Disorder and Two-Particle Interaction in Low-Dimensional Quantum Systems

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Abstract
We review some of the recent results on two-interacting particles (TIP) in low-dimensional disordered quantum models. Special attention is given to the mapping of the problem onto random band matrices. In particular, we construct two simple, seemingly closely related examples for which an analogous mapping leads to incorrect results. We briefly discuss possible reasons for this discrepancy based on the physical differences between the TIP problem and our examples.

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1. Introduction

Until 1994, the theoretical and experimental research on transport in disordered systems clearly supported the scaling hypothesis of localization for non-interacting electrons [1,2]. The systems studied usually fell into the predicted universality classes, and, if they didn't, then they could be shown not to be generic [2]. However, real electrons of course interact [3], and their interaction is of relevance for the transport properties of disordered systems [4,5], especially in 2D and 1D where screening [6] is less efficient than in 3D. The influence of *weak* interactions has been investigated extensively using perturbation theory and the perturbative renormalization group (RG) [7,8]. One of the key results is that the lower critical dimension of the MIT is $d_c^- = 2$ as it is for non-interacting electrons. The application of the perturbative RG in 1D [9–13] has lead to the prediction that all thermodynamic states remain localized in the presence of repulsive many-body interactions.

Due to the persistent current problem [14–19] and new experiments on 2D electron systems [20–22] which show striking signatures of a metal-insulator transition, these theoretical considerations received a lot of renewed attention. In order to theoretically study the effects of the interplay between disorder and interactions, one should in principle solve a problem with an exponentially growing number of states in the Hilbert space with increasing system size. At present, this can be achieved only for a few particles in 1D [23–26] and very few particles in 2D [27–31]. However, in 1994 Shepelyansky [32,33] proposed to simply look at *two* interacting particles (TIP) in a random environment. In particular, he suggested that the two particles would form pairs even for repulsive interactions such that the TIP pairs would have a larger localization length than the two single particles (SP) separately. Thus the interaction would lead to an enhanced possibility of transport [34]. The perhaps even more surprising part of the prediction is that the TIP pairs will have a localization length λ_2 such that at pair energy $E = 0$

$$\lambda_2 \propto U^2 {\lambda_1}^2, \tag{1}$$

where U represents the onsite interaction strength and λ_1 is the SP localization length. Since $\lambda_1 \propto 105/W^2$ [35–39] in 1D, this implies large values

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of λ_2 for small disorders W.

The first numerical studies devoted to the TIP problem used the transfer-matrix method (TMM) [2] to investigate the proposed enhancement of the pair localization length λ_2 [32,40]. The TMM of [32] contained an additional artificial infinitely-long-ranged interaction that tends to mask the onsite interaction [41]. The TMM of [40] avoids this problem, but is restricted to small system sizes and results for localization lengths of $\lambda_2 \approx 300$ had been deduced on systems of size $M = 100$. Therefore, two of us studied the TIP problem by a different TMM [42] at large system size $M \gtrsim 300$ and found that (i) the enhancement λ_2/λ_1 decreases with increasing M, (ii) the behavior of λ_2 for $U = 0$ is equal to λ_1 in the limit $M \to \infty$ only, and (iii) for $U \neq 0$ the enhancement λ_2/λ_1 also vanishes completely in this limit. Consequently, we concluded [42] that the TMM applied to the TIP problem in 1D measures an enhancement of the localisation length which is due to the finiteness of the systems considered. Although Ref. [42] has been criticized [43,44], we emphasize that subsequent publications have shown [45–47] that there are no variants of TMM that reproduce Eq. (1). Furthermore, in a later numerical approach [48], based on Green function methods, Song and v. Oppen argue that our extrapolations for $M \to \infty$ were off by $\approx 11\%$ only, whereas the original TMM of [40] deviated by about a factor of 3 [48]. Thus while our criticized TMM results are valid, various other numerical investigations by other groups [48–57] as well as ourselves [58–61] convincingly demonstrated some enhancement. The reason for the failure of the TMM approach of [40,42] has been explained by Song and v. Oppen [48] by arguing that the TMM measures a localization length $\lambda_f < \lambda_2$ due to the cigar-shape geometry [62] of the TIP states.

Reliable numerical approaches to the TIP problem are nowadays based on the computation of the decay of the Green function [48,50,57,59,63]. Other direct numerical approaches to the TIP problem have been based on the time evolution of wave packets [32,52,53,56], exact diagonalization [62], variants of level statistics [49,51] and analysis of multifractal properties [54,55], perturbative methods [64,65] and mappings to effective models [66–70]. In these investigations an enhancement of λ_2 compared to λ_1 has been found as remarked above but the quantitative results tend to differ both from the analytical prediction in Eq. (1), and, albeit less, from each other. Furthermore, a check of the functional dependence of λ_2 on λ_1 is numerically very expensive since it requires very large system sizes $M \gg \lambda_2 \gg \lambda_1$. Extensions of the original arguments have been proposed for TIP in 2D [27,34,61,69,71,72] and 3D [73], for TIP close to a Fermi sea [74], and for long-range interactions in 1D [41,42,52].

The basic idea leading to the prediction (1) is based on looking at the interaction matrix element between two eigenstates $\psi_{kl} = \psi_k\psi_l$ and $\psi_{nm} = \psi_n\psi_m$ of the non-interacting system [32,33]. Here $\psi_k, \psi_l, \psi_n, \psi_m$ denote SP eigenstates localized with localization length λ_1 around sites k, l, n, m. For an onsite interaction [75] $U\sum_{j=1}^{N} n_{j\downarrow}n_{j\uparrow}$ (with $n_{j\tau}$ denoting the number operator at site j for spin τ) only states with $|k-l| \leq \lambda_1$, $|n-m| \leq \lambda_1$, $|k-n| \leq \lambda_1$, $|l-m| \leq \lambda_1$ will give significant contributions to the interaction matrix element

$$\begin{aligned} u &= \langle \psi_{kl}|U|\psi_{nm}\rangle \\ &= U\sum_{j=1}^{N} \psi_k^\dagger(j)\psi_l^\dagger(j)\psi_n(j)\psi_m(j). \end{aligned} \tag{2}$$

These conditions are illustrated in Fig. 1. If one assumes [32,69] that the SP state is given as

$$\psi_k(j) \propto \frac{1}{\sqrt{\lambda_1}} \exp\left(-\frac{|j-k|}{\lambda_1} + i\theta(j)\right) \tag{3}$$

with $\theta(j)$ a random phase, one finds [32] that the typical interaction matrix element has a magnitude of

$$u \propto \lambda_1^{-3/2} \tag{4}$$

since it is the sum of λ_1 random contributions of magnitude λ_1^{-2}. Shepelyansky next calculated the decay rate Γ of a non-interacting eigenstate by means of Fermi's golden rule $\Gamma \sim U^2/\lambda_1 t$ [32,33,64]. Since the typical hopping distance is of the order of λ_1 the diffusion constant is $D \sim U^2\lambda_1/t$. Within a time τ the particle pair visits $N \sim U\lambda_1^{3/2}t^{-1/2}\tau^{1/2}$ states. Diffusion stops

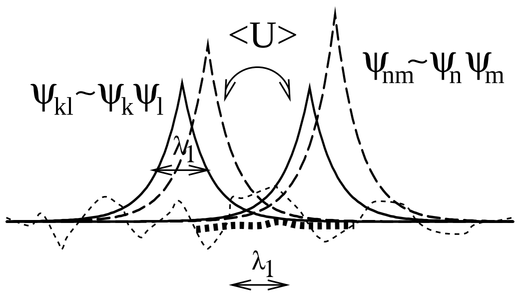

Figure 1. Schematic picture of the TIP arguments of Ref. [32]. The two-particle state ψ_{kl} (left solid and dashed exponentials indicate the envelopes of the constituents ψ_k and ψl) is localized within a distance λ_1 from the two-particle state ψ_{nm} (right solid and dashed curves). The resulting overlap-matrix element $u = \langle U \rangle \equiv \langle \psi_{kl} | U | \psi_{nm} \rangle$ leads to a longer decay length λ_2 for the TIP state as explained in the text. This effect can be visualized as an effective reduction (thick short-dashed line) of the original disorder potential (thin short-dashed line).

when the level spacing of the visited states is of the order of the frequency resolution $1/\tau$. This determines the cut-off time τ^* and the corresponding pair-localization length is obtained as $\lambda_2 \sim \sqrt{D\tau^*} \sim (U/t)^2 \lambda_1^2$ in agreement with Eq. (1). Applicability of Fermi's golden rule requires $\Gamma \gg t/\lambda_1^2$ which is equivalent to $U^2 \lambda_1 / t^2 \gg 1$. This is exactly the condition for an enhancement of λ_2 compared to λ_1. Alternatively, the model may be mapped to a random-matrix model (RMM) with entries chosen according to Eq. (4) [32,67,68].

2. Numerical results for the random-matrix model of TIP

The arguments presented in the last section are of a qualitatively nature and Eq. (1) must be checked for quantitative accuracy. Even before testing (1), it is already worthwhile to check the validity of (4) and the subsequent arguments or the RMM approach [32,67,68]. In Ref. [40], it had been shown that the assumption of a Gaussian distribution of the matrix elements u was oversimplified. The distribution showed long tails making the arithmetic average unsuitable to characterize the typical value. In Ref. [76] we have paid special attention to the exact dependence of u on λ_1 and system size. To this end, we diagonalized the 1D Anderson model for a given M and W and computed u by averaging over all suitable states and many disorder configurations. We showed that due to the strongly non-Gaussian distribution of u, one should rather use the logarithmic average than the arithmetic average as the typical value for the computation of $u(\lambda_1)$. But whereas the arithmetic average [54] gives $u_{\rm abs} \propto \lambda_1^{-1.5}$, the typical value obeys $u_{\rm typ} \propto \lambda_1^{-1.95}$. Following the arguments above, this would imply $\lambda_2 \propto \lambda_1^{1.1}$, i.e., a very small enhancement. We emphasize that this result does not mean that there is no enhancement of the localization length. Rather, the results of Ref. [76] indicate that the arguments of Ref. [32] capture the physics, but only in a somewhat simplified form. One step towards a better agreement between the analytical and the numerical approaches is to take into account the energy denominators in the computation of u, e.g., to consider only interaction matrix elements for states whose energy spacings are of the order of U or smaller [60]. In this case we find that there is a slight decrease in the value of the typical exponent and correspondingly a slight increase in TIP delocalization yielding $\lambda_2 \propto \lambda_1^{1.4\pm0.2}$. This suggests that higher orders in perturbation theory than the first order RMM approach [32] are important. Furthermore, the exponent 1.4 ± 0.2 is in reasonable agreement with previous results in the literature [40,41,45–55,57,59,63–65,69,70].

3. RMM approach for toy models

In this section we show that a naive application the RMM approach may give qualitatively incorrect results even if the RMM contains the correct dependence of the matrix elements on the disorder strength. To this end we consider two toy models which seem to be closely related to the TIP problem. For these models, viz. Anderson models of localization with additional perturbing random potentials, we show that mapping onto RMMs and estimating the localization length by

Fermi's golden rule leads to an incorrect enhancement of the localization length.

3.1. 2D Anderson model with perturbation on a line

The first example is set up to lead to the same RMM as the TIP problem. It consists of the usual 2D Anderson model perturbed by an additional weak random potential of strength U at the diagonal $x = y$ in real space. Since the perturbation increases the width of the disorder distribution at the diagonal we expect it to decrease the localization length. We map the model onto an RMM following the arguments for the TIP problem sketched in Sec. 1. Again, the eigenstates of the unperturbed system are localized with a localization length λ_1 and approximately given by

$$\psi_n(x,y) \sim \frac{1}{\lambda_1} \exp\left[-\frac{|\mathbf{r} - \mathbf{r}_n|}{\lambda_1} + i\theta_n(\mathbf{r}) \right] \qquad (5)$$

where $\mathbf{r} = (x,y)^T$ is the coordinate vector of the particle and θ is again assumed to be a random phase. The Hamiltonian of the 2D perturbed Anderson model differs from the TIP Hamiltonian in two points: (i) the diagonal elements (given by the random potential) are independent random numbers instead of being correlated as in the TIP problem and (ii) the perturbing potential $U(x,x) \in [-U,U]$ at each diagonal site is random instead of having a definite sign and modulus U as in the TIP problem. However, none of these points enters the mapping procedure outlined in Sec. 1. Thus, we find that the perturbation couples each state close to the diagonal ($|x_n - y_n| < \lambda_1$) to $\mathcal{O}(\lambda_1^2)$ other such states. The interaction matrix element is again a sum of $\mathcal{O}(\lambda_1)$ terms of magnitude U/λ_1^2 and random phases and as before $u \sim U\lambda_1^{-3/2}$. Consequently, our toy model is mapped onto exactly the same RMM as TIP in a random potential. Therefore, the resulting localization length along the diagonal is also given by Eq. (1). We thus arrive at the surprising conclusion, that adding a weak random potential at the diagonal of a 2D Anderson model leads to an enormous enhancement of the localization length along this diagonal, in contradiction to the expectation expressed above, viz. that increasing disorder leads to stronger localization.

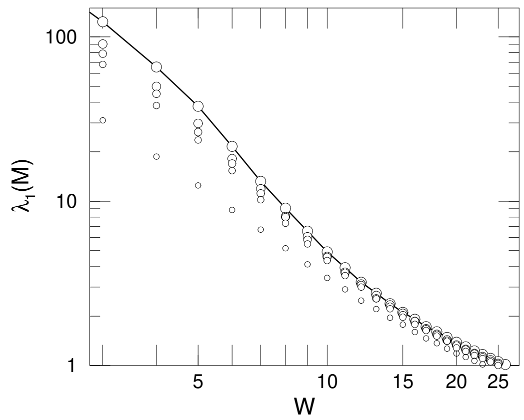

Figure 2. Dependence of $\lambda_1(M)$ on disorder W for the 2D Anderson model at $E = 0$ for $M = 10, 25, 30, 35$ and 50 indicated by increasing symbol size. We use the $M = 50$ data, emphasized by the solid line, as finite-size estimate of λ_1.

As for the TIP case [76] we now numerically check whether the relation $u \sim U\lambda_1^{-3/2}$ between the coupling matrix element u and the localization length λ_1 of the unperturbed system is correctly described by the RMM. Since in 2D a simple analytic formula for the dependence of λ_1 on the disorder W does not exist, we first compute estimates $\lambda_1(M)$ for quasi-1D strips of finite strip width M with 1% accuracy by TMM. In Fig. 2, we show data of $\lambda_1(M)$ as a function of W. In the following, we take $\lambda_1(50)$ to compute the coupling matrix elements.

Next, we calculate both the arithmetic average $u_{\rm abs} = \langle |u| \rangle$ and the logarithmic average $u_{\rm typ} = \exp[\langle \log(|u|) \rangle]$ for different values of W and various $M \times M$ squares. Disorder averaging is over 20 samples and we study $u_{\rm abs}$ and $u_{\rm typ}$ as functions of $\lambda_1(M)$. We emphasize that instead of the well-known extrapolations of $\lambda_1(M)$ to infinite system size by means of FSS [2], we take the finite-size approximants $\lambda_1(M)$ on purpose, since we compute λ_2 also for comparable finite sizes only.

As for the TIP model [76] the distribution $P_{\rm o}(u)$ of the (off-diagonal) coupling matrix elements is strongly non-Gaussian, suggesting that

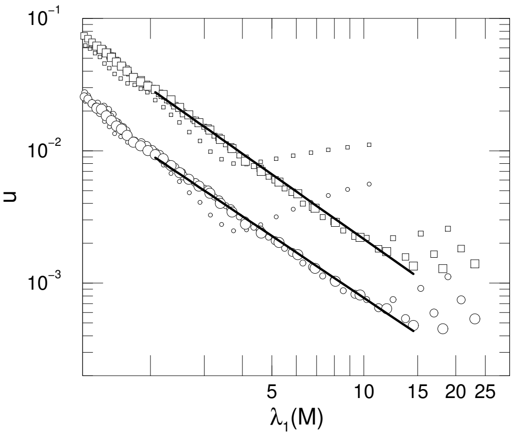

Figure 3. Dependence of $u_{\rm abs}$ (squares) and $u_{\rm typ}$ (circles) on $\lambda_1(M)$ for the perturbed 2D Anderson model with $U = 1$ and $M = 10, 25, 30$ and 35 indicated by increasing symbol size. The solid lines represent the power laws $u_{\rm abs} \sim \lambda_1^{-1.6}$ and $u_{\rm typ} \sim \lambda_1^{-1.5}$.

$u_{\rm typ}$ rather than $u_{\rm abs}$ is the relevant quantity. The results for $u_{\rm abs}$ and $u_{\rm typ}$ are presented in Fig. 3. The dependence of $u_{\rm abs}$ on $\lambda_1(M)$ for $2 \leq \lambda_1(M) \leq 12$ follows $u_{\rm abs} \propto \lambda_1(M)^{-1.6\pm0.1}$ in agreement with the RMM value of $3/2$ and with Ref. [76]. Furthermore, here we also have $u_{\rm typ} \propto \lambda_1(M)^{-1.5\pm0.1}$. We note that the change of the slopes of $u_{\rm abs}$ and $u_{\rm typ}$ at $\lambda_1(M) \approx M/2$ is entirely due to the finite sample sizes [76].

Consequently, in contrast to the TIP problem the RMM model for the 2D perturbed Anderson model of localization contains the correct dependence of the coupling matrix elements on the localization length of the unperturbed system, but still it leads to an incorrect enhancement of the localization length along the diagonal.

3.2. 1D Anderson model with perturbation

An even more striking contradiction can be obtained for an 1D Anderson model of localization. The eigenstates are again given by Eq. (3) with λ_1 known from second order perturbation theory [35–37] and numerical calculations [38,39] to vary as $\lambda_1 \sim t^2/W^2$ for small disorder. We now add a weak random potential of strength U at all sites. Since the result is obviously an 1D Anderson model with a slightly higher disorder strength $W_u > W$ the localization length will be reduced, $\lambda(U) \sim t^2/W_u^2$.

The mapping onto an RMM can be performed in complete analogy to the TIP problem and the 2D Anderson model discussed above. The perturbing potential leads to transitions between the unperturbed eigenstates ψ_n. Each such state is now coupled to $\mathcal{O}(\lambda_1)$ other states by coupling matrix elements $\langle\psi_n|U|\psi_{n'}\rangle$ with magnitude $u \sim U\lambda_1^{-1/2}$ since we sum over λ_1 contributions with magnitude U/λ_1 and supposedly random phases.

The application of Fermi's golden rule in this 1D case leads to a diffusion constant $D \sim U^2\lambda_1^2/t$. The number of states visited within a time τ is $N \sim U\lambda_1 t^{-1/2}\tau^{1/2}$. Again, diffusion stops at a time τ^* when the level spacing of the states visited equals the frequency resolution. This gives $\tau^* \sim U^2\lambda_1^2/t^3$. The localization length λ of the perturbed system thus reads $\lambda \sim \sqrt{D\tau^*} \sim U^2\lambda_1^2$ as in Eq. (1), in contradiction to the correct result.

Again we numerically check the relation between $u_{\rm abs}$ and $u_{\rm typ}$ and the unperturbed localization length λ_1. In Fig. 4, we show results obtained for chains with various lengths and 50 disorder configurations for each W. λ_1 is computed by TMM. For $10 \leq \lambda_1 \leq 250$, $u_{\rm abs}$ varies as $\lambda_1^{-0.48\pm0.10}$ as predicted above. $u_{\rm typ}$ varies as $\lambda_1^{-0.59\pm0.10}$. Both variations are compatible with the RMM value of $1/2$ for the exponent. Again we need $\lambda_1 < M/2$ in order to suppress finite size effects.

Consequently, although the RMM model for the 1D perturbed Anderson model of localization contains the correct dependence of the coupling matrix elements on the localization length of the unperturbed system, it still leads to an incorrect enhancement of the localization length.

4. Application of the block-scaling picture to toy models

Let us now discuss the relation of these results to Imry's block-scaling picture (BSP) [34,69] for the TIP problem. In this approach one considers blocks of linear size λ_1 and calculates the dimen-

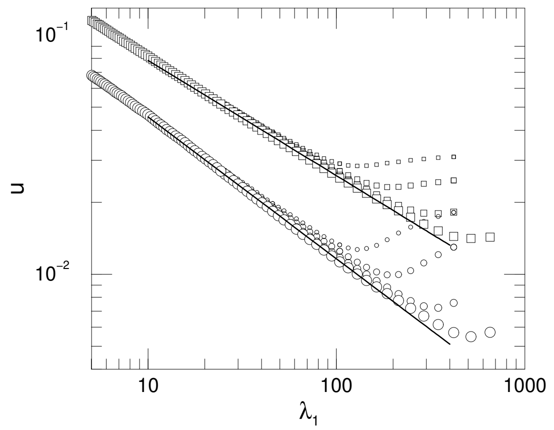

Figure 4. Dependence of $u_{\rm abs}$ (squares) and $u_{\rm typ}$ (circles) on λ_1 for the perturbed 1D Anderson model with $U = 1$ and $M = 200, 300, 500$ and 800 indicated by increasing symbol size. The solid lines represent the power laws $u_{\rm abs} \sim \lambda_1^{-0.48}$ and $u_{\rm typ} \sim \lambda_1^{-0.59}$.

sionless pair conductance on that scale,

$$g_2 \sim \frac{u^2}{\Delta^2}, \tag{6}$$

where u represents the typical interaction-induced coupling matrix element between states in neighboring blocks and $\Delta \sim t/\lambda_1^2$ is the level spacing within the block. If the typical coupling matrix element depends on λ_1 as $u \sim U\lambda_1^{-\alpha}$ the pair conductance obeys

$$g_2 \sim (U/t)^2 \lambda_1^{4-2\alpha}. \tag{7}$$

For the 2D Anderson model with perturbation considered above, the BSP can be applied analogously. Again, we consider blocks of linear size λ_1 and compute the typical perturbation-induced matrix elements between these blocks. We then find that according to the BSP the conductance of a 2D Anderson model with additional weak perturbing potential along the diagonal is given by Eq. (6). Using $\alpha = 1.5 \pm 0.1$ as obtained above from the numerical data for $u_{\rm abs}$ and $u_{\rm typ}$, we then have $g_2 \sim (U/t)^2\lambda_1$. Thus we conclude that the BSP does not work for our 2D toy model, because it yields the same unphysical result as the RMM approach of section 3.1.

Let us also apply the BSP to the 1D toy example. The level spacing in a 1D block of size λ_1 is $\Delta \sim t/\lambda_1$, and the coupling matrix element between states in neighboring blocks is $u \sim U\lambda_1^{-1/2}$. Thus, the conductance of the perturbed system on a scale λ_1 is obtained as $g_2 \sim (U/t)^2\lambda_1$. For large λ_1 this again contradicts the correct result, viz. a decrease of the conductance compared to the unperturbed system. Thus, the BSP applied to the two toy models introduced in the present work gives the same qualitatively incorrect results for the localization properties as the RMM. This is not surprising since the only ingredients of the BSP are the intra-block level spacing $\Delta \sim t/\lambda_1^2$ and the inter-block coupling matrix elements u which also enter the RMM.

5. Conclusions

We have presented two toy models which seem to be closely related to the TIP problem. For these toy models the usual analytical arguments given to support the delocalization of TIP, viz. the RMM and the BSP do not work. However, the large-scale numerical simulations [48,50,57, 59,61,71] have convincingly shown that an enhancement of the two-particle localization length due to the interaction exists, even though the detailed results are more complicated than the original prediction (1). This leads, of course, to the question, under which conditions the RMM mapping and the BSP give the correct result, at least qualitatively. While a general answer to this question is not known, it has been suggested [77] that the difference between the TIP and our toy models is an additional symmetry in the TIP problem.

In summary, the two examples suggest that additional physical insight is needed before applying the RMM. In addition, we expect that taking into account the energies of the states as in Ref. [60] for TIP will result in a reduced enhancement, i.e., a smaller value of α, in the analytical predictions. This will in turn give a better agreement with the numerically determined dependence of the TIP localization length on λ_1.

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