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Numerische Simulation auf massiv parallelen Rechnern

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On a criterion for asymptotic stability of differential-algebraic equations

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Abstract

This paper discusses Lyapunov stability of the trivial solution of linear differential-algebraic equations. As a criterion for the asymptotic stability we propose a numerical parameter $æ(A, B)$ characterizing the property of a regular matrix pencil $\lambda A - B$ to have all finite eigenvalues in the open left half-plane. Numerical aspects for computing this parameter are discussed.

Key words. differential-algebraic equations, asymptotic stability, Lyapunov equation, matrix pencils, deflating subspaces, projections.

AMS subject classification. 15A22, 34D20, 65F15

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1 Introduction

Differential-algebraic equations (DAEs)

$$f(x'(t), x(t), t) = 0$$

with nontrivial nullspace of the Jacobian $f'_y(y, x, t)$ arise naturally in many applications, e.g., in control problems, electrical networks and constrained mechanical systems [4, 12, 13, 14, 42]. The theoretical analysis and the numerical solution of DAEs has been the subject of intense research for many years (see [4, 27, 28] and the references therein). In the study of differential-algebraic equations one is often interested in the existence of stationary solutions and their asymptotic behavior [14, 27, 41, 43, 44, 51].

In this paper we propose an approach to study the asymptotic stability of the trivial solution of the linear homogeneous differential equation

$$Ax'(t) = Bx(t) \tag{1.1}$$

with constant matrix coefficients A and B. The case of a nonsingular matrix A is well studied, asymptotic stability of the trivial solution is equivalent to the condition for the matrix $A^{-1}B$ to have all eigenvalues in the left half-plane (see, e.g., [19]).

If the matrix A is singular, then the investigation of the spectrum of the matrix pencil $\lambda A - B$ is necessary. The trivial solution of (1.1) is asymptotically stable if all finite eigenvalues of $\lambda A - B$ have negative real part [14, 27]. However, it is well known that the generalized eigenvalue problem as well as the standard eigenvalue problem may be ill-conditioned in the sense that eigenvalues may change strongly even under small perturbations in A and B [50]. Recently the concept of ε-pseudospectra and spectral portraits (see, e.g., [25, 53]) was developed to better understand the influence of perturbations on the spectrum of matrices and matrix pencils. The application of the ε-pseudospectra in the study the asymptotic stability of differential equations arising in Computational Fluid Dynamics can be found in [17, 18, 52].

Another possible approach to investigate the asymptotic behavior of solutions of linear ordinary differential equations without explicitely computing the eigenvalues is the consideration of a dichotomy parameter that characterizes numerically the property of a matrix to have all eigenvalues in the left half-plane and that is efficiently computable [5, 6, 22, 24]. An analogous criterion for asymptotic stability of linear DAEs with index 1 was introduced in [49]. In this paper we generalize this criterion for higher index DAEs and discuss the computation of deflating subspaces for the matrix pencil $\lambda A - B$ corresponding to the finite eigenvalues with negative real part.

For the case of a nonsingular matrix A it was proposed in [1, 25] to reduce the computation of the deflating subspace of $\lambda A - B$ corresponding to the eigenvalues in the open left (right) half-plane to the computation of the deflating subspace of the Cayley-transformed matrix pencil $\lambda \mathcal{A} - \mathcal{B} = \lambda(A - B) - (A + B)$ corresponding to the eigenvalues inside (outside) the open unit circle. If the pencil $\lambda A - B$ has no eigenvalues on the imaginary axis, the pencil $\lambda \mathcal{A} - \mathcal{B}$ has no eigenvalues on the unit circle. Then the inverse free method based on Malyshev's algorithm [1, 39] can be applied to the pencil $\lambda\mathcal{A} - \mathcal{B}$ to compute the deflating subspaces of $\lambda\mathcal{A} - \mathcal{B}$ corresponding to the eigenvalues inside and outside the unit circle. However, if the matrix A is singular, the infinite eigenvalues of $\lambda A - B$ will be mapped by the Cayley transformation to the eigenvalues of $\lambda\mathcal{A} - \mathcal{B}$ on the unit circle. In this case the computation of the corresponding deflating subspaces is more complicated and still poorly understood.

This paper is organized as follows. In Section 2 we recall fundamental characteristics of matrix pencils, define a function of a pencil and obtain its some important properties. In Section 3 we consider an initial value problem for equation (1.1). Sections 4 and 5 present an extension of the Lyapunov stability theory for ordinary differential equations [24] to differential-algebraic equations. We introduce a numerical criterion æ(A, B) that can be used as a quantitative characteristic of the "quality" of the asymptotic stability of the trivial solution of (1.1). In Section 6 we consider a generalized Lyapunov equation that can be used in the stability analysis of DAEs. In Section 7 we describe an algorithm for computing the parameter æ(A, B) for the pencil $\lambda A - B$ with index at most one and the projection onto the deflating subspace of $\lambda A - B$ corresponding to the finite eigenvalues. Section 8 presents a perturbation analysis for this projection. The sensitivity analysis for the generalized Lyapunov equation is presented in Section 9. Section 10 contains numerical examples.

Throughout the paper the complex plane is denoted by $\mathbb{C}$, the open left half-plane is denoted by $\mathbb{C}^-$, the n-dimensional complex vector space is denoted by $\mathbb{C}^n$. The matrix A^T is the transpose of A, A^* is the complex conjugate transpose of A, and $A^{-*} = (A^*)^{-1}$. The inner product of vectors x and y is defined as $(x, y) = \sum_{j=1}^{n} x_j \overline{y}_j = y^* x$, $\|\cdot\|$ denotes the spectral matrix norm and the Euclidean vector norm, $\mathrm{cond}(A) = \|A\|\|A^{-1}\|$ is the condition number of the matrix A. We will denote the nullspace of the matrix A by $\ker A$ and the range of A by $\operatorname{im} A$.

2 Preliminaries

Let A and B be square complex matrices of order n. A matrix pencil $\lambda A - B$ is called *singular* if $\det(\lambda A - B) \equiv 0$ for all $\lambda \in \mathbb{C}$. Otherwise the pencil $\lambda A - B$ is called *regular* [50]. In the sequel, we will consider only regular matrix pencils.

A complex value $\lambda \neq \infty$ is said to be *a finite eigenvalue* of the matrix pencil $\lambda A - B$ if $\det(\lambda A - B) = 0$. The pencil $\lambda A - B$ has *infinite eigenvalue* if the matrix A is singular. We will denote the set of all eigenvalues of $\lambda A - B$ by $\mathrm{Sp}(A, B)$.

Vectors $x_1, \ldots, x_k$ form *a right Jordan chain of the pencil $\lambda A - B$ corresponding to an eigenvalue λ* if

$$(\lambda A - B)x_1 = 0, \quad (\lambda A - B)x_2 = -Ax_1, \qquad \ldots, \qquad (\lambda A - B)x_k = -Ax_{k-1}. \tag{2.1}$$

Vectors $y_1, \ldots, y_k$ form *a left Jordan chain of $\lambda A - B$ corresponding to an eigenvalue λ* if

$$y_1^*(\lambda A - B) = 0, \quad y_2^*(\lambda A - B) = -y_1^* A, \qquad \ldots, \qquad y_k^*(\lambda A - B) = -y_{k-1}^* A.$$

The vectors x_1 and y_1 are called, respectively, *right and left eigenvectors of* $\lambda A - B$ *corresponding to* λ.

A subspace $\mathcal{S}_\lambda \subseteq \mathbb{C}^n$ that is the span of all right (left) Jordan chains corresponding to an eigenvalue λ is called *the right (left) deflating subspace of* $\lambda A - B$ *corresponding to* λ. Let $\Lambda(A, B) = \{\lambda_1, \ldots, \lambda_s\}$ be a set of pairwise distinct eigenvalues of the pencil $\lambda A - B$ and let $\mathcal{S}_{\lambda_j}$ be the deflating subspace of $\lambda A - B$ corresponding to λ_j for $j = 1, \ldots, s$. Then the subspace

$$\mathcal{S}_\Lambda = \mathcal{S}_{\lambda_1} \dot{+} \ldots \dot{+} \mathcal{S}_{\lambda_s}$$

is *the deflating subspace of* $\lambda A - B$ *corresponding to* $\Lambda(A, B)$. Here $\dot{+}$ denotes the direct sum. Moreover, $\mathbb{C}^n$ admits a decomposition $\mathbb{C}^n = \mathcal{S}_\Lambda \dot{+} \mathcal{S}$, where $\mathcal{S}$ is the complementary deflating subspace of $\lambda A - B$ corresponding to $\mathrm{Sp}(A, B) \setminus \Lambda(A, B)$. A projection P onto the deflating subspace $\mathcal{S}_\Lambda$ along the deflating subspace $\mathcal{S}$ is called *the spectral projection onto* $\mathcal{S}_\Lambda$.

A matrix pencil $\lambda A - B$ with a singular matrix A can be reduced to the Weierstrass canonical form [50], i.e., there exist nonsingular matrices W and T such that

$$A = W \begin{pmatrix} I_{n_1} & 0 \\ 0 & N \end{pmatrix} T \qquad \text{and} \qquad B = W \begin{pmatrix} J & 0 \\ 0 & I_{n-n_1} \end{pmatrix} T, \tag{2.2}$$

where I_{n_1} is the identity matrix of order n_1, J and N are matrices in Jordan canonical form and N is nilpotent with index of nilpotency k. The number k is called *index* of the pencil $\lambda A - B$. The block J corresponds to the finite eigenvalues, the block N corresponds to the infinite eigenvalues of $\lambda A - B$.

The representation (2.2) defines the decomposition of $\mathbb{C}^n$ into complementary deflating subspaces of the matrix pencil $\lambda A - B$ corresponding to its finite and infinite eigenvalues. Then the matrices

$$P_f = T^{-1} \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} T, \qquad \Pi_f = W \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \tag{2.3}$$

are the spectral projections onto the right and left deflating subspaces of the pencil $\lambda A - B$ corresponding to the finite eigenvalues. For simplicity, the deflating subspace of $\lambda A - B$ corresponding to the finite (infinite) eigenvalues we will call *the finite (infinite) deflating subspace*.

Consider now a generalized resolvent $(\lambda A - B)^{-1}$ which is a rational matrix-valued function of λ defined on a domain not containing the spectrum of the matrix pencil $\lambda A - B$. At an eigenvalue (finite or infinite) $\lambda_j(A, B)$ of multiplicity m_j the resolvent has a pole of order m_j.

Let γ be a closed Jordan curve γ in $\mathbb{C}$ and let a function $f(t)$ be analytic within an open domain D bounded by γ and continuous on the closure $\overline{D}$. Assume that $\det(\lambda A - B) \neq 0$ for all λ lying on the curve γ. Similarly to the case $A = I$ we define *a function of the matrix pencil* via

$$f(A, B) := \frac{1}{2\pi i} \oint_\gamma f(\lambda)(\lambda A - B)^{-1} A \, d\lambda. \tag{2.4}$$

This definition is a matrix pencil version of the Cauchy's integral formula [48]. Obviously, if the domain D does not intersect with the spectrum of $\lambda A - B$, by Cauchy's theorem [48] we have $f(A,B) = 0$.

Remark 1. Note that the above definition of the function of a matrix pencil is slightly different from the one given in [15]. However, some familiar properties of scalar functions and functions of a matrix [38] can be extended to matrix pencils.

Lemma 1 *Let functions $f(t)$ and $g(t)$ be continuous on the curve γ and analytic inside γ. Then the following formula holds*

$$(fg)(A,B) := \frac{1}{2\pi i}\oint_{\gamma} f(\lambda)g(\lambda)(\lambda A - B)^{-1}A\,d\lambda = f(A,B)g(A,B). \tag{2.5}$$

PROOF. Consider a closed Jordan curve γ_1 that is situated in a domain bounded by γ and encloses all the eigenvalues of $\lambda A - B$ which lie inside γ. Obviously,

$$g(A,B) = \frac{1}{2\pi i}\oint_{\gamma} g(\mu)(\mu A - B)^{-1}A\,d\mu = \frac{1}{2\pi i}\oint_{\gamma_1} g(\mu)(\mu A - B)^{-1}A\,d\mu.$$

Then

$$f(A,B)g(A,B) = \frac{1}{(2\pi i)^2}\oint_{\gamma}\oint_{\gamma_1} f(\lambda)g(\mu)(\lambda A - B)^{-1}A(\mu A - B)^{-1}A\,d\mu\,d\lambda.$$

From the resolvent indentity

$$(\lambda A - B)^{-1} - (\mu A - B)^{-1} = (\mu - \lambda)(\lambda A - B)^{-1}A(\mu A - B)^{-1} \tag{2.6}$$

it follows that

$$\begin{aligned} f(A,B)g(A,B) &= \frac{1}{(2\pi i)^2}\oint_{\gamma} f(\lambda)(\lambda A - B)^{-1}A\left(\oint_{\gamma_1}\frac{g(\mu)}{\mu - \lambda}d\mu\right)d\lambda + \\ &+ \frac{1}{(2\pi i)^2}\oint_{\gamma_1} g(\mu)(\mu A - B)^{-1}A\left(\oint_{\gamma}\frac{f(\lambda)}{\lambda - \mu}d\lambda\right)d\mu. \end{aligned}$$

Since the curve γ_1 lies inside γ the first integral is zero. Applying Cauchy's integral formula [48] to the second integral we obtain

$$f(A,B)g(A,B) = \frac{1}{2\pi i}\oint_{\gamma_1} f(\mu)g(\mu)(\mu A - B)^{-1}A\,d\mu = (fg)(A,B).$$

□

Lemma 2 *Let γ be a closed Jordan curve surrounding some subset of the finite spectrum of the pencil $\lambda A - B$. Then the matrix*

$$P = \frac{1}{2\pi i}\oint_{\gamma}(\lambda A - B)^{-1}A\,d\lambda \tag{2.7}$$

is a spectral projection (known as Riesz projection) onto the right deflating subspace of the pencil $\lambda A - B$ corresponding to the eigenvalues inside the curve γ along the deflating subspace corresponding to the eigenvalues outside γ.

PROOF. The relation $P^2 = P$ immediately follows from Lemma 1 by $f(t) = g(t) = 1$. Let μ be an eigenvalue of the matrix pencil $\lambda A - B$ and let $x_1, \dots, x_k$ be a right Jordan chain corresponding to μ. Taking into account (2.1) it is easy to show by induction that

$$(\lambda A - B)^{-1} A\, x_j = \sum_{l=1}^{j} \frac{1}{(\lambda - \mu)^{j+1-l}}\, x_l, \qquad j = 1, 2, \dots, k \tag{2.8}$$

holds for all $\lambda \in \mathbb{C}^n$ such that $\det(\lambda A - B) \neq 0$. Integrating both sides of (2.8) along the curve γ and using Cauchy's integral formula [48],

$$\frac{1}{2\pi i} \oint_\gamma \frac{d\lambda}{(\lambda - \mu)^s} = \begin{cases} 1, & \text{if } s = 1 \text{ and } \mu \text{ is inside } \gamma, \\ 0, & \text{otherwise} \end{cases}$$

we obtain

$$P x_j = \frac{1}{2\pi i} \oint_\gamma (\lambda A - B)^{-1} A x_j \, d\lambda = \frac{1}{2\pi i} \oint_\gamma \frac{x_j}{\lambda - \mu} d\lambda = \begin{cases} x_j, & \text{if } \mu \text{ is inside } \gamma, \\ 0, & \text{if } \mu \text{ is outside } \gamma \end{cases}$$

for all $j = 1, \dots, k$. Thus, $\operatorname{im} P$ is the right deflating subspace of the pencil $\lambda A - B$ corresponding to the eigenvalues inside the curve γ and $\ker P$ is the right deflating subspace corresponding to the eigenvalues outside γ. □

3 Differential-algebraic equations

Consider the homogeneous differential-algebraic equation

$$A x'(t) = B x(t) \tag{3.1}$$

with constant square matrix coefficients A and B of order n. Equation (3.1) is said to be of index k if the matrix pencil $\lambda A - B$ is regular of index k [4, 28].

Transforming the pencil $\lambda A - B$ to Weierstrass canonical form (2.2) we have the decoupled system of equations

$$\begin{aligned} y_1'(t) &= J y_1(t), \\ N y_2'(t) &= y_2(t), \end{aligned}$$

where $y(t) = Tx(t) = [\, y_1^T(t), y_2^T(t)]^T$. A solution of the first equation can be determined in the explicit form $y_1(t) = e^{tJ} y_1(0)$. The nilpotency of the matrix N in the second equation implies that $y_2(t) \equiv 0$. Thus, the general solution of (3.1) can be written as

$$x(t) = T^{-1} y(t) = T^{-1} \begin{pmatrix} e^{tJ} y_1(0) \\ 0 \end{pmatrix}. \tag{3.2}$$

Hence, for each solution $x(t)$ of equation (3.1) we have $x(t) \in \operatorname{im} P_f$, where P_f is the spectral projection given in (2.3). In addition, (3.2) implies the existence and uniqueness of solutions of the initial value problem for (3.1).

Theorem 1 *Let $\Lambda(A,B)$ be a subset of the finite eigenvalues of the pencil $\lambda A - B$ and let P be the spectral projection onto the right deflating subspace of $\lambda A - B$ corresponding to $\Lambda(A,B)$. Then the initial value problem*

$$\begin{aligned} Ax'(t) - Bx(t) &= 0, \quad (3.3) \\ P(x(0) - x_0) &= 0 \quad (3.4) \end{aligned}$$

has a unique solution $x(t)$ in $\operatorname{im} P$ *for all $x_0 \in \mathbb{C}^n$.*

Remark 2. The initial condition (3.4) can be replaced by the equivalent condition

$$M(x(0) - x_0) = 0 \quad (3.5)$$

with a matrix M satisfying

$$\ker M = \ker P. \quad (3.6)$$

This fact is an immediate consequence of the relations $MP = M$ and $P = PM^+M$, where the matrix M^+ denotes the Moore-Penrose inverse of M [11]. Note that condition (3.6) is only sufficient for the initial value problem (3.3), (3.5) to be uniquely solvable in $\operatorname{im} P$. The following example shows that it is not necessary.

Example 1. Consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = I, \quad M = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the initial value problem (3.3), (3.5) has a unique solution $x(t)$ in $\operatorname{im} P$ for all $x_0 \in \mathbb{C}^n$, but $\ker M \neq \ker P$.

For equation (3.1) with solutions in $\operatorname{im} P$ we may define a fundamental solution matrix as follows.

Definition 1 *A matrix-valued function $\mathcal{F}(t) \equiv \mathcal{F}(t,A,B)$ is called* fundamental solution matrix of equation (3.1) in the subspace $\operatorname{im} P$ *if it is continuously differentiable and satisfies the initial value problem*

$$\begin{aligned} A\mathcal{F}'(t) &= B\mathcal{F}(t), \qquad t > 0, \quad (3.7) \\ \mathcal{F}(0) &= P. \quad (3.8) \end{aligned}$$

The matrix $\mathcal{F}(t,A,B)$ is a generalization of the matrix exponential that is the fundamental solution matrix for linear time-invariant ordinary differential equations [24].

Theorem 2 *There exists a unique fundamental solution matrix $\mathcal{F}(t,A,B)$ of equation (3.1) in the subspace $\operatorname{im} P$. Moreover, the unique solution $x(t)$ of the initial value problem (3.3), (3.4) is given by*

$$x(t) = \mathcal{F}(t)x_0. \quad (3.9)$$

Proof. Consider a closed Jordan curve γ not intersecting with the spectrum of the pencil $\lambda A - B$ and enclosing all eigenvalues from $\Lambda(A, B)$. Since the function $e^{t\lambda}$ of the variable λ is analytic everywhere in $\mathbb{C}$, we can define the function

$$\mathcal{F}(t) = \frac{1}{2\pi i} \oint_\gamma e^{t\lambda}(\lambda A - B)^{-1} A \, d\lambda. \tag{3.10}$$

It is easy to verify that this matrix satisfies equation (3.7). Indeed,

$$A\mathcal{F}'(t) - B\mathcal{F}(t) = \frac{1}{2\pi i} \oint_\gamma e^{t\lambda}(\lambda A - B)(\lambda A - B)^{-1} A \, d\lambda = \frac{1}{2\pi i} A \oint_\gamma e^{t\lambda} d\lambda = 0.$$

Moreover, it follows from (2.7) that

$$\mathcal{F}(0) = \frac{1}{2\pi i} \oint_\gamma (\lambda A - B)^{-1} A \, d\lambda = P.$$

In order to prove the uniqueness of the fundamental solution matrix we consider the homogeneous system

$$A\mathcal{F}'(t) = B\mathcal{F}(t), \qquad \mathcal{F}(0) = 0. \tag{3.11}$$

Using the Weierstrass canonical form (2.2) of the regular pencil $\lambda A - B$ it is easy to see that the problem (3.11) has only the trivial solution $\mathcal{F}(t) \equiv 0$. Let us now suppose that there exist two fundamental solution matrices $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$. Then their difference $\mathcal{F}(t) = \mathcal{F}_1(t) - \mathcal{F}_2(t)$ satisfying the homogeneous system (3.11) is identically equal to zero, i.e., $\mathcal{F}_1(t) = \mathcal{F}_2(t)$.

The straightforward verification that the function $x(t) = \mathcal{F}(t)x_0$ satisfies the initial value problem (3.3), (3.4) concludes the proof. □

Note that the above fundamental solution matrix $\mathcal{F}(t)$ is invariant with respect to the projection P, i.e.,

$$\ker \mathcal{F}(t) = \ker P, \qquad \operatorname{im} \mathcal{F}(t) = \operatorname{im} P$$

for all $t \geq 0$. These relations can be easily obtained from the equations

$$P\mathcal{F}(t) = \mathcal{F}(t) = \mathcal{F}(t)P,$$
$$\mathcal{F}(t)\mathcal{F}(-t) = P = \mathcal{F}(-t)\mathcal{F}(t),$$

which are obvious consequences of Lemma 1. Here

$$\mathcal{F}(-t) = \frac{1}{2\pi i} \oint_\gamma e^{-t\lambda}(\lambda A - B)^{-1} A \, d\lambda.$$

Remark 3. If $P = P_f$, then the initial condition $\mathcal{F}(0) = P_f$ can be replaced by the equivalent condition

$$\hat{P}(\mathcal{F}(0) - I) = 0 \tag{3.12}$$

with a certain projection $\hat{P}$ along $\ker P$ [27]. However, in the general case, when $\operatorname{im} P$ is the deflating subspace of $\lambda A - B$ associated only with a part of the finite spectrum, the initial value problems (3.7), (3.8) and (3.7), (3.12) are not equivalent. Indeed, if we consider the matrix $\mathcal{F}(t)$ given in (3.10), where γ is a closed curve surrounding all finite eigenvalues of $\lambda A - B$, then this matrix satisfies (3.7) and (3.12), but $\mathcal{F}(0) = P_f \neq P$.

4 Asymptotic stability

In this section we study the asymptotic stability of the trivial solution of equation (3.1). The following definitions describe Lyapunov stability in a subspace for the differential-algebraic equation (3.1).

Definition 2 *Let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$ and let P be a projection onto $\mathcal{S}$. The trivial solution $x(t) \equiv 0$ of (3.1) is* stable in the sense of Lyapunov in the subspace $\mathcal{S}$ *if for all $x_0 \in \mathbb{C}^n$ the initial value problem*

$$\begin{aligned} Ax'(t) - Bx(t) &= 0, \\ P(x(0) - x_0) &= 0 \end{aligned} \tag{4.1}$$

has a solution $x(t, x_0) \in \mathcal{S}$ defined on $[0, \infty)$. Moreover, for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(t, x_0)\| < \varepsilon$ for all $t \geq 0$ and for all $x_0 \in \mathbb{C}^n$ with $\|Px_0\| < \delta$.

Definition 3 *The trivial solution $x(t) \equiv 0$ of (3.1) is* asymptotically stable in the sense of Lyapunov in the subspace $\mathcal{S}$ *if it is stable in $\mathcal{S}$ and if there is a $\delta_0 > 0$ such that for all $x_0 \in \mathbb{C}^n$ with $\|Px_0\| < \delta_0$ the solution $x(t, x_0) \to 0$ for $t \to \infty$.*

The following theorem gives a necessary and sufficient condition for the trivial solution of (3.1) to be asymptotically stable on a subspace.

Theorem 3 *Let $\Lambda(A, B)$ be a subset of the finite eigenvalues of the pencil $\lambda A - B$ and let P be the spectral projection onto the right deflating subspace of $\lambda A - B$ corresponding to $\Lambda(A, B)$. Then the trivial solution $x(t) \equiv 0$ of equation (3.1) is asymptotically stable in the subspace $\operatorname{im} P$ if and only if $\Lambda(A, B)$ is contained in the open left half-plane.*

PROOF. Suppose that the matrix pencil $\lambda A - B$ has a finite eigenvalue $\lambda \in \Lambda(A, B)$ with nonnegative real part, and let $z \in \operatorname{im} P$ be the corresponding eigenvector with $\|z\| = 1$. Consider x_0 such that $Px_0 = \delta_0 z/2$. Then, obviously, the function $x(t) = \delta_0 e^{\lambda t} z/2$ belongs to $\operatorname{im} P$ and satisfies (4.1). We have $\|Px_0\| = \delta_0/2 < \delta_0$ and

$$\|x(t)\| = \frac{\delta_0}{2}|e^{\lambda t}| = \frac{\delta_0}{2} e^{t \Re e\, \lambda}.$$

Consequently, for $t \to \infty$ either the norm of $x(t)$ is constant or increases unboundedly, which implies that the trivial solution of (3.1) is not asymptotically stable.

In order to prove the sufficiency we rewrite the decomposition (2.2) as follows

$$\lambda A - B = W \begin{pmatrix} \lambda I_{n_0} - J_0 & 0 \\ 0 & \lambda A_2 - B_2 \end{pmatrix} T, \tag{4.2}$$

where J_0 contains the Jordan blocks associated with the eigenvalues from $\Lambda(A, B)$ and $\lambda A_2 - B_2$ contains the remaining spectrum. Here n_0 is the dimension of the matrix J_0. In this case the spectral projection P has the form

$$P = T^{-1} \begin{pmatrix} I_{n_0} & 0 \\ 0 & 0 \end{pmatrix} T. \tag{4.3}$$

Then the matrix $\mathcal{F}(t)$ given in (3.10) can be represented as

$$\mathcal{F}(t) = T^{-1}\begin{pmatrix} \frac{1}{2\pi i}\oint_\gamma e^{t\lambda}(\lambda I - J_0)^{-1}d\lambda & 0 \\ 0 & 0 \end{pmatrix} T = T^{-1}\begin{pmatrix} e^{tJ_0} & 0 \\ 0 & 0 \end{pmatrix} T, \tag{4.4}$$

where γ is a closed Jordan curve enclosing all eigenvalues of the matrix J_0. Since the eigenvalues of J_0 belong to the open left half-plane, i.e., $\Re e\, \lambda_j(J_0) \le -\eta < 0$, the estimate

$$\|e^{tJ_0}\| \le \theta(n_0)\Big(\frac{\|J_0\|}{\eta}\Big)^{n_0-1} e^{-t\eta/2} \tag{4.5}$$

holds [24]. Here $\theta(n_0)$ is a constant that depends on n_0 only. Furthermore, using the decomposition (4.4), we have the estimate

$$\|\mathcal{F}(t)\| \le \|T^{-1}\|\|T\|\|e^{tJ_0}\| \le \theta(n_0)\|T^{-1}\|\|T\|\Big(\frac{\|J_0\|}{\eta}\Big)^{n_0-1} e^{-t\eta/2}. \tag{4.6}$$

By Theorems 1 and 2 problem (4.1) has the unique solution $x(t) = \mathcal{F}(t)x_0$. Then for all $\varepsilon > 0$ choosing

$$\delta = \frac{\varepsilon\eta^{n_0-1}}{\theta(n_0)\|T^{-1}\|\|T\|\|J_0\|^{n_0-1}}$$

we obtain that

$$\|x(t)\| = \|\mathcal{F}(t)Px_0\| < e^{-t\eta/2}\varepsilon \le \varepsilon$$

for all $t \ge 0$ and all $x_0 \in \mathbb{C}^n$ with $\|Px_0\| < \delta$. This means that the trivial solution of (3.1) is stable in the subspace $\operatorname{im} P$. Moreover, $x(t) \to 0$ for $t \to \infty$, i.e., the solution $x(t) \equiv 0$ is asymptotically stable in $\operatorname{im} P$. □

Remark 4. According to Remark 2, Lyapunov stability does not depend on the special choice of the projection P which can be replaced by any matrix M with the property that $\ker M = \ker P$.

For simplicity, the (asymptotically) stable solution of (3.1) in the subspace $\operatorname{im} P_f$ will be called *(asymptotically) stable.*

Corollary 1 *The trivial solution of (3.1) is asymptotically stable if and only if all finite eigenvalues of the matrix pencil $\lambda A - B$ lie in $\mathbb{C}^-$.*

This result is well known, see [14, 27].

Example 2. [14] Consider the differential-algebraic equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x'(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x(t). \tag{4.7}$$

We have $\det(\lambda A - B) = \lambda^2 + \lambda + 1 = 0$, i.e., $\lambda_{1,2} = -1/2 \pm i\sqrt{3}/2$ are the finite eigenvalues of $\lambda A - B$. Thus, the trivial solution of (4.7) is asymptotically stable.

5 A numerical criterion for asymptotic stability

In this section we are concerned with the asymptotic stability in the subspace $\operatorname{im} P_f$ and consider the problem to numerically check whether all finite eigenvalues of the pencil $\lambda A - B$ belong to the open left half-plane. This problem arises in the study of the asymptotic properties of stationary solutions of not only linear differential-algebraic equations, but also autonomous quasilinear and nonlinear DAEs [40, 51] and nonautonomous DAEs with constant coefficient linear part and small nonlinearity [41].

Definition 4 *A matrix pencil $\lambda A - B$ is called* stable *if all its finite eigenvalues lie in the open left half-plane.*

We will omit the index $_f$ in P_f, that is, P denotes in the sequel the spectral projection onto the right deflating subspace of $\lambda A - B$ corresponding to the finite spectrum. Let $Q = I - P$ and define

$$æ(A, B) = 2\|A\|\|B\|\|Z\|,$$

where the matrix Z has the form

$$Z = (AP + BQ)^{-*} \left(\int_0^\infty \mathcal{F}^*(t)\mathcal{F}(t)dt \right) (AP + BQ)^{-1} \tag{5.1}$$

and $\mathcal{F}(t)$ is the fundamental solution matrix in (3.10). If the pencil $\lambda A - B$ is stable, then by (4.6) the integral in (5.1) is convergent. Moreover, it follows from the decompositions (2.2) and (2.3) that

$$AP + BQ = W \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} T,$$

i.e., the matrix $AP + BQ$ is nonsingular. Hence, $æ(A, B)$ is bounded. We set $æ(A, B) = \infty$ if the pencil $\lambda A - B$ has at least one finite eigenvalue with nonnegative real part.

It is interesting that the parameter $æ(A, B)$ can be used for pointwise estimation of the solution of problem (4.1). We will use a similar technique as in [25].

Theorem 4 *Let $x(t)$ be a solution of the initial value problem (4.1). Then*

$$\|x(t)\| \leq \sqrt{æ(A, B)\|A\|\|(AP + BQ)^{-1}\|}\, e^{-t\|B\|/(\|A\|æ(A,B))}\|Px_0\|. \tag{5.2}$$

Proof. If $æ(A, B) = \infty$ then inequality (5.2) is fulfilled. Assume that $æ(A, B) < \infty$. Let us consider for $t \geq 0$ the matrix-valued function

$$Y(t) = \int_t^\infty \mathcal{F}^*(s)\mathcal{F}(s)ds.$$

Using the properties of the fundamental matrix

$$\mathcal{F}(t + s) = \mathcal{F}(t)\mathcal{F}(s) = \mathcal{F}(s)\mathcal{F}(t)$$

we have

$$\begin{aligned} Y(t) &= \int_t^\infty \mathcal{F}^*(s)\mathcal{F}(s)ds = \mathcal{F}^*(t)\left\{\int_0^\infty \mathcal{F}^*(s)\mathcal{F}(s)ds\right\}\mathcal{F}(t) = \\ &= \mathcal{F}^*(t)(AP+BQ)^*Z(AP+BQ)\mathcal{F}(t) = \mathcal{F}^*(t)A^*ZA\mathcal{F}(t). \end{aligned}$$

Differentiating the matrix $Y(t)$ we obtain

$$\frac{d}{dt}Y(t) = -\mathcal{F}^*(t)\mathcal{F}(t).$$

Then for an arbitrary vector z, from $(A^*ZAz, z) \le \|A\|^2\|Z\|(z,z)$, we have the estimate

$$\frac{d}{dt}(Y(t)z,z) = -(\mathcal{F}(t)z, \mathcal{F}(t)z) \le -\frac{(ZA\mathcal{F}(t)z, A\mathcal{F}(t)z)}{\|A\|^2\|Z\|} = -\frac{(Y(t)z,z)}{\|A\|^2\|Z\|},$$

which implies that

$$\frac{d}{dt}\left(e^{t/(\|A\|^2\|Z\|)}(Y(t)z,z)\right) \le 0,$$

and, consequently,

$$\begin{aligned} (\mathcal{F}^*(t)A^*ZA\mathcal{F}(t)z,z) &= (Y(t)z,z) \le e^{-t/(\|A\|^2\|Z\|)}(Y(0)z,z) = \\ &= e^{-t/(\|A\|^2\|Z\|)}(A^*ZAPz, Pz). \end{aligned} \tag{5.3}$$

Furthermore, it is not difficult to verify that

$$\mathcal{F}(t) = T^{-1}\begin{pmatrix} e^{tJ} & 0 \\ 0 & 0 \end{pmatrix} T = e^{t(AP+BQ)^{-1}B}P = Pe^{t(AP+BQ)^{-1}B}. \tag{5.4}$$

Then, taking into account the inequality $\|e^{t(AP+BQ)^{-1}B}Pz\| \ge e^{-|t|\|(AP+BQ)^{-1}\|\|B\|}\|Pz\|$, see [24, p. 24], we have

$$\begin{aligned} (A^*ZAPz, Pz) &= ((AP+BQ)^*Z(AP+BQ)Pz, Pz) = \int_0^\infty \|\mathcal{F}(t)Pz\|^2 dt \ge \\ &\ge \|Pz\|^2 \int_0^\infty e^{-2t\|(AP+BQ)^{-1}\|\|B\|}dt = \frac{\|Pz\|^2}{2\|(AP+BQ)^{-1}\|\|B\|}. \end{aligned} \tag{5.5}$$

Substituting in (5.5) the vector $z = \mathcal{F}(t)x_0$ we obtain

$$\|x(t)\|^2 = \|\mathcal{F}(t)x_0\|^2 \le 2\|(AP+BQ)^{-1}\|\|B\|\left(A^*ZA\mathcal{F}(t)x_0, \mathcal{F}(t)x_0\right).$$

Finally, using (5.3) with $z = Px_0$ we obtain

$$\begin{aligned} \|x(t)\|^2 &\le 2\|(AP+BQ)^{-1}\|\|B\|e^{-t/(\|A\|^2\|Z\|)}(A^*ZAPx_0, Px_0) \le \\ &\le æ(A,B)\|A\|\|(AP+BQ)^{-1}\|\, e^{-2t\|B\|/(\|A\|æ(A,B))}\|Px_0\|^2. \end{aligned}$$

□

We see that if $\text{æ}(A, B)$ is bounded, then by (5.2) the trivial solution of (3.1) is asymptotically stable. On the other hand, from the asymptotic stability of the trivial solution of equation (3.1) it follows that $\text{æ}(A, B) < \infty$. The parameter $\text{æ}(A, B)$ together with $\|A\|$ and $\|B\|$ describes the rate of decrease in the solution of (4.1). The larger $\text{æ}(A, B)$ is the slower the solution $x(t)$ of (4.1) converges to zero for $t \to \infty$. Note that for $A = I$ the parameter $\text{æ}(A, B)$ coincides with the parameter $\text{æ}(B)$ for the asymptotic stability of linear ordinary differential equations introduced in [5, 6, 24]. Therefore, analogously to [5], the number $\text{æ}(A, B)$ may be called *a quality criterion for the asymptotic stability* of the differential-algebraic equation (3.1).

The matrix Z in (5.1) can be also represented as

$$Z = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* P (i\xi A - B)^{-1} d\xi. \tag{5.6}$$

Here we mean the Cauchy principal value of the integral. Indeed, taking into account (5.4) and the relation

$$e^{tJ} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} (i\xi I - J)^{-1} d\xi$$

(see, e.g., [24]), we obtain

$$\begin{aligned} \mathcal{F}(t)(AP + BQ)^{-1} &= T^{-1} \begin{pmatrix} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} (i\xi I - J)^{-1} d\xi & 0 \\ 0 & 0 \end{pmatrix} W^{-1} = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} P (i\xi A - B)^{-1} d\xi, \end{aligned} \tag{5.7}$$

i.e., the entries of the matrix $P(i\xi A - B)^{-1}$ are the Fourier transformations of the entries of $\mathcal{F}(t)(AP + BQ)^{-1}$. Then (5.6) immediately follows from the Parseval's identity [48, p. 85].

Concluding this section we estimate the spectral norm of $P(i\xi A - B)^{-1}$ by means of $\text{æ}(A, B)$.

Lemma 3 *Let $\text{æ}(A, B) < \infty$. Then for all $\xi \in \mathbb{R}$ the estimate*

$$\|P(i\xi A - B)^{-1}\| \leq \frac{5\pi}{2\|B\|} \text{æ}(A, B) \tag{5.8}$$

holds.

Proof. Note that the matrix Z is Hermitian and positive semidefinite. Then for any vector z of unit length we have

$$\frac{\text{æ}(A, B)}{2\|A\|\|B\|} = \|Z\| \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|P(i\xi A - B)^{-1} z\|^2 d\xi \tag{5.9}$$

Let ξ_0 be a point on the real line where the norm $\|P(i\xi A-B)^{-1}\|$ achieves its maximal value. Using the relation

$$P(i\xi A-B)^{-1}A = P(i\xi A-B)^{-1}AP = (i\xi A-B)^{-1}AP$$

we obtain

$$P(i\xi A-B)^{-1} = P(i\xi_0 A-B)^{-1} - i(\xi-\xi_0)P(i\xi A-B)^{-1}AP(i\xi_0 A-B)^{-1}.$$

Then we have the estimate

$$\|P(i\xi A-B)^{-1}\| \le \frac{\|P(i\xi_0 A-B)^{-1}\|}{1-|\xi-\xi_0|\|A\|\|P(i\xi_0 A-B)^{-1}\|},$$

which is valid for all ξ such that $|\xi-\xi_0|\|A\|\|P(i\xi_0 A-B)^{-1}\|<1$. Furthermore, choosing a vector z such that

$$\|P(i\xi_0 A-B)^{-1}z\| = \|P(i\xi_0 A-B)^{-1}\|,$$

we obtain the estimate

$$\begin{aligned}\|P(i\xi A-B)^{-1}z\| &\ge \|P(i\xi_0 A-B)^{-1}z\|\,(1-|\xi-\xi_0|\|A\|\|P(i\xi A-B)^{-1}\|)\\ &\ge \|P(i\xi_0 A-B)^{-1}\|\frac{1-2|\xi-\xi_0|\|A\|\|P(i\xi_0 A-B)^{-1}\|}{1-|\xi-\xi_0|\|A\|\|P(i\xi_0 A-B)^{-1}\|}.\end{aligned}$$

Setting $r=\|A\|\|(i\xi_0 A-B)^{-1}AP\|$, we obtain from (5.9) that

$$\begin{aligned}\frac{\pi\|A\|æ(A,B)}{\|B\|} &\ge \int\limits_{\xi_0-1/(2r)}^{\xi_0+1/(2r)} r^2\left(\frac{1-2|\xi-\xi_0|r}{1-|\xi-\xi_0|r}\right)^2 d\xi\\ &= r^2\int\limits_{\xi_0-1/(2r)}^{\xi_0}\left(\frac{1-2(\xi_0-\xi)r}{1-(\xi_0-\xi)r}\right)^2 d\xi + r^2\int\limits_{\xi_0}^{\xi_0+1/(2r)}\left(\frac{1-2(\xi-\xi_0)r}{1-(\xi-\xi_0)r}\right)^2 d\xi\\ &= 2r\int_0^{1/2}\left(\frac{1-2t}{1-t}\right)^2 dt = 2r(3-4\ln 2)\ge\frac{2r}{5}.\end{aligned}$$

Therefore,

$$\|P(i\xi_0 A-B)^{-1}\| \le \frac{5\pi}{2\|B\|}æ(A,B).$$

□

The estimate (5.8) implies that the finite eigenvalues of the pencil $\lambda A-B$ are separated from the imaginary axis by a distance not less than $2\|B\|/(5\pi æ(A,B))$. In other words, (5.8) yields a lower bound for perturbations which preserve the dimension of the deflating subspace of $\lambda A-B$ corresponding to the finite eigenvalues and cause the pencil to obtain a finite eigenvalue on the imaginary axis. Thus, the parameter $æ(A,B)$ characterizes the absence of eigenvalues of the pencil $\lambda A-B$ not only on the imaginary axis but in a neighbourhood of it. An analogous result for $A=I$ has been obtained in [6].

To measure the smallest real (complex) perturbation to a stable matrix required to make the perturbed matrix unstable, the real (complex) stability radius can be used [32, 54]. For numerical methods for the computation of the stability radius see, e.g., [8, 31, 47] and the references therein. Unfortunately, these results are not immediately applicable to matrix pencils. The general problem to measure or estimate the distance to instability for the matrix pencil, i.e., the distance from the given pencil to the "nearest" pencil that is singular or has an eigenvalue in the closed right half-plane, is more difficult. Only partial solutions are known. A lower bound of the stability radius for the matrix pencil $\lambda A - B$ allowing perturbations in B only is given in [46]. A computable expression for the stability radius for the regular matrix pencil of index less than or equal to one is studied in [10]. Computationally attractive upper and lower bounds of smallest norm de-regularizing perturbation are discussed in [9].

6 Generalized Lyapunov equations

It is well known that the study the asymptotic behavior of solutions of ordinary differential equations is directly related to the analysis of Lyapunov matrix equations, see [19, 24]. In this section we present a generalized Lyapunov equation that can be used to investigate the asymptotic stability of the differential-algebraic equation (3.1).

Consider the generalized Lyapunov equation

$$A^*ZB + B^*ZA = -P^*CP, \tag{6.1}$$

where A, B and C are given matrices, P is the spectral projection onto the right finite deflating subspace of $\lambda A - B$ and Z is the unknown matrix. If A is nonsingular then P=I and (6.1) is equivalent to the standard Lyapunov equation

$$ZBA^{-1} + (BA^{-1})^*X = -A^{-*}CA^{-1}. \tag{6.2}$$

In this case Lyapunov theorem [33] on the existence and uniqueness of the Hermitian, positive definite solution of (6.2) can be generalized to equation (6.1).

Theorem 5 *Let $\lambda A - B$ be a matrix pencil with a nonsingular matrix A. The generalized Lyapunov equation (6.1) has a unique Hermitian, positive definite solution Z for each Hermitian, positive definite matrix C if and only if all eigenvalues of $\lambda A - B$ lie in the open left half-plane.*

For a singular matrix A the solvability of (6.1) depends only on the structure of the finite spectrum of the pencil $\lambda A - B$. The following theorem gives a necessary and sufficient condition for the existence of solutions of the generalized Lyapunov equation (6.1) with the singular matrix A.

Theorem 6 *Let $\lambda A - B$ be a regular matrix pencil and let P be the spectral projection onto the right deflating subspace of $\lambda A - B$ corresponding to the finite eigenvalues. There exists* *a Hermitian, positive semidefinite matrix Z satisfying the generalized Lyapunov equation (6.1) with a Hermitian, positive definite matrix C if and only if all finite eigenvalues of $\lambda A - B$ lie in the open left half-plane.*

PROOF. Let the matrix pencil $\lambda A - B$ be in Weierstrass canonical form (2.2), where all eigenvalues of J have negative real part. Let the matrices

$$T^{-*}CT^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix} \quad \text{and} \quad W^*ZW = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \tag{6.3}$$

be partitioned in blocks accordingly to A and B. We have from (6.1) the decoupled system of equations

$$Z_{11}J + J^*Z_{11} = -T_{11}, \tag{6.4}$$
$$Z_{12} + J^*Z_{12}N = 0, \tag{6.5}$$
$$N^*Z_{21}J + Z_{21} = 0, \tag{6.6}$$
$$N^*Z_{22} + Z_{22}N = 0. \tag{6.7}$$

Since the eigenvalues of J lie in $\mathbb{C}^-$, the Lyapunov equation (6.4) with the Hermitian, positive definite T_{11} has a unique Hermitian, positive definite solution Z_{11} [33, p. 96]. Because the matrices J^{-*} and $-N$ have disjoint spectra, the homogeneous equations (6.5) and (6.6) are uniquely solvable [33, p. 270] and have the trivial solutions $Z_{12} = 0$ and $Z_{21} = 0$. Finally, for $Z_{22} = 0$ equation (6.7) is fulfilled. Thus, the the generalized Lyapunov equation (6.1) has at least one Hermitian, positive semidefinite solution Z.

Assume now that a matrix

$$Z = W^{-*}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} W^{-1}$$

satisfies (6.1), where Z_{11} and Z_{22} are Hermitian, positive semidefinite. Then equations (6.4)-(6.7) are fulfilled. Since T_{11} is positive definite and Z_{11} is positive semidefinite, we have from (6.4) that the eigenvalues of J have negative real part, i.e., all finite eigenvalues of the pencil $\lambda A - B$ lie in the left half-plane.

□

It follows from the proof of Theorem 6 that any solution Z of the generalized Lyapunov equation (6.1) can be represented as

$$Z = W^{-*}\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix} W^{-1}, \tag{6.8}$$

where Z_{11} and Z_{22} satisfy equations (6.4) and (6.7), respectively. Whenever the solution of (6.4) exists, it is unique, whereas (6.7) in general has many solutions [33]. Hence, the solution of the generalized Lyapunov equation (6.1) with singular A is not unique. As usual for linear systems we may resolve the nonuniqueness of the solution by requiring extra conditions. This may be the solution of minimum norm, or we may require $Z_{22} = 0$. In terms of the original data this latter requirement can be expressed as $Z = Z\Pi$, where Π is the spectral projection onto the left finite deflating subspace of the pencil $\lambda A - B$. Then we have the existence and uniqueness theorem for solutions of the generalized Lyapunov equation (6.1).

Theorem 7 *If the matrix pencil $\lambda A - B$ is stable, then the generalized Lyapunov equation (6.1) with Hermitian, positive definite C has a unique Hermitian, positive semidefinite solution Z such that $Z = Z\Pi$. This solution is given by*

$$Z = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^{*}CP(i\xi A - B)^{-1}d\xi. \tag{6.9}$$

Proof. It follows from (6.8) that any solution of (6.1) satisfying $Z = Z\Pi$ has the form

$$Z = W^{-*}\begin{pmatrix} Z_{11} & 0 \\ 0 & 0 \end{pmatrix}W^{-1},$$

where Z_{11} is the unique Hermitian, positive definite solution of (6.4) that is given by

$$Z_{11} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi I - J)^{-*}T_{11}(i\xi I - J)^{-1}d\xi$$

(see [24]). Therefore,

$$\begin{aligned} Z &= W^{-*}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix} (i\xi I - J)^{-*} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{12}^{*} & T_{22} \end{pmatrix}\begin{pmatrix} (i\xi I - J)^{-1} & 0 \\ 0 & 0 \end{pmatrix}d\xi\right\}W^{-1} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^{*}CP(i\xi A - B)^{-1}d\xi. \end{aligned}$$

□

Remark 5. Note that the assertions of Theorems 6 and 7 remain valid if the matrix C is positive definite only on the subspace $\operatorname{im} P$, i.e., $(Cz, z) > 0$ for all nonzero $z \in \operatorname{im} P$, since in this case the property for the matrix T_{11} in (6.3) to be positive definite is preserved. Indeed, for any nonzero vector y we have

$$(T_{11}y, y) = \left(\begin{bmatrix} T_{11} & T_{12} \\ T_{12}^{*} & T_{22} \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix}\right) = (CT^{-1}\begin{bmatrix} y \\ 0 \end{bmatrix}, T^{-1}\begin{bmatrix} y \\ 0 \end{bmatrix}) = (Cz, z) > 0,$$

since $z = T^{-1}(y^T\ 0)^T \in \operatorname{im} P$.

We will now establish a connection between the solution of the generalized Lyapunov equation and the differential-algebraic equation (3.1). This connection is well-known for the standard Lyapunov equation ($A = I$) and the linear ordinary differential equation, see [24, 30].

Let the matrix pencil $\lambda A - B$ be stable and Z be the solution of the generalized Lyapunov equation

$$A^{*}ZB + B^{*}ZA = -P^{*}P \tag{6.10}$$

such that $Z = Z\Pi$. Taking into account (6.9) with $C = I$ and (5.7) we obtain from Parseval's identity [48, p. 85] that

$$A^*ZA = \int_0^\infty \mathcal{F}^*(t)\mathcal{F}(t)dt,$$

where $\mathcal{F}(t)$ is the fundamental solution matrix of the differential-algebraic equation (3.1). Then

$$\|A^*ZA\| = \max_{x_0\neq 0} \frac{\int_0^\infty \|\mathcal{F}(t)x_0\|^2 dt}{\|x_0\|^2} = \max_{Px_0=x_0\neq 0} \frac{\int_0^\infty \|x(t)\|^2 dt}{\|Px_0\|^2},$$

i.e., the norm of A^*ZA is the square of the maximum L_2-norm of the solution of the initial value problem (4.1). Moreover, for all nonzero solution $x(t)$ of the differential-algebraic equation (3.1) we have

$$\begin{aligned}\frac{d}{dt}(A^*ZAx(t), x(t)) &= (ZAx'(t), Ax(t)) + (ZAx(t), Ax'(t)) = \\ &= ((A^*ZB + B^*ZA)x(t), x(t)) = -(Px(t), Px(t)) = -(x(t), x(t)).\end{aligned}$$

The quadratic form (A^*ZAx, x) is an extension of the Lyapunov function for ordinary differential equations [24] to differential-algebraic equations and the matrix equation (6.10) generalizes the standard Lyapunov equation to matrix pencils. The norm of its solution multiplied by $2\|A\|\|B\|$ is equal to the parameter $æ(A, B)$ that characterizes the asymptotic stability of the differential-algebraic equation (3.1), see the estimate (5.2).

Thus, we may compute $æ(A, B)$ by determining the spectral projection P and solving the generalized Lyapunov equation (6.10). The numerical solution of the standard Lyapunov equation has been studied in numerous publications (see, e.q., [2, 29] and the references therein). Numerical methods for the generalized Lyapunov equation with nonsingular A have been considered in [3, 20, 21, 45]. However, the case of singular A is more complicated, since the solution of the generalized Lyapunov equation is not unique. We need the special solution Z of (6.10), namely, such that $Z = Z\Pi$. In the next section we present an algorithm for computing the projections P, Π and the desired matrix Z for the matrix pencil $\lambda A - B$ with index at most one.

7 Computing spectral projections and the matrix Z

We now assume that the matrix pencil $\lambda A - B$ is a regular of index at most one. Recall that $\lambda A - B$ has index one if and only if the matrix $A + BQ_1$ is nonsingular for any projection Q_1 onto the nullspace of A [27]. In this case the spectral projections P and Π onto the right and left finite deflating subspaces of $\lambda A - B$ can be represented as

$$P = I - Q_1(A + BQ_1)^{-1}B, \qquad \Pi = I - BQ_1(A + BQ_1)^{-1} \tag{7.1}$$

(see [27]). The norm of these projections characterize the conditioning of the deflating subspaces of $\lambda A - B$ associated with the finite and infinite eigenvalues and the property of $\lambda A - B$ to be regular of index one. The large value of $\|P\|$ or $\|\Pi\|$ indicates that the problem to find the finite deflating subspace of the pencil $\lambda A - B$ with index one is ill-conditioned, i.e., either the finite and infinite eigenvalues are hard to be separated from each other, or $\lambda A - B$ is nearly a pencil of index greater than one or a singular pencil.

We now describe an algorithm for computing the projections P and Π for the matrix pencil $\lambda A - B$ of index one. Let $r = \operatorname{rank} A$ and let

$$A = V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^* \tag{7.2}$$

be the singular value decomposition of A [23, 26], where U and V are unitary matrices and Σ is a nonsingular diagonal $(r \times r)$-matrix with positive diagonal elements

$$\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_{r-1}(A) \geq \sigma_r(A) > 0,$$

which are nonzero singular values of A. Then

$$Q^\perp = U \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} U^*$$

is the orthogonal projection onto $\ker A$. Let the matrix

$$V^* B U = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{7.3}$$

be partitioned in blocks in accordance with $V^* A U$. Then

$$A + BQ^\perp = V \begin{pmatrix} \Sigma & B_{12} \\ 0 & B_{22} \end{pmatrix} U^*$$

and by (7.1) with $Q_1 = Q^\perp$ we have

$$P = U \begin{pmatrix} I & 0 \\ -B_{22}^{-1} B_{21} & 0 \end{pmatrix} U^*, \qquad \Pi = V \begin{pmatrix} I & -B_{12} B_{22}^{-1} \\ 0 & 0 \end{pmatrix} V^*. \tag{7.4}$$

The accuracy in the computation of the projections P and Π will clearly depend on the condition number of B_{22} with respect to inversion. But it also depends on the condition number of Σ as is shown in the following example.

Example 3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \delta \end{pmatrix}$$

For nonzero ε we have $B_{22} = \delta$. If δ is small, then the pencil $\lambda A - B$ is nearly a pencil of index two. If $\varepsilon = 0$, then we have that the 2×2-matrix B_{22} is well-conditioned for δ not too large. But for $\varepsilon = 0$ the dimension of the finite deflating subspace changes.

If the matrices Σ and B_{22} are well-conditioned, then we can easily compute the important parts of the Weierstrass canonical form (2.2). The transformation matrices W and T are given by

$$W = V \begin{pmatrix} \Sigma & B_{12} \\ 0 & B_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} I & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} U^*,$$

and the block J associated with the finite eigenvalues has the form

$$J = \Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21}),$$

see [49]. Note that a transformation of J to Jordan form is not necessary.

Consider now the generalized Lyapunov equation (6.10). Let the matrix

$$Z = V \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix} V^* \tag{7.5}$$

partitioned conformally with A in (7.2) be the solution of (6.10). Inserting (7.2)-(7.5) in (6.10) we obtain

$$\Sigma^* Z_{11}B_{11} + \Sigma^* Z_{12}B_{21} + B_{11}^* Z_{11}\Sigma + B_{21}^* Z_{12}^*\Sigma = -(I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21}), \tag{7.6}$$

$$\Sigma^* Z_{11}B_{12} + \Sigma^* Z_{12}B_{22} = 0. \tag{7.7}$$

It follows from (7.7) that $Z_{12} = -Z_{11}B_{12}B_{22}^{-1}$. Inserting Z_{12} in (7.6) we have

$$\Sigma^* Z_{11}(B_{11} - B_{12}B_{22}^{-1}B_{21}) + (B_{11} - B_{12}B_{22}^{-1}B_{21})^* Z_{11}\Sigma = -(I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21}). \tag{7.8}$$

Thus, the matrix

$$Z = V \begin{pmatrix} Z_{11} & -Z_{11}B_{12}B_{22}^{-1} \\ -B_{22}^{-*}B_{12}^*Z_{11} & B_{22}^{-*}B_{12}^*Z_{11}B_{12}B_{22}^{-1} \end{pmatrix} V^*,$$

where Z_{11} is the solution of (7.8), satisfies (6.10) and $Z = Z\Pi$.

We rewrite the generalized Lyapunov equation (7.8) as

$$\Sigma^* Z_{11} F + F^* Z_{11}\Sigma = -G, \tag{7.9}$$

where $F = B_{11} - B_{12}B_{22}^{-1}B_{21}$ and $G = I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21}$. This equation with nonsingular Σ can be solved using the generalized Bartels-Stewart algorithm [20, 21, 45] or the sign function method [3].

Note that for computing the matrix G we have to multiply the matrices $B_{21}^* B_{22}^{-*}$ and $B_{22}^{-1}B_{21}$. This may lead to a larger sensitivity, in the worst case the condition number may be squared. In fact, this multiplication is not necessary. The matrix G can be represented as

$$G = I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21} = \begin{bmatrix} I \\ B_{22}^{-1}B_{21} \end{bmatrix}^* \begin{bmatrix} I \\ B_{22}^{-1}B_{21} \end{bmatrix}.$$

Then using the generalized Hammarling method [45] to solve (7.9) we compute the solution in factored form $Z_{11} = Y^*Y$. In this case the solution Z of (6.10) can be written as

$$Z = V \begin{bmatrix} I & -B_{12}B_{22}^{-1} \end{bmatrix}^* Y^*Y \begin{bmatrix} I & -B_{12}B_{22}^{-1} \end{bmatrix} V^*$$

and it has the norm $\|Z\| = \|Y \begin{bmatrix} I & -B_{12}B_{22}^{-1} \end{bmatrix}\|^2$.

The described methods to compute the projections P, Π and to solve the generalized Lyapunov equation (6.10) can be used if the matrices Σ and B_{22} are well-conditioned. Since $\mathrm{cond}(\Sigma) \leq \mathrm{cond}(A + BQ^{\perp})$ and $\mathrm{cond}(B_{22}) \leq \mathrm{cond}(A + BQ^{\perp})$, we can take the condition number of the matrix $A + BQ^{\perp}$ as a measure for the sensitivity of the projection onto the finite deflating subspace of the pencil $\lambda A - B$ of index at most one.

8 Perturbation analysis for the projection P

The numerical computation of the deflating subspaces associated with specified eigenvalues of a regular matrix pencil and condition estimations for this problem have been studied extensively in recent years, e.g. [1, 16, 36, 39]. Unfortunately, this problem may be ill-conditioned, since arbitrary small perturbations may change the structure of subspaces and even their dimension. In this section we present an error and perturbation analysis for the spectral projection P onto the right finite deflating subspace of the pencil $\lambda A - B$ of index at most one computed by the method described in Section 7.

The computation of the projection P requires as first step the decision about the numerical rank of A. The usual procedure is to compute the singular value decomposition of A and to set all singular values satisfying $\sigma_j < \epsilon c\|A\|$ to zero, where c is a constant and ϵ is the machine precision. If A and B are perturbed then the same procedure is performed. Due to the perturbation the numerical rank of A may change and hence also the spectral projection P may change to $\widetilde{P}$. Even if we assume that the rank decision yields the same result r in both cases, then the accuracy of P depends on the gap between σ_r and σ_{r+1} which is defined as

$$d_r = \frac{\|A\|}{\sigma_r(A) - \sigma_{r+1}(A)}. \tag{8.1}$$

Consider the perturbed matrices $\widetilde{A} = A + \Delta A$, $\widetilde{B} = B + \Delta B$, where $\|\Delta A\| \leq \varepsilon\|A\|$ and $\|\Delta B\| \leq \varepsilon\|B\|$. Let r be the numerical rank of A and let $P^{\perp}$ and $\widetilde{P}^{\perp}$ be the orthogonal projections onto the spans of the right singular vectors of A and $\widetilde{A}$, respectively, corresponding to their largest r singular values. Set $A_r = AP^{\perp}$ and $\widetilde{A}_r = \widetilde{A}\widetilde{P}^{\perp}$. Then $Q^{\perp} = I - P^{\perp}$ and $\widetilde{Q}^{\perp} = I - \widetilde{P}^{\perp}$ are the orthogonal projections onto $\ker A_r$ and $\ker \widetilde{A}_r$ with the same r. We will show that if the matrix pencil $\lambda A_r - B$ is regular of index one, then for sufficiently small ε the pencil $\lambda \widetilde{A}_r - \widetilde{B}$ is regular of index one as well.

Lemma 4 *Let d_r be as in (8.1). If the matrix $(A_r + BQ^{\perp})$ is nonsingular and*

$$\varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|) < 1,$$

where $\varepsilon_r = \varepsilon(1+2d_r)$*, then the matrix* $(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})$ *is nonsingular and*

$$\|(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1} - (A_r + BQ^{\perp})^{-1}\| \leq \frac{\varepsilon_r \|(A_r + BQ^{\perp})^{-1}\|^2(\|A\| + \|B\|)}{1 - \varepsilon_r \|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|)}. \tag{8.2}$$

PROOF. From the relation

$$(\widetilde{A}_r+\widetilde{B}\widetilde{Q}^{\perp})^{-1} = (A_r+BQ^{\perp})^{-1} - (\widetilde{A}_r+\widetilde{B}\widetilde{Q}^{\perp})^{-1}(\widetilde{A}_r - A_r + \widetilde{B}\widetilde{Q}^{\perp} - BQ^{\perp})(A_r+BQ^{\perp})^{-1} \tag{8.3}$$

we obtain the estimate

$$\|(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1}\| \leq \frac{\|(A_r + BQ^{\perp})^{-1}\|}{1 - \|\widetilde{A}_r - A_r + \widetilde{B}\widetilde{Q}^{\perp} - BQ^{\perp}\|\|(A_r + BQ^{\perp})^{-1}\|}. \tag{8.4}$$

For $2\varepsilon d_r < 1$ one has the bound

$$\|\widetilde{P}^{\perp} - P^{\perp}\| = \|\widetilde{Q}^{\perp} - Q^{\perp}\| \leq \frac{\varepsilon d_r}{1 - \varepsilon d_r}, \tag{8.5}$$

see [23]. Then using the identities $A_r = AP^{\perp}$, $\widetilde{A}_r = \widetilde{A}\widetilde{P}^{\perp}$ and $\|\widetilde{P}^{\perp}\| = \|\widetilde{Q}^{\perp}\| = 1$ we have

$$\begin{aligned} \|\widetilde{A}_r - A_r + \widetilde{B}\widetilde{Q}^{\perp} - BQ^{\perp}\| &= \|(\widetilde{A} - A)\widetilde{P}^{\perp} + A(\widetilde{P}^{\perp} - P^{\perp}) + (\widetilde{B} - B)\widetilde{Q}^{\perp} + B(\widetilde{Q}^{\perp} - Q^{\perp})\| \\ &\leq \|\widetilde{A} - A\| + \|A\|\|\widetilde{P}^{\perp} - P^{\perp}\| + \|\widetilde{B} - B\| + \|B\|\|\widetilde{Q}^{\perp} - Q^{\perp}\| \\ &\leq \varepsilon(1 + \frac{d_r}{1 - \varepsilon d_r})(\|A\| + \|B\|) \\ &\leq \varepsilon(1 + 2d_r)(\|A\| + \|B\|) = \varepsilon_r(\|A\| + \|B\|). \end{aligned} \tag{8.6}$$

Combining (8.4) and (8.6) we obtain

$$\|(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1}\| \leq \frac{\|(A_r + BQ^{\perp})^{-1}\|}{1 - \varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|)}$$

under the condition that

$$\varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|) < 1.$$

Hence $(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})$ is nonsingular if $(A_r + BQ^{\perp})$ is nonsingular. Moreover, from (8.3) and (8.6) we obtain

$$\|(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1} - (A_r + BQ^{\perp})^{-1}\| \leq \frac{\varepsilon_r\|(A + BQ^{\perp})^{-1}\|^2(\|A\| + \|B\|)}{1 - \varepsilon_r\|(A + BQ^{\perp})^{-1}\|(\|A\| + \|B\|)}.$$

□

As a consequence of Lemma 4 we have the following theorem.

Theorem 8 *Let r be the numerical rank of the matrix A and let $P^{\perp}$ be the orthogonal projection onto the span of the right singular vectors of A corresponding to its largest r singular values. Assume that the pencil $\lambda A_r - B$ is of index one, where $A_r = AP^{\perp}$. Then for $\varepsilon_r cond(A_r + BQ^{\perp})(\|A\| + \|B\|) < \|A_r + BQ^{\perp}\|$, the perturbed pencil $\lambda \widetilde{A}_r - \widetilde{B}$ is of index one. Moreover, for the spectral projections P and $\widetilde{P}$ onto the right finite deflating subspaces of $\lambda A_r - B$ and $\lambda \widetilde{A}_r - \widetilde{B}$, respectively, one has the bound*

$$\|\widetilde{P} - P\| \leq \frac{3\varepsilon_r cond^2(A_r + BQ^{\perp})(\|A\| + \|B\|)\|B\|}{\|A_r + BQ^{\perp}\|(\|A_r + BQ^{\perp}\| - \varepsilon_r cond(A_r + BQ^{\perp})(\|A\| + \|B\|))}. \tag{8.7}$$

Proof. If the pencil $\lambda A_r - B$ has index one, then $(A_r + BQ^{\perp})$ is nonsingular and the spectral projection onto the right finite deflating subspace of $\lambda A_r - B$ can be computed as

$$P = I - Q^{\perp}(A_r + BQ^{\perp})^{-1}B,$$

see [27]. Then by Lemma 4 the matrix $(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})$ is nonsingular and, hence, the matrix pencil $\lambda \widetilde{A}_r - \widetilde{B}$ is of index one and the spectral projection $\widetilde{P}$ onto the right finite deflating subspace of $\lambda \widetilde{A}_r - \widetilde{B}$ has the form

$$\widetilde{P} = I - \widetilde{Q}^{\perp}(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1}\widetilde{B}.$$

Then by adding and subtracting equal terms we obtain

$$\begin{aligned}
\|\widetilde{P} - P\| &= \|\widetilde{Q}^{\perp}(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1}\widetilde{B} - Q^{\perp}(A_r + BQ^{\perp})^{-1}B\| \leq \\
&\leq \|\widetilde{Q}^{\perp}(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1}\widetilde{B} - \widetilde{Q}^{\perp}(A_r + BQ^{\perp})^{-1}\widetilde{B}\| + \\
&+ \|\widetilde{Q}^{\perp}(A_r + BQ^{\perp})^{-1}\widetilde{B} - \widetilde{Q}^{\perp}(A_r + BQ^{\perp})^{-1}B\| + \\
&+ \|\widetilde{Q}^{\perp}(A_r + BQ^{\perp})^{-1}B - Q^{\perp}(A_r + BQ^{\perp})^{-1}B\| \leq \\
&\leq \|(\widetilde{A}_r + \widetilde{B}\widetilde{Q}^{\perp})^{-1} - (A_r + BQ^{\perp})^{-1}\|\|\widetilde{B}\| + \|(A_r + BQ^{\perp})^{-1}\|\|\widetilde{B} - B\| + \\
&+ \|\widetilde{Q}^{\perp} - Q^{\perp}\|\|(A_r + BQ^{\perp})^{-1}\|\|B\|.
\end{aligned}$$

Using the bounds (8.2), (8.5) and $\text{cond}(A_r + BQ^{\perp}) = \|A_r + BQ^{\perp}\|\|(A_r + BQ^{\perp})^{-1}\| > 1$ we obtain

$$\begin{aligned}
\|\widetilde{P} - P\| &\leq \frac{\varepsilon_r(1+\varepsilon)\|(A_r + BQ^{\perp})^{-1}\|^2(\|A\| + \|B\|)\|B\|}{1 - \varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|)} + \\
&+ \varepsilon\|(A_r + BQ^{\perp})^{-1}\|\|B\| + \frac{\varepsilon d_r}{1 - \varepsilon d_r}\|(A_r + BQ^{\perp})^{-1}\|\|B\| \\
&\leq \varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|\|B\|\frac{(1+\varepsilon)\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|) + 1}{1 - \varepsilon_r\|(A_r + BQ^{\perp})^{-1}\|(\|A\| + \|B\|)} \\
&\leq \frac{\varepsilon_r\text{cond}(A_r + BQ^{\perp})(2\text{cond}(A_r + BQ^{\perp})(\|A\| + \|B\|) + \|A_r + BQ^{\perp}\|)\|B\|}{\|A_r + BQ^{\perp}\|(\|A_r + BQ^{\perp}\| - \varepsilon_r\text{cond}(A_r + BQ^{\perp})(\|A\| + \|B\|))} \\
&\leq \frac{3\varepsilon_r\text{cond}^2(A_r + BQ^{\perp})(\|A\| + \|B\|)\|B\|}{\|A_r + BQ^{\perp}\|(\|A_r + BQ^{\perp}\| - \varepsilon_r\text{cond}(A_r + BQ^{\perp})(\|A\| + \|B\|))}.
\end{aligned}$$

□

The bound (8.7) implies that if the gap between the singular values σ_r and σ_{r+1} of the matrix A is not small, i.e., the value d_r is not large, and if the condition number of $A_r + BQ^{\perp}$ is not large then the error of the projection P is small for enough small ε. Large values of d_r and $\mathrm{cond}(A_r + BQ^{\perp})$ indicate that either the deflating subspace of the matrix pencil $\lambda A_r - B$ corresponding to the finite eigenvalues is ill-conditioned or $\lambda A_r - B$ is near to a pencil with index greater than one.

9 Sensitivity analysis for the generalized Lyapunov equation

In this section we present a bound on the sensitivity of the solution Z of the generalized Lyapunov equation (6.10). The perturbation analysis for the standard Lyapunov equation was the topic of numerous papers [24, 29, 30, 34]. The sensitivity of the generalized Lyapunov equation with nonsingular A is studied in [37]. The analysis of the general problem with a singular matrix A is very complicated and still not completely known. The difficulty is that small perturbations in the stable pencil $\lambda A - B$ may alter strongly its eigenstructure. This may lead to the change of the dimension of the finite deflating subspace, loss of the regularity or jumping of eigenvalues to the closed right half-plane [10].

In the sequel we consider only perturbations which exclude the case when the dimension of the finite deflating subspace of the pencil is changed. In many practical applications this is justified. Consider, for example, semi-explicit differential-algebraic equations

$$A_{11}x_1'(t) = B_{11}x_1(t) + B_{12}x_2(t), \tag{9.1}$$

$$0 = B_{21}x_1(t) + B_{22}x_2(t), \tag{9.2}$$

with a nonsingular matrix A_{11} [4, 42]. Equation (9.1) describes the dynamic behavior of the system, while equation (9.2) gives algebraic constraints on the states. Obviously, it is unreasonable to consider perturbations which cause the algebraic constraints to become differential.

Note that in the study of the asymptotic stability of the differential-algebraic equation (3.1) it is allowed for the index of the matrix pencil $\lambda A - B$ to be changed by perturbations. It is important only that finite eigenvalues stay finite and infinite eigenvalues must stay infinite. However, the perturbation analysis in this case is very complicated. We will deal only with perturbations which preserve the nilpotency structure of the pencil $\lambda A - B$, i.e., the right and left infinite deflating subspaces of $\lambda A - B$ are not changed. In this case

$$\ker P = \ker \widetilde{P}, \qquad \ker \Pi = \ker \widetilde{\Pi}, \tag{9.3}$$

where P and $\widetilde{P}$ (Π and $\widetilde{\Pi}$) are the spectral projections onto the right (left) finite deflating subspaces of the pencil $\lambda A - B$ and the perturbed pencil $\lambda\widetilde{A} - \widetilde{B}$, respectively. It follows from (9.3) that

$$\begin{array}{ccc} \widetilde{P}P = \widetilde{P} & \text{and} & P\widetilde{P} = P, \\ \widetilde{\Pi}\Pi = \widetilde{\Pi} & \text{and} & \Pi\widetilde{\Pi} = \Pi. \end{array} \tag{9.4}$$

Moreover, we will assume that for the allowable perturbations ΔA and ΔB of the matrix pencil $\lambda A - B$ such that $\|\Delta A\| \le \varepsilon\|A\|$ and $\|\Delta B\| \le \varepsilon\|B\|$ we have an error bound $\|\widetilde{P} - P\| \le \varepsilon K\|P\|$ with some constant K. This estimate implies that the right deflating subspace to the finite eigenvalues of the perturbed pencil $\lambda\widetilde{A} - \widetilde{B} = \lambda(A+\Delta A) - (B+\Delta B)$ is close to the corresponding right deflating subspace of $\lambda A - B$. For example, in the case of a matrix pencil $\lambda A - B$ of index at most one the bound (8.7) implies that

$$K = \frac{3(1+2d_r)\mathrm{cond}^2(A+BQ^{\perp})(\|A\|+\|B\|)\|B\|}{\|P\|\|A+BQ^{\perp}\|(\|A+BQ^{\perp}\| - \varepsilon_r \mathrm{cond}(A+BQ^{\perp})(\|A\|+\|B\|))}.$$

Nevertheless, under allowable perturbations the perturbed pencil may have a finite eigenvalue in the closed right half-plane. We will show that if all finite eigenvalues of the regular pencil $\lambda A - B$ lie in the open left half-plane then for small enough ε the pencil $\lambda\widetilde{A} - \widetilde{B}$ is regular and it has no finite eigenvalues with nonnegative real part.

Consider the integral equation

$$X = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*(\widetilde{P}^*\widetilde{P} - D(X))P(i\xi A - B)^{-1}d\xi, \tag{9.5}$$

with $D(X) = (\Delta A)^* X\widetilde{B} + A^* X\Delta B + (\Delta B)^* XA + \widetilde{B}^* X\Delta A$. We will show that this equation has a unique solution X.

Lemma 5 *Let* $æ(A,B) \le \infty$ *and let* ΔA, ΔB *be the perturbations of the pencil* $\lambda A - B$ *such that* $\|\Delta A\| \le \varepsilon\|A\|$ *and* $\|\Delta B\| \le \varepsilon\|B\|$. *Assume that relations (9.4) are satisfied and* $\|\widetilde{P} - P\| \le \varepsilon K\|P\|$ *with some constant* K. *If* $3\varepsilon æ(A,B) \le 1/2 < 1$, *then the integral equation (9.5) has a unique solution.*

Proof. The solution of equation (9.5) can be obtained by the method of successive approximations [35]. Define a sequence of matrices X_j by the recurrent formula

$$X_j = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*(\widetilde{P}^*\widetilde{P} - D(X_{j-1}))P(i\xi A - B)^{-1}d\xi, \tag{9.6}$$

with $X_0 = 0$. We have

$$\begin{aligned} X_1 - X_0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*\widetilde{P}^*\widetilde{P}P(i\xi A - B)^{-1}d\xi, \\ X_j - X_{j-1} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*D(X_{j-2} - X_{j-1})P(i\xi A - B)^{-1}d\xi, \quad j = 2,3,\ldots. \end{aligned}$$

Then

$$\begin{aligned} \|X_1\| &\le \left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*\widetilde{P}^*\widetilde{P}P(i\xi A - B)^{-1}d\xi\right\| \le \\ &\le \|\widetilde{P}\|^2\left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A - B)^{-*}P^*P(i\xi A - B)^{-1}d\xi\right\| = \|\widetilde{P}\|^2\|Z\| \end{aligned} \tag{9.7}$$

and

$$\|X_j - X_{j-1}\| \le \|D(X_{j-2} - X_{j-1})\|\|Z\|.$$

From

$$\begin{aligned} \|D(X)\| &= \|(\Delta A)^* X \widetilde{B} + A^* X \Delta B + (\Delta B)^* X A + \widetilde{B}^* X \Delta A\| \le \\ &\le 2(\|\Delta A\|\|\widetilde{B}\| + \|\Delta B\|\|A\|)\|X\| \le 6\varepsilon \|A\|\|B\|\|X\| \end{aligned} \tag{9.8}$$

we obtain

$$\begin{aligned} \|X_j - X_{j-1}\| &\le 6\varepsilon\|A\|\|B\|\|X_{j-1} - X_{j-2}\|\|Z\| = 3\varepsilon æ(A,B)\|X_{j-1} - X_{j-2}\| \le \dots \le \\ &\le (3\varepsilon æ(A,B))^{j-1}\|X_1\| \le (3\varepsilon æ(A,B))^{j-1}\|\widetilde{P}\|^2\|Z\|. \end{aligned} \tag{9.9}$$

Since $3\varepsilon æ(A,B) < 1$, the sequence X_j converges to a matrix X_∞ satifying

$$\begin{aligned} \|X_\infty\| &\le \sum_{j=1}^{\infty} \|X_j - X_{j-1}\| \le \|\widetilde{P}\|^2\|Z\| \sum_{j=1}^{\infty} (3\varepsilon æ(A,B))^{j-1} = \frac{\|\widetilde{P}\|^2\|Z\|}{1 - 3\varepsilon æ(A,B)}, \\ \|X_\infty - X_j\| &\le \sum_{l=j+1}^{\infty} \|X_l - X_{l-1}\| \le \|\widetilde{P}\|^2\|Z\| \sum_{l=j+1}^{\infty} (\varepsilon æ(A,B))^{l-1} = \\ &= \frac{\|\widetilde{P}\|^2\|Z\|\,(\delta æ(A,B))^j}{1 - 3\varepsilon æ(A,B)}. \end{aligned}$$

Therefore, we take $j \to \infty$ in both sides of (9.6) and we obtain that the matrix $X = X_\infty$ satisfies the integral equation (9.5).

Assume now that the solution of (9.5) is not unique, i.e., there exist two matrices $X^{(1)}$ and $X^{(2)}$ satisfying (9.5). We have

$$X^{(1)} - X^{(2)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* D(X^{(2)} - X^{(1)}) P (i\xi A - B)^{-1} d\xi$$

and, hence,

$$\|X^{(1)} - X^{(2)}\| \le 3\varepsilon æ(A,B)\|X^{(1)} - X^{(2)}\|.$$

Then $3\varepsilon æ(A,B) \ge 1$, which contradicts the assumption of the theorem. Thus, the integral equation (9.5) has a unique solution. □

Using the Weierstrass canonical form (2.2) for the pencil $\lambda A - B$ it is easy to verify that $\Pi A = \Pi A P$ and $\Pi B = \Pi B P$. Analogously, for the perturbed pencil $\lambda \widetilde{A} - \widetilde{B}$ we have $\widetilde{\Pi}\widetilde{A} = \widetilde{\Pi}\widetilde{A}\widetilde{P}$ and $\widetilde{\Pi}\widetilde{B} = \widetilde{\Pi}\widetilde{B}\widetilde{P}$. Then by (9.4) we obtain $X = X\Pi = X\Pi\widetilde{\Pi} = X\widetilde{\Pi}$ and

$$\begin{aligned} XA = X\Pi A = XAP = X\Pi A P \widetilde{P} = XA\widetilde{P}, \\ X\widetilde{A} = X\widetilde{\Pi}\widetilde{A} = X\widetilde{A}\widetilde{P} = X\widetilde{\Pi}\widetilde{A}\widetilde{P}P = X\widetilde{A}P. \end{aligned}$$

Similarly, we have $XB = XBP = XB\widetilde{P}$ and $X\widetilde{B} = X\widetilde{B}\widetilde{P} = X\widetilde{B}P$ and hence

$$P^* D(X) P = D(X) = \widetilde{P}^* D(X) \widetilde{P}. \tag{9.10}$$

Then, for all nonzero $z \in \operatorname{im} P$, we get

$$\begin{aligned} ((\widetilde{P}^*\widetilde{P} - D(X))z, z) &= ((\widetilde{P}^*\widetilde{P} - \widetilde{P}^*D(X)\widetilde{P})z, z) \geq (\widetilde{P}z, \widetilde{P}z) - \|D(X)\|(\widetilde{P}z, \widetilde{P}z) = \\ &= (1 - \|D(X)\|)\|\widetilde{P}z\|^2. \end{aligned} \tag{9.11}$$

Assuming that $\widetilde{P}z = 0$ we obtain from (9.4) that $z \in \ker P$, but $z \in \operatorname{im} P$ and $z \neq 0$. Hence, $\widetilde{P}z \neq 0$. Furthermore, from (9.5) and (9.8) we have $\|X\| \leq (\|\widetilde{P}\|^2 + 6\varepsilon\|A\|\|B\|\|X\|)\|Z\|$. Then using the estimate $\|\widetilde{P}\| \leq \|\widetilde{P} - P\| + \|P\| \leq (1 + \varepsilon K)\|P\|$ we obtain

$$\|X\| \leq \frac{(1+\varepsilon K)^2\|P\|^2\|Z\|}{1 - 6\varepsilon\|A\|\|B\|\|Z\|} = \frac{(1+\varepsilon K)^2\|P\|^2\|Z\|}{1 - 3\varepsilon æ(A, B)} \tag{9.12}$$

and

$$\begin{aligned} \|D(X)\| &\leq \frac{6\varepsilon(1+\varepsilon K)^2\|A\|\|B\|\|Z\|\|P\|^2}{1 - 3\varepsilon æ(A, B)} = \frac{3\varepsilon(1+\varepsilon K)^2 æ(A, B)\|P\|^2}{1 - 3\varepsilon æ(A, B)} \leq \\ &\leq 6\varepsilon(1+\varepsilon K)^2 æ(A, B)\|P\|^2. \end{aligned}$$

If $6\varepsilon(1+\varepsilon K)^2 æ(A, B)\|P\|^2 < 1$, then it follows from (9.11) that

$$((\widetilde{P}^*\widetilde{P} - D(X))z, z) > 0 \tag{9.13}$$

for all nonzero $z \in \operatorname{im} P$, i.e., the matrix $\widetilde{P}^*\widetilde{P} - D(X)$ is positive definite on the subspace $\operatorname{im} P$ and the solution X of (9.5) is positive semidefinite. If all finite eigenvalues of the pencil $\lambda A - B$ have negative real part, by Theorem 7 and Remark 6 the matrix X satisfies the matrix equation

$$A^*XB + B^*XA = -P^*(\widetilde{P}^*\widetilde{P} - D(X))P.$$

Using (9.4) and (9.10) we rewrite this equation as

$$\widetilde{A}^*X\widetilde{B} + \widetilde{B}^*X\widetilde{A} = -\widetilde{P}^*\widetilde{P}, \tag{9.14}$$

and we have that the Hermitian, positive semidefinite matrix X satisfies equation (9.14) and $X = X\widetilde{\Pi}$. Then by Theorem 6 all finite eigenvalues of the pencil $\lambda\widetilde{A} - \widetilde{B}$ lie in the left half-plane. Moreover, we can estimate the error of the matrix Z via

$$\begin{aligned} \|X - Z\| &= \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*}(\widetilde{P}^*\widetilde{P} - D(X) - P^*P)(i\xi A - B)^{-1} d\xi \right\| \leq \\ &\leq (\|\widetilde{P}^*\widetilde{P} - P^*P\| + \|D(X)\|)\|Z\|. \end{aligned}$$

Since

$$\|\widetilde{P}^*\widetilde{P} - P^*P\| \leq 3\|\widetilde{P} - P\|\|P\| \leq 3\varepsilon K\|P\|^2,$$

we obtain

$$\|X - Z\| \leq \frac{3\varepsilon(K + (1+\varepsilon K)^2 æ(A, B))\|P\|^2\|Z\|}{1 - 3\varepsilon æ(A, B)}.$$

Thus, we proved the following theorem.

Theorem 9 *Let $\lambda A - B$ be a regular matrix pencil and let Z be the Hermitian, positive semidefinite solution of the generalized Lyapunov equation (6.10) satisfying $Z = Z\Pi$. Consider a perturbed pencil $\lambda\widetilde{A} - \widetilde{B}$ such that (9.4) holds, $\|\widetilde{A} - A\| \le \varepsilon\|A\|$, $\|\widetilde{B} - B\| \le \varepsilon\|B\|$ and $\|\widetilde{P} - P\| \le \varepsilon K\|P\|$ with some constant K. If $6\varepsilon(1+\varepsilon K)^2 æ(A,B)\|P\|^2 < 1$, then (9.14) has a unique Hermitian, positive semidefinite solution X such that $X = X\widetilde{\Pi}$ and*

$$\frac{\|X - Z\|}{\|Z\|} \le \frac{3\varepsilon(K + (1+\varepsilon K)^2 æ(A,B))\|P\|^2}{1 - 3\varepsilon æ(A,B)}. \tag{9.15}$$

Now it is possible to derive a relative error bound for the parameter æ(A,B) under allowable perturbations.

Corollary 2 *Under the assumptions of Theorem 9 we have $æ(\widetilde{A},\widetilde{B}) < \infty$ and*

$$\frac{|æ(\widetilde{A},\widetilde{B}) - æ(A,B)|}{æ(A,B)} \le \frac{3\varepsilon(K + 2(1+\varepsilon K)^2 æ(A,B))æ(A,B)}{1 - 3\varepsilon æ(A,B)}. \tag{9.16}$$

PROOF. The boundedness of $æ(\widetilde{A},\widetilde{B})$ immediately follows from Theorem 9. Using (9.12) and (9.15) we have

$$\begin{aligned}
|æ(\widetilde{A},\widetilde{B}) - æ(A,B)| &= 2|\,\|\widetilde{A}\|\|\widetilde{B}\|\|X\| - \|A\|\|B\|\|Z\|\,| \\
&\le 2(\|\widetilde{A} - A\|\|\widetilde{B}\|\|X\| + \|A\|\|\widetilde{B} - B\|\|X\| + \|A\|\|B\|\|X - Z\|) \\
&\le \frac{3\varepsilon æ^2(A,B)((1+\varepsilon K)^2 + K + (1+\varepsilon K)^2 æ(A,B))\|P\|^2}{1 - 3\varepsilon æ(A,B)}.
\end{aligned}$$

Taking into account that

$$1 \le \|P\|^2 = \|P^*P\| = \|A^*ZB + B^*ZA\| \le 2\|A\|\|B\|\|Z\| = æ(A,B),$$

we obtain the estimate (9.16). □

10 Numerical experiments

In this section we present results of numerical experiments of computing the projection P and the parameter $æ(A,B)$. Computations were performed in MATLAB 5.2 on HP-UX 10.20 workstation using double precision arithmetic with machine precision $\epsilon \approx 2.2\cdot 10^{-16}$. In the rank decision problem we set the computed singular value σ_j to zero if $\sigma_j \le \epsilon n\|A\|$. The number of remaining nonzero singular values is taken to be the numerical rank of the matrix. To solve the generalized Lyapunov equation (7.9) we use the generalized Bartels-Stewart method from [45]. The normalized residual

$$\Delta = \frac{\|A^*ZB + B^*ZA + P^*P\|}{2\|Z\|\,\|A\|\,\|B\|}$$

is a measure of the quality of the computed solution of the generalized Lyapunov equation (6.10).

Example 4.[14, Example 1-3.1], [7, Example 2] Consider the system

$$Ax'(t) = Bx(t) + Fv_s(t) \tag{10.1}$$

with the measured output $y(t) = Gx(t)$, where

$$A = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \tag{10.2}$$

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \qquad x(t) = \begin{pmatrix} I(t) & v_L(t) & v_C(t) & v_R(t) \end{pmatrix}^T.$$

Equation (10.1) with (10.2) describes a simple RLC electrical circuit. The voltage source $v_s(t)$ is the control input, $R = 2$, $L = 1.1$ and $C = 10^{-4}$ are the resistance, inductance and capacitance, respectively, $v_R(t)$, $v_L(t)$ and $v_C(t)$ are the corresponding voltage drops and $I(t)$ is the current. For the proportional output feedback control $v_s(t) = Ky(t) = KGx(t)$ we have the closed loop system

$$Ax'(t) = (B + FKG)x(t).$$

The finite eigenvalues of the matrix pencil $\lambda A - B_K$ with $B_K = B + FKG$ are given by

$$\lambda_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{K-1}{CL}}.$$

It is easy to see that if $K \geq 1$ then the pencil $\lambda A - B_K$ has one eigenvalue in the closed right half-plane and both its finite eigenvalues lie in $\mathbb{C}^-$, otherwise.

The following table gives the numerical results for different values of K. For all K the gap is $d_2 = 1.1$.

K	$\mathrm{cond}(A_2 + B_K Q^{\perp})$	Δ	$\Vert Z\Vert$	$æ(A, B_K)$
0	3.9022	$3.57 \cdot 10^{-20}$	$1.5006 \cdot 10^4$	$3.3013 \cdot 10^8$
$1 - 10^{-2}$	3.9022	$2.36 \cdot 10^{-20}$	$7.5008 \cdot 10^5$	$1.6502 \cdot 10^{10}$
$1 - 10^{-4}$	3.9022	$2.53 \cdot 10^{-20}$	$7.5000 \cdot 10^7$	$1.6500 \cdot 10^{12}$
$1 - 10^{-6}$	3.9022	$1.73 \cdot 10^{-20}$	$7.5000 \cdot 10^9$	$1.6500 \cdot 10^{14}$
1	3.9022	–	∞	∞

We see that as K approaches to 1, the values of $\Vert Z\Vert$ and, respectively, $æ(A, B_K)$ increase. For $K = 1$ the Lyapunov equation (6.10) is not solvable.

Example 5.[28] The following example is a model for the transistor amplifier. The equation has the form

$$A\frac{dy}{dt} = f(y), \tag{10.3}$$

where

$$A = \begin{bmatrix} -C_1 & C_1 & 0 & 0 & 0 \\ C_1 & -C_1 & 0 & 0 & 0 \\ 0 & 0 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & -C_3 & C_3 \\ 0 & 0 & 0 & C_3 & -C_3 \end{bmatrix}, \; f(y) = \begin{bmatrix} -\frac{U_e(t)}{R_0} + \frac{y_0}{R_0} \\ -\frac{6}{R_2} + y_2(\frac{1}{R_1} + \frac{1}{R_2}) + 0.01g(y_2 - y_3) \\ -g(y_2 - y_3) + \frac{y_3}{R_3} \\ -\frac{6}{R_4} + \frac{y_4}{R_4} + 0.99g(y_2 - y_3) \\ \frac{y_4}{R_5} \end{bmatrix}$$

with

$$g(x) = 10^{-6}\left(e^{x/0.026} - 1\right); \qquad U_e(t) = 0.1\sin(200\pi t); \qquad R_0 = 1000;$$
$$C_k = k \cdot 10^{-6}, \quad k = 1,2,3; \qquad R_k = 9000, \quad k = 1,\ldots,5.$$

Asymptotic stability of the stationary solution y_* of (10.3) is equivalent to asymptotic stability of the trivial solution of the linearized system $Ax'(t) = Bx(t)$ with $B = f'(y_*)$ [40].

The stationary solution is given by $y_* = (\,0,\ 2.98582,\ 2.83616,\ 3.19220,\ 0\,)^T$. The following computed parameters

$$d_3 = 3, \qquad \mathrm{cond}(A_3 + BQ^{\perp}) = 7.9915 \cdot 10^4, \qquad \|P\| = 80.9228,$$
$$\Delta = 2.3526 \cdot 10^{-18}, \qquad æ(A,B) = 2.0789 \cdot 10^6.$$

show that the pencil $\lambda A - B$ is of index 1 and has no finite eigenvalues in the closed right half-plane, i.e., the stationary solution y_* of (10.3) is asymptotically stable.

Conclusion

We have derived a parameter that can be used to investigate the asymptotic stability of the trivial solution of linear DAE without explicit computing the eigenvalues of the corresponding matrix pencil. To compute this parameter it is necessary to compute the spectral projections onto the right and left deflating subspaces of the pencil corresponding to the finite eigenvalues and to solve a generalized Lyapunov equation. We have described a method for computing such projections and for solving the generalized Lyapunov equation for the matrix pencil of index at most one. This method is based on the singular value decomposition and admits error analysis for the computed projection. The sensitivity of the generalized Lyapunov equation under allowable perturbations which preserve the nilpotency structure of the pencil has been discuss. The computation of the projection onto the finite deflating subspace and the solution of the generalized Lyapunov equation for a pencil of higher index together with a complete perturbation analysis are still open problems and currently under investigation.

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References

[1] Zh. Bai, J. Demmel, M. Gu. *An inverse free parallel spectral divide and conquer algorithm for nonsymmetric eigenproblems.* – Numer. Math. 76, 1997, pp.279-308.

[2] R.H. Bartels, G.W. Stewart. *Solution of the Equation $AX + XB = C$.* – Comm. A.C.M. 15, No. 9, 1972, pp. 820-826.

[3] P. Benner, E.S. Quintana-Ortí. *Solving Stable Generalized Lyapunov Equations with the matrix Sign Function.* – Numerical Algorithms 20, No. 1, 1999, pp. 75-100.

[4] K.E. Brenan, S.L. Campbell, L.R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* – Classics in Applied Mathematics. 14. Philadelphia, PA: SIAM, 1996.

[5] A. Ja. Bulgakov. *An efficiently calculable parameter for the stability property of a system of linear differential equations with constant coefficients.* – Siberian Math. J. 21, No.3, 1980, pp.339-347.

[6] A. Ja. Bulgakov. *The basis of guaranteed accuracy in the problem of separation of invariant subspaces for non-self-adjoint matrices.* – Sib. Adv. Math. 1, No.1, 2, 1991, pp. 64-108, 1-56; transl. from Tr. Inst. Mat. 15, 1989, pp. 12-93.

[7] A.Bunse-Gerstner, R. Byers, V. Mehrmann, N.K. Nichols. *Feedback Design for Regularizing Descriptor Systems.* – to appear in Linear Algebra Appl.

[8] R. Byers. *A bisection method for measuring the distance of a stable matrix to the unstable matrices.* – SIAM J. Sci. Stat. Comput. 9, 1988, pp. 875-881.

[9] R. Byers, C. He, V. Mehrmann. *Where is the Nearest Non-Regular Pencil* – Linear Algebra Appl., Vol. 285, 1998, pp. 81-105.

[10] R. Byers, N.K. Nichols. *On the Stability Radius of a Generalized State-Space System.* – Linear Algebra Appl., Vol. 188/189, 1993, pp. 113-134.

[11] S.L. Campbell, C.D. Meyer. *Generalized Inverses of Linear Transformations.* – Pitman, London, 1979.

[12] S.L. Campbell. *Singular Systems of Differential Equation, I.* – Pitman, San Francisco, 1980.

[13] S.L. Campbell. *Singular Systems of Differential Equation, II.* – Research Notes in Mathematics, 61. Pitman, San Francisco, 1982.

[14] L. Dai. *Singular Control Systems.* – Lecture Notes in Control and Information Sciences, 118. Springer-Verlag, Berlin, Heidelberg, 1989.

[15] F. Delebecque. *On the Resolvent Approach to the Spectral Decomposition of a Regular Matrix Pencil.* – Linear Algebra Appl., Vol. 129, 1990, pp. 63-75.

[16] J.W. Demmel, B. Kågström. *Computing Stable Eigendecomposition of Matrix Pencils.* – Linear Algebra Appl., Vol. 88/89, 1987, pp. 139-186.

[17] J.L.M. van Dorsselaer. *Pseudospectra for matrix pencils and stability of equilibria.* – BIT 37, No. 4, 1997, pp. 833-845.

[18] V. Frayssé, M. Gueury, F. Nicoud, V. Toumazou. *Spectral portraits for matrix pencils.* – CERFACS Technical Report TR/PA/96/19, CERFACS, 42av. Coriolis, 31057 Toulouse Cedex, France, 1996.

[19] F.R. Gantmacher. *The Theory of Matrices.* – New York: Chelsea, 1959.

[20] J.D. Gardiner, A.J. Laub, J.J. Amato, C.B. Moler. *Solution of the Sylvester Matrix Equation* $AXB^T + CXD^T = E$. – ACM Trans. Math. Software 18, 1992, pp. 223-231.

[21] J.D. Gardiner, M.R. Wette, A.J. Laub, J.J. Amato, C.B. Moler. *Algorithm 705: A Fortran-77 Software Package for Solving the Sylvester Matrix Equation* $AXB^T + CXD^T = E$. – ACM Trans. Math. Software 18, No. 2, 1992, pp. 232-238.

[22] S.K. Godunov. *Problem of the dichotomy of the spectrum of a matrix.* – Siberian Math. J. 27, No. 5, 1986, pp.649-660.

[23] S.K. Godunov, A.G. Antonov, O.P. Kirilyuk, V.I. Kostin. *Guaranteed accuracy in numerical linear algebra. Updated and rev. transl. of the original work 'The guaranteed precision of the linear equations solutions in Euclidean spaces'.* – Mathematics and its Applications, Dordrecht: Kluwer Academic Publishers, 1993.

[24] S.K. Godunov. *Ordinary differential equations with constant coefficients. Vol. 1. Boundary value problems.* – Novosibirsk University, Novosibirsk, 1994.

[25] S.K. Godunov. *Modern aspects of linear algebra.* – Translations of Mathematical Monographs 175. Providence, RI: American Mathematical Society, 1998.

[26] G.H. Golub, C.F. Van Loan. *Matrix Computations. 3rd ed.* – Johns Hopkins University Press, Baltimore, London, 1996.

[27] E. Griepentrog, R. März. *Differential-algebraic equations and their numerical treatment.* – Teubner-Texte Math. 88, Leipzig, 1986.

[28] E. Hairer, G. Wanner. *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems.* – Springer Verlag, Berlin, Heidelberg, New York, 1996.

[29] S.J. Hammarling. *Numerical solution of the stable, non-negative definite Lyapunov equation.* – IMA J. Numer. Anal. 2, 1982, pp. 303-323.

[30] G. Hewer, C. Kenney. *The sensitivity of the stable Lyapunov equation.* – SIAM J. Control Optim. 26, No. 2, 1988, pp. 321-344.

[31] C. He, G. A. Watson *An algorithm for computing the distance to instability.* – SIAM J. Matrix Anal. Appl. 20, No. 1, 1998, pp. 101-116.

[32] D. Hinrichsen, A.J. Pritchard. *Real and complex stability radii: a survey,* – in D. Hinrichsen and B. Mårtensson (eds.) Control of Uncertain Systems, Proc. Int. Workshop, Bremen 1989. Prog. Syst. Control Theory 6, 1990, pp. 119-162.

[33] R.A. Horn, C.R. Johnson. *Topics in matrix analysis.* – Cambridge University Press, Cambridge, 1991.

[34] E. Jonckheere. *New bound on the sensitivity of the solution of the Lyapunov equation.* – Linear Algebra Appl., Vol. 60, 1984, pp. 57-64.

[35] L.V. Kantorovich, G.P. Akilov. *Functional analysis.* – Pergamon Press, Oxford, 1982.

[36] B. Kågström, P. Poromaa. *Computing eigenspaces with specified eigenvalues of a regular matrix pencil* (A, B) *and condition estimation: theory, algorithms and software.* – Numer. Algorithms 12, 1996, pp. 369-407.

[37] M.M. Konstantinov, P.Hr. Petkov, D.W. Gu, V. Mehrmann, *Sensitivity of general Lyapunov equations.* – Technical Report 98-15, Depart. of Engineering, Leicester University, Leicester LE1 7RH, UK, 1998.

[38] P. Lancaster and M. Tismenetsky. *The Theory of Matrices. 2nd ed.* – Computer Science and Applied Mathematics, Academic Press, Orlando, 1985.

[39] A.N. Malyshev. *Parallel Algorithm for Solving Some Spectral Problems of Linear Algebra.* – Linear Algebra Appl., Vol. 188/189, 1993, pp. 489-520.

[40] R. März. *Practical Lyapunov stability criteria for differential algebraic equations.* – Numerical analysis and mathematical modelling. Banach Cent. Publ. 29, Inst. of Mathematics, Polish Academy of Sciences, Warszawa, 1994, pp. 245-266.

[41] R. März. *Criteria for the trivial solution of differential algebraic equations with small nonlinearities to be asymptotically stable.* – J. Math. Anal. Appl. 225, No.2, 1998, pp.587-607.

[42] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution.* – Lecture Notes in Control and Information Sciences, 163, Springer-Verlag Berlin, Heidelberg, 1991.

[43] P.C. Müller. *Stability of linear mechanical systems with holonomic constraints.* – Appl. Mech. Rev. 46, No. 11, part 2, 1993, pp.160-164.

[44] P.C. Müller. *Stability of Nonlinear Descriptor Systems.* – Z. Angew. Math. Mech. 76, Supplement 4, 1996, pp. 6-12.

[45] T. Penzl. *Numerical solution of generalized Lyapunov equations.* – Adv. Comput. Math. 8, No. 1-2, 1998, pp. 33-48.

[46] L. Qiu, E.J. Davison. *The stability robustness of generalized eigenvalues.* – IEEE Trans. Autom. Control 37, No. 6, 1992, pp. 886-891.

[47] L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young, J.C. Doyle. *A formula for computation of the real stability radius.* – Automatica 31, No. 6, 1995, pp. 879-890.

[48] W. Rudin. *Real and Complex Analysis.* – Int. Student Edition, McGraw-Hill, 1970.

[49] T. Shtykel. *On the criterion of asymptotical stability for index-1 tractable DAEs.* – Preprint Nr. 98-6, Humboldt-Universität Berlin, Math.-Naturwissenschaftliche Fakultät II, Institut für Mathematik, D-10099 Berlin, 1998.

[50] G.W. Stewart, J. Sun. *Matrix perturbation theory.* – Academic Press, Inc., 1990.

[51] C. Tischendorf. *On the Stability of Solutions of Autonomous Index-1 Tractable and Quasilinear Index-2 Tractable DAEs.* – Circuits Systems Signal Process, Vol. 13, No. 2-3, 1994, pp. 139-154.

[52] L.N. Trefethen, A.E. Trefethen, S.C. Reddy, T.A. Driscoll. *Hydrodynamic Stability without Eigenvalues.* – Science 261, 1993, pp. 578-584.

[53] L.N. Trefethen. *Spectra and Pseudospectra: The Behavior of Non-Normal Matrices and Operators.* – The Graduate Student's Guide to Numerical Analysis, Springer, Berlin, to appear.

[54] C.F. Van Loan. *How near is a stable matrix to an unstable matrix?* – Contemp. Math. 47, 1985, pp. 465-478.