# Lax pair formulation for a small-polaron chain with integrable boundaries

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**Abstract.** Using a fermionic version of the Lax pair formulation, we construct an integrable small-polaron model with general open boundary conditions. The Lax pair and the boundary supermatrices $K_{\pm}$ for the model are obtained. This provides a direct proof of the integrability of the model.

**Keywords:** Correlated electrons; Yang-Baxter equation; Integrable boundary conditions; Lax pair formulation

## 1 Introduction

Recently, there has been renewed interest in integrable boundary systems [1, 2] due to their connection to the Kondo problem [3] and boundary conformal field theory [4] in low-dimensional quantum many-body systems. Sklyanin [5] proposed a systematic approach to construct and solve integrable quantum spin systems with open boundary conditions (BC). Central to his method is the introduction of the so-called reflection equations (RE), which are the boundary analogues of the Yang-Baxter equations (YBE) [6]. For integrable fermion systems, in particular models of strongly correlated electrons, graded versions of the YBE and the RE [7] have been applied. However, solutions to the graded RE have so far been found only for certain models due to the mathematical complexity of the graded RE. An alternative method to Sklyanin's approach is the Lax pair formulation for integrable quantum systems with open BC [8]. In this article, we use the Lax pair formulation to construct an integrable model of correlated fermions with general BC. The Lax pair and the boundary supermatrices $K_{\pm}$ are given explicitly.

## 2 Fermionic version of the Lax pair formulation

We first recall the Lax pair formulation for integrable fermion models with general open BC in one dimension [8]. We consider an operator version of the auxiliary linear problem

$$\Phi_{j+1} \quad = \quad L_j(u)\Phi_j, \quad j = 1, 2, \ldots, N,$$

$$\frac{d}{dt}\Phi_j = M_j(u)\Phi_j, \quad j = 2, 3, \ldots, N, \tag{1}$$

with boundary equations

$$\begin{aligned} \frac{d}{dt}\Phi_{N+1} &= M_+(u)\Phi_{N+1} \\ \frac{d}{dt}\Phi_1 &= M_-(u)\Phi_1. \end{aligned} \tag{2}$$

Here, $L_j(u)$, $M_j(u)$, and $M_\pm(u)$ are matrices depending on the spectral parameter $u$; $u$ itself does neither depend on the time $t$, nor on dynamical variables. Evidently, the consistency conditions for Eqs. (1) and (2) yield the Lax equations

$$\frac{d}{dt}L_j(u) = M_{j+1}(u)L_j(u) - L_j(u)M_j(u), \quad j = 2, 3, \ldots, N-1, \tag{3}$$

and

$$\begin{aligned} \frac{d}{dt}L_N(u) &= M_+(u)L_N(u) - L_N(u)M_N(u), \\ \frac{d}{dt}L_1(u) &= M_2(u)L_1(u) - L_1(u)M_-(u). \end{aligned} \tag{4}$$

Let $T(u) = L_N(u)\cdots L_2(u)L_1(u)$ be the usual monodromy matrix [6] of the system and $K_-(u)$, $K_+(u)$ the supermatrices for the left and the right boundary, respectively. The transfer matrix $\tau(u)$ is defined as the supertrace on the auxiliary space $V_0$ as

$$\tau(u) = \mathrm{str}_0\left[K_+(u)T(u)K_-(u)T^{-1}(-u)\right]. \tag{5}$$

From the Lax equations (3) and (4), it follows that the transfer matrix $\tau(u)$ does not depend on time $t$, provided the boundary matrices satisfy the conditions

$$\begin{aligned} K_-(u)M_-(-u) &= M_-(u)K_-(u) \\ \mathrm{str}_0\left[K_+(u)M_+(u)\tilde{T}(u)\right] &= \mathrm{str}_0\left[K_+(u)\tilde{T}(u)M_+(-u)\right], \end{aligned} \tag{6}$$

where $\tilde{T}(u) = T(u)K_-(u)T^{-1}(-u)$. This implies that the system possesses an infinite number of independent conserved quantities, and thus is completely integrable.

## 3 The small-polaron model with general open BC

We consider the small-polaron model [9], which describes the motion of an additional electron in a polar crystal with external field parallel to the transverse direction. The Hamiltonian reads

$$\begin{aligned} H &= -J\sum_{j=2}^{N}(a_j^\dagger a_{j-1} + a_{j-1}^\dagger a_j) + V\sum_{j=2}^{N} n_j n_{j-1} + W\sum_{j=1}^{N} n_j \\ &\quad + p_+ n_N + \alpha_+ a_N^\dagger + \beta_+ a_N + p_- n_1 + \alpha_- a_1^\dagger + \beta_- a_1, \end{aligned} \tag{7}$$

where in the boundary terms $p_\pm$, $\alpha_\pm$ and $\beta_\pm$ are Grassmann variables, with $p_\pm$ even and $\alpha_\pm$, $\beta_\pm$ odd. Hermiticity of the Hamiltonian requires $\alpha_\pm^\dagger = \beta_\pm$ and $p_\pm^\dagger =$

$p_\pm$ The fermionic creation and annihilation operators $a_j^\dagger$ and $a_j$ satisfy the usual anticommutation relations, and $n_j = a_j^\dagger a_j$. We parametrize the coupling parameters $J$, $V$, and $W$ as

$$\begin{aligned} J &= 1, \\ V &= -2\cos(2\eta), \\ W &= 2\sin(2\eta)\tan(\omega) + \cos(2\eta), \end{aligned} \tag{8}$$

in terms of $\eta$ and $\omega$. The monodromy matrix corresponding to the Hamiltonian (7) is given in Ref. [10], and obeys the graded YBE. This ensures the integrability of the model with periodic BC. In what follows, we shall use this monodromy matrix and construct the boundary $K$-matrices as described in Sec. 2 in order to show the integrability of the Hamiltonian.

## 4 Constructing the Lax pair at the boundaries

The equations of motion $-i\frac{d}{dt}O = [H, O]$ for the fermionic operators at the left boundary are

$$\begin{aligned} -i\frac{d}{dt}a_1^\dagger &= -Ja_2^\dagger + Vn_2a_1^\dagger + (W + p_-)a_1^\dagger + \beta_-, \\ -i\frac{d}{dt}a_1 &= Ja_2 - Vn_2a_1 - (W + p_-)a_1 + \alpha_-, \\ -i\frac{d}{dt}n_1 &= -J(a_2^\dagger a_1 + a_2a_1^\dagger) - \alpha_-a_1^\dagger + \beta_-a_1. \end{aligned} \tag{9}$$

At the right boundary, they are given by

$$\begin{aligned} -i\frac{d}{dt}a_N^\dagger &= -Ja_{N-1}^\dagger + Vn_{N-1}a_N^\dagger + (W + p_+)a_N^\dagger + \beta_+, \\ -i\frac{d}{dt}a_N &= Ja_{N-1} - Vn_{N-1}a_N - (W + p_+)a_N + \alpha_+, \\ -i\frac{d}{dt}n_N &= J(a_N^\dagger a_{N-1} + a_Na_{N-1}^\dagger) - \alpha_+a_N^\dagger + \beta_+a_N. \end{aligned} \tag{10}$$

Using the Lax matrices $L_j(u)$ and $M_j(u)$ for the bulk part as given in Ref. [10], together with Eq. (4), we can construct the boundary Lax matrices

$$M_\pm(u) = \begin{pmatrix} A_\pm(u) & B_\pm(u) \\ C_\pm(u) & D_\pm(u) \end{pmatrix}. \tag{11}$$

After some algebra, we find

$$\begin{aligned} A_\pm(u) &= \frac{1}{s_{+2}s_{-2}}\Big[i\sin^2 2\eta\,(p_\pm + \cos 2\eta)(1 - n_\pm) + s_{\pm 2}\left(is_{\mp 2} \pm \xi_+^{\pm 2}\sin u\right)\alpha_\pm a_\pm^\dagger \\ &\qquad + s_{\mp 2}\left(is_{\pm 2} \mp \xi_+^{\mp 2}\sin u\right)\beta_\pm a_\pm\Big] - i\sin 2\eta\tan\omega, \qquad (12) \\ D_\pm(u) &= \frac{1}{s_{+2}s_{-2}}\Big[-i\sin^2 2\eta\,(p_\pm - \cos 2\eta)n_\pm + s_{\mp 2}\left(is_{\pm 2} \mp \xi_+^{\pm 2}\sin u\right)\alpha_\pm a_\pm^\dagger \end{aligned}$$

$$+s_{\pm 2}\left(is_{\mp 2}\pm\xi_+^{\mp 2}\sin u\right)\beta_\pm a_\pm\Big]+i\sin 2\eta\tan\omega, \tag{13}$$

$$\begin{aligned}
B_\pm(u) &= \frac{i^{\frac{1}{2}\pm\frac{1}{2}}}{s_{+2}s_{-2}}\Big[-\xi_+^{\pm 1}\sin u\sin 4\eta\,\alpha_\pm n_\pm\mp i\xi_+^{\mp 1}\sin 2\eta\,(p_\pm\sin u\pm\sin 2\eta\cos u)a_\pm \\
&\quad +\xi_+^{\pm 1}\sin 2\eta\, s_{\mp 2}\alpha_\pm\Big], &(14)\\
C_\pm(u) &= \frac{i^{\frac{1}{2}\pm\frac{1}{2}}}{s_{+2}s_{-2}}\Big[\mp\xi_+^{\mp 1}\sin u\sin 4\eta\,\beta_\pm n_\pm - i\xi_+^{\pm 1}\sin 2\eta\,\left(p_\pm\sin u\mp\sin 2\eta\cos u\right)a_\pm^\dagger \\
&\quad \pm\xi_+^{\mp 1}\sin 2\eta\, s_{\pm 2}\beta_\pm\Big]. &(15)
\end{aligned}$$

For convenience, we introduced the abbreviations

$$\begin{aligned}
s_{\pm n} &= \sin(u\pm n\eta), & c_{\pm n} &= \cos(u\pm n\eta),\\
a_+ &= a_N, & a_- &= a_1, \qquad (16)
\end{aligned}$$

and

$$\xi_\pm=\xi(\pm u)=\frac{\cos(u\pm\omega)}{\cos u\,\cos\omega}. \tag{17}$$

## 5 Boundary reflection supermatrices

We are now ready to compute the boundary $K$-matrices

$$K_\pm(u)=\begin{pmatrix} K_{11}^\pm & K_{12}^\pm \\ K_{21}^\pm & K_{22}^\pm\end{pmatrix} \tag{18}$$

by using Eq. (6). This yields 16 equations for the four matrix elements $K_{ij}^\pm$, only six of which are independent. After a lengthy calculation, we find

$$\begin{aligned}
K_{11}^- &= \xi_+\sin(u-\psi_-)\left(\xi_-^2 s_{+2}-\xi_+^2 s_{-2}\right), &(19)\\
K_{22}^- &= -\xi_-\sin(u+\psi_-)\left(\xi_+^2 s_{+2}-\xi_-^2 s_{-2}\right), &(20)\\
K_{12}^- &= -\frac{\alpha_-\sin\psi_-\sin u}{i\xi_+\xi_-\sin^2 2\eta}\left(\xi_-^2 s_{+2}-\xi_+^2 s_{-2}\right)\left(\xi_+^2 s_{+2}-\xi_-^2 s_{-2}\right), &(21)\\
K_{21}^- &= -\frac{\beta_-\sin\psi_-\sin u}{i\sin^2 2\eta}\left(\xi_-^2 s_{+2}-\xi_+^2 s_{-2}\right)\left(\xi_+^2 s_{+2}-\xi_-^2 s_{-2}\right), &(22)\\
K_{11}^+ &= \xi_+\sin(u+2\eta-\psi_+)\left(\xi_-^2 s_{+4}-\xi_+^2\sin u\right), &(23)\\
K_{22}^+ &= \xi_-\sin(u+2\eta+\psi_+)\left(\xi_+^2 s_{+4}-\xi_-^2\sin u\right), &(24)\\
K_{12}^+ &= -\frac{\alpha_+\sin\psi_+ s_{+2}}{i\xi_+\xi_-\sin^2 2\eta}\left(\xi_-^2 s_{+4}-\xi_+^2\sin u\right)\left(\xi_+^2 s_{+4}-\xi_-^2\sin u\right), &(25)\\
K_{21}^+ &= -\frac{\beta_+\sin\psi_+ s_{+2}}{i\sin^2 2\eta}\left(\xi_-^2 s_{+4}-\xi_+^2\sin u\right)\left(\xi_+^2 s_{+4}-\xi_-^2\sin u\right), &(26)
\end{aligned}$$

where we have parametrized $p_\pm=\sin 2\eta\cot\psi_\pm$. The Hamiltonian (7) can be found as usual as an invariant of the commuting family of transfer matrices $\tau(u)$ (5) by taking the derivative at a special value of the spectral parameter $u$. Namely,

$$-\sin 2\eta\;\frac{d}{du}\tau(u)\bigg|_{u=0}=2H\tau(0)+\mathrm{str}_0\left(\frac{d}{du}K_+(u)\bigg|_{u=0}\right). \tag{27}$$

## 6 Conclusion

We have presented the Lax pair formulation for the 1D small-polaron model with general boundary conditions, providing a direct demonstration for the integrability of the model. The Grassmann variables at the boundary terms play a crucial role in the construction of the boundary $K$-supermatrices. They may be interpreted in physical terms as sources and sinks for injecting additional particles into the system. In particular, we can choose as a special representation at the left end $\alpha_- = c_-^{(\alpha)} a_0$, $\beta_- = c_-^{(\beta)} a_0^\dagger$, and $p_- = c_-^{(p)}$, and at the right end $\alpha_+ = c_+^{(\alpha)} a_{N+1}$, $\beta_+ = c_+^{(\beta)} a_{N+1}^\dagger$, and $p_+ = c_+^{(p)}$, with complex numbers $c_\pm^{(\cdot)}$ and two additional boundary fermions $a_0$ and $a_{N+1}$. We note that the boundaries considered here act as pure back-scatterers [1, 2]. Thus a combination with the forward-scattering impurities of Refs. [11, 12] may result in physically relevant, yet completely integrable impurities. Corresponding Hamiltonians may be constructed by the methods outlined here [13].

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## References

[1] H. Frahm, A. Zvyagin, J. Phys. Condens. Matter **9** (1997) 9939; G. Bedürftig, H. Frahm, J. Phys. A **30** (1997) 4139; G. Bedürftig, B. Brendel, H. Frahm, R.M. Noack, Report cond-mat/9805123, to appear in Phys. Rev. B **58**

[2] Y.P. Wang, J.H. Dai, Z.N. Hu, F.C. Pu, Phys. Rev. Lett. **79** (1997) 1901; Y.P. Wang, Phys. Rev. B **56** (1997) 14045; J. Dai, Y. Wang, U. Eckern, Report cond-mat/9808079

[3] I. Affleck, Nucl. Phys., B **336** (1990) 517; P. Fröjdh, H. Johannesson, Phys. Rev. Lett., **75** (1995) 300

[4] J.L. Cardy, Nucl. Phys., B **324** (1989) 581

[5] E.K. Sklyanin, J. Phys. A **21** (1988) 2375

[6] R.J. Baxter, Exactly solved Models in Statistical Mechanics, Academic Press, London, 1982; V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, New York, 1993

[7] P.P. Kulish, Report hep-th/9507070; M. Shiroishi, M Wadati, J.Phys. Soc. Jpn. **66** (1997) 2288; A.J. Bracken, X.Y. Ge, Y.Z. Zhang, H.-Q. Zhou, Nucl. Phys. B **516** (1998) 588

[8] H.Q. Zhou, J. Phys. A **29** (1996) L607; X.W. Guan, M.S. Wang, S.D. Yang, Nucl. Phys. B **485** (1997) 685

[9] V.K. Fedyanin, V. Yushankhay, Theor. Mat. Fiz. **35** (1978) 240; V. G. Makhankov, V. K. Fedyanin, Phys. Rep. **104** (1984) 1

[10] H.Q. Zhou, L.J. Jiang, J.G. Tang, J. Phys. A **23** (1990) 213

[11] N. Andrei, H. Johannesson, Phys. Lett. A **100** (1984) 108

[12] P. Schlottmann, J. Phys. Condens. Matter **3** (1991) 6619; P. Schmittechert, P. Schwab, U. Eckern, Europhys. Lett. **30** (1995) 534; H.-P. Eckle, A. Punnoose, R.A. Römer, Europhys. Lett. **39** (1997) 293

[13] X.-W. Guan, U. Grimm, R. A. Römer, M. Schreiber, preprint in preparation (1998)