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Multiscale Preconditioning for the Coupling of FEM-BEM

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Contents

MULTISCALE PRECONDITIONING FOR THE COUPLING OF FEM-BEM

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Abstract. We apply multiscale methods to the coupling of finite and boundary element methods to solve an exterior two dimensional Laplacian. The matrices belonging to the boundary terms of the coupled FEM-BEM system are compressed using biorthogonal wavelet bases developed from A. Cohen, I. Daubechies and J.-C. Feauveau [5]. We describe different solving and preconditioning techniques. Through numerical experiments we provide results which corroborate the theory of [19] and the present paper.

AMS subject classification: 65F35, 65M55, 65M60, 65N30, 65N55, 65R20.

Introduction

During the past decade the coupling of finite element methods (FEM) with boundary element methods (BEM) has been developed to combine the advantages of both methods, mentioning only [4, 6, 12, 19]. Due to the large complexity of the boundary element part, an application of modern fast methods for integral equations, like *Fast Multipole Method* [14], *Panel Clustering* [17] or *wavelet approaches* [1, 7, 9, 25, 23] seems to be highly attractive. However, a rigorous investigation of these methods in conjunction with FEM-BEM coupling was missing. In [19] we have studied the corresponding wavelet matrix compression for two dimensional boundary value problems. In particular, the boundary integral equation can be formulated on an artificial interface Γ. Even if a circle or a sphere is not preferable, the geometry of this interface can be chosen fairly simple. By such a choice the wavelet approaches on Γ become much more efficient. The questions of solving the compressed linear system has been defered to the present paper. Herein, we will focus mainly practical issues, like preconditioning and practical implementation.

The present approach is set up as follows. First, the exterior Dirichlet boundary problem is reduced to an equivalent one in a bounded domain using the so-called *two integral formulation* of the coupling [6, 12, 18]. A smooth parameterization of the artificial boundary Γ is used to simplify the analysis and numerical solution of the discrete Galerkin scheme [19, 22]. More precisely, the normal derivative on the boundary is substituted by a new unknown on the interval $[0,1]$. The finite elements on the domain are supposed

Key words and phrases. Finite element, boundary element, multiscale methods, biorthogonal wavelet bases, norm equivalences, matrix compression, preconditioning, fast solution.

to be piecewise linear and continuous. Consequently their traces are also piecewise linear on the interval $[0, 1]$ with respect to the same parameterization as above. In the focused case it is sufficient to discretize the new unknown on the interval by piecewise constant functions. This suggests the use of piecewise linear and piecewise constant biorthogonal wavelet bases with sufficiently many vanishing moments as introduced in [5]. Then, the wavelet matrix compression strategy proposed in [19] is performed on the BEM matrices. It is worth remarking that this procedure can be efficiently implemented, and without loss of stability and accuracy [19] of the Galerkin scheme. We confirm this by the given numerical results. These results also demonstrate, compared to the traditional boundary element approach, a dramatical saving of the required memory and computing time.

Since in general, the corresponding linear system is difficult to solve due to its complicated structure and ill-conditioning, we study several preconditioning techniques. The constructed triangulation is suitable to precondition the matrix arising from the FEM discretization by the BPX preconditioner [3]. Moreover, for operators of nonzero order, a simple diagonal preconditioner for the BEM matrices discretized in wavelet bases is available [10, 25]. We apply these results combined with the discrete wavelet transform to construct, similar to [20, 21], a global 3-block-preconditioner avoiding above mentioned ill-conditioning. We use Krylov subspace methods, namely GMRES [24] and MINRES [23], as iterative solvers for the preconditioned nonsymmetric system and its symmetric counterpart, respectively. Alternatively, we apply Bramble-Pasciack's CG [2], where a new bilinear form enables the application of the CG algorithm. We show that these methods have almost optimal complexity and in combination with nested iterations we achieve optimal complexity [15].

The paper is organized as follows. In section 1, we introduce the model problem and transform it via the two integral formulation into an equivalent variational formulation. Then, the discretization of the variational formulation is described. In section 2 we briefly recall the biorthogonal wavelet approximation for the coupling proposed in [19]. Here, aspects of the computational implementation of our method are described: We show that the change of bases in each iteration step for the unknown on the interval is not required, and that, similar to the classical approach, we have to compute only two BEM matrices, now with respect to wavelet bases. In Section 3, we propose preconditioning techniques for the resulting linear equation system. In Section 4, through numerical experiments, we explore the biorthogonal wavelet approximation for the coupling of FEM-BEM. We demonstrate that the accuracy of the Galerkin scheme has not been deteriorated by the compression strategy. The different solving and preconditioning techniques are discussed and, moreover, a nested iteration algorithm is performed. Finally, in Section 5, we state concluding remarks.

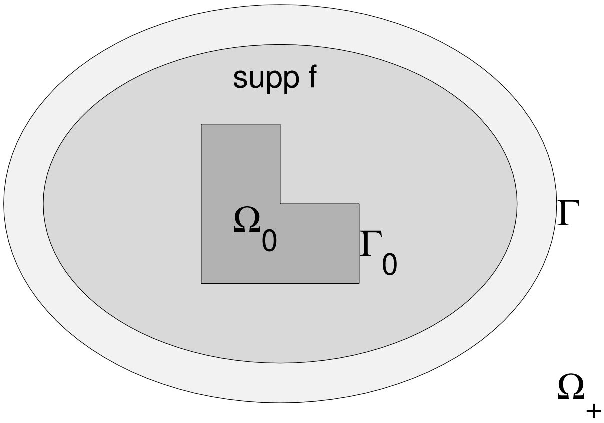

FIGURE 1.1. The model problem.

Throughout this paper $a \lesssim b$ expresses that a can be bounded by some constant multiple of b uniformly in any parameters on which a and b may depend. Likewise $a \sim b$ means that $a \lesssim b$ and $a \gtrsim b$.

1. THE COUPLING OF FEM-BEM

In this section we introduce the given exterior boundary value problem and transform it via the *two integral formulation*, cf. [6, 12, 18], into an equivalent variational formulation, for which we show uniqueness and existence of the solution. The subsequent discretization is described in the last subsection yielding a Galerkin scheme without further restrictions.

1.1. **The model problem.**

We consider as model problem an exterior Dirichlet problem for the Laplacian in the plane. Let $\Omega_0 \in \mathbb{R}^2$ be a bounded and simply connected domain with Lipschitz boundary Γ_0. Then, for a given compactly supported function $f \in L^2(\mathbb{R}^2 \setminus \Omega_0)$ we seek u such that

$$\begin{aligned} -\triangle u &= f && \text{in } \mathbb{R}^2 \setminus \Omega_0, \\ u &= 0 && \text{on } \Gamma_0, \\ u(x) &= \mathcal{O}(1) && \text{as } |x| \to \infty. \end{aligned} \tag{1.1}$$

According to the hypothesis on f we choose a smooth bounded second domain Ω_1 containing $\overline{\Omega}_0$ and $\operatorname{supp} f$. Its boundary $\Gamma := \partial\Omega_1$ divides $\mathbb{R}^2 \setminus \overline{\Omega}_0$ into an annular region Ω (bounded by Γ_0 and Γ) and an unbounded exterior domain $\Omega_+ := \mathbb{R}^2 \setminus \overline{\Omega}_1$, as shown in FIGURE 1.1.

With this setup, (1.1) can be split in a coupled interior and exterior boundary value problem

$$
\begin{aligned}
-\triangle u &= f && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_0,\\
\triangle u &= 0 && \text{in } \Omega_+,\\
u(x) &= \mathcal{O}(1) && \text{as } |x|\to\infty,\\
\lim_{\substack{x\to x_0\\ x\in\Omega}} u(x) &= \lim_{\substack{x\to x_0\\ x\in\Omega_+}} u(x) && \text{for all } x_0\in\Gamma,\\
\lim_{\substack{x\to x_0\\ x\in\Omega}} \frac{\partial u}{\partial \nu}(x) &= \lim_{\substack{x\to x_0\\ x\in\Omega_+}} \frac{\partial u}{\partial \nu}(x) && \text{for all } x_0\in\Gamma.
\end{aligned}
\tag{1.2}
$$

We introduce the *single layer operator* $\mathcal{V}$, the *double layer operator* $\mathcal{K}$, its adjoint $\mathcal{K}^\star$ and the *hypersingular operator* $\mathcal{W}$ defined by

$$
\begin{aligned}
(\mathcal{V}u)(x) &:= \int_\Gamma E(x,y)v(y)ds_y,\\
(\mathcal{K}u)(x) &:= \int_\Gamma \frac{\partial}{\partial \nu_y}E(x,y)v(y)ds_y,\\
(\mathcal{K}^\star u)(x) &:= \int_\Gamma \frac{\partial}{\partial \nu_x}E(x,y)v(y)ds_y,\\
(\mathcal{W}u)(x) &:= -\frac{\partial}{\partial \nu_x}\int_\Gamma \frac{\partial}{\partial \nu_y}E(x,y)v(y)ds_y,
\end{aligned}
$$

where the fundamental solution $E(x,y)$ is given by

$$
E(x,y) = -\frac{1}{2\pi}\log|x-y|. \tag{1.3}
$$

If we denote by $L^2(\Gamma)$ the function space of all squared integrable functions on Γ with respect to the canonical inner product

$$
(u,v)_{L^2(\Gamma)} = \int_\Gamma u(x)v(x)ds_x
$$

and by $H^s(\Gamma)$ ($s\in\mathbb{R}$) the corresponding Sobolev spaces, then, in this context, $\mathcal{V}$ defines an operator of order -1

$$
\mathcal{V}: H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma),
$$

$\mathcal{K}$, $\mathcal{K}^\star$ are zero order operators

$$
\mathcal{K}: H^{1/2}(\Gamma)\to H^{1/2}(\Gamma), \qquad \mathcal{K}^\star: H^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma),
$$

and $\mathcal{W}$ is an operator of order $+1$

$$
\mathcal{W}: H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma).
$$

Finally, introducing the variables $\sigma := \frac{\partial u}{\partial \nu}$ and $c := \lim_{|x|\to\infty} u$ the coupled system (1.2) lead us to the following nonlocal boundary value problem:

Find (u, σ, c) such that

$$
\begin{aligned}
-\triangle u &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma_0, \\
\Big(\frac{1}{2} - \mathcal{K}\Big) u + \mathcal{V}\sigma - c &= 0 && \text{on } \Gamma, \\
-\mathcal{W} u + \Big(\frac{1}{2} - \mathcal{K}\Big)\sigma &= \sigma && \text{on } \Gamma, \\
\int_\Gamma \sigma(x)\, ds_x &= 0. &&
\end{aligned} \tag{1.4}
$$

This system is the so-called *two integral formulation*, which is equivalent to our original model problem (1.1), see for example [6, 12, 18].

1.2. **The variational formulation.** The smooth boundary Γ can be parameterized by a 1-periodic function $\gamma : [0,1] \to \Gamma$ such that for all $t \in [0,1]$ there holds

$$\alpha(t) := |\gamma'(t)| > 0. \tag{1.5}$$

In addition to the spaces $L^2(\Gamma)$ and $H^s(\Gamma)$ we introduce the (1-periodic) spaces $L^2(0,1)$ and $H^s(0,1)$, respectively. Precisely, let $L^2(0,1)$ be the space of all 1-periodic squared integrable functions. Its inner product is denoted by

$$\langle v, w \rangle = \int_0^1 v(t) w(t)\, dt. \tag{1.6}$$

Then, for any real number s the 1-periodic Sobolev space $H^s(0,1)$ is defined as the closure with respect to the norm

$$\|v\|^2_{H^s(0,1)} = \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |\hat{v}(n)|^2$$

of the space of all 1-periodic C^∞-functions. Here, $\hat{v}(n)$ indicate the Fourier coefficients

$$\hat{v}(n) = \int_0^1 e^{-2\pi i n s} v\, ds, \quad n \in \mathbb{Z}.$$

Then, clearly, since (1.5) is provided, there holds for $v \in H^s(\Gamma)$ the norm equivalence

$$\|v \circ \gamma\|_{H^s(0,1)} \sim \|v\|_{H^s(\Gamma)}. \tag{1.7}$$

Next, we introduce product spaces $M := H^{1/2}(0,1) \times H^{-1/2}(0,1)$ and $N := H^{-1/2}(0,1) \times \mathbb{R}$ equipped by the product norms

$$
\begin{aligned}
\|(v,w)\|^2_M &:= \|v\|^2_{H^{1/2}(0,1)} + \|w\|^2_{H^{-1/2}(0,1)} && \forall\ (v,w) \in M, \\
\|(v,w)\|^2_N &:= \|v\|^2_{H^{-1/2}(0,1)} + |w|^2 && \forall\ (v,w) \in N.
\end{aligned}
$$

Further, let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $B : M \times M \to \mathbb{R}$ and $b : N \times N \to \mathbb{R}$ be bilinear forms defined by

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \nabla v dx, \\ B((\mu,\lambda),(\omega,\delta)) &= \langle \omega, W\mu\rangle - \langle (\tfrac{1}{2} - K)\omega, \lambda\rangle + \langle (\tfrac{1}{2} - K)\mu, \delta\rangle + \langle V\lambda, \delta\rangle, \\ b((\lambda,c),(\delta,d)) &= \langle d, \lambda\rangle - \langle c, \delta\rangle, \end{aligned} \tag{1.8}$$

where the integral operators $V : H^{-1/2}(0,1) \to H^{1/2}(0,1)$, $K : H^{1/2}(0,1) \to H^{1/2}(0,1)$ and $W : H^{1/2}(0,1) \to H^{-1/2}(0,1)$ are given by

$$\begin{aligned} (Vu)(s) &:= \int_0^1 E(\gamma(s),\gamma(t))u(t)dt, \\ (Ku)(s) &:= \int_0^1 \frac{\partial}{\partial \nu_y} E(\gamma(s),\gamma(t))u(t)\alpha(t)dt, \\ (Wu)(s) &:= -\frac{\partial}{\partial \nu_x}\int_0^1 \frac{\partial}{\partial \nu_y} E(\gamma(s),\gamma(t))u(t)\alpha(s)\alpha(t)dt, \end{aligned}$$

with $E(\cdot,\cdot)$ from (1.3). We set $H := H^1(\Omega) \times H^{-1/2}(\Gamma) \times \mathbb{R}$ and define a bilinear form $A : H \times H \to \mathbb{R}$ by

$$\begin{aligned} A((u,\sigma,c),(v,\rho,d)) &:= a(u,v) + B((u\circ\gamma,(\sigma\circ\gamma)\alpha),(v\circ\gamma,(\rho\circ\gamma)\alpha)) \\ &\quad + b(((\sigma\circ\gamma)\alpha,c),((\rho\circ\gamma)\alpha,d)). \end{aligned} \tag{1.9}$$

Introducing the linear functional $F : H \to \mathbb{R}$,

$$F(v,\rho,d) = \int_\Omega f v dx,$$

one readily verifies that the variational formulation of (1.4) is given by:

Seek $(u,\sigma,c) \in H$ such that

$$A((u,\sigma,c),(v,\rho,d)) = F(v,\rho,d) \tag{1.10}$$

for all $(v,\rho,d) \in H$.

Existence and uniqueness of the solution of this variational formulation is shown by the following lemma.

Lemma 1.1. *Under the assumption that Γ has a conformal radius < 1, the variational formulation* (1.10) *has a unique solution $(u,\sigma,c) \in H$ for all $F \in H'$.*

Proof. 1. $A : H \times H \to \mathbb{R}$ *is continuous:* It is well known that the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is continuous

$$a(u,v) = \int_\Omega \nabla u \nabla v dx \lesssim \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Moreover, since the operators V, W and K are continuous [16], one has

$$B((\mu,\lambda),(\omega,\delta)) \lesssim \|(\mu,\lambda)\|_M \|(\omega,\delta)\|_M.$$

Next, involving Hölder's inequality, for $b : N \times N \to \mathbb{R}$ there holds

$$\begin{aligned}|b((\lambda,c),(\delta,d))| &= |\langle d,\lambda\rangle - \langle c,\delta\rangle| \\ &\leq |d|\ \|\lambda\|_{H^{-1/2}(0,1)} + |c|\ \|\delta\|_{H^{-1/2}(0,1)} \\ &\leq \|(\lambda,c)\|_N \|(\delta,d)\|_N.\end{aligned}$$

Setting

$$\mu := u\circ\gamma, \qquad \omega := v\circ\gamma, \qquad \lambda := (\sigma\circ\gamma)\alpha, \qquad \delta := (\rho\circ\gamma)\alpha, \tag{1.11}$$

observing (1.7) and by the trace theorem

$$\begin{aligned}\|\mu\|_{H^{1/2}(0,1)} &= \|u\circ\gamma\|_{H^{1/2}(0,1)} \lesssim \|u\|_{H^1(\Omega)}, \\ \|\omega\|_{H^{1/2}(0,1)} &= \|v\circ\gamma\|_{H^{1/2}(0,1)} \lesssim \|v\|_{H^1(\Omega)},\end{aligned}$$

one has proved the continuity of $A : H \times H \to \mathbb{R}$

$$A((u,\sigma,c),(v,\rho,d)) \lesssim \|(u,\sigma,c)\|_H \|(v,\rho,d)\|_H.$$

2. *$A : H\times H \to \mathbb{R}$ is H-coercive:* The bilinear form $a : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is elliptic

$$a(u,u) = \int_\Omega \nabla u \nabla u dx \gtrsim \|u\|^2_{H^1(\Omega)}.$$

Moreover, we find

$$\begin{aligned}B((\mu,\lambda),(\mu,\lambda)) &= \langle\mu, W\mu\rangle - \langle(\tfrac12 - K)\mu,\lambda\rangle + \langle(\tfrac12 - K)\mu,\lambda\rangle + \langle V\lambda,\lambda\rangle \\ &= \langle\mu, W\mu\rangle + \langle V\lambda,\lambda\rangle.\end{aligned}$$

If Γ has a conformal radius < 1, the single layer potential $V : H^{-1/2}(0,1) \to H^{1/2}(0,1)$ is a symmetric, positive definite operator [16], hence,

$$\langle V\lambda,\lambda\rangle \gtrsim \|\lambda\|^2_{H^{-1/2}(0,1)}.$$

Besides, the hypersingular operator $W : H^{1/2}(0,1) \to H^{-1/2}(0,1)$ is symmetric and positive semidefinite [16]

$$\|\langle W\mu,\mu\rangle\|^2_{H^{-1/2}(0,1)} \geq 0.$$

Combining both yields

$$B((\mu,\lambda),(\mu,\lambda)) \gtrsim \|\lambda\|^2_{H^{-1/2}(0,1)}.$$

For the bilinear form $b : N\times N\to\mathbb{R}$ we obtain

$$b((\lambda,c),(\lambda,c)) = -\langle c,\lambda\rangle + \langle c,\lambda\rangle = 0.$$

Summerized one gets with the settings (1.11)

$$\left|A((u,\sigma,c),(u,\sigma,c)) + k|c|^2\right| \gtrsim \|(u,\sigma,c)\|^2_H, \qquad k>0,$$

which signifies the H-coercivity. Note, for the latter step we again employed the norm equivalence (1.7).

3. $A : H \times H \to \mathbb{R}$ *is injective:* Let be $(u_1, \sigma_1, c_1), (u_2, \sigma_2, c_2) \in H$ arbitrarily but fixed and assume
$$A((u_1 - u_2, \sigma_1 - \sigma_2, c_1 - c_2), (v, \rho, d)) = 0 \quad \forall\ (v, \rho, d) \in H.$$
Since it holds
$$\begin{aligned} A((u_1 - u_2, \sigma_1 - \sigma_2, c_1 - c_2), (u_1 - u_2, \sigma_1 - \sigma_2, c_1 - c_2) \\ \gtrsim \|u_1 - u_2\|^2_{H^1(\Omega)} + \|\sigma_1 - \sigma_2\|^2_{H^{-1/2}(\Gamma)}, \end{aligned}$$
we find $u_1 = u_2$ and $\sigma_1 = \sigma_2$. Next, testing with $(1, 1, 1) \in H$ yields
$$A((u_1 - u_2, \sigma_1 - \sigma_2, c_1 - c_2), (1, 1, 1)) = c_1 - c_2,$$
i.e., $c1 = c2$. Therefore, A is injective.
4. According to items 1, 2 and 3 the bilinear form $A : H \times H \to \mathbb{R}$ is continuous, H-coercive and injective. Hence, one concludes existence and uniqueness of the solution by the Riesz-Schauder theory.

□

1.3. **The Galerkin Scheme.** In [19] a regular triangular mesh with curved triangles along Γ was proposed as triangulation of the annular domain Ω. Since we need for the application of the BPX preconditioner a sequence of nested spaces, we employ the initial triangulation as a parameterization of Ω. The refinement step is then obtained from subdividing the reference triangle which leads to a sequence of nested spaces on Ω, cf. [22]. More precisely, let $0 = t_0^{(0)} < t_1^{(0)} < \ldots < t_{N_0^\Gamma}^{(0)} = 1$, $N_0^\Gamma \in \mathbb{N}$, be a uniform partition of $[0, 1]$ with $t_i^{(0)} - t_{i-1}^{(0)} = h_0 := 1/N_0^\Gamma$, $i = 1, \ldots, N_0^\Gamma$. We denote by Ω_{h_0} the polygonal annular domain whose vertices on Γ are $\gamma(t_0^{(0)}), \gamma(t_1^{(0)}), \ldots, \gamma(t_{N_0^\Gamma-1}^{(0)})$. Let τ_0 be a regular triangulation of $\overline{\Omega}_{h_0}$ by triangles of diameter satisfying $\operatorname{diam} T_i \le h_0 \sup_{t\in[0,1]} \alpha(t)$ for all $T_i \in \tau_0$. If $\widehat{T} = \triangle\big((0, 0), (1, 0), (0, 1)\big)$ denotes the reference triangle there exist $|\tau_0|$ affine mappings F_i with $F_i(\widehat{T}) = T_i$. Now, each triangle $T_i \in \tau_0$ with two vertices on Γ is replaced by the corresponding curved triangle. Without loss of generality we may suppose that the vertices p_0, p_1, p_2 of a curved triangle T_i satisfy $p_1 = \gamma(t_{i'}^{(0)})$, $p_2 = \gamma(t_{i'+1}^{(0)})$, respectively. Then, a C^∞-mapping $\widetilde{F}_i$ with $\widetilde{F}_i(\widehat{T}) = T_i$ is given by
$$\widetilde{F}_i = F_i + G_i$$
with
$$G_i(\widehat{x}) = \frac{\widehat{x}_1}{1 - \widehat{x}_2}\big[\gamma((1 - \widehat{x}_2)t_{i'}^{(0)} + \widehat{x}_2 t_{i'+1}^{(0)}) - (1 - \widehat{x}_2)\gamma(t_{i'}^{(0)}) - \widehat{x}_2\gamma(t_{i'+1}^{(0)})\big],$$
see [26] for further details. We will indicate this initial triangulation with noncurved and curved triangles by $\widetilde{\tau}_0$. Subdividing in the usual way the reference triangle $\widehat{T}$ in $4, 16, 64, \ldots$

triangles yields a sequence of meshes

$$\widetilde{\tau}_0 \subset \widetilde{\tau}_1 \subset \widetilde{\tau}_2 \subset \dots$$

with step width $\operatorname{diam} T_i \leq h_j \sup_{t\in[0,1]} \alpha(t)$ for all $T_i \in \widetilde{\tau}_j$, where $h_j = 2^{-j} h_0$, cf. FIGURE 1.2.

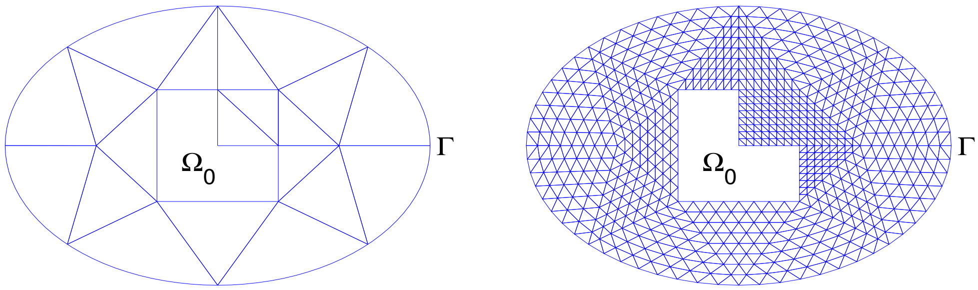

FIGURE 1.2. The initial triangulation $\widetilde{\tau}_0$ and the triangulation $\widetilde{\tau}_3$ obtained after three refinement steps.

The finite element spaces on these meshes are considered to be piecewise linear and continuous. The Ansatz and test functions on the given meshes are assumed to be Lagrange tent functions, which are equal 1 in one knot and equal 0 in all others knots. Supposing N_j^Ω degrees of freedom on the mesh $\widetilde{\tau}_j$ these functions will be called by

$$\phi_{j,k}^\Omega(x), \quad x \in \Omega, \quad k = \triangle_j^\Omega := \{0, 1, \dots, N_j^\Omega - 1\}.$$

Denoting the space on $\widetilde{\tau}_j$ by

$$V_j^\Omega = \operatorname{span}\{\phi_{j,k}^\Omega : k \in \triangle_j^\Omega\} \subset H^1(\Omega).$$

we obtain

$$V_0^\Omega \subset V_1^\Omega \subset V_2^\Omega \subset \dots .$$

The above introduced refinement strategy leads in the i-th step to an equidistant partition $0 = t_0^{(j)} < t_1^{(j)} < \dots < t_{N_j^\Gamma}^{(j)} = 1$ of $[0,1]$ with $N_j^\Gamma = 2^j N_0^\Gamma$. Therefore, if we discretize the space $H^{-1/2}(0,1)$ by piecewise constant Ansatz functions on the given partition

$$\phi_{j,k}^\Gamma(t) = \chi_{[t_k^{(j)}, t_{k+1}^{(j)}]}, \quad t \in [0,1], \quad k \in \triangle_j^\Gamma := \{0, 1, \dots, N_j^\Gamma - 1\}.$$

this again yields a nested sequence

$$V_0^\Gamma \subset V_1^\Gamma \subset V_2^\Gamma \subset \dots$$

of subspaces $V_j^\Gamma := \operatorname{span}\{\phi_{i,k}^\Gamma : k \in \triangle_j^\Gamma\} \subset H^{-1/2}(0,1)$.

In order to simplify our Galerkin scheme we do not approximate in (1.10) the unknown $\sigma \in H^{-1/2}(\Gamma)$ but $\lambda := (\sigma \circ \gamma)\alpha \in H^{-1/2}(0,1)$, see [19] for details. Then, from the definitions of the bilinear forms (1.8), setting

$$
\begin{aligned}
&\mathbf{A} = \Big[\int_\Omega \nabla\phi^\Omega_{j,k}\nabla\phi^\Omega_{j,k'}dx\Big]_{k,k'\in\triangle^\Omega_j}, &&\mathbf{W} = \Big[\langle W(\phi^\Omega_{j,k'}\circ\gamma),\phi^\Omega_{j,k}\circ\gamma\rangle\Big]_{k,k'\in\triangle^\Omega_j},\\
&\mathbf{B} = \Big[\langle\phi^\Omega_{j,k'}\circ\gamma,\phi^\Gamma_{j,k}\rangle\Big]_{k\in\triangle^\Gamma_j,k'\in\triangle^\Omega_j}, &&\mathbf{K} = \Big[\langle K(\phi^\Omega_{j,k'}\circ\gamma),\phi^\Gamma_{j,k}\rangle\Big]_{k\in\triangle^\Gamma_j,k'\in\triangle^\Omega_j},\\
&\mathbf{M} = \Big[\langle\phi^\Gamma_{j,k'},1\rangle\Big]_{k'\in\triangle^\Gamma_j}, &&\mathbf{V} = \Big[\langle V\phi^\Gamma_{k',l},\phi^\Gamma_{j,k}\rangle\Big]_{k,k'\in\triangle^\Gamma_j},
\end{aligned}
\tag{1.12}
$$

we obtain the following linear equation system

$$
\begin{bmatrix} \mathbf{A}+\mathbf{W} & \mathbf{K}^T-\mathbf{B}^T & \mathbf{0}\\ \mathbf{B}-\mathbf{K} & \mathbf{V} & -\mathbf{M}^T\\ \mathbf{0} & \mathbf{M} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\\ c\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{0}\\ 0\end{bmatrix}. \tag{1.13}
$$

2. Wavelet Approximation for the Coupling

By the constructed triangulation the boundary Γ is partitioned in the j-th step via an equidistant partition $0 = t^{(j)}_0 < t^{(j)}_1 < \ldots < t^{(j)}_{N^\Gamma_j} = 1$ of the interval $[0,1]$, where the step width h_j satisfies $h_j = 2^{-j}h_0$. On this partition the unknowns are discretized via (periodic) piecewise constant and linear functions, respectively. Instead of using these *single-scale bases* we want to apply wavelets with vanishing moments (more precisely: biorthogonal wavelet bases) yielding numerically sparse system matrices, cf. [25, 19]. For sake of simplicity in representation, we skip the given numbering and the suffices Γ and Ω, assuming an equidistant partition with $t^{(j)}_k = 2^{-j}k$.

The outline is as follows. We first introduce biorthogonal wavelet bases on $\mathbb{R}$, obtaining then the wavelet bases on the interval $[0,1]$ by periodization. According to [19] we give in the third subsection a briefly recall to the compression strategy of the matrices arising from the BEM. The last two subsections are dedicated to the realization of the wavelet Galerkin scheme for the coupling.

2.1. **Biorthogonal Multiresolution on $\mathbb{R}$.**

On $\mathbb{R}$ piecewise polynomial functions of degree $d-1$ can be defined as follows. Denoting by $[x_0,\ldots,x_d]f$ the d-th order *divided difference* at the points $x_0,\ldots,x_d\in\mathbb{R}$ (see e.g. [11]) the (centered) cardinal B-spline of order d is given by

$$
\phi^{(d)}(x) = d[0,1,\ldots,d]\Big(\cdot - x - \Big\lfloor\frac{d}{2}\Big\rfloor\Big)^{d-1}_+.
$$

where $x^l_+ := (\max\{0,x\})^l$ and $\lfloor x\rfloor$ ($\lceil x\rceil$) is the largest (smallest) integer less (greater) than or equal to x. This *scaling function* $\phi^{(d)}$ is normalized

$$
\|\phi^{(d)}\|_{L^1(\mathbb{R})} = 1,
$$

compactly supported

$$
\operatorname{diam}(\operatorname{supp}\phi^{(d)}) \sim 1
$$

and refinable

$$\phi^{(d)}(x) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} a_k \phi^{(d)}(2x-k)$$

with *mask coefficients*

$$a_k = \begin{cases} 2^{1-d}\binom{d}{k}, & -\lfloor \frac{d}{2} \rfloor \le k \le \lceil \frac{d}{2} \rceil, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.14}$$

Introducing for $j,k\in\mathbb{Z}$ translates and dilates of the scaling function $\phi_{j,k}^{(d)} := 2^{-j/2}\phi^{(d)}(2^j \cdot -k)$, the sets $\Phi_j^{(d)} := \{\phi_{j,k}^{(d)} : k \in \mathbb{Z}\}$ generate a sequence of spaces $V_j := \mathrm{clos}_{L^2}(\mathrm{span}\, \Phi_j^{(d)})$, which is nested

$$\ldots \subset V_j \subset V_{j+1} \subset \ldots$$

and dense in $L^2(\mathbb{R})$

$$\mathrm{clos}_{L^2}\Big(\bigcup_{j\in\mathbb{Z}} V_j\Big) = L^2(\mathbb{R}), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

Since the basis functions are piecewise polynomials, the spaces V_j are exact of order d, i.e., we find for a given $0 \le r < d$ some $x_k = x_k(j,r) \in \mathbb{R}$ with

$$x^r = \sum_{k\in\mathbb{Z}} x_k \phi_{j,k}^{(d)}.$$

Furthermore, $\Phi_j^{(d)}$ forms a stable basis in V_j

$$\big\|\Phi_j^{(d)}\mathbf{c}\big\|_{L^2(\mathbb{R})} \sim \|\mathbf{c}\|_{l^2(\mathbb{Z})} \qquad \forall\, \mathbf{c} \in l^2(\mathbb{Z}).$$

Due to [5] it exists for every integer $\tilde{d} \ge d$ with $\tilde{d}+d$ even a *dual scaling function* $\widetilde{\phi}^{(d,\tilde{d})} \in L^2(\mathbb{R})$ which is biorthogonal to the first scaling function

$$\big(\phi^{(d)}, \widetilde{\phi}^{(d,\tilde{d})}(\cdot - k)\big)_{L^2(\mathbb{R})} = \delta_{0,k}, \quad k \in \mathbb{Z}.$$

Moreover, similarly to the primal scaling function, this function is normalized, compactly supported and refinable

$$\widetilde{\phi}^{(d,\tilde{d})}(x) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \tilde{a}_k \widetilde{\phi}^{(d,\tilde{d})}(2x-k). \tag{2.15}$$

The mask coefficients in (2.15) can be defined by the *z-notation*, i. e., in the undermentioned sequences $\tilde{\mathbf{a}}(z) = \sum_k \tilde{a}_k z_k$ the coefficient $\tilde{a}_k$ of z^k corresponds to the mask coefficient $\tilde{a}_k$. It holds $\tilde{\mathbf{a}}(z) = p(z)q(z)$ with

$$p(z) = 2^{1-\tilde{d}} \sum_{k=0}^{\tilde{d}} \binom{\tilde{d}}{k} z^{k-\lfloor \frac{\tilde{d}}{2} \rfloor},$$

$$q(z) = \sum_{k=0}^{\frac{d+\tilde{d}}{2}-1} 2^{-k} \binom{\frac{d+\tilde{d}}{2}-1+k}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-z)^{l-k},$$

cf. [5]. Exactly like the primal side, the translates and dilates of the dual scaling function $\widetilde{\phi}_{j,k}^{(d,\tilde{d})} := 2^{-j/2}\widetilde{\phi}^{(d,\tilde{d})}(2^j\cdot -k)$ $(j,k\in\mathbb{Z})$ generate collections of stable bases $\widetilde{\Phi}_j^{(d,\tilde{d})} := \{\widetilde{\phi}_{j,k}^{(d,\tilde{d})} : k\in\mathbb{Z}\}$ in spaces $\widetilde{V}_j := \operatorname{clos}_{L^2}(\operatorname{span}\Phi_j^{(d,\tilde{d})})$ which are also nested, dense in $L^2(\mathbb{R})$ and exact of order $\tilde{d}$.

According to [5] a dual pair of *wavelets* $\psi^{(d,\tilde{d})}(x)$, $\widetilde{\psi}^{(d,\tilde{d})}(x)\in L^2(\mathbb{R})$ satisfying

$$\big(\psi^{(d,\tilde{d})},\widetilde{\psi}^{(d,\tilde{d})}(\cdot-k)\big)_{L^2(\mathbb{R})} = \delta_{0,k}, \qquad k\in\mathbb{Z},$$

is defined by

$$\psi^{(d,\tilde{d})}(x) := \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} b_k\phi^{(d)}(2x-k), \qquad \widetilde{\psi}^{(d,\tilde{d})}(x) := \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \tilde{b}_k\widetilde{\phi}^{(d,\tilde{d})}(2x-k), \tag{2.16}$$

where the masks $\mathbf{b}$, $\tilde{\mathbf{b}}$ are given by

$$b_k = (-1)^k\tilde{a}_{1-k}, \qquad \tilde{b}_k = (-1)^k a_{1-k}, \qquad k\in\mathbb{Z} \tag{2.17}$$

with $\mathbf{a}$ from (2.14) and $\tilde{\mathbf{a}}$ from (2.15). As a consequence of finite masks and compact supports of the scaling functions both wavelets are compactly supported

$$\operatorname{diam}(\operatorname{supp}\psi^{(d,\tilde{d})}) \sim \operatorname{diam}(\operatorname{supp}\widetilde{\psi}^{(d,\tilde{d})}) \sim 1.$$

Setting analogously to the scaling functions

$$\psi_{j,k}^{(d,\tilde{d})} := 2^{j/2}\psi^{(d,\tilde{d})}(2^j\cdot -k), \qquad \widetilde{\psi}_{j,k}^{(d,\tilde{d})} := 2^{j/2}\widetilde{\psi}^{(d,\tilde{d})}(2^j\cdot -k)$$

the sets

$$\Psi_j^{(d,\tilde{d})} := \{\psi_{j,k}^{(d,\tilde{d})} : k\in\mathbb{Z}\}, \qquad \widetilde{\Psi}_j^{(d,\tilde{d})} := \{\widetilde{\psi}_{j,k}^{(d,\tilde{d})} : k\in\mathbb{Z}\},$$

generate complement spaces $W_j := \operatorname{clos}_{L^2}(\operatorname{span}\Psi_j^{(d,\tilde{d})})$, $\widetilde{W}_j = \operatorname{clos}_{L^2}(\operatorname{span}\widetilde{\Psi}_j^{(d,\tilde{d})})$ with

$$V_j\oplus W_j = V_{j+1}, \qquad \widetilde{V}_j\oplus\widetilde{W}_j = \widetilde{V}_{j+1},$$

where $\oplus$ denotes the direct sum. Thus, recursively one obtains

$$\operatorname{clos}_{L^2}\Big(\bigoplus_{j\in\mathbb{Z}} W_j\Big) = \operatorname{clos}_{L^2}\Big(\bigoplus_{j\in\mathbb{Z}}\widetilde{W}_j\Big) = L^2(\mathbb{R}).$$

Since biorthogonality implies $W_j\perp\widetilde{V}_j$ the primal wavelets have vanishing moments of order $\tilde{d}$, i. e.,

$$\big((\cdot)^\alpha,\psi_{j,k}^{(d,\tilde{d})}\big)_{L^2(\mathbb{R})} = 0, \quad 0\le\alpha<\tilde{d}.$$

Moreover, the collections

$$\Psi^{(d,\tilde{d})} := \bigcup_{j\in\mathbb{Z}}\Psi_j^{(d,\tilde{d})}, \qquad \widetilde{\Psi}^{(d,\tilde{d})} := \bigcup_{j\in\mathbb{Z}}\widetilde{\Psi}_j^{(d,\tilde{d})},$$

form Riesz bases in $L^2(\mathbb{R})$

$$\|\mathbf{c}\|_{l^2(\mathbb{Z}\times\mathbb{Z})}^2 \sim \big\|\Psi^{(d,\tilde{d})}\mathbf{c}\big\|_{L^2(\mathbb{R})}^2 \sim \big\|\widetilde{\Psi}^{(d,\tilde{d})}\mathbf{c}\big\|_{L^2(\mathbb{R})}^2 \qquad \forall\,\mathbf{c}\in l^2(\mathbb{Z}\times\mathbb{Z}). \tag{2.18}$$

2.2. **Periodization.** The above setting is clearly not suitable for the treatment of equations which are defined on bounded domains. In the sequel we define a periodic version of a multiscale resolution. It essentially retains all the structural and computational advantages of the stationary and shift-invariant case considered in the previous subsection.

To this end, the simple trick is to replace the meaning of $u_{j,k} := 2^{\frac{j}{2}} u(2^j \cdot -k)$, $k \in \mathbb{Z}$, for compactly supported $u \in L^2(\mathbb{R})$ by its periodized counterpart

$$u_{j,k} := 2^{\frac{j}{2}} \sum_{n\in\mathbb{Z}} u(2^j(\cdot + n) - k).$$

In this way, given any dual pair $\phi^{(d)}$ and $\widetilde{\phi}^{(d,\tilde{d})}$ on $\mathbb{R}$ of compactly supported scaling functions, and setting $\triangle_j := \mathbb{Z} \setminus 2^j\mathbb{Z}$, the corresponding sets

$$\Phi_j^{(d)} := \{\phi_{j,k} : k \in \triangle_j\}, \qquad \Psi_j^{(d,\tilde{d})} := \{\psi_{j,k}^{(d,\tilde{d})} : k \in \triangle_j\}, \qquad j \geq j_0,$$

and likewise $\widetilde{\Phi}_j^{(d,\tilde{d})}$ and $\widetilde{\Psi}_j^{(d,\tilde{d})}$, have finite cardinality 2^j and consist of functions which are 1-periodic. Note that these definition preserves biorthogonality relations. One easily checks that the scaling functions are biorthogonal

$$\langle \Phi_j^{(d)}, \widetilde{\Phi}_j^{(d,\tilde{d})} \rangle = \mathbf{I}.$$

Moreover, the wavelet bases

$$\Psi^{(d,\tilde{d})} := \Phi_{j_0}^{(d)} \bigcup_{j\geq j_0} \Psi_j^{(d,\tilde{d})}, \qquad \widetilde{\Psi}^{(d,\tilde{d})} := \widetilde{\Phi}_{j_0}^{(d,\tilde{d})} \bigcup_{j\geq j_0} \widetilde{\Psi}_j^{(d,\tilde{d})}$$

are biorthogonal, i.e.,

$$\langle \Psi^{(d,\tilde{d})}, \widetilde{\Psi}^{(d,\tilde{d})} \rangle = \mathbf{I},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(0,1)$ defined by (1.6). For sake of simplicity in representation, we will indicate in the sequel the scaling functions on the coarsest level of the wavelet bases by $\Psi_{j_0-1}^{(d,\tilde{d})} := \Phi_{j_0}^{(d)}$ and $\widetilde{\Psi}_{j_0-1}^{(d,\tilde{d})} := \widetilde{\Phi}_{j_0}^{(d,\tilde{d})}$, where, clearly, the coarsest level j_0 has to be chosen sufficiently large. The spaces $V_j := \operatorname{clos}_{L^2}(\operatorname{span} \Phi_j^{(d)})$ and $\widetilde{V}_j := \operatorname{clos}_{L^2}(\operatorname{span} \widetilde{\Phi}_j^{(d,\tilde{d})})$ form two multiscale decompositions of $L^2(\mathbb{R}\setminus\mathbb{Z}) = L^2(0,1)$. One readily verifies

$$\phi_{j,k}^{(d)} = \sum_{k'\in\triangle_{j+1}} \Big(\sum_{n\in\mathbb{Z}} a_{k'-2k+2^{j+1}n}\Big) \phi_{j+1,k'}^{(d)}, \qquad \widetilde{\phi}_{j,k}^{(d,\tilde{d})} = \sum_{k'\in\triangle_{j+1}} \Big(\sum_{n\in\mathbb{Z}} \tilde{a}_{k'-2k+2^{j+1}n}\Big) \widetilde{\phi}_{j+1,k'}^{(d,\tilde{d})}$$

with $\mathbf{a}, \tilde{\mathbf{a}}$ from (2.14), (2.15), respectively, and

$$\psi_{j,k}^{(d,\tilde{d})} = \sum_{k'\in\triangle_{j+1}} \Big(\sum_{n\in\mathbb{Z}} a_{k'-2k+2^{j+1}n}\Big) \psi_{j+1,k'}^{(d,\tilde{d})}, \qquad \widetilde{\psi}_{j,k}^{(d,\tilde{d})} = \sum_{k'\in\triangle_{j+1}} \Big(\sum_{n\in\mathbb{Z}} \tilde{a}_{k'-2k+2^{j+1}n}\Big) \widetilde{\psi}_{j+1,k'}^{(d,\tilde{d})}$$

with $\mathbf{b}, \tilde{\mathbf{b}}$ from (2.17). That is, the mask sequences are obtained by 2^{j+1}-periodization.

2.3. **Matrix compression.** Discretizing the boundary integral operators V, W and K by biorthogonal wavelet bases one obtains quasi-sparse matrices. These can be compressed without loss of accuracy, see [19] for details. All matrix entries, for which the distances of the supports (on the given boundary Γ) of the corresponding Ansatz- and test functions are bigger than a level depending cut-off parameter $\mathcal{B}_{j,j'}$, are set to zero. Hence, one only has to calculate those values, for which these distances are smaller than the cut-off parameter. More precisely, abbreviating

$$\Theta_{j,k}^{(d,\tilde{d})} := \operatorname{supp}_\Gamma \psi_{j,k}^{(d,\tilde{d})},$$

the compressed system matrices are given by

$$[\mathbf{V}]_{(j,k),(j',k')} = \begin{cases} 0, \text{ if } \operatorname{dist}_\Gamma\left(\Theta_{j,k}^{(1,\tilde{d}_1)}, \Theta_{j',k'}^{(1,\tilde{d}_1)}\right) > \mathcal{B}_{j,j'}^V, \\ \langle V\psi_{j',k'}^{(1,\tilde{d}_1)}, \psi_{j,k}^{(1,\tilde{d}_1)}\rangle, \text{ otherwise}, \end{cases}$$

$$[\mathbf{W}]_{(j,k),(j',k')} = \begin{cases} 0, \text{ if } \operatorname{dist}_\Gamma\left(\Theta_{j,k}^{(2,\tilde{d}_2)}, \Theta_{j',k'}^{(2,\tilde{d}_2)}\right) > \mathcal{B}_{j,j'}^W, \\ \langle W(\psi_{j',k'}^{(2,\tilde{d}_2)} \circ \gamma), \psi_{j,k}^{(2,\tilde{d}_2)} \circ \gamma\rangle, \text{ otherwise}, \end{cases}$$

$$[\mathbf{K}]_{(j,k),(j',k')} = \begin{cases} 0, \text{ if } \operatorname{dist}_\Gamma\left(\Theta_{j,k}^{(2,\tilde{d}_3+1)}, \Theta_{j',k'}^{(1,\tilde{d}_3)}\right) > \mathcal{B}_{j,j'}^K, \\ \langle K(\psi_{j',k'}^{(2,\tilde{d}_3+1)} \circ \gamma), \psi_{j,k}^{(1,\tilde{d}_3)}\rangle, \text{ otherwise}, \end{cases}$$

where the cut-off parameters $B_{j,j'}^V, B_{j,j'}^W, B_{j,j'}^K$ are set as follows

$$\begin{aligned} B_{j,j'}^V &= a_1 \max\left\{2^{-j}, 2^{-j'}, 2^{\frac{J(2d_1'+1)-(j+j')(d_1'+\tilde{d}_1)}{2\tilde{d}_1-1}}\right\}, \\ B_{j,j'}^W &= a_2 \max\left\{2^{-j}, 2^{-j'}, 2^{\frac{J(2d_2'-1)-(j+j')(d_2'+\tilde{d}_2)}{2\tilde{d}_2+1}}\right\}, \\ B_{j,j'}^K &= a_3 \max\left\{2^{-j}, 2^{-j'}, 2^{\frac{J(2d_3'+1)-(j+j')(d_3'+\tilde{d}_3)-2j'}{2\tilde{d}_3+1}}\right\}, \end{aligned} \tag{2.19}$$

with

$$a_1, a_2, a_3 > 1, \quad 1 < d_1' < \tilde{d}_1 - 1, \quad 2 < d_2' < \tilde{d}_2 + 1, \quad 1 < d_3' < \tilde{d}_3. \tag{2.20}$$

It has been shown in [19] that this compression strategy reduces the number of nonzero entries to $\mathcal{O}(N_J \log N_J)$. In this expression, $N_J = |\triangle_J|$ denotes the number of unknowns.

To satisfy (2.20) we have to choose wavelets with enough vanishing moments $\tilde{d}_1, \tilde{d}_2$ and $\tilde{d}_3$. Since on the other hand the supports of the wavelets increase proportionally with the number of vanishing moments (which reduces in our experiences the compression rates) one has to apply wavelets with minimal number of vanishing moments, respectively supports. That is

- $\Psi^{(1,3)}$ for the discretization of the single layer operator V,
- $\Psi^{(2,2)}$ for the discretization of the hypersingular operator$W$,
- $\Psi^{(1,3)}$ and $\Psi^{(2,4)}$ for the discretization of the double layer operator K.

An important result is that in this setup there holds $\tilde{d}_2 := \tilde{d}_1 + 1$. Hence, choosing $a_2 := a_1 > 1$ and $d_1' \in \mathbb{R}$ such that $1 < d_1' < \tilde{d}_1 - 1$ we find that for $d_2' := d_1' + 1$ also the estimate $2 < d_2' < \tilde{d}_2 + 1$ is valid, i.e., we are allowed to set $B^V_{j,j'} = B^W_{j,j'}$. This will be exploited in the next but one subsection.

2.4. **Changing bases.** In the single-scale basis the whole system is given by

$$\begin{bmatrix} \mathbf{A} + \mathbf{W}_\phi & \mathbf{K}_\phi^T - \mathbf{B}_\phi^T & \mathbf{0} \\ \mathbf{B}_\phi - \mathbf{K}_\phi & \mathbf{V}_\phi & -\mathbf{M}_\phi^T \\ \mathbf{0} & \mathbf{M}_\phi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda}_\phi \\ c \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ 0 \end{bmatrix}, \tag{2.21}$$

where the suffix ϕ indicates the single-scale matrices. Since we want to compute the BEM matrices with respect to wavelet bases, we have to rewrite the system. For this let $\mathbf{T}_{d,\tilde{d}}$ denote the fast wavelet transform, i.e., the matrix which gives the change of bases

$$\sum_{k \in \triangle_J} a_k \phi_{k,J}^{(d)} = \sum_{j_0 - 1 \le j < J} \sum_{k \in \triangle_j} b_{j,k} \psi_{j,k}^{(d,\tilde{d})}$$

where

$$[b_{j,k}]^T_{j_0-1 \le j < J, k \in \triangle_j} = [a_k]^T_{k \in \triangle_J} \mathbf{T}_{d,\tilde{d}}.$$

Then, the system matrix in (2.21) with respect to the multiscale bases is given by

$$\begin{bmatrix} \mathbf{A} + \mathbf{T}_{2,2}\mathbf{W}_\psi \mathbf{T}_{2,2}^T & \mathbf{T}_{2,4}\mathbf{K}_\psi^T \mathbf{T}_{1,3}^T - \mathbf{B}_\phi^T & \mathbf{0} \\ \mathbf{B}_\phi - \mathbf{T}_{1,3}\mathbf{K}_\psi \mathbf{T}_{2,4}^T & \mathbf{T}_{1,3}\mathbf{V}_\psi \mathbf{T}_{1,3}^T & -\mathbf{M}_\phi^T \\ \mathbf{0} & \mathbf{M}_\phi & 0 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{1,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} + \mathbf{T}_{2,2}\mathbf{W}_\psi \mathbf{T}_{2,2}^T & \mathbf{T}_{2,4}\mathbf{K}_\psi^T - \mathbf{B}_\phi^T \mathbf{T}_{1,3}^{-T} & \mathbf{0} \\ \mathbf{T}_{1,3}^{-1}\mathbf{B}_\phi - \mathbf{K}_\psi \mathbf{T}_{2,4}^T & \mathbf{V}_\psi & -\mathbf{M}_\psi^T \\ \mathbf{0} & \mathbf{M}_\psi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{1,3}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}.$$

Hence, to apply the wavelet preconditioner for $\mathbf{V}_\psi$, the linear system (2.21) is transformed to

$$\begin{bmatrix} \mathbf{A} + \mathbf{T}_{2,2}\mathbf{W}_\psi \mathbf{T}_{2,2}^T & \mathbf{T}_{2,4}\mathbf{K}_\psi^T - \mathbf{B}_\phi^T \mathbf{T}_{1,3}^{-T} & \mathbf{0} \\ \mathbf{T}_{1,3}^{-1}\mathbf{B}_\phi - \mathbf{K}_\psi \mathbf{T}_{2,4}^T & \mathbf{V}_\psi & -\mathbf{M}_\psi^T \\ \mathbf{0} & \mathbf{M}_\psi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda}_\psi \\ c \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ 0 \end{bmatrix}$$

where the unknown λ is now given in the multiscale bases.

2.5. **The hypersingular operator.** We want to discuss the properties of the matrix $\mathbf{W}$ corresponding to the hypersingular operator. For sake of simplicity, we skip the suffices Γ and Ω and assume corresponding index sets $\triangle_j$ for piecewise constant and linear functions on the interval $[0, 1]$.

Since for piecewise linear functions there holds (see [19])

$$\langle W\phi_{J,k}^{(2)}, \phi_{J,k'}^{(2)} \rangle = \langle V\big(\phi_{J,k}^{(2)}\big)', \big(\phi_{J,k'}^{(2)}\big)' \rangle, \qquad \forall\, k, k' \in \triangle_J,$$

we obtain in the single-scale basis the identity

$$\mathbf{W}_\phi = \frac{1}{h_J^2}\mathbf{H}_J\mathbf{V}_\phi\mathbf{H}_J^T, \tag{2.22}$$

where $\mathbf{H}_j$, $j_0 \le j \le J$, is the matrix given by

$$\mathbf{H}_j := \begin{bmatrix} -1 & & & & 1 \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{N_j \times N_j}.$$

Since the application of $\mathbf{H}_J$ on a vector $\mathbf{x} \in \mathbb{R}^{N_J}$ requires only $\mathcal{O}(2^J)$ operations, we are not computing $\mathbf{W}_\phi$ but using the right hand side of (2.22) in the iterative solver. Consequently, in the single-scale scheme we only have to compute two BEM matrices, namely $\mathbf{V}_\phi$ and $\mathbf{K}_\phi$.

For the compressed wavelet matrices we cannot use relation (2.22) without further examinations on the compression errors. Hence, while using compression it seems that we have to compute the discretization of three integral operators instead of two. But, as we will see, this is not necessary since for the multiscale basis there exists another approach.

We assume piecewise constant wavelets $\Psi^{(1,\tilde{d})}$ for the discretization of single layer operator V and piecewise linear wavelets $\Psi^{(2,\tilde{d}-1)}$ for the discretization of hypersingular operator W. Of course, $\tilde{d}$ has to be ≥ 3. As remarked in subsection 2.3 under this assumption one can choose identical cut-off parameters for the compression yielding identical sets of nonzero entries in the corresponding matrices. We make use of (cf. [5])

$$\big(\psi^{(d,\tilde{d})}\big)' = \psi^{(d-1,\tilde{d}+1)}.$$

Consequently we obtain

$$\mathbf{W}_\psi = \begin{bmatrix} \mathbf{H}_{j_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{D}\mathbf{V}_\psi\mathbf{D} \begin{bmatrix} \mathbf{H}_{j_0}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{2.23}$$

where $\mathbf{D}$ is a diagonal matrix given by

$$[\mathbf{D}]_{(j,k),(j',k')} = \frac{\delta_{j,j'}\delta_{k,k'}}{h_{j+1}}, \qquad j_0 - 1 \le j, j' < J, \quad k \in \triangle_j, \quad k' \in \triangle_{j'}.$$

Hence, employing (2.23) in the iterative solver, we also have to compute only the BEM matrices $\mathbf{V}_\psi$ and $\mathbf{K}_\psi$.

3. Preconditioning

Since the system matrix of the Galerkin scheme (1.13) includes matrices belonging to operators of positive and negative order, preconditioning techniques are very important for an efficient iterative solution of the linear system. Based on the BPX preconditioner [3] and the wavelet preconditioner [7, 8, 10, 25] we discuss several precondition methods. Note, the discretization level J is assumed to be sufficiently large $J \geq J_0$ and, moreover, all given estimates have to be understood uniformly in J.

3.1. **The BPX preconditioner.** We have constructed on Ω a sequence of regular and quasi uniform triangulations $\widetilde{\tau}_j$ and piecewise linear and continuous spaces V_j^Ω, such that

$$V_0^\Omega \subset V_1^\Omega \subset \dots, \qquad \operatorname{clos}_{H^1}\Big(\bigcup_{j\geq 0} V_j^\Omega\Big) = H^1(\Omega), \qquad \bigcap_{j\geq 0} V_j^\Omega = V_0^\Omega.$$

The basis $\Phi_j^\Omega = \{\phi_{j,k}^\Omega : k \in \triangle_j^\Omega\}$ of V_j^Ω was chosen as the set of Lagrange tent functions, which we may consider to be L^2-normed. This implies

$$\|\Phi_j^\Omega \mathbf{u}\|_{L^2(\Omega)} \sim \|\mathbf{u}\|_{l^2(\triangle_J^\Omega)} \qquad \forall\, \mathbf{u} \in l^2(\triangle_J^\Omega). \tag{3.24}$$

For the L^2-normed Lagrange tent functions the (simplified) BPX preconditioner C_J can be defined by

$$C_J u = \sum_{j=0}^{J} 2^{-2j} \sum_{k\in\triangle_j^\Omega} (u, \phi_{j,k}^\Omega)\phi_{j,k}^\Omega, \qquad u \in H^1(\Omega), \tag{3.25}$$

see [7]. C_J defines a symmetric and positive definite operator, hence the square root $C_J^{1/2}$ is well defined. The application of $C_J^{1/2}$ to a given function $u \in V_J^\Omega$ yields a norm equivalence

$$\|C_J^{1/2} u\|_{H^s(\Omega)} \sim \|u\|_{H^{s-1}(\Omega)}, \qquad s \in (-\tfrac{1}{2}, \tfrac{3}{2}). \tag{3.26}$$

We denote by $A : H^1(\Omega) \to H^{-1}(\Omega)$ the operator corresponding to the FEM bilinear form $a(\cdot,\cdot)$ from (1.8)

$$(Au, v) = a(u, v) \qquad \forall\, u, v \in H^1(\Omega).$$

Based on the bijectivity of A, the norm equivalence (3.26) implies for all $u \in V_J^\Omega$

$$\|u\|_{L^2(\Omega)} \sim \|C_J^{1/2} u\|_{H^1(\Omega)} \sim \|AP_J C_J^{1/2} u\|_{H^{-1}(\Omega)} \sim \|C_J^{1/2} P_J A P_J C_J^{1/2} u\|_{L^2(\Omega)}, \tag{3.27}$$

where P_J denotes the orthogonal projection onto V_J^Ω.

Next, we consider the discrete system. We denote by $\mathbf{I}_{j+1}^j$ the restriction from level $j+1$ onto level j and by $\mathbf{I}_j^{j+1}$ the prolongation from level j onto level $j+1$. Restriction and prolongation are described by the well known stencils

$$\mathbf{I}_j^{j-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad \mathbf{I}_{j-1}^j = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The product $\mathbf{I}_J^j := \mathbf{I}_{j+1}^j \mathbf{I}_{j+2}^{j+1} \cdots \mathbf{I}_J^{J-1}$ signifies the restriction from level J onto level j and $\mathbf{I}_j^J = \mathbf{I}_{J-1}^J \mathbf{I}_{J-2}^{J-1} \cdots \mathbf{I}_j^{j+1}$ the prolongation from level j onto level J. Then, the discrete BPX preconditioner (cf. [3]) is given by

$$\mathbf{C} = \sum_{j=0}^{J} 2^{-2j} \mathbf{I}_j^J \mathbf{I}_J^j. \tag{3.28}$$

We denote by $\mathbf{A} = a(\Phi_J^\Omega, \Phi_J^\Omega)$ the Galerkin matrix corresponding to A. Of course, there holds for all $u = \Phi_J^\Omega \mathbf{u} \in V_J^\Omega$

$$\Phi_J^\Omega \mathbf{A}\mathbf{u} = P_J A P_J u,$$

hence, (3.27) can be rewritten as

$$\left\|\Phi_J^\Omega \mathbf{u}\right\|_{L^2(\Omega)} \sim \left\|\Phi_J^\Omega \mathbf{C}^{1/2}\mathbf{u}\right\|_{H^1(\Omega)} \sim \left\|\Phi_J^\Omega \mathbf{A}\mathbf{C}^{1/2}\mathbf{u}\right\|_{H^{-1}(\Omega)} \sim \left\|\Phi_J^\Omega \mathbf{C}^{1/2}\mathbf{A}\mathbf{C}^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}.$$

By (3.24) this implies

$$\|\mathbf{u}\|_{l^2(\triangle_J^\Omega)} \sim \left\|\mathbf{C}^{1/2}\mathbf{A}\mathbf{C}^{1/2}\mathbf{u}\right\|_{l^2(\triangle_J^\Omega)},$$

or, in other words, the well conditioning of the matrix $\mathbf{C}^{1/2}\mathbf{A}\mathbf{C}^{1/2}$

$$\operatorname{cond}_{l^2}(\mathbf{C}^{1/2}\mathbf{A}\mathbf{C}^{1/2}) \sim 1.$$

3.2. **Wavelet preconditioning.** Let us define two numbers $\underline{s}$ and $\overline{s}$ which signify the regularity of the primal and dual wavelet basis, respectively,

$$\overline{s} = \sup\{s \in \mathbb{R} : \psi^{(d,\tilde{d})} \in H^s(0,1)\}, \qquad \underline{s} = \sup\{s \in \mathbb{R} : \widetilde{\psi}^{(d,\tilde{d})} \in H^s(0,1)\}.$$

According, for example, to [7, 8, 10, 25] there holds uniformly the norm equivalence

$$\|\lambda\|_{H^{s+t}(0,1)} \sim \left\|D_J^s \lambda\right\|_{H^t(0,1)}, \qquad \lambda \in V_J^\Gamma, \qquad s+t, t \in (-\underline{s}, \overline{s}), \tag{3.29}$$

where D_J^s denotes the operator given by

$$D_J^s \lambda = \sum_{j=j_0-1}^{J-1} \sum_{k \in \triangle_j^\Gamma} 2^{sj} \langle \lambda, \widetilde{\psi}_{j,k}^{(d,\tilde{d})} \rangle \psi_{j,k}^{(d,\tilde{d})}. \tag{3.30}$$

Let Q_J^Γ be the projection onto V_J^Γ defined by

$$Q_J \lambda = \sum_{j=j_0-1}^{J-1} \sum_{k \in \triangle_j^\Gamma} \langle \lambda, \widetilde{\psi}_{j,k}^{(d,\tilde{d})} \rangle \psi_{j,k}^{(d,\tilde{d})}.$$

Then, as one easily checks, the adjoint $Q_J^\star$ is given by

$$Q_J^\star \lambda = \sum_{j=j_0}^{J-1} \sum_{k \in \triangle_j^\Gamma} \langle \lambda, \psi_{j,k}^{(d,\tilde{d})} \rangle \widetilde{\psi}_{j,k}^{(d,\tilde{d})}.$$

Since the single layer operator $V : H^{-1/2}(0,1) \to H^{1/2}(0,1)$ is an operator of order -1, we utilize the norm equivalences (3.29) to find that for $\underline{s} \geq 1/2$ there holds for all $\lambda \in V_J^\Gamma$

(3.31)
$$\|\lambda\|_{L^2(\Gamma)} \sim \|D_J^{1/2}\lambda\|_{H^{-1/2}(0,1)} \sim \|VQ_JD_J^{1/2}\lambda\|_{H^{1/2}(0,1)} \sim \|D_J^{1/2}Q_J^\star VQ_JD_J^{1/2}\lambda\|_{L^2(\Gamma)}.$$

Next, we consider the discrete operators. For this, let be $\Psi_J^\Gamma = \bigcup_{j=j_0-1}^{J-1} \Psi_j^{(1,\tilde{d})}$ the wavelet basis in V_J^Γ. Moreover, we introduce the Galerkin matrix

$$\mathbf{V} = \langle V\Psi_J^\Gamma, \Psi_J^\Gamma\rangle$$

and define the diagonal matrix $\mathbf{D}^s$ by

$$\big[\mathbf{D}^s\big]_{(j,k),(j',k')} = 2^{sj}\delta_{j,j'}\delta_{k,k'}, \tag{3.32}$$

which is obviously the discrete analog of (3.30). Then, since

$$\Psi_J^\Gamma \mathbf{V}\boldsymbol{\lambda} = Q_J^\star VQ_J\lambda \qquad \forall\ \lambda = \Psi_J^\Gamma\boldsymbol{\lambda} \in V_J^\Gamma,$$

one has similarly to subsection 3.1 the relation

$$\|\Psi_J^\Gamma\boldsymbol{\lambda}\|_{L^2(0,1)} \sim \|\Psi_J^\Gamma\mathbf{D}^{1/2}\boldsymbol{\lambda}\|_{H^{-1/2}(0,1)} \sim \|\Psi_J^\Gamma\mathbf{V}\mathbf{D}^{1/2}\boldsymbol{\lambda}\|_{H^{1/2}(0,1)} \sim \|\Psi_J^\Gamma\mathbf{D}^{1/2}\mathbf{V}\mathbf{D}^{1/2}\boldsymbol{\lambda}\|_{L^2(0,1)}.$$

By the Riesz property (2.18) there follows

$$\|\boldsymbol{\lambda}\|_{l^2(\triangle_J^\Gamma)} \sim \|\mathbf{D}^{1/2}\mathbf{V}\mathbf{D}^{1/2}\boldsymbol{\lambda}\|_{l^2(\triangle_J^\Gamma)},$$

that is the well conditioning of the matrix $\mathbf{D}^{1/2}\mathbf{V}\mathbf{D}^{1/2}$.

3.3. **The preconditionend system.** It is shown in subsection 1.3 that the linear system of the introduced Galerkin scheme for the coupling is given by

$$\mathbf{S}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\\ c\end{bmatrix} \equiv \begin{bmatrix}\mathbf{A}+\mathbf{W} & \mathbf{K}^T-\mathbf{B}^T & \mathbf{0}\\ \mathbf{B}-\mathbf{K} & \mathbf{V} & -\mathbf{M}^T\\ \mathbf{0} & \mathbf{M} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\\ c\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{0}\\ 0\end{bmatrix}. \tag{3.33}$$

The system matrix $\mathbf{S}$ is nonsymmetric, but can be (orthogonal) transformed into a symmetric, but indefinite, matrix $\tilde{\mathbf{S}}$ by multiplying the second line with -1. Clearly, the obtaind linear system

$$\tilde{\mathbf{S}}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\\ c\end{bmatrix} \equiv \begin{bmatrix}\mathbf{A}+\mathbf{W} & \mathbf{K}^T-\mathbf{B}^T & \mathbf{0}\\ \mathbf{K}-\mathbf{B} & -\mathbf{V} & \mathbf{M}^T\\ \mathbf{0} & \mathbf{M} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\\ c\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{0}\\ 0\end{bmatrix}. \tag{3.34}$$

is equivalent to (3.33). Motivated by the above facts we define

$$\mathbf{T} := \begin{bmatrix}\mathbf{C}^{1/2} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{1/2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 1\end{bmatrix}, \tag{3.35}$$

where $\mathbf{C}^{1/2}$ denotes the square root of $\mathbf{C}$ from (3.28) and $\mathbf{D}^{1/2}$ is given by (3.32). The following lemma holds:

Theorem 3.1. *The condition numbers of the matrices* $\mathbf{TST}$ *and* $\mathbf{T\tilde{S}T}$, *respectively, are uniformly bounded*

$$\operatorname{cond}_{l^2}(\mathbf{TST}) = \operatorname{cond}_{l^2}(\mathbf{T\tilde{S}T}) \sim 1.$$

Proof. We abbreviate the underlying function spaces by

$$\begin{aligned} H &:= H^1(\Omega) \times H^{-1/2}(0,1) \times \mathbb{R}, \\ L &:= L^2(\Omega) \times L^2(0,1) \times \mathbb{R}, \\ H' &:= H^{-1}(\Omega) \times H^{1/2}(0,1) \times \mathbb{R}, \end{aligned}$$

all equipped by product norms. Moreover, the set of basis functions is called

$$\Xi_J = \begin{bmatrix} \Phi_J^\Omega & \Psi_J^\Gamma & 1 \end{bmatrix}.$$

One concludes from the frame properties (2.18) and (3.24) the validity of

$$\left\| \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{l^2} \sim \left\| \Xi_J \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{L}.$$

On the other hand the proved bijectivity of the underlying operator, cf. lemma 1.1, yields

$$\left\| \Xi_J \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{H} \sim \left\| \Xi_J \mathbf{S} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{H'}.$$

Invoking the norm equivalences

$$\left\| \Xi_J \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{L} \sim \left\| \Xi_J \mathbf{T} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{H}, \qquad \left\| \Xi_J \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{H'} \sim \left\| \Xi_J \mathbf{T} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} \right\|_{L},$$

this implies the assertion. ∎

3.4. **The Bramble-Pasciack-CG.** As above we assume that Γ has a conformal radius < 1 to ensure that the single layer operator is positive definite. Then, of course, the discrete single layer operator $\mathbf{V}$ is also positive definite, i.e.,

$$\mathbf{V} > \mathbf{0}.$$

We may assume the wavelet preconditioner (3.32) $\mathbf{D}^{-1}$ scaled such that $\mathbf{V} > \mathbf{D}^{-1}$, in other words,

$$\mathbf{V} - \mathbf{D}^{-1} > \mathbf{0}. \tag{3.36}$$

To apply the algorithm of Bramble-Pasciack in [2] we reorder the given linear equation system (3.33) obtaining an equivalent one

$$\hat{\mathbf{S}} \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ c \end{bmatrix} \equiv \begin{bmatrix} \mathbf{V} & \mathbf{B} - \mathbf{K} & -\mathbf{M}^T \\ (\mathbf{B} - \mathbf{K})^T & -\mathbf{A} - \mathbf{W} & \mathbf{0} \\ -\mathbf{M} & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{f} \\ 0 \end{bmatrix}. \tag{3.37}$$

Abbreviating

$$\mathbf{U} := \begin{bmatrix} (\mathbf{B}-\mathbf{K})^T \\ -\mathbf{M} \end{bmatrix}, \qquad \mathbf{Q} := \begin{bmatrix} \mathbf{A}+\mathbf{W} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \geq \mathbf{0},$$

the system (3.37) can be rewritten as

$$\begin{bmatrix} \mathbf{V} & \mathbf{U}^T \\ \mathbf{U} & -\mathbf{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{f} \\ 0 \end{bmatrix}.$$

Following [2] we multiply $\hat{\mathbf{S}}$ from the left hand side by

$$\mathbf{E} := \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{U}^T\mathbf{D} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} \\ (\mathbf{B}-\mathbf{K})^T\mathbf{D} & \mathbf{I} & \mathbf{0} \\ -\mathbf{M}\mathbf{D} & \mathbf{0} & 1 \end{bmatrix} \tag{3.38}$$

obtaining the linear equation system

$$\mathbf{E}\hat{\mathbf{S}} \left[\begin{smallmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ c \end{smallmatrix} \right] = \mathbf{E} \left[\begin{smallmatrix} \mathbf{0} \\ -\mathbf{f} \\ 0 \end{smallmatrix} \right]. \tag{3.39}$$

Since $\mathbf{E}$ is nonsingular, which is easily confirmed by execution of a block Gauss-Jordan step with the diagonal matrix $\mathbf{D}$, the linear system (3.39) is equivalent to the given system (3.37). Observing (3.36),

$$\mathbf{F} := \begin{bmatrix} \mathbf{V}-\mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

is a symmetric and positive definite matrix. Hence, we may define a new inner product by

$$\left\langle \left[\begin{smallmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ c \end{smallmatrix} \right], \left[\begin{smallmatrix} \boldsymbol{\delta} \\ \mathbf{v} \\ d \end{smallmatrix} \right] \right\rangle := \left[\begin{smallmatrix} \boldsymbol{\lambda}^T & \mathbf{u}^T & c \end{smallmatrix} \right] \mathbf{F} \left[\begin{smallmatrix} \boldsymbol{\delta} \\ \mathbf{v} \\ d \end{smallmatrix} \right]. \tag{3.40}$$

According to [2] the system matrix $\mathbf{E}\hat{\mathbf{S}}$ of (3.39) is symmetric and positive definite with respect to the inner product $\langle \cdot, \cdot \rangle$ from (3.40) if

$$\mathbf{R} := \begin{bmatrix} \mathbf{A}+\mathbf{W}+(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & -(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}\mathbf{M}^T \\ -\mathbf{M}\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & \mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix} = \mathbf{Q}+\mathbf{U}\mathbf{D}\mathbf{U}^T > \mathbf{0}.$$

In this case one may apply the CG algorithm to (3.39) based on the modified inner product (3.40).

Theorem 3.2. *If Γ has a conformal radius < 1 there holds*

$$\mathbf{R} > \mathbf{0}.$$

Moreover, $\mathbf{R}$ is spectrally equivalent to $\mathbf{X}$ given by

$$\mathbf{X} := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix}.$$

Proof. The proof is done in five steps.

1. We show first the inequality

$$\mathbf{V} \geq \frac{\mathbf{M}^T\mathbf{M}}{\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T}, \tag{3.41}$$

which is equivalent to

$$(\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T)\mathbf{V} \geq \mathbf{M}^T\mathbf{M}.$$

Since there holds $\mathbf{V}^{-1} > \mathbf{0}$ the square root $\mathbf{V}^{-1/2} > \mathbf{0}$ is well defined. Therefore, we may multiply the inequality from both sides by $\mathbf{V}^{-1/2}$ obtaining

$$(\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T)\mathbf{I} \geq \mathbf{V}^{-1/2}\mathbf{M}^T\mathbf{M}\mathbf{V}^{-1/2}.$$

Abbreviating $\mathbf{b} := \mathbf{V}^{-1/2}\mathbf{M}^T \in \mathbb{R}^{N_J^\Gamma}$ gives

$$(\mathbf{b}^T\mathbf{b})\mathbf{I} \geq \mathbf{b}\mathbf{b}^T.$$

We employ the fact that $\mathbf{b}\mathbf{b}^T$ is a matrix of rank 1, which implies that there is only one Eigenvalue $\neq 0$. From

$$(\mathbf{b}\mathbf{b}^T)\mathbf{b} = (\mathbf{b}^T\mathbf{b})\mathbf{b}$$

we find that this Eigenpair is given by $(\mathbf{b}^T\mathbf{b}, \mathbf{b})$, and, consequently, there holds

$$(\mathbf{b}\mathbf{b}^T)\mathbf{a} = \mathbf{0} \qquad \forall\, \mathbf{a} \perp \mathbf{b}.$$

Thus, we obtain

$$\mathbf{a}^T\big((\mathbf{b}^T\mathbf{b})\mathbf{I} - \mathbf{b}\mathbf{b}^T\big)\mathbf{a} \begin{cases} = 0 & \text{for } \mathbf{a} = \mathbf{b}, \\ \geq 0 & \text{for } \mathbf{a} \perp \mathbf{b}, \end{cases}$$

which proves (3.41).

2. Let be

$$\mathbf{Y}(\alpha) := \begin{bmatrix} \frac{1}{\alpha}(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & -(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}\mathbf{M}^T \\ -\mathbf{M}\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & \alpha\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix}$$

and

$$\mathbf{Z}(\alpha) := \begin{bmatrix} \mathbf{I} & \frac{\mathbf{M}\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K})}{\alpha\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T} \\ \mathbf{0} & 1 \end{bmatrix}.$$

By (3.41) and the identity

$$\mathbf{Z}(\alpha)^T\mathbf{Y}(\alpha)\mathbf{Z}(\alpha) = \begin{bmatrix} \frac{1}{\alpha}(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}\Big(\mathbf{V} - \frac{\mathbf{M}^T\mathbf{M}}{\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T}\Big)\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & \mathbf{0} \\ \mathbf{0} & \alpha\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix}$$

application of Sylvester's law of inertia, cf. for example [13], gives

$$\mathbf{Y}(\alpha) \begin{cases} \geq \mathbf{0} & \text{for } \alpha > 0, \\ \leq \mathbf{0} & \text{for } \alpha < 0. \end{cases} \tag{3.42}$$

3. Since $W : H^{1/2}(0,1) \to H^{-1/2}(0,1)$ is an symmetric, positive definite and continuous operator [16], we find a constant $\beta > 0$ such that

$$\mathbf{0} \leq \mathbf{W} \leq \beta\mathbf{A}. \tag{3.43}$$

Note that this estimate is a consequence of the trace theorem as used in the proof of lemma 1.1. Since $\mathbf{V}$ results from an operator of order -1, its invers corresponds to an operator of order $+1$. Hence, since $\mathbf{B}-\mathbf{K}$ corresponds to an operator of order 0, the product $(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K})$ corresponds to a (continous) operator $R : H^{1/2}(0,1) \to H^{-1/2}(0,1)$. Consequently, similarly as the latter estimate, there exists a constant $\gamma > 0$, such that

$$\mathbf{0} \leq (\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) \leq \gamma\mathbf{A}. \tag{3.44}$$

4. It is straightforward manipulation to confirm

$$0 < \frac{\sqrt{4x^2+4x}-x}{2} < 1, \qquad x > 0.$$

Thus, from (3.44), we deduce that it holds for $\underline{\alpha} = \frac{\sqrt{4x^2+4x}-x}{2}$

$$\underline{\alpha}\mathbf{A} + (\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) \geq \frac{1}{\underline{\alpha}}(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}),$$

and hence, since $\mathbf{W} \geq \mathbf{0}$ (3.44) and $\mathbf{Y}(\underline{\alpha}) \geq \mathbf{0}$ (3.42), we find the lower bound

$$\begin{aligned}\mathbf{R} &= (1-\underline{\alpha})\mathbf{X} + \begin{bmatrix} \underline{\alpha}\mathbf{A} + \mathbf{W} + (\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & -(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}\mathbf{M}^T \\ -\mathbf{M}\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & \underline{\alpha}\mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix} \\ &\geq (1-\underline{\alpha})\mathbf{X} + \mathbf{Y}(\underline{\alpha}) \geq (1-\underline{\alpha})\mathbf{X} > \mathbf{0}.\end{aligned}$$

5. On the other hand we find for $\overline{\alpha} = \frac{1-\sqrt{4\gamma+1}}{2\gamma} < 0$

$$-\overline{\alpha}\mathbf{A} + \frac{1}{\overline{\alpha}}(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) \geq (\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}),$$

and hence, since $\mathbf{Y}(\overline{\alpha}) \leq \mathbf{0}$ (3.42),

$$(1-\overline{\alpha})\mathbf{X} \geq (1-\overline{\alpha})\mathbf{X} + \mathbf{Y}(\overline{\alpha}) \geq \begin{bmatrix} \mathbf{A} + (\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & -(\mathbf{B}-\mathbf{K})^T\mathbf{V}^{-1}\mathbf{M}^T \\ -\mathbf{M}\mathbf{V}^{-1}(\mathbf{B}-\mathbf{K}) & \mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix}.$$

Combined with (3.43) this gives the upper bound

$$(1-\overline{\alpha}+\beta)\mathbf{X} \geq \mathbf{R}.$$

□

As a consequence, this theorem combined with [2] proves that there holds in the modified inner product $\langle \cdot, \cdot \rangle$

$$\mathbf{E}\hat{\mathbf{S}} \sim \mathbf{Y} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}\mathbf{V}^{-1}\mathbf{M}^T \end{bmatrix},$$

i.e., $\mathbf{E}\hat{\mathbf{S}}$ is sprectrally equivalent to $\mathbf{Y}$. Therefore, we have the following proposition.

Proposition 3.3. *The system matrix of the preconditioned system*

$$\mathbf{T}\mathbf{E}\hat{\mathbf{S}}\mathbf{T} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{c} \end{bmatrix} = \mathbf{T}\mathbf{E} \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ c \end{bmatrix} = \mathbf{T} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{c} \end{bmatrix}$$

with

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$$

is well conditioned with respect to the modified inner product

$$\operatorname{cond}_{\langle \cdot, \cdot \rangle} \sim 1.$$

Remark 3.4.
1. *Since there is no restriction on the used basis functions in the proofs, the given algorithm is also practicable for the single-scale scheme, if a sprectrally equivalent preconditioner of* $\mathbf{V}$ *is available.*
2. *The implementation of the Bramble-Pasciack-CG is possible without the explicit knowledge of* $\mathbf{V}^{-1}$ *and* $\mathbf{D}^{-1}$*, cf.* [2].

4. Numerical results

This section is dedicated to numerical experiments on the given theory. For the comparison of the traditional scheme and the compressed multiscale scheme we construct an example for which an analytic solution is known. We first investigate the effects of matrix compression on the accuracy of the numercal solution, the computing times and the required memory. Then we analyse the different solving and preconditioning techniques. In the last subsection we study a full multigrid scheme based on the multiscale scheme.

4.1. **An analytical example.** For the numerical results we choose Ω_0 as the two dimensional L-shape $[-\frac{1}{10}, \frac{1}{10}]^2 \setminus [0, \frac{1}{10}]^2$. Similar to [19] we construct a problem for which an analytic solution is known. For sake of simplicity, we choose nonhomogeneous Dirichlet data on Γ_0. We split

$$u(x, y) = u_1(x, y) + u_2(x, y) \in C^2(\mathbb{R} \setminus \left[\begin{smallmatrix} -1/20 \\ 0 \end{smallmatrix}\right])$$

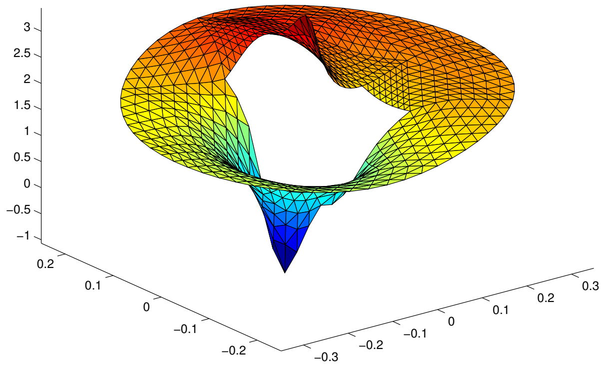

FIGURE 4.3. The analytic solution u on Ω.

with the harmonic function

$$u_1(x,y) = \frac{1}{100} \cdot \frac{(x+\frac{1}{20}) + y}{(x+\frac{1}{20})^2 + y^2} \in C^\infty(\mathbb{R} \setminus \left[\begin{smallmatrix} -1/20 \\ 0 \end{smallmatrix}\right])$$

and the nonharmonic function $u_2 \in C^2(\mathbb{R})$ defined by

$$u_2(x,y) = 2 + \begin{cases} \left(\frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} - 1\right)^3, & \text{if } \frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} \le 1, \\ 0, & \text{if } \frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} > 1. \end{cases}$$

The function f

$$f := -\triangle u_2(x,y) \in C^1(\mathbb{R})$$

is compactly supported in the ellipse with semiaxis 0.3 and 0.2

$$\operatorname{supp} f = \Big\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} \le 1\Big\}.$$

Thus, setting

$$g(x,y) := u|_{\Gamma_0}(x,y)$$

we obtain the following exterior boundary value problem

$$\begin{aligned} -\triangle u &= f && \text{in } \mathbb{R}^2 \setminus \Omega_0 \\ u &= g && \text{on } \Gamma_0 \\ u(x) &= \mathcal{O}(1) && \text{as } |x| \to \infty. \end{aligned}$$

We choose Ω_1 as the ellipse with semiaxis 0.35 and 0.25. The whole configuration is plotted in FIGURE 1.1. The function u is plotted in FIGURE 4.3

4.2. **Single-scale versus multiscale scheme.** We list in TABLE 4.1 and TABLE 4.2 the effect of compression on the computing times of the BEM matrices $\mathbf{K}_\phi$, $\mathbf{V}_\phi$, $\mathbf{K}_\psi$ and $\mathbf{V}_\psi$. One can figure out of TABLE 4.1 that the break even point for the double layer operator K is due to 1024 unknowns. For the single layer operator V the break even point occurs much earlier, namely due to 256 unknowns. The reason is, that the wavelets required for the discretization of K have comparatively large supports, which makes the computation of an entry more expensive. Anyhow, the economy of required main memory is enormous in comparison to the classical scheme, as one can see from the compression rates. Moreover, one has to complete that the compression would be even better by applying a second compression, see [25]. Combining both compressions asymptotically only $\mathcal{O}(N_J^\Gamma)$ entries have to be computed.

N_J^Γ	nnz($\mathbf{K}_\psi$) in %	$T(\mathbf{K}_\phi)$ in sec.	$T(\mathbf{K}_\psi)$ in sec.
32	100.00	0.01	0.01
64	86.426	0.02	0.04
128	59.033	0.06	0.16
256	36.688	0.23	0.45
512	21.698	0.98	1.16
1024	12.534	4.57	2.94
2048	7.0668	18.9	7.54

TABLE 4.1. Number of nonzero elements in % of the BEM matrix $\mathbf{K}_\psi$ and computing times of $\mathbf{K}_\phi$ / $\mathbf{K}_\psi$ in seconds.

N_J^Γ	nnz($\mathbf{V}_\psi$) in %	$T(\mathbf{V}_\phi)$ in sec.	$T(\mathbf{V}_\psi)$ in sec.
32	94.531	0.01	0.03
64	68.457	0.04	0.06
128	43.726	0.16	0.20
256	26.685	0.66	0.59
512	15.768	2.75	1.47
1024	8.9458	11.3	3.75
2048	5.0101	47.1	9.31

TABLE 4.2. Number of nonzero elements in % of the BEM matrix $\mathbf{V}_\psi$ and computing times of $\mathbf{V}_\phi$ / $\mathbf{V}_\psi$ in seconds.

Next, we compare the accuracy of the solutions obtained from the single-scale scheme and from the compressed multiscale scheme. One can figure out of TABLE 4.3 the L^2-errors of the numerical solutions u_ϕ, λ_ϕ, respectively, and the absolute error of the constant c_ϕ. The corresponding errors of u_ψ, λ_ψ and c_ψ obtained from the compressed multiscale scheme are given in TABLE 4.4. Comparing both tables one finds that our compression strategy does not decrease the accuracy of the solution.

N_J^Ω	N_J^Γ	$\|u - u_\phi\|_{L^2(\Omega)}$	$\|\lambda - \lambda_\phi\|_{L^2(0,1)}$	$\|c - c_\phi\|$
188	32	9.3063e-03	5.9979e-02	3.1150e-03
696	64	3.7929e-03	1.6400e-02	4.3119e-03
2672	128	7.7040e-04	4.6377e-03	7.7151e-04
10464	256	1.3317e-04	1.3902e-03	5.6473e-05
41408	512	3.9053e-05	4.1934e-04	2.9662e-05
164736	1024	8.6537e-06	1.3361e-04	4.7211e-06
657152	2048	2.0511e-06	4.4152e-05	6.4501e-07

TABLE 4.3. Accuracy of the traditional scheme.

N_J^Ω	N_J^Γ	$\|u - u_\psi\|_{L^2(\Omega)}$	$\|\lambda - \lambda_\psi\|_{L^2(0,1)}$	$\|c - c_\psi\|$
188	32	9.3063e-03	5.9980e-02	3.1150e-03
696	64	3.7929e-03	1.6402e-02	4.3119e-03
2672	128	7.7040e-04	4.6409e-03	7.7152e-04
10464	256	1.3317e-04	1.3907e-03	5.6466e-05
41408	512	3.9051e-05	4.2303e-04	2.9680e-05
164736	1024	8.6512e-06	1.4341e-04	4.7292e-06
657152	2048	2.0487e-06	7.5080e-05	6.5146e-07

TABLE 4.4. Accuracy of the compressed scheme.

4.3. **Comparison of the iterative solvers.** In TABLE 4.5 we list the number of iterations required by GMRES [24] and MINRES [23] to solve the nonsymmetric system (3.33) and the symmetric system (3.34), respectively. We compare the particularly preconditioned single-scale scheme and the full preconditioned multiscale scheme, more precisely:

- GMRES$_\phi$ and MINRES$_\phi$: Single-scale scheme, where only the FEM part $\mathbf{A}_\phi + \mathbf{W}_\phi$ is preconditioned by BPX, the BEM part $\mathbf{V}_\phi$ is not preconditioned.
- GMRES$_\psi$ and MINRES$_\psi$: Multiscale scheme, the FEM and the BEM part are preconditioned by BPX and a diagonal scaling of $\mathbf{V}_\psi$. Note, a diagonal scaling of $\mathbf{V}_\psi$ is easier to implement than the application of $\mathbf{D}$ (3.32) and improves the wavelet preconditioner.

The iteration starts always with the initial vector $\mathbf{0}$ and stops when the residual norm is less than 1e-6. Since the system becomes very large, we limited the required main memory by restarting the GMRES algorithm after 20 iteration steps. As one can figure out of TABLE 4.5, the number of iterations of GMRES are not improved by the full preconditioned multiscale scheme. We want to remark, that this is not really astonishing, since for nonsymmetric systems a bounded condition number does not prove optimal convergency of an iterative solver. Nevertheless, in the case of MINRES, preconditioning the single layer potential improves the number of iterations quite a lot.

Remark 4.1. *In all the above cases we apply the BPX preconditioner described in section 3.1. Of course, by a diagonal scaling on each level one would reach somewhat lower iteration numbers, but the implementation is much more difficult.*

N_J^Ω	N_J^Γ	GMRES$_\phi$	GMRES$_\psi$	MINRES$_\phi$	MINRES$_\psi$
188	32	48	50	79	72
696	64	56	55	119	95
2672	128	58	61	163	109
10464	256	62	69	212	117
41408	512	69	73	283	124
164736	1024	73	77	358	130
657152	2048	81	79	457	136

TABLE 4.5. Number of iterations required by GMRES and MINRES to attain a residual norm less than 1e-6.

The speed-up by using the different wavelet Galerkin schemes is shown in TABLE 4.6. We measure the time for building up all matrices and solving the system. Since the differences in computation time of the BEM matrices (TABLE 4.1 and TABLE 4.2) and in number of iterations of GMRES and MINRES (TABLE 4.5) is not as big as the speed-up, the big speed-up of the compressed wavelet Galerkin schemes is obviously based on the much cheaper matrix-vector-multiplication for the BEM matrices.

N_J^Ω	N_J^Γ	T(GMRES$_\phi$)	T(GMRES$_\psi$)	T(MINRES$_\phi$)	T(MINRES$_\psi$)
188	32	0.520	0.860	0.590	0.960
696	64	1.110	1.960	1.560	2.700
2672	128	3.250	5.520	5.170	7.480
10464	256	19.55	19.94	45.83	24.37
41408	512	137.1	74.21	422.1	85.14
164736	1024	774.1	294.1	2964	316.7
657152	2048	3703	1297	15967	1403

TABLE 4.6. Overall CPU time in seconds required to build up all matrices and to solve the linear system with GMRES and MINRES.

The next table gives the numbers of iterations and the overall computing times required by Bramble-Pasciack's CG. Since we do not have a preconditioner for the single layer operator in the single-scale bases, we only list the results of the multiscale scheme. As one can figure out of TABLE 4.7 this solver is that one, which requires least iterations steps. Consequently, it is the fasted one.

N_J^Ω	N_J^Γ	CG_ψ	$T(\mathrm{CG}_\psi)$
188	32	37	0.570
696	64	43	1.380
2672	128	46	3.740
10464	256	46	10.69
41408	512	47	35.12
164736	1024	48	121.7
657152	2048	49	484.8

TABLE 4.7. Number of iterations and overall CPU time in seconds of the Bramble-Pasciack-CG.

4.4. **Nested iteration.** As we have seen in subsection 1.3 our discretization yields two sequences of nested subspaces

$$V_0^\Omega \subset V_1^\Omega \subset \ldots \subset H^1(\Omega), \qquad V_0^\Gamma \subset V_1^\Gamma \subset \ldots \subset H^{-1/2}(0,1).$$

Hence, to reduce the computing time for solving the given linear equation system we may apply multigrid techniques. The crucial idea of nested iterations is to employ the solution attained on the level j as initial guess for the solving step on level $j+1$. More precisely, to get the numerical solution $(u^{(J)}, \lambda^{(J)}, c^{(J)})$ on level J one follows the algorithm:

1. Solve the linear system on the coarsest level 0 with initial guess $(0,0,0)$ to obtain the numerical solution $(u^{(0)}, \lambda^{(0)}, c^{(0)})$. Set $j := 0$.
2. Increase $j := j+1$. Prolongate the solution $(u^{(j-1)}, \lambda^{(j-1)}, c^{(j-1)})$ of level $j-1$ onto level j to get $(\tilde{u}^{(j)}, \tilde{\lambda}^{(j)}, \tilde{c}^{(j)})$. Solve the linear system on level j with initial guess $(\tilde{u}^{(j)}, \tilde{\lambda}^{(j)}, \tilde{c}^{(j)})$ which yields the solution $(u^{(j)}, \lambda^{(j)}, c^{(j)})$.
3. If $j = J$ then stop else goto item 2.

According to [15], by this algorithm one obtains combined with a preconditioned iterative solver an optimal scheme, i.e., for the computation of the solution on level J only $\mathcal{O}(N_J^\Omega)$ operations are necessary.

Since the Bramble-Pasciack-CG is the fasted solver, we have implemented the nested iteration algorithm based on this solver. In TABLE 4.8 we list the results obtained for the multiscale scheme for $N_J^\Omega = 657152$ and $N_J^\Gamma = 2048$. In the fourth column we list the number of iterations required to solve (with the prolongated solution from the last level as initial guess) the system until the residual norm is less than an level depending $\varepsilon = \varepsilon(j)$. This ε should behave like the discretization error, i.e., one may solve the system on coarser levels with less accuracy than on higher levels. Of course, for a fair comparison we choose $\varepsilon(J) = 1e-6$ like in the above schemes. In the fifth column we give the time for building up the system on level j and solving them while in the last column we sum up them. Hence, for solving the whole system on level $J = 7$ the nested iteration takes 336.8

seconds. Since the Bramble-Pasciack-CG requires 484.8 seconds this means a speed-up of 30%.

j	N_J^Ω	N_J^Γ	CG_ψ	$T(\mathrm{CG}_\psi)$	$\Sigma T(\mathrm{CG}_\psi)$
0	54	16	14	0.270	0.270
1	188	32	13	0.240	0.510
2	696	64	12	0.540	1.050
3	2672	128	16	1.720	2.770
4	10464	256	18	5.640	8.410
5	41408	512	16	17.28	25.69
6	164736	1024	18	63.81	89.50
7	657152	2048	18	247.4	336.8

TABLE 4.8. Nested iteration for $J = 7$.

5. Concluding Remarks

In the present paper we explore a biorthogonal wavelet based approximation for the coupling of FEM-BEM to solve two-dimensional exterior Dirichlet boundary problems for the Laplacian. We also discuss several iterative solvers and preconditioning techniques based on the norm equivalences of wavelet bases. From all numerical experiments, for the wavelet compression performed and for the preconditioning, satisfactory results are obtained. These prove the possibilities of this method.

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