to generalize our results of section 3 for all sort of open boundary conditions and for arbitrary values of n. In this case, probably the most suitable tool would be instead the algebraic Bethe ansatz approach. This method would allow us to show that indeed the Bethe ansatz states are highest weight states of the underlying quantum algebra in the case of the first non-diagonal family at $\xi_{\pm} = 0$.

Other interesting issue is to apply the notion of the quantum group twisting [19] to find out slightly different D_n^2 R-matrices. As a result, this might lead us to integrable models with very different behaviour, for an example see ref.[20]. The practical implementation of twisting, however, seems to be quite involving specially for an algebra such D_n^2. To shed some light to this problem we proceed in a much more phenomenological way. Motived by the structure of the non-diagonal solutions, we add extra Boltzmann weights to the Jimbo's R-matrix to account for such boundary terms at the level of the associated bulk Hamiltonian. Next step is to try to solve the Yang-Baxter equation for this novel R-matrix structure. It turns out that we succeed to find a new R-matrix solution for the D_2^2 model. Since this involves many technicalities, we have summarized it in appendix B together with the study of the corresponding solutions of the reflection equation. We hope that these results will be useful to motivate further progress in this problem.

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Appendix A : R matrix properties and reflection equations

In this appendix we briefly discuss some useful properties of the D_n^2 R-matrix [10]. We also present for $n = 2$ some relevant relations derived from the reflection equation. The D_n^2 R-matrix satisfies, besides the unitarity and regularity, extra relations denominated PT invariance and crossing symmetry, PT-Symmetry:

$$P_{12}R_{12}(\lambda)P_{12} = R_{12}^{t_1t_2}(\lambda) \tag{A.1}$$

Crossing-symmetry:

$$R_{12}(\lambda) = \frac{\zeta[\lambda]}{\zeta[(n-1)\ln[q]-\lambda]} \overset{1}{V} R_{12}^{t_2}[(n-1)\ln[q]-\lambda) \overset{1}{V}^{-1} \tag{A.2}$$

where $\zeta(\lambda)$ is a normalization function and V is the following crossing matrix

$$V = \mathrm{antidiag}\left\{q^{-(2n-3)/2}, q^{-(2n-5)/2}, \cdots, \frac{1}{\sqrt{q}}, 1, 1, \sqrt{q}, \cdots, q^{(2n-5)/2}, q^{(2n-3)/2}\right\}. \tag{A.3}$$

Here we find convenient to normalize the original Jimbo's R-matrix by an overall factor $e^{2\lambda}q^n$ and the function $\zeta(\lambda)$ is given by

$$\zeta(\lambda) = (e^{\lambda} - e^{-\lambda})(\frac{e^{\lambda}}{q^{(n-1)}} - \frac{q^{(n-1)}}{e^{\lambda}}) \tag{A.4}$$

Next we present the simplest relations derived from the reflection equation we used in section 2. For sake of simplicity we shall use the following notation $Y_i(x) \equiv Y_i, Y_i(y) \equiv Y_i^{'}, w_j(x-y) \equiv w_j, w_j(x+y) = w_j^{'}$. Considering this notation, the relations we have selected from the reflection equation are given by

$$-w_2^{'}\left[-w_3Y_1^{'}Y_2 + w_4Y_1Y_2^{'} - w_5Y_1^{'}Y_5 + w_6Y_1Y_6^{'}\right] =$$

$$w_2\left[-w_4'Y_1'Y_1+w_5'(Y_2'Y_5+Y_2Y_6')+w_3'(Y_2'Y_2+Y_5Y_6')\right] \tag{A.5}$$

$$-w_2'\left[-w_3Y_1'Y_3+w_4Y_1Y_3'+w_6Y_1Y_5'-w_5Y_1'Y_6\right]=$$

$$w_2\left[-w_4'Y_1'Y_1+w_5'(Y_3Y_5'+Y_3'Y_6)+w_3'(Y_3'Y_3+Y_5'Y_6)\right] \tag{A.6}$$

$$-w_2'\left[w_3Y_3'Y_4-w_4Y_3Y_4'+w_5Y_4Y_5'-w_6Y_4'Y_6\right]=$$

$$w_2\left[-w_3'Y_4'Y_4+w_6'(Y_3Y_5'+Y_3'Y_6)+w_4'(Y_3'Y_3+Y_5'Y_6)\right] \tag{A.7}$$

$$-w_5'\left\{w_5(Y_6'Y_6-Y_5Y_5')+w_3\left[Y_2'(Y_5-Y_6)+Y_2(Y_6'-Y_5')\right]\right\}=$$

$$w_3'\left\{w_3(Y_6'Y_5-Y_6Y_5')+w_5\left[Y_2'(Y_6-Y_5)+Y_3(Y_6'-Y_5')\right]\right\} \tag{A.8}$$

$$-w_3'\left\{w_5(Y_3'Y_3-Y_2Y_2')+w_3\left[Y_5(Y_3'-Y_2')+Y_5'(Y_2-Y_3)\right]\right\}=$$

$$w_5'\left\{w_3(Y_2Y_3'-Y_3Y_2')+w_5\left[Y_5'(Y_3-Y_2)+Y_6(Y_3'-Y_2')\right]\right\} \tag{A.9}$$

$$-w_6'\left\{w_6(Y_6'Y_6-Y_5Y_5')+w_4\left[Y_2'(Y_5-Y_6)+Y_2(Y_6'-Y_5')\right]\right\}=$$

$$w_4'\left\{w_4(Y_6'Y_5-Y_6Y_5')+w_6\left[Y_2'(Y_6-Y_5)+Y_3(Y_6'-Y_5')\right]\right\} \tag{A.10}$$

$$-w_4'\left\{w_6(Y_3'Y_3-Y_2Y_2')+w_4\left[Y_5(Y_3'-Y_2')+Y_5'(Y_2-Y_3)\right]\right\}=$$

$$w_6'\left\{w_4(Y_3'Y_2-Y_3Y_2')+w_6\left[Y_5'(Y_3-Y_2)+Y_6(Y_3'-Y_2')\right]\right\} \tag{A.11}$$

The functions $w_j(\lambda)$ are some of the Boltzmann weights of D_2^2 model and are given by [10]

$$w_2(\lambda)=(e^{\lambda}-e^{-\lambda})(\frac{e^{\lambda}}{q}-\frac{q}{e^{\lambda}}),\quad w_3(\lambda)=-\frac{1}{2}(q-\frac{1}{q})(\frac{e^{\lambda}}{q}-\frac{q}{e^{\lambda}})(e^{-\lambda}+1) \tag{A.12}$$

$$w_4(\lambda)=-\frac{1}{2}(q-\frac{1}{q})(\frac{e^{\lambda}}{q}-\frac{q}{e^{\lambda}})(e^{\lambda}+1),\quad w_5(\lambda)=\frac{1}{2}(q-\frac{1}{q})(\frac{e^{\lambda}}{q}-\frac{q}{e^{\lambda}})(-e^{\lambda}+1) \tag{A.13}$$

$$w_6(\lambda)=-\frac{1}{2}(q-\frac{1}{q})(\frac{e^{\lambda}}{q}-\frac{q}{e^{\lambda}})(e^{\lambda}-1) \tag{A.14}$$

Appendix B : A new D_2^2 R-matrix

We begin by presenting the new D_2^2 R-matrix

$$\begin{aligned} R(\lambda) &= (e^{2\lambda}-q^2)^2\sum_{\alpha\neq 2,3}E_{\alpha\alpha}\otimes E_{\alpha\alpha}+q(e^{2\lambda}-1)(e^{2\lambda}-q^2)\sum_{\substack{\alpha\neq\beta,\beta' \\ \alpha\mathrm{or}\beta\neq 2,3}}E_{\alpha\alpha}\otimes E_{\beta\beta} \\ &-\frac{(q^2-1)(e^{2\lambda}-q^2)}{2}[(e^{\lambda}+1)\sum_{\substack{\alpha<2 \\ \beta=2,3}}+e^{\lambda}(e^{\lambda}+1)\sum_{\substack{\alpha>3 \\ \beta=2,3}}](E_{\alpha\beta}\otimes E_{\beta\alpha}+E_{\beta'\alpha'}\otimes E_{\alpha'\beta'}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{(q^2-1)(e^{2\lambda}-q^2)}{2}\Big[(1-e^{\lambda})\sum_{\substack{\alpha<2\\ \beta=2,3}} + e^{\lambda}(e^{\lambda}-1)\sum_{\substack{\alpha>3\\ \beta=2,3}}\Big](E_{\alpha\beta}\otimes E_{\beta'\alpha}+E_{\beta'\alpha'}\otimes E_{\alpha'\beta})\\
&+\sum_{\alpha,\beta\neq 2,3} a_{\alpha\beta}(\lambda)E_{\alpha\beta}\otimes E_{\alpha'\beta'}\\
&+\sum_{\substack{\alpha\neq 2,3\\ \beta=2,3}} b^{+}_{\alpha}(\lambda)E_{\alpha\beta}\otimes E_{\alpha'\beta'}+\tilde{b}^{+}_{\alpha}(\lambda)E_{\beta'\alpha'}\otimes E_{\beta\alpha}+b^{-}_{\alpha}(\lambda)E_{\alpha\beta}\otimes E_{\alpha'\beta}+\tilde{b}^{-}_{\alpha}(\lambda)E_{\beta\alpha'}\otimes E_{\beta\alpha}\\
&+\sum_{\alpha=2,3} c^{+}(\lambda)E_{\alpha\alpha}\otimes E_{\alpha'\alpha'}+c^{-}(\lambda)E_{\alpha\alpha}\otimes E_{\alpha\alpha}+d^{+}(\lambda)E_{\alpha\alpha'}\otimes E_{\alpha'\alpha}+d^{-}(\lambda)E_{\alpha\alpha'}\otimes E_{\alpha\alpha'}\\
&+\sum_{\alpha=2,3} f(\lambda)\left[E_{\alpha\alpha'}\otimes E_{\alpha\alpha}+E_{\alpha\alpha}\otimes E_{\alpha\alpha'}-E_{\alpha\alpha}\otimes E_{\alpha'\alpha}-E_{\alpha'\alpha}\otimes E_{\alpha\alpha}\right]
\end{aligned}
\tag{B.1}
$$

where $E_{\alpha\beta}$ are the elementary 4×4 matrices and we set $\alpha' = 5-\alpha$. The Boltzmann weights are given by

$$a_{11}(\lambda)=a_{44}(\lambda)=q^2(e^{2\lambda}-1)^2, \quad a_{14}(\lambda)=a_{41}(\lambda)e^{-2\lambda}=(q-1)(q^2-1)(e^{2\lambda}+q) \tag{B.2}$$

$$b^{\pm}_1=\pm\frac{q^{3/2}}{2}(q^2-1)(e^{2\lambda}-1)(e^{\lambda}\pm 1), \quad \tilde{b}^{\pm}_1=\pm\frac{q^{-1/2}}{2}(q^2-1)(e^{2\lambda}-1)(e^{\lambda}\pm q^2) \tag{B.3}$$

$$b^{\pm}_4=\frac{q^{1/2}}{2}e^{\lambda}(q^2-1)(e^{2\lambda}-1)(e^{\lambda}\pm q^2), \quad \tilde{b}^{\pm}_4=\frac{q^{1/2}}{2}e^{\lambda}(q^2-1)(e^{2\lambda}-1)(e^{\lambda}\pm 1) \tag{B.4}$$

$$d^{\pm}=\pm\frac{e^{\lambda}}{4}(q^2-1)^2(e^{\lambda}\pm 1)^2, \quad f(\lambda)=\frac{e^{\lambda}}{4}(e^{2\lambda}-1)(q^2-1)^2 \tag{B.5}$$

$$c^{\pm}=\pm\frac{e^{\lambda}}{4}(q^2-1)(e^{\lambda}\mp 1)[e^{\lambda}(3+q^2)\pm(1+3q^2)]+q(e^{2\lambda}-1)(e^{2\lambda}-q^2) \tag{B.6}$$

This R-matrix has additional Boltzmann weights, the last term in equation (B.1), as compare to the standard D^2_2 R-matrix [10]. In addition, several other weights have also a different functional dependence on the spectral paramater λ. For periodic boundary conditions, such differences are not important since we verified, by using the algebraic Bethe ansatz approach [15], that the corresponding Bethe ansatz equations and eigenvalues are the same as those found for the standard D^2_2 model [16, 15]. This result is a strong indication that indeed the R-matrix (B.1) can be obtained by twisting the usual D^2_2 R-matrix. However, the situation for open boundary conditions turns out to be a bit different. In fact, we did not find any diagonal solution of the corresponding reflection equation. The basic K-matrices are non-diagonal and we managed to find two classes of such solutions. The first family depends only on a discrete parameter $\varepsilon = \pm$ and is given by

$$
\begin{aligned}
Y_1^{(1,\varepsilon)}(\lambda,\xi_-) &= Y_4^{(1,\varepsilon)}(\lambda,\xi_-) = (e^{2\lambda}+\varepsilon\, q) && \text{(B.7)}\\
Y_2^{(1,\varepsilon)}(\lambda,\xi_-) &= Y_3^{(1,\varepsilon)}(\lambda,\xi_-) = \frac{1}{2}(1+\varepsilon\, q)(1+e^{2\lambda}) && \text{(B.8)}\\
Y_5^{(1,\varepsilon)}(\lambda,\xi_-) &= Y_6^{(1,\varepsilon)}(\lambda,\xi_-) = \frac{1}{2}(1-\varepsilon\, q)(-1+e^{2\lambda}) && \text{(B.9)}
\end{aligned}
$$

while the second family has an extra continuous parameter ξ_-

$$
\begin{aligned}
Y_1^{(2,\varepsilon)}(\lambda,\xi_-) &= (e^{2\lambda}-\xi_-^2)(q+\varepsilon\, e^{2\lambda})e^{-\lambda}, && \text{(B.10)}\\
Y_4^{(2,\varepsilon)}(\lambda,\xi_-) &= (1-\xi_-^2\, e^{2\lambda})(q+\varepsilon\, e^{2\lambda})e^{\lambda}, && \text{(B.11)}\\
Y_2^{(2,\varepsilon)}(\lambda,\xi_-) &= \frac{1}{2}(1-\xi_- e^{\lambda})(\xi_- + e^{\lambda})(1+e^{2\lambda})(\varepsilon+q), && \text{(B.12)}\\
Y_3^{(2,\varepsilon)}(\lambda,\xi_-) &= \frac{1}{2}(1+\xi_- e^{\lambda})(-\xi_- + e^{\lambda})(1+e^{2\lambda})(\varepsilon+q), && \text{(B.13)}\\
Y_5^{(2,\varepsilon)}(\lambda,\xi_-) &= \frac{1}{2}(e^{2\lambda}-1)(\varepsilon-q)(\xi_- - e^{\lambda})(\xi_- e^{\lambda}-1), && \text{(B.14)}\\
Y_6^{(2,\varepsilon)}(\lambda,\xi_-) &= \frac{1}{2}(e^{2\lambda}-1)(\varepsilon-q)(\xi_- + e^{\lambda})(\xi_- e^{\lambda}+1), && \text{(B.15)}
\end{aligned}
$$

Finally, we remark that since this new R-matrix is only unitary, the associated $K_+(\lambda)$ matrices can not be directly obtained by an isomorphism of the type described in (20). However, as shown in ref. [21], unitarity is a sufficient condition to allow one to construct commutative transfer matrices leading to open spin chains. In this case one has to solve an extra reflection equation to obtain the $K_+(\lambda)$ matrix [21].

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