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Nitsche- and Fourier-finite-element method for the Poisson equation in axisymmetric domains with reentrant edges

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Abstract

The paper deals with a combination of the Fourier method with the Nitsche-finite-element method (as a mortar method). The approach is applied to the Dirichlet problem of the Poisson equation in three-dimensional axisymmetric domains with reentrant edges generating singularities. The approximating Fourier method yields a splitting of the 3D problem into 2D problems on the meridian plane of the given domain. For solving the 2D problems bearing corner singularities, the Nitsche-finite-element method with non-matching meshes and mesh grading near reentrant corners is applied. Using the explicit representation of singular functions, the rate of convergence of the Fourier-Nitsche-mortaring is estimated in some H^1-like norm as well as in the L_2-norm. Finally, some numerical results are presented.

Keywords. finite-element method, Fourier method, non-matching meshes, Nitsche-mortaring, Poisson equation, edge singularities
AMS subject classification. 65N30, 65N35

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1 Introduction

For the efficient numerical treatment of boundary value problems (BVPs) in science and engineering, domain decomposition methods are widely used. The Nitsche-finite-element method as a mortar method enables the discretization of the BVP in the subdomains to be done in an flexible way, e.g. in presence of non-matching meshes and discontinuities of the finite element appproximation at the interface of domain decomposition, see e.g. [2, 3, 4, 9, 24].
Nevertheless, domain decomposition with non-matching meshes in 3D is, in general, much more complicated and expensive than in 2D. In order to bound the effort, in [15] it was proposed to take the Fourier method for reducing the dimension from 3D to 2D and to combine it with the Nitsche-finite-element method for getting a domain decomposition method in 3D, at least for domains $\widehat{\Omega} \subset \mathbb{R}^3$ having a uniform extension in one direction, like prismatic or axisymmetric domains.
Thus, in [15] this combined method was presented and investigated as a discretization scheme for regular solutions $\hat{u}$ belonging to the Sobolev space $H^2(\widehat{\Omega})$ ($H^s(\widehat{\Omega})$: the usual Sobolev-Slobodetskiĭ space, s real, $H^0 = L_2$). The requirement $\hat{u} \in H^2(\widehat{\Omega})$ restricts the geometry of the domain $\widehat{\Omega}$ and the range of applicability of the method in real problems, since reentrant edges of $\widehat{\Omega}$ are excluded, in general.
In this paper we shall present the extension of the combined method to axisymmetric domains with reentrant edges and weak regularity of the solution $\hat{u} \in H^{1+\delta}(\widehat{\Omega})$, $\delta > \frac{1}{2}$, caused by edge singularities of $\widehat{\Omega}$. The approach is applied to the Dirichlet problem of the Poisson equation, $-\Delta \hat{u} = \hat{f}$ in $\widehat{\Omega}$, $\hat{u} = 0$ on $\partial\widehat{\Omega}$, where the axisymmetric domain $\widehat{\Omega} \subset \mathbb{R}^3$ is generated by rotation of the corresponding meridian domain Ω_a about the x_3-axis.
The combined method can be characterized as follows. The first component, the approximating Fourier method (cf. [5, 7, 17, 18]) uses trigonometric polynomials of degree $\leq N$ in one space direction, here with respect to the rotational angle $\varphi \in (\pi, -\pi]$. This yields an approximate splitting of the 3D problem into $2N + 1$ problems in 2D for the parameter $k = 0, \pm 1, ..., \pm N$. The solutions u_k of these 2D problems are just the first $2N + 1$ Fourier coefficients of the solution $\hat{u}$. The second component comprises the Nitsche-finite-element discretization as a mortar method, cf. [1, 2, 9, 13, 14, 16, 21], for solving numerically the 2D problems on the meridian domain Ω_a and approximating the Fourier coefficients u_k. Along the interface of the domain decomposition of Ω_a, non-matching meshes as well as discontinuities of the approximated solutions are admitted. Compared with the papers cited previously, the differential operator depends now on the parameter k and has a more general form. The method arising by combination of these two components was proposed and investigated for regular solutions in [15].

The aim of this paper is to extend this new method to problems with singularities of the solution generated by reentrant edges. The efficient numerical treatment of such BVPs requires a careful representation of the edge singularity in 3D by the corresponding corner singularities of the Fourier coefficients acting on the meridian domain Ω_a in 2D. The weak assumption $\hat{f} \in L_2(\widehat{\Omega})$ does not allow a tensor product representation of the edge singularity function. So it will be necessary to apply a non-tensor product singularity function, cf. [11, 12]. Furthermore, we employ a triangulation with shape regular triangles and piecewise linear finite elements on the meridian domain Ω_a, with a global mesh parameter $h > 0$. In order to improve the rate of convergence under the influence of edge singularities, the mesh is provided with a local grading near the reentrant corners on the boundary of Ω_a. Important properties of the approximation scheme in 3D are derived and, moreover, the convergence $u_{hN} \to u$ of the Nitsche-Fourier-finite element approximation u_{hN} with respect to $N \to \infty$ and $h \to 0$ is shown. In an H^1-like norm and in the L_2-norm, the convergence rates on quasi-uniform meshes are proved to be of the type $\mathcal{O}(h^\lambda + N^{-1})$ and $\mathcal{O}(h^{2\lambda} + N^{-2})$, respectively, where λ is the typical singularity exponent with $1/2 < \lambda < 1$. Using an appropriate local mesh grading near reentrant corners, the improved rates $\mathcal{O}(h + N^{-1})$ and $\mathcal{O}(h^2 + N^{-2})$ are derived. The results are valid in the important case when h and N are chosen independently from each other (anisotropic discretization).

The paper is organized as follows. First we describe the BVP, its transformation into cylindrical coordinates, the corner singularities in 2D and a non-tensor product edge singularity in 3D. Then the Nitsche mortaring for the domain decomposition of the meridian domain Ω_a in 2D is founded for weak regularity of the Fourier coefficients $u_k(r, z)$ of the solution $\hat{u}$. Using a mesh with grading for approximating corner singularities of $u_k(r, z)$, optimal error estimates for the Nitsche-finite-element approximation of the Fourier coefficients are derived. Finally, the combined method acting in 3D is presented and estimates of the error $u - u_{hN}$ with respect to $h \to 0$ and $N \to \infty$ are given. A numerical example exhibiting an edge singularity and the observed rates of convergence of the Nitsche-Fourier-finite-element approximation are given.

2 Analytical framework

Let $\widehat{\Omega} \subset \mathbb{R}^3$ be a bounded domain which is axisymmetric with respect to the x_3-axis. The part of the x_3-axis contained in $\widehat{\Omega}$ is denoted by Γ_0. Then the set $\widehat{\Omega} \setminus \Gamma_0$ is generated by rotation of the corresponding plane meridian domain Ω_a about the x_3-axis. The set Γ_a is defined by $\Gamma_a := \partial\Omega_a \setminus \overline{\Gamma}_0$, where $\partial\Omega_a$ is the boundary of Ω_a. In the following we assume that Ω_a is polygonally bounded. Further let P_i, $i = 1, \ldots, m$ (m: the total number of corners of $\overline{\Omega}_a$), denote the corners of the polygon $\overline{\Omega}_a$ such that $P_1,\ P_m \in \overline{\Gamma}_0 \cap \overline{\Gamma}_a$, cf. Figure 1. Then we require that for the interior angles β_1, β_m at the corners P_1, P_m holds: $\beta_1, \beta_m < 0.72616\pi$, i.e., the conical vertices of the three-dimensional domain $\widehat{\Omega}$ do not generate singularities at the x_3-axis, cf. [5]. Since the treatment of reentrant corners is to be done locally, we consider for the sake of simplicity only one reentrant corner, i.e. $\beta_i > \pi$ holds for one index $i \in \{2, \ldots, m-1\}$. Thus, the axisymmetric domain $\widehat{\Omega}$ has only one reentrant edge, and several reentrant edges can be treated analogously.
Let $H^s(X)$ ($s \geq 0$, s real, $H^0 = L_2$) denote the usual Sobolev-Slobodetskiĭ space of functions defined on X. Subsequently we consider the Dirichlet problem for the Poisson equation on $\widehat{\Omega}$:

$$-\Delta\hat{u} := -\sum_{i=1}^{3} \frac{\partial^2 \hat{u}}{\partial x_i^2} = \hat{f} \quad \text{in } \widehat{\Omega}, \quad \hat{u} = 0 \quad \text{on } \partial\widehat{\Omega}, \tag{1}$$

with $\hat{f} \in L_2(\widehat{\Omega})$. Since the domain $\widehat{\Omega}$ is assumed to be axisymmetric, it is natural to employ cylindrical coordinates r, φ, z ($x_1 = r\cos\varphi, x_2 = r\sin\varphi, x_3 = z$), with $\varphi \in (-\pi, \pi]$. Then

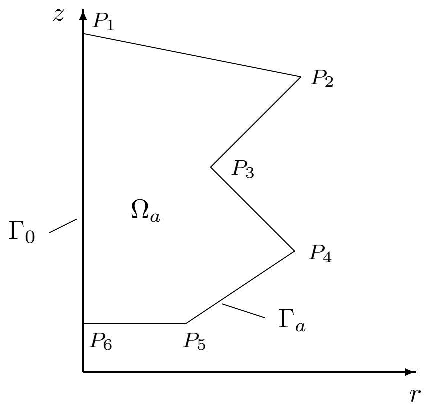

Figure 1: Meridian domain Ω_a

we get one-to-one mappings: $\widehat{\Omega} \setminus \Gamma_0 \to \Omega := \Omega_a \times (-\pi, \pi]$ and $\partial\widehat{\Omega} \setminus \overline{\Gamma}_0 \to \Gamma_a \times (-\pi, \pi]$. Consequently, for each function $\hat{v}(x)$ with $x \in \widehat{\Omega} \setminus \Gamma_0$, the associated function v on Ω is defined by

$$v(r, \varphi, z) := \hat{v}(r\cos\varphi, r\sin\varphi, z). \tag{2}$$

Using this, we can define spaces $X^l_{1/2}(\Omega)$ of Sobolev-type of functions periodic with respect to $\varphi \in (-\pi, \pi]$ as follows: $H^l(\widehat{\Omega} \setminus \Gamma_0) \to X^l_{1/2}(\Omega)$ $(l = 0, 1, 2)$. Since Γ_0 is one-dimensional, $H^l(\widehat{\Omega} \setminus \Gamma_0)$ and $H^l(\widehat{\Omega})$ can be identified. The spaces $X^l_{1/2}(\Omega)$ are equipped with the natural norms and seminorms given by the relations

$$|u|_{X^l_{1/2}(\Omega)} = |\hat{u}|_{H^l(\widehat{\Omega})}, \quad \|u\|_{X^l_{1/2}(\Omega)} = \|\hat{u}\|_{H^l(\widehat{\Omega})}, \quad l = 0, 1, 2, \tag{3}$$

with u, $\hat{u}$ related by (2). In [11, 17, 23] the spaces $X^l_{1/2}(\Omega)$ are described in more detail.

According to (2), the variational formulation of (1) in cylindrical coordinates can be stated as follows. Find $u \in V_0(\Omega) := \{u \in X^1_{1/2}(\Omega) : u|_{\Gamma_a \times (-\pi,\pi]} = 0\}$ such that

$$\begin{aligned} b(u, v) &= f(v) \quad \forall v \in V_0(\Omega), \qquad (4)\\ \text{with} \quad b(u, v) &:= \int\limits_{\Omega} \Big\{ \frac{\partial u}{\partial r}\overline{\frac{\partial v}{\partial r}} + \frac{1}{r^2}\frac{\partial u}{\partial \varphi}\overline{\frac{\partial v}{\partial \varphi}} + \frac{\partial u}{\partial z}\overline{\frac{\partial v}{\partial z}} \Big\}\, r dr d\varphi dz, \quad f(v) := \int\limits_{\Omega} f\,\overline{v}\, r dr d\varphi dz. \end{aligned}$$

For $u(r, \varphi, z)$, $u \in X^1_{1/2}(\Omega)$, (and for $f(r, \varphi, z)$, $f \in X^0_{1/2}(\Omega)$, resp.) we employ partial Fourier analysis with respect to the rotational angle φ:

$$u(r, \varphi, z) = \sum_{k \in \mathbb{Z}} u_k(r, z)\, \mathrm{e}^{ik\varphi}, \qquad u_k(r, z) := \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u(r, \varphi, z)\, \mathrm{e}^{-ik\varphi} d\varphi \quad \text{for } k \in \mathbb{Z} \tag{5}$$

($\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$; $i^2 = -1$). Inserting (5) into (4) it can be shown that by means of the forms

$$b_k(u_k, v_k) = \int\limits_{\Omega_a} \Big\{ \frac{\partial u_k}{\partial r}\overline{\frac{\partial v_k}{\partial r}} + \frac{\partial u_k}{\partial z}\overline{\frac{\partial v_k}{\partial z}} + \frac{k^2}{r^2} u_k \overline{v}_k \Big\}\, r dr dz, \quad f_k(v_k) = \int\limits_{\Omega_a} f_k \overline{v}_k\, r dr dz \ (k \in \mathbb{Z}),$$

the BVP (4) can be decomposed into a family of decoupled BVPs in 2D written in the variational form as follows (see e.g. [5, 11, 17]):

$$
\begin{aligned}
k = 0 : \text{find } u_0 \in V_0^a := \{v \in H^1_{1/2}(\Omega_a) \colon v|_{\Gamma_a} = 0\} \colon \; b_0(u_0, w) = f_0(w) \;\; \forall w \in V_0^a, \\
k \in \mathbb{Z}\backslash\{0\} \colon \text{find } u_k \in W_0^a := \{v \in V_0^a \colon v \in L_{2,-1/2}(\Omega_a)\} \colon \; b_k(u_k, w) = f_k(w) \;\; \forall w \in W_0^a.
\end{aligned}
\tag{6}
$$

It is well-known that the solutions u_k ($k \in \mathbb{Z}$) of (6) are the Fourier coefficients of u defined by (5). In (6), $H^l_\alpha(\Omega_a)$ (resp. $L_{2,\alpha}(\Omega_a)$) denote the following spaces of functions with power weights r^α (α real):

$$
H^l_\alpha(\Omega_a) \; := \; \{w = w(r,z) : r^\alpha D^\beta w \in L_2(\Omega_a), \; 0 \le |\beta| \le l\} \quad \text{for } l \in \{0,1,2\}; \tag{7}
$$

$$
D^\beta w := \frac{\partial^{|\beta|} w}{\partial r^{\beta_1} \partial z^{\beta_2}}, \; \beta = (\beta_1, \beta_2), \; |\beta| = \beta_1 + \beta_2; \quad H^0_\alpha(\Omega_a) = L_{2,\alpha}(\Omega_a).
$$

The canonical scalar product and norm in $L_{2,\alpha}(\Omega_a)$ are given by

$$
(v,w)_{\alpha,\Omega_a} := \int\limits_{\Omega_a} v \bar{w} \, r^{2\alpha} \, dr dz, \quad \|w\|_{L_{2,\alpha}(\Omega_a)} := \Big\{ \int\limits_{\Omega_a} |r^\alpha w|^2 \, dr dz \Big\}^{1/2}. \tag{8}
$$

The spaces $H^l_\alpha(\Omega_a)$, $l \in \{1,2\}$, are provided with the seminorms and norms

$$
|w|_{H^l_\alpha(\Omega_a)} := \Big\{ \sum_{|\beta|=l} \|r^\alpha D^\beta w\|^2_{L_2(\Omega_a)} \Big\}^{1/2}, \; \|w\|_{H^l_\alpha(\Omega_a)} := \Big\{ \sum_{|\beta|\le l} \|r^\alpha D^\beta w\|^2_{L_2(\Omega_a)} \Big\}^{1/2}. \tag{9}
$$

Subsequently, these spaces, scalar products, and norms will also be used with Ω_a^i ($i = 1,2$) instead of Ω_a, where Ω_a^i are subdomains of Ω_a.

If u_k ($k \in \mathbb{Z}$) is sufficiently regular, the following differential equations and boundary conditions for the Fourier coefficients u_k correspond to the variational equations (6):

$$
\begin{aligned}
-\Big\{ \frac{\partial^2 u_k}{\partial r^2} + \frac{\partial^2 u_k}{\partial z^2} + \frac{1}{r}\frac{\partial u_k}{\partial r} \Big\} + \frac{k^2}{r^2} u_k \; &= \; f_k \;\text{ in } \Omega_a \;\; \forall k \in \mathbb{Z}, \\
u_k = 0 \;\text{ on } \Gamma_a \;\; \forall k \in \mathbb{Z}, \quad u_k \; &= \; 0 \;\text{ on } \Gamma_0 \;\; \forall k \in \mathbb{Z}\backslash\{0\}.
\end{aligned}
\tag{10}
$$

The boundary condition for u_0 on Γ_0 is formulated in the context of the variational problem.

Now we describe the regularity of the solutions u_k ($k \in \mathbb{Z}$) and u of the BVPs (6) and (4), respectively. According to the abovementioned assumptions on the geometry of Ω_a, let E_a denote the corner of Ω_a generating the reentrant edge of $\widehat{\Omega}$. It is well-known that the regularity of solutions of elliptic BVPs is a local problem. Therefore we introduce local polar coordinates with respect to the non-convex corner $E_a = (r_{E_a}, z_{E_a})$ as follows: $r - r_{E_a} = R\cos(\theta + \theta_r)$, $z - z_{E_a} = R\sin(\theta + \theta_r)$. Then we define some circular sector G_a (see Figure 2) with the radius R'_0 and the angle θ_0:

$$
\overline{G}_a := \{(r,z) \in \mathbb{R}^2 : 0 \le R \le R'_0, \; 0 \le \theta \le \theta_0\}, \quad G_a := \overline{G}_a \setminus \partial G_a, \tag{11}
$$

∂G_a: boundary of G_a. Consider also the domain $\widehat{G} \in \mathbb{R}^3$ generated by rotation of G_a about the x_3-axis, with boundary $\partial\widehat{G}$. Then $G := G_a \times (-\pi, \pi]$ and $\partial_0 G := \partial G_a \times (-\pi, \pi]$ are

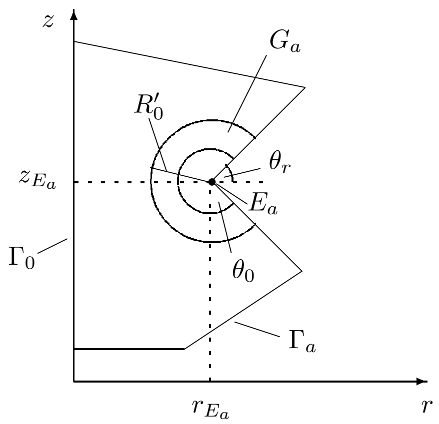

Figure 2: Domain Ω_a with the sector G_a

the images of $\widehat{G}$ and $\partial\widehat{G}$ in the (r,φ,z)-system. The spaces $X^l_{1/2}$ and H^l_α ($l = 0,1,2$; α real) from (3), (7) can also be defined for G instead of Ω and G_a instead of Ω_a. Owing to the boundedness of the weighting factor r on the subdomain G_a, the norms on $H^l_\alpha(G_a)$ are equivalent to the norms on the usual Sobolev spaces $H^l(G_a)$, $l = 0,1,2$. Further we define a smooth cut-off function $\eta = \eta(r,\varphi,z) = \tilde{\eta}(R)$ with

$$0 \le \eta \le 1,\ \ \tilde{\eta} \in C^\infty[0,\infty),\ \ \tilde{\eta} = 1 \ \text{ for } \ 0 \le R \le \frac{R_0'}{3},\ \ \tilde{\eta} = 0 \ \text{ for } \ R \ge \frac{2R_0'}{3}.$$

Then, the regularity of the solution u to (4) can be characterized as follows.

Theorem 1 *Assume $f \in X^0_{1/2}(\Omega)$ and that one edge is reentrant with the angle $\theta_0 \in (\pi, 2\pi)$, i.e., $\lambda := \frac{\pi}{\theta_0}$ satisfies $\frac{1}{2} < \lambda < 1$. Then the solution u to the BVP (4) can be represented by*

$$u(r,\varphi,z) = \sum_{k\in\mathbb{Z}} u_k(r,z)\,\mathrm{e}^{ik\varphi} = \begin{cases} u_s(r,\varphi,z) + w(r,\varphi,z) & \text{for } 0 < R < \frac{2R_0'}{3}, \\ & \qquad 0 < \theta < \theta_0 \\ w(r,\varphi,z) & \text{otherwise in } \Omega. \end{cases} \tag{12}$$

The singular part u_s and its Fourier coefficients $s_k(r,z)$ satisfy the relations

$$\begin{aligned} u_s(r,\varphi,z) &:= \sum_{k\in\mathbb{Z}} s_k(r,z)\mathrm{e}^{ik\varphi} = \eta R^\lambda \sin(\lambda\theta)\Psi(\varphi,R),\ \ \Psi(\varphi,R) := \sum_{k\in\mathbb{Z}} \delta_k \mathrm{e}^{-|k|R}\mathrm{e}^{ik\varphi}, \\ s_k(r,z) &:= \eta\delta_k \mathrm{e}^{-|k|R} R^\lambda \sin(\lambda\theta),\ \ (1+k^2)^{\frac{1-\lambda}{2}}|\delta_k| \le M_1\|f_k\|_{L_{2,1/2}(\Omega_a)} \ \text{for } k \in \mathbb{Z}, \end{aligned} \tag{13}$$

and the regular part w fulfills

$$\begin{aligned} & w(r,\varphi,z) = \sum_{k\in\mathbb{Z}} w_k(r,z)\mathrm{e}^{ik\varphi} \quad \text{with } \ w_k \in H^2_{1/2}(\Omega_a) \ \text{ for } k \in \mathbb{Z}, \\ & w \in V_0(\Omega) \cap X^2_{1/2}(\Omega),\ \ \|w\|_{X^2_{1/2}(\Omega)} \le M_2\|f\|_{X^0_{1/2}(\Omega)}. \end{aligned} \tag{14}$$

For the proof of this theorem, we refer to [11, Lemma 6.2 and Theorem 6.3] or [12, Theorem 2.7].

The function u_s from (13) is called a non-tensor product singularity function since the factor Ψ depends on both φ and R. In [11, Section 5], we have additionally a tensor product representation of the singularity of the solution to the BVP (4). But this representation needs the additional assumption $\frac{\partial^l f}{\partial \varphi^l} \in X^0_{1/2}(G)$ $(l = 1, 2)$. Since the non-tensor product singularity function requires only $f \in X^0_{1/2}(\Omega)$, we prefer that one in the present paper. Clearly, if the domain $\widehat{\Omega}$ has several reentrant edges, the singular part u_s in (12) is to be replaced by a sum of the corresponding singular parts.

3 Nitsche mortaring on locally graded meshes in 2D

For the Nitsche-finite-element discretization (cf. also [2, 9, 13, 14, 21]) of the BVPs (6) we shall need a subdivision of Ω_a into subdomains. Throughout this paper we restrict ourselves to the case of two subdomains Ω^1_a, Ω^2_a with

$$\overline{\Omega}_a = \overline{\Omega}^1_a \cup \overline{\Omega}^2_a, \quad \Omega^1_a \cap \Omega^2_a = \emptyset, \quad \Gamma = \overline{\Omega}^1_a \cap \overline{\Omega}^2_a,$$

and being polygonally bounded. We also introduce the 'broken' spaces

$$V_a := V^1_a \times V^2_a,\ W_a := W^1_a \times W^2_a, \tag{15}$$

with $V^i_a = \{\, w \in H^1_{1/2}(\Omega^i_a) : w|_{\partial\Omega^i_a \cap \Gamma_a} = 0 \,\}$, $W^i_a = \{\, w \in V^i_a : w \in L_{2,-1/2}(\Omega^i_a) \,\}$ for $i = 1, 2$. There are different cases regarding the position of the two subdomains: Figure 3(a) shows the case $\partial\Omega^i_a \cap \Gamma_a \neq \emptyset$ for $i = 1, 2$ as well as $\Gamma \cap \Gamma_0 \neq \emptyset$ and in Figure 3(b) we have $\partial\Omega^2_a \cap \Gamma_a = \emptyset$, $\Gamma = \partial\Omega^2_a$, and $\Gamma \cap \Gamma_0 = \emptyset$. In view of the subdivision of Ω_a we introduce the restrictions $v^i := v|_{\Omega^i_a}$ of a function v on Ω^i_a as well as the vectorized form $v = (v^1, v^2)$, i.e., $v^i(x) = v(x)$ holds for $x \in \Omega^i_a$ $(i = 1, 2)$. It should be noted that for simplicity we use here the same symbol v for denoting the function on Ω_a as well as the vector (v^1, v^2).

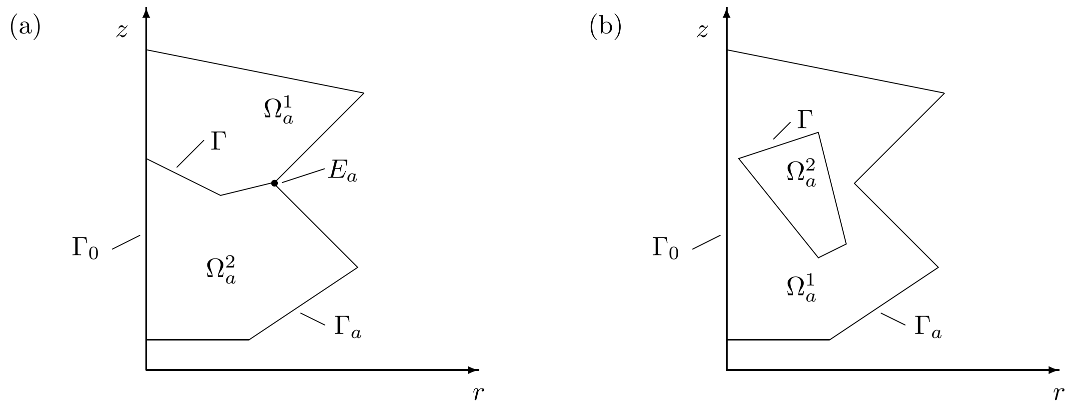

Figure 3: Meridian domain Ω_a with subdomains Ω^1_a, Ω^2_a

Using this notation we deduce that for each $k \in \mathbb{Z}$ the solution of the BVP (10) is equivalent to the solution of the following problem: Find (u^1_k, u^2_k) such that

$$\begin{aligned} -\Big\{ \frac{\partial^2 u^i_k}{\partial r^2} + \frac{\partial^2 u^i_k}{\partial z^2} + \frac{1}{r}\frac{\partial u^i_k}{\partial r} \Big\} + \frac{k^2}{r^2} u^i_k &= f_k \quad \text{in } \Omega^i_a, \quad i = 1, 2 \\ u^i_k = 0 \quad \text{on } \partial\Omega^i_a \cap \Gamma_a, \quad u^i_k &= 0 \quad \text{on } \partial\Omega^i_a \cap \Gamma_0 \ \text{ (only for } k \in \mathbb{Z}\backslash\{0\}) \\ \frac{\partial u^1_k}{\partial n_1} + \frac{\partial u^2_k}{\partial n_2} &= 0 \quad \text{on } \Gamma, \quad u^1_k = u^2_k \quad \text{on } \Gamma \end{aligned} \tag{16}$$

are satisfied, where n_i $(i=1,2)$ denotes the outward normal to $\partial\Omega_a^i \cap \Gamma$.

In order to define the finite-element discretization with non-matching meshes, we cover Ω_a^i $(i=1,2)$ by a triangulation $\mathcal{T}_h^i$ $(i=1,2)$ consisting of triangles T $(T=\overline{T})$, where $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent of each other. Moreover, the compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along the mortar interface $\Gamma = \partial\Omega_a^1 \cap \partial\Omega_a^2$ is not required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, with $0 < h \le h_0$ and sufficiently small h_0. Take e.g. $h = \max\{h_T : T \in \mathcal{T}_h\}$, where h_T denotes the diameter of the triangle T. In the sequel, positive constants C occurring in the inequalities are generic constants.

Throughout this paper we suppose that the following assumption on the triangulations $\mathcal{T}_h^i$ $(i=1,2)$ is fulfilled.

Assumption 1

(i) For $i=1,2$, we have $\overline{\Omega}_a^i = \cup_{T\in\mathcal{T}_h^i} T$, and two arbitrary triangles $T, T' \in \mathcal{T}_h^i$ $(T \ne T')$ are either disjoint or have a common vertex, or a common edge.

(ii) The mesh in $\overline{\Omega}_a^i$ $(i=1,2)$ is shape regular, i.e., the following relation holds:

$$\frac{h_T}{\rho_T} \le C \text{ for any } T \in \mathcal{T}_h^i \text{ and } h:\ 0 < h \le h_0 \ \ (\rho_T : \text{radius of inscribed circle of } T). \tag{17}$$

Relation (17) means that the triangulations $\mathcal{T}_h^i$ $(i=1,2)$ do not have to be quasi-uniform in general. The error of the finite element approximation shall be estimated on quasi-uniform meshes (partially) as well as on meshes with appropriate local grading at the reentrant corner. In [13], Nitsche-mortaring on meshes with local grading has been studied for elliptic BVPs in two-dimensional domains with reentrant corners, but the type of the corresponding singularity functions in 2D differs from the functions s_k given by (13).

In order to provide a framework for graded meshes, we introduce the real grading parameter μ, $0 < \mu \le 1$, the grading function R_i $(i=0,1,\dots,n)$ with some real constant $b > 0$, and the step size h_i for the mesh associated with layers $[R_{i-1}, R_i] \times [0, \theta_0]$ around E_a:

$$R_i := b(ih)^{\frac{1}{\mu}} \ \ (i=0,1,\dots,n), \quad h_i := R_i - R_{i-1} \ \ (i=1,2,\dots,n). \tag{18}$$

Here $n := n(h)$ denotes an integer of the order h^{-1}, $n := [\beta h^{-1}]$ for some real $\beta > 0$ ($[\cdot]$ means the integer part). We shall choose b and β such that $\frac{2}{3}R_0' < R_n < R_0'$ holds, i.e. the mesh grading is located within $\overline{G}_a$ from (11).

Using the step size h_i $(i=0,1,\dots,n)$ we define a mesh which is graded in the neighbourhood of the vertex E_a of the reentrant corner and quasi-uniform in the remaining part of the domain Ω_a. The triangulation $\mathcal{T}_h$ is now characterized by the mesh size h, $0 < h \le h_0$, and the grading parameter μ, with fixed μ: $0 < \mu \le 1$. The properties of $\mathcal{T}_h$ are summarized in the following assumption.

Assumption 2 *Let the triangulation $\mathcal{T}_h$ satisfy Assumption 1. Furthermore, $\mathcal{T}_h$ is provided with a grading around the vertex E_a of the reentrant corner such that $h_T := \operatorname{diam} T$ depends* *on the distance* R_T *of* T *from* E_a, $R_T := \mathrm{dist}\,(T, E_a) := \inf_{P\in T} |E_a - P|$, *in the following way:*

$$\begin{aligned} \rho_1 h^{\frac{1}{\mu}} &\le h_T \le \rho_1^{-1} h^{\frac{1}{\mu}} && \textit{for } T \in \mathcal{T}_h : R_T = 0, \\ \rho_2 h R_T^{1-\mu} &\le h_T \le \rho_2^{-1} h R_T^{1-\mu} && \textit{for } T \in \mathcal{T}_h : 0 < R_T < R_g, \\ \rho_3 h &\le h_T \le \rho_3^{-1} h && \textit{for } T \in \mathcal{T}_h : R_g \le R_T \end{aligned} \tag{19}$$

with some constants ρ_i, $0 < \rho_i \le 1$ $(i = 1, 2, 3)$ *and some real* R_g, $0 < \underline{R}_g < R_g < \overline{R}_g$, *where* $\underline{R}_g, \overline{R}_g$ *are fixed and independent of* h.

Here, R_g is the radius of the sector with mesh grading, and w.l.o.g. we may assume $R_g = R_n$. The value $\mu = 1$ yields a quasi-uniform mesh in the whole domain Ω_a, i.e. the relation $\frac{\max_{T\in\mathcal{T}_h^i} h_T}{\min_{T\in\mathcal{T}_h^i} \rho_T} \le C$ $(i = 1, 2)$ holds instead of (17). Owing to Assumption 2, the asymptotic behaviour of h_T is determined by the relations

$$\begin{aligned} &\varepsilon_1 h_j \le h_T \le \varepsilon_1^{-1} h_j \quad \text{for } T \in \mathcal{T}_h : R_{j-1} \le R_T \le R_j \ \ (j = 1, 2, \ldots, n), \\ &\varepsilon_2 h \le h_T \le \varepsilon_2^{-1} h \quad \text{for } T \in \mathcal{T}_h : R_n \le R_T, \end{aligned} \tag{20}$$

with $0 < \varepsilon_l \le 1$ $(l = 1, 2)$, and h_j, R_j as well as n from (18). An example of a mesh with local grading as described in Assumption 2 is given in Figure 4.

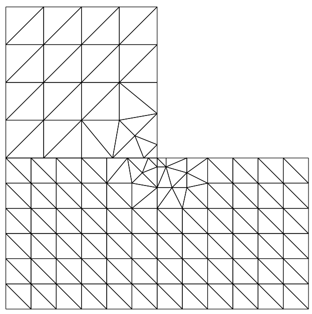

Figure 4: Locally graded mesh with parameter $\mu = 0.6$

It should be noted that the total number of nodes of $\mathcal{T}_h$ satisfying Assumption 2 is always of the order $\mathcal{O}(h^{-2})$. In [13, 19, 20] related types of mesh grading are given.

In accordance with V_a^i, W_a^i from (15) introduce finite element spaces V_{ah}^i, W_{ah}^i of functions v_h^i on $\overline{\Omega}_a^i$:

$$\begin{aligned} V_{ah}^i &:= \{\, v_h^i \in C(\overline{\Omega}_a^i) : \ v_h^i \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h^i, \ v_h^i|_{\partial\Omega_a^i \cap \Gamma_a} = 0 \,\}, \\ W_{ah}^i &:= \{\, v_h^i \in V_{ah}^i \quad \text{and} \quad v_h^i|_{\partial\Omega_a^i \cap \Gamma_0} = 0 \,\}, \ \text{ for } i = 1, 2, \end{aligned} \tag{21}$$

i.e. employ linear finite elements. It should be noted that $w \in W_a^i$ implies $w|_{\partial\Omega_a^i \cap \Gamma_0} = 0$ (cf. [17]) so that we require this also for $v_h^i \in W_{ah}^i$. The finite element spaces V_{ah} and W_{ah} of vectorized functions v_h with components v_h^i on Ω_a^i are given by

$$V_{ah} := V_{ah}^1 \times V_{ah}^2, \quad W_{ah} := W_{ah}^1 \times W_{ah}^2. \tag{22}$$

It should be noted that the functions v_h in V_{ah} and in W_{ah} are in general not continuous across Γ. Since we focus our interest on the treatment of edge singularities with non-matching meshes, we restrict ourselves to the case that one endpoint of Γ coincides with the vertex E_a of the reentrant corner (cf. Figure 3(a)).

Further we introduce some triangulation $\mathcal{E}_h$ of the mortar interface Γ by intervals E ($E = \overline{E}$), i.e., $\Gamma = \cup_{E\in\mathcal{E}_h} E$. Let h_E denote the diameter of E. We suppose that two segments E, E' are either disjoint or have a common endpoint. A natural choice for the triangulation $\mathcal{E}_h$ is $\mathcal{E}_h := \mathcal{E}_h^1$ or $\mathcal{E}_h := \mathcal{E}_h^2$, where $\mathcal{E}_h^1$ and $\mathcal{E}_h^2$ denote the triangulations of Γ defined by the traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ, respectively:

$$\mathcal{E}_h^i := \{E : E = \partial T \cap \Gamma, \text{ if } E \text{ is a segment}, T \in \mathcal{T}_h^i\} \quad \text{for } i = 1, 2.$$

Subsequently we use real parameters α_1, α_2 with

$$0 \le \alpha_i \le 1 \quad (i = 1, 2), \quad \alpha_1 + \alpha_2 = 1. \tag{23}$$

The asymptotic behaviour of the triangulations $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and of $\mathcal{E}_h$ should be consistent on Γ in the sense of the following assumption (cf. also [13, 14]).

Assumption 3

1. *For $E \in \mathcal{E}_h$ and $T \in \mathcal{T}_h^i$ with $\partial T \cap E \ne \emptyset$, $i = 1$ and $i = 2$, there are positive constants C_1 and C_2 independent of h_T, h_E and h ($0 < h \le h_0$) such that the following condition is satisfied*
$$C_1 h_T \le h_E \le C_2 h_T. \tag{24}$$
2. *In the special case $\mathcal{E}_h := \mathcal{E}_h^i$ and $\alpha_i := 1$ (cf. (23)), where $i = 1$ or $i = 2$, for $E \in \mathcal{E}_h$ and $T \in \mathcal{T}_h^{3-i}$ with $\partial T \cap E \ne \emptyset$, instead of relation (24) the following condition is required:*
$$C_1 h_T \le h_E. \tag{25}$$

Relation (24) means that the diameter h_T of the triangle T touching the interface Γ at E is asymptotically equivalent to the diameter of the segment E, i.e. the equivalence of h_T, h_E is required only locally. Condition (25) is weaker and admits even locally at Γ different asymptotics of the triangles $T_1 \in \mathcal{T}_h^1$, $T_2 \in \mathcal{T}_h^2$: $T_1 \cap T_2 \ne \emptyset$.

In order to define the Nitsche-finite-element approximation of the solutions of the BVPs (16), we introduce sesquilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$ and linear forms $\mathcal{F}_{h,k}(\cdot)$, $k \in \mathbb{Z}$. The definition of $\mathcal{B}_{h,k}(\cdot,\cdot)$ and $\mathcal{F}_{h,k}(\cdot)$ is also given in [15] where the Fourier-Nitsche-finite element approximation of the BVP (4) for regular solutions $u \in X^2_{1/2}(\Omega)$ with Fourier coefficients $u_k \in H^2_{1/2}(\Omega_a)$ has been studied (i.e. reentrant edges are not considered). It should be noted that in comparison with the mortar methods in [2, 9, 13, 14, 21], we now have to take into account the spaces with power weights r^α as well as the Fourier parameter k.
For $k \in \mathbb{Z}\backslash\{0\}$ and $u_h, v_h \in W_{ah}$ as well as for $k = 0$ and $u_h, v_h \in V_{ah}$, resp., $\mathcal{B}_{h,k}(\cdot,\cdot)$ and $\mathcal{F}_{h,k}(\cdot)$ are defined as follows:

$$\mathcal{B}_{h,k}(u_h, v_h) := \sum_{i=1}^{2} \Big\{ (\nabla u_h^i, \nabla v_h^i)_{1/2,\Omega_a^i} + k^2 (u_h^i, v_h^i)_{-1/2,\Omega_a^i} \Big\} - \Big\langle \alpha_1 \frac{\partial u_h^1}{\partial n_1} - \alpha_2 \frac{\partial u_h^2}{\partial n_2}, v_h^1 - v_h^2 \Big\rangle_{1/2,\Gamma}$$

$$-\Big\langle \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}, u_h^1 - u_h^2 \Big\rangle_{1/2,\Gamma} + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} (u_h^1 - u_h^2, v_h^1 - v_h^2)_{1/2,E} \tag{26}$$

$$\mathcal{F}_{h,k}(v_h) \ := \ \sum_{i=1}^{2} (f_k^i, v_h^i)_{1/2,\Omega_a^i}.$$

Here, $\langle \cdot,\cdot \rangle_{1/2,\Gamma}$ denotes the $[H^{1/2}_{1/2,*}(\Gamma)]' \times H^{1/2}_{1/2,*}(\Gamma)$-duality pairing, with the space $H^{1/2}_{1/2,*}(\Gamma)$ from [15, p.5], and $(\cdot,\cdot)_{1/2,E}$ means the weighted $L_{2,1/2}(E)$ scalar product defined by $(u,v)_{1/2,E} := \int_E u\overline{v} r ds$. Moreover, γ is a sufficiently large positive constant (the restriction of γ will be given subsequently) and α_1 as well as α_2 are as in (23). For $v_h = (v_h^1, v_h^2) \in V_{ah}$, we have $\frac{\partial v_h^i}{\partial n_i}|_\Gamma \in L_{2,1/2}(\Gamma)$. This will be used subsequently for evaluating $\langle \cdot,\cdot \rangle_{1/2,\Gamma}$ by the $L_{2,1/2}(\Gamma)$-scalar product.
The Nitsche-finite-element approximations $u_{0h} = (u_{0h}^1, u_{0h}^2) \in V_{ah}$ and $u_{kh} = (u_{kh}^1, u_{kh}^2) \in W_{ah}$, $k \in \mathbb{Z}\setminus\{0\}$, of the functions $u_k = (u_k^1, u_k^2)$ are defined to be the solutions of the equations

$$\mathcal{B}_{h,0}(u_{0h}, v_h) = \mathcal{F}_{h,0}(v_h) \ \forall v_h \in V_{ah}\,; \ \ \mathcal{B}_{h,k}(u_{kh}, v_h) = \mathcal{F}_{h,k}(v_h) \ \forall v_h \in W_{ah}, \ k \in \mathbb{Z}\setminus\{0\}. \tag{27}$$

We now summarize some important properties of the sesquilinear forms $\mathcal{B}_{h,k}(\cdot,\cdot)$. The following lemma states the consistency of the solutions u_k ($k \in \mathbb{Z}$) from (6) with the variational equations (27).

Lemma 1 *Let u_k ($k \in \mathbb{Z}$) be the solution of the BVPs (6). Then $u_k = (u_k^1, u_k^2)$ satisfies:*

$$\mathcal{B}_{h,0}(u_0, v_h) = \mathcal{F}_{h,0}(v_h) \ \ \forall v_h \in V_{ah}; \quad \mathcal{B}_{h,k}(u_k, v_h) = \mathcal{F}_{h,k}(v_h) \ \ \forall v_h \in W_{ah}, \ \ k \in \mathbb{Z}\setminus\{0\}. \tag{28}$$

Proof: For regular solutions $u_k \in H^2_{1/2}(\Omega_a)$ of the BVPs (6), relations (28) are proved in [15]. In our case, the singular part s_k (see (13)) of u_k satisfies $s_k \in H^{\frac{3}{2}+\varepsilon}(\Omega_a)$ for any $\varepsilon \in (0,\varepsilon_0)$, with sufficiently small ε_0. From this we conclude $u_k \in H^{\frac{3}{2}+\varepsilon}(\Omega_a)$ and $\frac{\partial u_k^i}{\partial n_i} \in L_{2,1/2}(\Gamma)$, $i = 1,2$. Since additionally $\Delta_{r,z} u_k := \left\{ \frac{\partial^2 u_k}{\partial r^2} + \frac{\partial^2 u_k}{\partial z^2} + \frac{1}{r}\frac{\partial u_k}{\partial r} \right\} \in L_{2,1/2}(\Omega_a)$ holds, the proof of (28) is analogous to [15, Proof of Lemma 1]. ■

In order to state the boundedness and ellipticity of the forms $\mathcal{B}_{h,k}(\cdot,\cdot)$ we impose the restriction $\gamma > C_I$ on the parameter γ from (26), where the constant C_I is taken from the estimate

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|^2_{L_{2,1/2}(E)} \le C_I \sum_{i=1}^{2} \alpha_i^2 \|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)} \quad \text{for } v_h \in V_{ah},$$

with α_1, α_2 from (23). This estimate is obtained from [15, Ineq. (24)] valid on shape-regular meshes. Moreover, we shall need the weighted discrete norms $\|\cdot\|_{1,h,k}$ ($k \in \mathbb{Z}$) defined by

$$\|v_h\|^2_{1,h,k} := \sum_{i=1}^{2} \Big\{ \|\nabla v_h^i\|^2_{L_{2,1/2}(\Omega_a^i)} + k^2 \|v_h^i\|^2_{L_{2,-1/2}(\Omega_a^i)} \Big\} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|v_h^1 - v_h^2\|^2_{L_{2,1/2}(E)}. \tag{29}$$

Lemma 2 *Let Assumptions 1-3 be satisfied for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$. Then there exists a constant $\mu_1 > 0$ such that the following estimate holds,*

$$|\mathcal{B}_{h,k}(w_h, v_h)| \leq \mu_1 \|w_h\|_{1,h,k} \|v_h\|_{1,h,k} \;\; \forall w_h, v_h \in W_{ah}, k \in \mathbb{Z}\backslash\{0\} \; (w_h, v_h \in V_{ah}, k = 0, \textit{resp.}).$$

If the constant γ in (26) is independent of h and k and fulfills $\gamma > C_I$, then the inequality

$$\mathcal{B}_{h,k}(v_h, v_h) \geq \mu_2 \|v_h\|_{1,h,k}^2 \quad \forall v_h \in W_{ah}, k \in \mathbb{Z}\backslash\{0\} \; (v_h \in V_{ah}, k = 0, \textit{resp.})$$

holds with a positive constant μ_2. The constants μ_1, μ_2 are independent of h and k.

For the proof we refer to [15, Proof of Theorem 1] which is also valid under weaker assumptions on the mesh (shape regularity but not necessarily quasi-uniformity).

4 Error estimates in 2D

The aim of this section is to estimate the approximation error for solutions to BVPs in 2D. We have to take into account that we deal with a family of 2D problems depending on the parameter $k \in \mathbb{Z}$ and with norms containing power weights r^α, $\alpha \in \{1, 2\}$, cf. Section 2. In [17, 12, 23], an interpolation operator and a projection-interpolation operator are employed to estimate the approximation error of the 2D solutions arising from the decomposition of the BVP in 3D. Since we now consider the FEM with mortaring, these operators have to be slightly adapted.
For estimating the approximation error for the regular part w_k (see (14)) of the Fourier coefficients u_k with $|k| \leq 1$, the interpolation operator Π_h will be employed. It is now defined as follows:

$$\Pi_h w_k := (\Pi_h w_k^1, \Pi_h w_k^2), \tag{30}$$

where $\Pi_h w_k^i$ ($i = 1, 2$) denotes the usual Lagrange interpolant of w_k^i in the space V_{ah}^i.
The use of $\Pi_h w_k$ for $|k| \geq 2$ does not lead to optimal error estimates with respect to the discretization parameters h and N^{-1}, cf. [17]. Therefore, for w_k with $|k| \geq 2$ we shall apply a projection-interpolation operator P_h (cf. also [8, 15, 17]) which is defined subsequently.
On the other hand, for estimating the approximation error for the singular part s_k of the Fourier coefficients u_k we can employ the operator Π_h from (30) (now with s_k instead of w_k) for all $k \in \mathbb{Z}$, without loss of optimality of the error estimate. For this estimate, we take advantage of the fact that the singular part s_k is explicitely known, see (13).
The projection-interpolation operator P_h is defined as follows (see [15, Section 5] for more details):

$$P_h u_k := (P_h^1 u_k^1, P_h^2 u_k^2) \quad \text{with} \;\; P_h^i v := \sum_{Q \in \Sigma_h^{i,\star}} v_Q^i \Phi_Q^i, \;\; i = 1, 2. \tag{31}$$

In (31), $\Sigma_h^{i,\star}$ denotes the set of all nodes $Q \in \mathcal{T}_h^i$ with $Q \notin (\partial\Omega_a^i \cap \Gamma_a)$ and $T \cap \overline{\Gamma}_0 = \emptyset$ for any $T \in \mathcal{T}_h^i$ having Q as vertex. Further, v_Q^i is given by $v_Q^i := (P_Q^i v)(Q)$ with the orthogonal

projection operator $P_Q^i: L_2(S_Q^i) \longrightarrow \mathbb{P}_1(S_Q^i)$ defined by the relation $(v - P_Q^i v, p)_{L_2(S_Q^i)} = 0$ $\forall p \in \mathbb{P}_1(S_Q^i)$, and Φ_Q^i denotes Courant's basis function associated with the node Q.
In addition to $\|\cdot\|_{1,h,k}^2$ at (29) and adapted to $\mathcal{B}_{h,k}(\cdot,\cdot)$, we introduce the weighted mesh-dependent norm $\|\cdot\|_{h,k,\Omega_a}$:

$$\begin{aligned} \|v\|_{h,k,\Omega_a}^2 := \; & \sum_{i=1}^{2} \Big\{ \|\nabla v^i\|_{L_{2,1/2}(\Omega_a^i)}^2 + k^2 \|v^i\|_{L_{2,-1/2}(\Omega_a^i)}^2 + \sum_{E\in\mathcal{E}_h} h_E \Big\| \alpha_i \frac{\partial v^i}{\partial n_i} \Big\|_{1/2,E}^2 \Big\} \\ & + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|v^1 - v^2\|_{L_{2,1/2}(E)}^2 \end{aligned} \tag{32}$$

for functions v satisfying $v \in V_a$ for $k = 0$, $v \in W_a$ for $k \in \mathbb{Z} \setminus \{0\}$, and $\frac{\partial v^i}{\partial n_i}|_\Gamma \in L_{2,1/2}(\Gamma)$ for $i \in \{1,2\}:\ \alpha_i \neq 0$.
The norm of the error $u_k - u_{kh}$ ($k \in \mathbb{Z}$) of the Nitsche finite-element approximation u_{kh} can be bounded by means of the norms of $w_k - P^* w_k$ (with $P^* w_k := \Pi_h w_k$ for $|k| \le 1$, $P^* w_k := P_h w_k$ for $|k| \ge 2$) and $s_k - \Pi_h s_k$ ($k \in \mathbb{Z}$). This is stated in the following lemma.

Lemma 3 *Let Assumptions 1-3 be satisfied for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$, moreover, let $\gamma > C_I$. Then the following estimates hold for the error $u_k - u_{kh}$ (u_k, u_{kh} from (10), (27)):*

$$\begin{aligned} \|u_k - u_{kh}\|_{1,h,k} &\le C \left(\|w_k - \Pi_h w_k\|_{h,k,\Omega_a} + \|s_k - \Pi_h s_k\|_{h,k,\Omega_a} \right) \ \textit{for } |k| \le 1 \\ \|u_k - u_{kh}\|_{1,h,k} &\le C \left(\|w_k - P_h w_k\|_{h,k,\Omega_a} + \|s_k - \Pi_h s_k\|_{h,k,\Omega_a} \right) \ \textit{for } |k| \ge 2, \end{aligned} \tag{33}$$

with w_k, s_k from (14), (13).

Proof: We employ the representation $u_k = w_k + s_k$ (cf. Theorem 1) of the Fourier coefficients u_k. Then, by means of Lemmas 1 and 2 and the linearity of the operator Π_h, the proof of the first inequality in (33) can be carried out by analogy to the proof of [13, Lemma 3]. In order to prove the second inequality in (33), we use the relation $\|v\|_{1,h,k} \le \|v\|_{h,k,\Omega_a}$ (cf. (29) and (32)) leading to

$$\begin{aligned} \|u_k - u_{kh}\|_{1,h,k} \;&=\; \|w_k + s_k - u_{kh}\|_{1,h,k} \\ &\le\; \|w_k - P_h w_k\|_{h,k,\Omega_a} + \|P_h w_k + \Pi_h s_k - u_{kh}\|_{1,h,k} + \|s_k - \Pi_h s_k\|_{h,k,\Omega_a}. \end{aligned} \tag{34}$$

Further, thanks to Lemmas 1 and 2 we obtain for the second term on the right-hand side of (34):

$$\begin{aligned} \|P_h w_k + \Pi_h s_k - u_{kh}\|_{1,h,k}^2 &\le \mu_2^{-1} \mathcal{B}_{h,k}(P_h w_k + \Pi_h s_k - u_{kh}, P_h w_k + \Pi_h s_k - u_{kh}) \\ &= \mu_2^{-1} \mathcal{B}_{h,k}(P_h w_k + \Pi_h s_k - w_k - s_k, P_h w_k + \Pi_h s_k - u_{kh}). \end{aligned} \tag{35}$$

Employing the Hölder and Cauchy–Schwarz inequalities we get the estimate

$$\begin{aligned} & |\mathcal{B}_{h,k}(P_h w_k + \Pi_h s_k - w_k - s_k, P_h w_k + \Pi_h s_k - u_{kh})| \\ & \quad \le C \, \|P_h w_k + \Pi_h s_k - w_k - s_k\|_{h,k,\Omega_a} \|P_h w_k + \Pi_h s_k - u_{kh}\|_{1,h,k}. \end{aligned}$$

This, together with (35) leads to

$$\|P_h w_k + \Pi_h s_k - u_{kh}\|_{1,h,k} \le C \, \|P_h w_k + \Pi_h s_k - w_k - s_k\|_{h,k,\Omega_a}, \tag{36}$$

and the second estimate in (33) is a consequence of (34) and (36). ■

Lemma 3 implies that we need estimates for the interpolation and projection-interpolation errors of the regular part w_k of u_k as well for the interpolation error of the singular part s_k. First we give estimates for the error of the regular part.

Theorem 2 *Let the Assumptions 1-3 be satisfied. Furthermore, for each $F \in \mathcal{E}_h^i$ we require that the triangle $T_F \in \mathcal{T}_h^i$ with $T_F \cap \Gamma = F$ has at most one common point with Γ_0. Then for the regular parts w_k of the Fourier coefficients u_k of u, the following error estimate holds:*

$$\|w_k - \Pi_h w_k\|_{h,k,\Omega_a} \;\leq\; Ch\|w_k\|_{H^2_{1/2}(\Omega_a)} \qquad \textit{for } |k| \leq 1, \tag{37}$$

$$\|w_k - P_h w_k\|_{h,k,\Omega_a} \;\leq\; Ch\Big\{k^2\|w_k\|^2_{H^1_{-1/2}(\Omega_a)} + \|w_k\|^2_{H^2_{1/2}(\Omega_a)}\Big\}^{1/2} \qquad \textit{for } |k| \geq 2, \tag{38}$$

where Π_h and P_h are defined in (30) and (31), respectively.

Proof: Owing to $w_k \in H^2_{1/2}(\Omega_a)$, $k \in \mathbb{Z}$, the estimate (37) follows from [15, Theorem 3], and the estimate (38) is a consequence of [15, Theorem 4]. ■

For the error estimate of the singular part s_k we introduce some notations. Let the subset $\mathcal{C}_{0h}$ of the triangulation $\mathcal{T}_h$ be given by: $\mathcal{C}_{0h} := \{T \in \mathcal{T}_h : R_T < R_n\}$, with $R_T := \operatorname{dist}(T, E_a)$ and R_n from (18), i.e. $\mathcal{C}_{0h}$ consists of the triangles near the vertex E_a of the reentrant corner. The set $\mathcal{C}_{0h}$ can be decomposed into layers $\mathcal{D}_{jh}$ $(j = 0, 1, \ldots, n)$ of triangles, such that $\mathcal{C}_{0h} := \cup_{j=0}^n \mathcal{D}_{jh}$ holds:

$$\mathcal{D}_{0h} := \{T \in \mathcal{T}_h : R_T = 0\}, \quad \mathcal{D}_{jh} := \{T \in \mathcal{T}_h : R_{j-1} \leq R_T < R_j\} \text{ for } j = 1, \ldots, n, \tag{39}$$

where R_j is given in (18). Furthermore, we define for $i = 1, 2$ and $j = 0, \ldots, n$: $\mathcal{D}^i_{jh} := \{T \in \mathcal{D}_{jh} : T \subset \overline{\Omega}^i_a\}$. According to $\frac{2}{3}R'_0 < R_n < R'_0$, the triangles $T \in \mathcal{C}_{0h}$ are located in $\overline{G}_a$. The number n_j of all triangles $T \in \mathcal{D}_{jh}$ $(j = 1, \ldots, n)$ is bounded by $C \cdot j$, and $n_0 < C$ holds for the number n_0 of all triangles $T \in \mathcal{D}_{0h}$, cf. [12, 19].

Concerning the error norm of the singular part, we may write by means of (32):

$$\begin{aligned}
\|s_k - \Pi_h s_k\|^2_{h,k,\Omega_a} := &\sum_{i=1}^{2}\Big\{\|\nabla(s^i_k - \Pi_h s^i_k)\|^2_{L_{2,1/2}(\Omega^i_a)} + k^2\|s^i_k - \Pi_h s^i_k\|^2_{L_{2,-1/2}(\Omega^i_a)} \\
&+ \sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_i \frac{\partial(s^i_k - \Pi_h s^i_k)}{\partial n_i}\right\|^2_{1/2,E}\Big\} + \sum_{E\in\mathcal{E}_h} h_E^{-1}\|s^1_k - \Pi_h s^1_k - (s^2_k - \Pi_h s^2_k)\|^2_{L_{2,1/2}(E)}.
\end{aligned} \tag{40}$$

In order to deal with the first two terms on the right-hand side of (40), we take into account that the singular part s_k, $k \in \mathbb{Z}$, vanishes outside of the subdomain G_a given by (11). Therefore, the norms on $H^l_\alpha(G_a)$ (α real) are equivalent to the norms on the usual Sobolev spaces $H^l(G_a)$. Moreover, let G^i_a $(i = 1, 2)$ be defined by $G^i_a := G_a \cap \Omega^i_a$. Then we have the following estimate.

Lemma 4 *Under the Assumptions 1-3 on the triangulation $\mathcal{T}_h$, the interpolation error $s_k^i - \Pi_h s_k^i$ for the function $s_k^i = s_k|_{\Omega_a^i}$ $(i = 1, 2)$, with s_k from (13), satisfies the estimates*

$$\|s_k^i - \Pi_h s_k^i\|^2_{L_2(G_a^i)} \le C\,|\delta_k|^2 \begin{cases} h^{2+2\lambda} & \text{for } k = 0 \\ k^{-2} h^{\frac{2\lambda}{\mu}} + |k|^{-2\lambda} h^2 & \text{for } k \in \mathbb{Z}\setminus\{0\}, \end{cases} \tag{41}$$

$$|s_k^i - \Pi_h s_k^i|^2_{H^1(G_a^i)} \le C\,|\delta_k|^2 (\kappa^2(h,\mu) + |k|^{2-2\lambda} h^2) \quad \text{for } k \in \mathbb{Z}, \tag{42}$$

$$\text{with} \quad \kappa^2(h,\mu) = \begin{cases} h^{\frac{2\lambda}{\mu}} & \text{for } \lambda < \mu \le 1 \\ h^2 |\ln h| & \text{for } \mu = \lambda \\ h^2 & \text{for } 0 < \mu < \lambda. \end{cases} \tag{43}$$

The proof of this lemma (with s_k instead of s_k^i and G_a instead of G_a^i) is given by [12, Proof of Lemma 4.2].

In the next lemma, we give estimates for the last two terms on the right-hand side of (40).

Lemma 5 *Let the Assumptions 1-3 on the triangulation $\mathcal{T}_h$ be satisfied. Then for the interpolation error $s_k^i - \Pi_h s_k^i$ $(k \in \mathbb{Z};\ i = 1, 2)$, the following estimates hold:*

$$\sum_{E \in \mathcal{E}_h} h_E \Big\| \alpha_i \frac{\partial (s_k^i - \Pi_h s_k^i)}{\partial n_i} \Big\|^2_{L_{2,1/2}(E)} \le C |\alpha_i|^2 |\delta_k|^2 (\kappa^2(h,\mu) + |k|^{2-2\lambda} h^2) \tag{44}$$

$$\sum_{E \in \mathcal{E}_h} h_E^{-1} \|s_k^i - \Pi_h s_k^i\|^2_{L_{2,1/2}(E)} \le C |\delta_k|^2 (\kappa^2(h,\mu) + |k|^{2-2\lambda} h^2), \tag{45}$$

where $\kappa^2(h,\mu)$ is taken from (43).

It should be noted that the sums on the left-hand side of (44) (respectively (45)) could be taken over $E \in \mathcal{E}_h : E \cap \overline{G}_a \neq \emptyset$ instead of $E \in \mathcal{E}_h$ because the summands for $E \not\subset \overline{G}_a$ vanish.

Proof: For the sake of brevity we set $v_k^i := s_k^i - \Pi_h s_k^i$ $(i = 1, 2)$. By analogy to [15, Eqs. (57), (58)] we may write

$$\sum_{E \in \mathcal{E}_h} h_E \Big\| \alpha_i \frac{\partial v_k^i}{\partial n_i} \Big\|^2_{L_{2,1/2}(E)} \le C |\alpha_i|^2 \sum_{F \in \mathcal{E}_h^i} h_F \|\nabla v_k^i\|^2_{L_{2,1/2}(F)} \quad \text{for } i \in \{1,2\}\colon \alpha_i > 0, \tag{46}$$

$$\sum_{E \in \mathcal{E}_h} h_E^{-1} \|v_k^i\|^2_{L_{2,1/2}(E)} \le C \sum_{F \in \mathcal{E}_h^i} h_F^{-1} \|v_k^i\|^2_{L_{2,1/2}(F)} \quad \text{for } i = 1, 2, \tag{47}$$

i.e. the summation over $E \in \mathcal{E}_h$ can be replaced by a summation over $F \in \mathcal{E}_h^i$, and $\|\frac{\partial v_k^i}{\partial n_i}\|_{L_{2,1/2}(E)}$ can be bounded by $\|\nabla v_k^i\|_{L_{2,1/2}(E)}$.

For $F \in \mathcal{E}_h^i$, let T_F be the triangle such that $T_F \cap \Gamma = F$. Since the functions s_k vanish outside of G_a, for proving inequalities (44), (45) it suffices to consider the triangles $T = T_F$ with $T \in \mathcal{C}_{0h}$, where we distinguish two cases concerning the position of the triangle T.

Case 1: We suppose that $T \in \mathcal{D}_{0h}^i$, $i \in \{1, 2\}$, holds with $\mathcal{D}_{0h}^i$ from (39). Taking into account the estimate (46), we have to find a bound of $\|\nabla v_k^i\|^2_{L_{2,1/2}(F)}$. Clearly, we have

$$\begin{aligned} \|\nabla v_k^i\|^2_{L_{2,1/2}(F)} &\le C(\|\nabla s_k^i\|^2_{L_{2,1/2}(F)} + \|\nabla(\Pi_h s_k^i)\|^2_{L_{2,1/2}(F)}) \\ &\le C(\|\nabla s_k^i\|^2_{L_2(F)} + \|\nabla(\Pi_h s_k^i)\|^2_{L_2(F)}). \end{aligned} \tag{48}$$

We now use the explicit representation of the singular functions s_k (see (13)) and get

$$\|\nabla s_k^i\|^2_{L_2(F)} = \int\limits_F \Big\{ \Big|\frac{\partial s_k^i}{\partial R}\Big|^2 + \frac{1}{R^2}\Big|\frac{\partial s_k^i}{\partial \theta}\Big|^2 \Big\}\, dR \le C|\delta_k|^2 \int\limits_0^{h_F} (|k|^2 R^{2\lambda} \mathrm{e}^{-2|k|R} + R^{2\lambda-2}\mathrm{e}^{-2|k|R})\, dR. \quad (49)$$

Since $\lambda > \frac{1}{2}$ holds, the first term on the right-hand side of (49) (without the factor $|\delta_k|^2$) can be bounded as follows:

$$|k|^2 \int\limits_0^{h_F} R^{2\lambda} \mathrm{e}^{-2|k|R} dR \le |k|^2 h_F^{2\lambda-1} \int\limits_0^{h_F} R\, \mathrm{e}^{-2|k|R} dR \le |k|^2 h_F^{2\lambda-1} \frac{\Gamma(2)}{(2|k|)^2} \le C h_F^{2\lambda-1}, \qquad (50)$$

with the Gamma function $\Gamma(\cdot)$. The second term on the right-hand side of (49) (without the factor $|\delta_k|^2$) satisfies the estimate

$$\int\limits_0^{h_F} R^{2\lambda-2} \mathrm{e}^{-2|k|R} dR \le \int\limits_0^{h_F} R^{2\lambda-2} dR \le C h_F^{2\lambda-1}. \qquad (51)$$

Furthermore, by means of the properties of the linear interpolant $\Pi_h s_k^i$ we obtain $\|\nabla(\Pi_h s_k^i)\|^2_{L_2(F)} \le C|\delta_k|^2 h_F{}^{2\lambda-1}$. This, together with (48)-(51), and with the estimate $h_F \le h_T \le C h^{\frac{1}{\mu}}$ (see (19), case $R_T = 0$) yields

$$h_F \|\nabla v_k^i\|^2_{L_{2,1/2}(F)} \le C|\delta_k|^2 h^{\frac{2\lambda}{\mu}} \qquad (52)$$

for all triangles $T \in \mathcal{D}^i_{0h}$, $i \in \{1, 2\}$.
The estimate for the norm $\|v_k^i\|^2_{L_{2,1/2}(F)}$ (cf. (47)) can be derived by analogy to the aforementioned estimates. In this way we obtain

$$h_F^{-1} \|v_k^i\|^2_{L_{2,1/2}(F)} \le C|\delta_k|^2 h^{\frac{2\lambda}{\mu}}. \qquad (53)$$

Case 2: We consider $T \in \mathcal{D}^i_{jh}$, $i \in \{1, 2\}$ and $j \neq 0$. Then, the triangle has a positive distance to the singular corner and, consequently, $s_k \in H^2_{1/2}(T)$ holds. Moreover, according to [15, Theorem 2] (applied to $v := \nabla v_k^i$) , we obtain

$$\|\nabla v_k^i\|^2_{L_{2,1/2}(F)} \le C\left(h_T^{-1} |v_k^i|^2_{H^1_{1/2}(T)} + |v_k^i|_{H^1_{1/2}(T)}\, |v_k^i|_{H^2_{1/2}(T)} \right).$$

Owing to $s_k \in H^2_{1/2}(T)$, this yields together with [15, Lemma 4] and [17, Lemma 6.2]:

$$\|\nabla v_k^i\|^2_{L_{2,1/2}(F)} \le C\left(h_T^{-1} h_T^2 |s_k^i|^2_{H^2_{1/2}(T)} + h_T |s_k^i|^2_{H^2_{1/2}(T)} \right) \le C h_T |s_k^i|^2_{H^2_{1/2}(T)}.$$

Owing to the boundedness of the weighting factor r, the seminorm $|\cdot|_{H^2_{1/2}(T)}$ can be replaced by $|\cdot|_{H^2(T)}$. Taking into account $h_F \le h_T$ and summing up over all triangles $T \in \mathcal{D}^i_{jh}$ ($j = 1, \ldots, n$; $i \in \{1, 2\}$) satisfying $T \cap \Gamma \neq \emptyset$ we obtain with the help of [12, Ineq. (4.18)]:

$$\sum_{j=1}^{n} \sum_{\substack{F \in \mathcal{E}^i_h: \\ T_F \in \mathcal{D}^i_{jh}}} h_F \|\nabla v_k^i\|^2_{L_{2,1/2}(F)} \le C \sum_{j=1}^{n} \sum_{\substack{T \in \mathcal{D}^i_{jh}: \\ T \cap \Gamma \neq \emptyset}} h_T^2 \|s_k^i\|^2_{H^2_{1/2}(T)} \le C|\delta_k|^2 \sum_{j=1}^{n} h_j^2 \int\limits_{\underline{R}_{j-1}}^{\overline{R}_j} (\Phi_1 + \Phi_2)\, dR, \quad (54)$$

with $\underline{R}_{j-1} := R_{j-1}$ ($j = 2, \dots, n$), $\overline{R}_j := R_j + \varepsilon_1^{-1} h_j$ (ε_1 from (20); $j = 1, \dots, n$) and $\Phi_1 = \Phi_1(R, \lambda, k) := R^{2\lambda-3}(k^2 R^2 + k^4 R^4)\,\mathrm{e}^{-2|k|R}$, $\Phi_2 = \Phi_2(R, \lambda, k) := R^{2\lambda-3}\mathrm{e}^{-2|k|R}$, cf. [12, Proof of Lemma 4.2]. For estimating the integral on the right-hand side of (54) we employ [12, Ineq. (4.21), (4.22)]:

$$\sum_{j=1}^{n} h_j^2 \int_{\underline{R}_{j-1}}^{\overline{R}_j} \Phi_1 dR \leq C h^2 |k|^{2-2\lambda}, \quad k \in \mathbb{Z} \setminus \{0\}, \quad \sum_{j=1}^{n} h_j^2 \int_{\underline{R}_{j-1}}^{\overline{R}_j} \Phi_2 dR \leq C \kappa^2(h, \mu), \quad k \in \mathbb{Z}, \tag{55}$$

with $\kappa^2(h, \mu)$ from (43), which leads to

$$\sum_{j=1}^{n} \sum_{\substack{T \in \mathcal{D}^i_{jh}: \\ T \cap \Gamma \neq \emptyset}} h_T^2 \|s_k^i\|^2_{H^2_{1/2}(T)} \leq C|\delta_k|^2 (h^2 |k|^{2-2\lambda} + \kappa^2(h, \mu)). \tag{56}$$

In order to derive a bound for the norm $\|v_k^i\|^2_{L_{2,1/2}(F)}$ occurring on the right-hand side of (47), we use again [15, Theorem 2, Lemma 4] and [17, Lemma 6.2]. This yields

$$\|v_k^i\|^2_{L_{2,1/2}(F)} \leq C \left(h_T^{-1} \|v_k^i\|^2_{L_{2,1/2}(T)} + \|v_k^i\|_{L_{2,1/2}(T)} \|\nabla v_k^i\|_{L_{2,1/2}(T)} \right) \leq C h_T^3 |s_k^i|^2_{H^2_{1/2}(T)},$$

and by means of inequalities analogous to (54), (55) we are led to

$$\sum_{j=1}^{n} \sum_{\substack{T \in \mathcal{D}^i_{jh}: \\ T \cap \Gamma \neq \emptyset}} h_F^{-1} \|v_k^i\|^2_{L_{2,1/2}(T)} \leq C|\delta_k|^2 (h^2 |k|^{2-2\lambda} + \kappa^2(h, \mu)). \tag{57}$$

Finally, collecting inequalities (46), (52), (54), and (56) (resp. (47), (53), and (57)) we obtain the assertion of Lemma 5. ■

5 The Fourier-Nitsche-finite-element approximation and convergence results in 3D

In order to define the Fourier-Nitsche-finite-element approximation of the solution to the BVP (4) in 3D, we employ the space V_{hN} depending on the parameters h and N,

$$V_{hN} := \Big\{ v \colon v(r, \varphi, z) = \sum_{|k| \leq N} v_{kh}(r, z)\, \mathrm{e}^{ik\varphi} \;\text{ with }\; v_{0h} \in V_{ah}, v_{kh} \in W_{ah} \;\text{ for }\; 1 \leq |k| \leq N \Big\}, \tag{58}$$

with V_{ah} and W_{ah} from (22). Furthermore, by means of $\mathcal{B}_{h,k}(\cdot,\cdot)$ and $\mathcal{F}_{h,k}(\cdot)$ from (26) we introduce the forms

$$\mathcal{B}_h^N(u, v) := 2\pi \sum_{|k| \leq N} \mathcal{B}_{h,k}(u_k, v_k)\,, \quad \mathcal{F}_h^N(v) := 2\pi \sum_{|k| \leq N} \mathcal{F}_{h,k}(v_k), \tag{59}$$

for $u, v \in X^1_{1/2}(\Omega^1) \times X^1_{1/2}(\Omega^2)$. Note that the decomposition of Ω_a in 2D yields the corresponding decomposition in 3D, with $\Omega^j := \Omega^j_a \times (-\pi, \pi]$, $j = 1, 2$. For treating the

BVP in 3D, the combined Fourier-Nitsche-finite-element method is now defined by the Galerkin approach

$$\text{find } u_{hN} \in V_{hN} \text{ such that } \mathcal{B}_h^N(u_{hN}, v_{hN}) = \mathcal{F}_h^N(v_{hN}) \ \forall v_{hN} \in V_{hN}. \tag{60}$$

By analogy to [15], we can state that the solution u_{hN} to (60) is given by

$$u_{hN} = (u_{hN}^1, u_{hN}^2) \quad \text{with } u_{hN}^j = \sum_{|k| \le N} u_{kh}^j(r,z)\, \mathrm{e}^{ik\varphi} \quad \text{for } j = 1,2, \tag{61}$$

where $u_{kh} = (u_{kh}^1, u_{kh}^2)$ $(k = 0, \pm 1, ..., \pm N)$ can be calculated as the solution of the 2D problem (27). The Fourier-Nitsche-finite-element approximation u_{hN} of u obviously depends on h and N.

In order to derive estimates of the error $u - u_{hN}$, with u and u_{hN} from (4) and (60), respectively, we introduce for elements of the 'broken' space $X_{1/2}^1(\Omega^1) \times X_{1/2}^1(\Omega^2)$ a suitable H^1-like norm:

$$\|v\|_{1,h,\Omega}^2 := \sum_{j=1}^2 |v^j|_{X_{1/2}^1(\Omega^j)}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|v^1 - v^2\|_{X_{1/2}^0(E \times (-\pi,\pi])}^2, \tag{62}$$

where the H^1-seminorm part $|\cdot|_{X_{1/2}^1(\Omega^j)}$ is defined by analogy to $|\cdot|_{X_{1/2}^1(\Omega)}$ at (3), and the L_2-norm assigned to $E \times (-\pi, \pi] \subset \overline{\Omega}^1 \cap \overline{\Omega}^2$ is determined by the completeness relation

$$\|v\|_{X_{1/2}^0(E \times (-\pi,\pi])}^2 := 2\pi \sum_{k \in \mathbb{Z}} \|v_k\|_{L_{2,1/2}(E)}^2. \tag{63}$$

It should be noted that we have $u_{hN} \in X_{1/2}^1(\Omega^1) \times X_{1/2}^1(\Omega^2)$ and, in general, $u_{hN} \notin X_{1/2}^1(\Omega)$.

Now we are in a position to give the error estimate in the norm $\|\cdot\|_{1,h,\Omega}$.

Theorem 3 *Assume that $\hat{f} \in L_2(\widehat{\Omega})$ ($\widehat{\Omega}$: axisymmetric domain) and that there is only one reentrant edge on $\partial\widehat{\Omega}$, u is the solution of the BVP (4), u_{hN} its Fourier-Nitsche-finite-element approximation on V_{hN}. Then, under the assumptions of Theorem 2 the following error estimate holds,*

$$\|u - u_{hN}\|_{1,h,\Omega} \le C(\kappa(h,\mu) + N^{-1}) \|f\|_{X_{1/2}^0(\Omega)} \tag{64}$$

$$\text{with } \kappa(h,\mu) = \begin{cases} h^{\frac{\lambda}{\mu}} & \text{for } \lambda < \mu \le 1 \\ h|\ln h|^{\frac{1}{2}} & \text{for } \mu = \lambda \\ h & \text{for } 0 < \mu < \lambda. \end{cases}$$

Clearly, relation (64) implies also the convergence $u_{hN} \to u$ as $h \to 0$, $N \to \infty$. In particular, h and N can be chosen independently from each other.

Proof: By means of the auxiliary function $u_N = (u_N^1, u_N^2)$ defined by

$$u_N^j = \sum_{|k| \le N} u_k^j(r,z)\, \mathrm{e}^{ik\varphi} \quad j = 1,2, \tag{65}$$

we easily get

$$\|u-u_{hN}\|_{1,h,\Omega} \le \|u-u_N\|_{1,h,\Omega} + \|u_N-u_{hN}\|_{1,h,\Omega} =: S_1+S_2, \tag{66}$$

where S_1 and S_2 denote the corresponding norm terms. We shall now find estimates of S_1 and S_2 in terms of powers of h and N. According to (62) we have

$$S_1^2 = \sum_{j=1}^{2} |u^j - u_N^j|^2_{X^1_{1/2}(\Omega^j)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|u^1-u_N^1-(u^2-u_N^2)\|^2_{X^0_{1/2}(E\times(-\pi,\pi])}. \tag{67}$$

Owing to $u-u_N \in X^1_{1/2}(\Omega)$, the first term on the right-hand side of (67) is equal to $|u-u_N|^2_{X^1_{1/2}(\Omega)}$. Further, by means of completeness relations from [11, Lemma 3.2] and the following a priori estimate (see [11, Ineq. (4.4(c))])

$$\|u_0\|^2_{V_0^a} + \sum_{|k|>N} k^2 \Big\{ \Big\|\frac{\partial u_k}{\partial r}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial u_k}{\partial z}\Big\|^2_{L_{2,1/2}(\Omega_a)} + k^2\Big\|\frac{u_k}{r}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} \le C\|f\|^2_{X^0_{1/2}(\Omega)} \tag{68}$$

we get

$$\begin{aligned} |u-u_N|^2_{X^1_{1/2}(\Omega)} &= 2\pi \sum_{|k|>N} \Big\{ \Big\|\frac{\partial u_k}{\partial r}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial u_k}{\partial z}\Big\|^2_{L_{2,1/2}(\Omega_a)} + k^2\Big\|\frac{u_k}{r}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} \\ &\le 2\pi N^{-2} \sum_{|k|>N} k^2 \Big\{ \Big\|\frac{\partial u_k}{\partial r}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial u_k}{\partial z}\Big\|^2_{L_{2,1/2}(\Omega_a)} + k^2\Big\|\frac{u_k}{r}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} \\ &\le CN^{-2}\|f\|^2_{X^0_{1/2}(\Omega)}. \end{aligned} \tag{69}$$

The second term on the right-hand side of (67) vanishes. This is clear by $u^1|_{E\times(-\pi,\pi]} = u^2|_{E\times(-\pi,\pi]}$; the same holds for u_N^1, u_N^2. This, together with (67) and (69) completes the estimate for S_1:

$$S_1 \le CN^{-1}\|f\|_{X^0_{1/2}(\Omega)}. \tag{70}$$

Using relation (63) we obtain for S_2 the relation

$$\begin{aligned} S_2^2 &= \sum_{j=1}^{2} |u_N^j - u_{hN}^j|^2_{X^1_{1/2}(\Omega^j)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|u_N^1-u_{hN}^1-(u_N^2-u_{hN}^2)\|^2_{X^0_{1/2}(E\times(-\pi,\pi])} \\ &= 2\pi \sum_{j=1}^{2}\sum_{|k|\le N} \Big\{ \|\nabla(u_k^j-u_{kh}^j)\|^2_{L_{2,1/2}(\Omega_a^j)} + k^2\|u_k^j-u_{kh}^j\|^2_{L_{2,-1/2}(\Omega_a^j)} \Big\} \\ &\quad + 2\pi \sum_{E\in\mathcal{E}_h} h_E^{-1} \Big\{ \sum_{|k|\le N} \|u_k^1-u_{kh}^1-(u_k^2-u_{kh}^2)\|^2_{L_{2,1/2}(E)} \Big\}. \end{aligned} \tag{71}$$

Changing the order of summation and applying (29) as well as Lemma 3 we are led to

$$\begin{aligned} S_2^2 &= 2\pi \sum_{|k|\le N} \|u_k-u_{kh}\|^2_{1,h,k} \\ &\le C\Big\{ \sum_{|k|\le 1} \|w_k-\Pi_h w_k\|^2_{h,k,\Omega_a} + \sum_{2\le|k|\le N} \|w_k-P_h w_k\|^2_{h,k,\Omega_a} + \sum_{|k|\le N} \|s_k-\Pi_h s_k\|^2_{h,k,\Omega_a}\Big\}. \end{aligned} \tag{72}$$

The first two terms on the right-hand side of this inequality can be estimated by means of Theorem 2 and the last inequality in (14):

$$\sum_{|k|\le 1} \|w_k - \Pi_h w_k\|^2_{h,k,\Omega_a} + \sum_{2\le|k|\le N} \|w_k - P_h w_k\|^2_{h,k,\Omega_a} \tag{73}$$

$$\le Ch^2\Big\{\sum_{|k|\le N} \|w_k\|^2_{H^2_{1/2}(\Omega_a)} + \sum_{2\le|k|\le N} k^2\|w_k\|^2_{H^1_{-1/2}(\Omega_a)}\Big\} \le Ch^2\|w\|^2_{X^2_{1/2}(\Omega)} \le Ch^2\|f\|^2_{X^0_{1/2}(\Omega)},$$

such that it remains to find an estimate for the last term on the right-hand side of inequality (72). Here, relation (40) together with Lemmas 4 and 5 as well as estimate (13) yields:

$$\sum_{|k|\le N} \|s_k - \Pi_h s_k\|^2_{h,k,\Omega_a} \le C \sum_{|k|\le N} |\delta_k|^2(\kappa^2(h,\mu) + h^2|k|^{2-2\lambda}) \tag{74}$$

$$\le C\kappa^2(h,\mu)\sum_{|k|\le N} |\delta_k|^2(1+|k|^{2-2\lambda}) \le C\kappa^2(h,\mu)\sum_{|k|\le N} \|f_k\|^2_{L_{2,1/2}(\Omega_a)} \le C\kappa^2(h,\mu)\|f\|^2_{X^0_{1/2}(\Omega)}.$$

Finally, the assertion of Theorem 3 can be concluded from (66), (70), and (72)-(74). ■

The error estimate in the norm $\|\cdot\|_{X^0_{1/2}(\Omega)}$ is given in the next theorem.

Theorem 4 *Let the assumptions of Theorem 3 be fulfilled. Then, for u and its approximation u_{hN} the following error estimate is satisfied:*

$$\|u - u_{hN}\|_{X^0_{1/2}(\Omega)} \le C(\kappa^2(h,\mu) + N^{-2})\|f\|_{X^0_{1/2}(\Omega)}, \tag{75}$$

where $\kappa^2(h,\mu)$ is given by (43).

Proof: We consider the BVP (4) with $u - u_{hN}$ instead of f, i.e., find $u^e \in V_0(\Omega)$ such that

$$b(u^e, v) = \int_\Omega (u - u_{hN})\,\overline{v}\,r dr d\varphi dz =: (u - u_{hN}, v)_{X^0_{1/2}(\Omega)} \quad \forall v \in V_0(\Omega). \tag{76}$$

Owing to $u - u_{hN} \in X^0_{1/2}(\Omega)$ and to the assumptions on Ω (cf. Section 2), the solution u^e can be decomposed into a singular and a regular part as mentioned in Theorem 1. In order to distinguish the decompositions of u and u^e, we use in context with u^e notations with index e such as u^e_s instead of u_s (and analogously: w^e, s^e_k, w^e_k, and δ^e_k). By analogy to the last inequality in (13) we have the estimate

$$(1+k^2)^{\frac{1-\lambda}{2}}|\delta^e_k| \le M'_1\,\|(u-u_{hN})_k\|_{L_2(\Omega_a)} \ \text{ for } \ k \in \mathbb{Z}, \tag{77}$$

where $(u - u_{hN})_k$ denotes the kth Fourier coefficient of the error $u - u_{hN}$, i.e.

$$(u - u_{hN})_k = \begin{cases} u_k - u_{kh} & \text{for } |k| \le N \\ u_k & \text{for } |k| > N. \end{cases} \tag{78}$$

Moreover, we have the following decomposition (cf. also (6)) of the BVP (76):

$$\begin{aligned} k = 0\text{:}\quad &\text{find } u^e_0 \in V^a_0 : \ b_0(u^e_0, v) = ((u-u_{hN})_0, v)_{1/2,\Omega_a} \ \ \forall v \in V^a_0, \\ k \in \mathbb{Z}\backslash\{0\}\text{:}\quad &\text{find } u^e_k \in W^a_0 : \ b_k(u^e_k, v) = ((u-u_{hN})_k, v)_{1/2,\Omega_a} \ \ \forall v \in W^a_0. \end{aligned} \tag{79}$$

Further, using the definition (26) of $\mathcal{B}_{h,k}(\cdot,\cdot)$ and applying Green's formula with the weight r we obtain

$$\mathcal{B}_{h,k}(u_k^e, u_k) = b_k(u_k^e, u_k) \quad \forall k \in \mathbb{Z}. \tag{80}$$

By means of (80), the completeness relation $b(u^e, u) = 2\pi \sum_{k\in\mathbb{Z}} b_k(u_k^e, u_k)$ (cf. [11, Lemma 4.1]) as well as (76) we are led to

$$2\pi \sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, u_k) = b(u^e, u) = (u - u_{hN}, u)_{X^0_{1/2}(\Omega)}. \tag{81}$$

This, together with $\|u-u_{hN}\|^2_{X^0_{1/2}(\Omega)} = (u-u_{hN}, u-u_{hN})_{X^0_{1/2}(\Omega)}$ and $(u-u_{hN}, u_{hN})_{X^0_{1/2}(\Omega)} = 2\pi \sum_{k\in\mathbb{Z}} ((u-u_{hN})_k, u_{kh})_{1/2,\Omega_a} = 2\pi \sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, u_{kh})$, cf. (28) and (79), yields

$$\|u - u_{hN}\|^2_{X^0_{1/2}(\Omega)} = 2\pi\Big(\sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, u_k) - \sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, u_{kh})\Big) = 2\pi \sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, (u-u_{hN})_k). \tag{82}$$

Further we introduce the function

$$\tilde{u}^e_{hN} := \sum_{|k|\le N} \tilde{u}^e_{kh}(r,z)\, \mathrm{e}^{ik\varphi} \quad \text{with} \quad \tilde{u}^e_{kh} = \begin{cases} \Pi_h w_k^e + \Pi_h s_k^e & \text{for } |k| \le 1 \\ P_h w_k^e + \Pi_h s_k^e & \text{for } 2 \le |k| \le N, \end{cases} \tag{83}$$

where Π_h and P_h are taken from (30), (31). In the following, for $|k| > N$ we set $\tilde{u}^e_{kh} = 0$. Owing to $\tilde{u}^e_{0h} \in V_{ah}$, $\tilde{u}^e_{kh} \in W_{ah}$, $1 \le |k| \le N$, we get by means of Lemma 1 (with $v_h := \tilde{u}^e_{kh}$): $\mathcal{B}_{h,k}(u_k - u_{kh}, \tilde{u}^e_{kh}) = 0$ for $0 \le |k| \le N$. Combining this with (59), (78), (82), and (83) and using the symmetry of $\mathcal{B}_h$ yields

$$\begin{aligned} \|u - u_{hN}\|^2_{X^0_{1/2}(\Omega)} &= 2\pi\Big(\sum_{k\in\mathbb{Z}} \mathcal{B}_{h,k}(u_k^e, (u-u_{hN})_k) - \sum_{|k|\le N} \mathcal{B}_{h,k}(\tilde{u}^e_{kh}, (u-u_{hN})_k)\Big) \qquad (84) \\ &= 2\pi\Big(\sum_{k\le N} \mathcal{B}_{h,k}(u_k^e - \tilde{u}^e_{kh}, u_k - u_{kh}) + \sum_{k>N} \mathcal{B}_{h,k}(u_k^e, u_k)\Big) =: 2\pi(S_1 + S_2). \end{aligned}$$

Employing the Hölder and Cauchy–Schwarz inequalities, the terms occurring in the sum S_1 can be bounded as follows,

$$|\mathcal{B}_{h,k}(u_k^e - \tilde{u}^e_{kh}, u_k - u_{kh})| \le C\|u_k^e - \tilde{u}^e_{kh}\|_{h,k,\Omega_a} \|u_k - u_{kh}\|_{h,k,\Omega_a} \ \text{ for } |k| \le N. \tag{85}$$

In order to estimate the second factor in (85), we define the function $\tilde{u}_{kh}$ by analogy to $\tilde{u}^e_{kh}$ in (83). Then, using the equivalence of the norms $\|\cdot\|_{h,k,\Omega_a}$ and $\|\cdot\|_{1,h,k}$ on the spaces V_{ah}, W_{ah} as well as the inequality $\|u_{kh} - \tilde{u}_{kh}\|_{1,h,k} \le \|u_k - \tilde{u}_{kh}\|_{h,k,\Omega_a}$ being analogous to [13, estimate (22)], we arrive at

$$\|u_k - u_{kh}\|_{h,k,\Omega_a} \le C(\|u_k - \tilde{u}_{kh}\|_{h,k,\Omega_a} + \|u_{kh} - \tilde{u}_{kh}\|_{1,h,k}) \le C\|u_k - \tilde{u}_{kh}\|_{h,k,\Omega_a}. \tag{86}$$

Owing to the definitions of $\tilde{u}^e_{kh}$ and $\tilde{u}_{kh}$, both factors on the right-hand side of (85) can be estimated by means of Theorem 2 and Lemmas 4, 5. Then we obtain

$$\begin{aligned} &|\mathcal{B}_{h,k}(u_k^e - \tilde{u}^e_{kh}, u_k - u_{kh})| \\ &\quad \le C\,\{h\|w_k\|_{H^2_{1/2}(\Omega_a)} + |\delta_k| S(h,k,\kappa)\}\{h\|w_k^e\|_{H^2_{1/2}(\Omega_a)} + |\delta_k^e| S(h,k,\kappa)\} \ \text{ for } |k| \le 1, \\ &|\mathcal{B}_{h,k}(u_k^e - \tilde{u}^e_{kh}, u_k - u_{kh})| \\ &\quad \le C\,\{\Sigma(h,k,w_k) + |\delta_k| S(h,k,\kappa)\}\{\Sigma(h,k,w_k^e) + |\delta_k^e| S(h,k,\kappa)\} \ \text{ for } 2 \le |k| \le N, \end{aligned}$$

with the abbreviations $S(h,k,\kappa) := (\kappa^2(h,\mu) + |k|^{2-2\lambda}h^2)^{1/2}$, $\Sigma(h,k,w_k) = h(k^2\|w_k\|^2_{H^1_{-1/2}(\Omega_a)} + \|w_k\|^2_{H^2_{1/2}(\Omega_a)})^{1/2}$ and analogously with w_k^e instead of w_k. Summing up these inequalities and using Hölder's inequality, completeness relations, the last inequality from (13) as well as (77), we get for S_1 from (84)

$$S_1 \leq C(h^2\|w\|^2_{X^2_{1/2}(\Omega)} + \kappa^2(h,\mu)\|f\|^2_{X^0_{1/2}(\Omega)})^{1/2}(h^2\|w^e\|^2_{X^2_{1/2}(\Omega)} + \kappa^2(h,\mu)\|u-u_{hN}\|^2_{X^0_{1/2}(\Omega)})^{1/2}.$$

Now, employing the last inequality from (14) as well as the analogous estimate $\|w^e\|_{X^2_{1/2}(\Omega)} \leq C\|u-u_{hN}\|_{X^0_{1/2}(\Omega)}$ yields the following bound of S_1:

$$S_1 \leq C\kappa^2(h,\mu)\|f\|_{X^0_{1/2}(\Omega)}\|u-u_{hN}\|_{X^0_{1/2}(\Omega)}. \tag{87}$$

In order to deal with the term S_2 from (84), we take (80), (79) (with $v := u_k$) and employ the Cauchy–Schwarz and Hölder inequalities as well as [11, Lemma 3.2] and the estimate (68) to deduce that

$$\begin{aligned} S_2 \;&\leq \sum_{|k|>N} |\mathcal{B}_{h,k}(u_k^e, u_k)| = \sum_{|k|>N} |b_k(u_k^e, u_k)| = \sum_{|k|>N} |((u-u_{hN})_k, u_k)_{1/2,\Omega_a}| \\ &\leq N^{-2}\Big(\sum_{|k|>N} \|(u-u_{hN})_k\|^2_{L_{2,1/2}(\Omega_a)}\Big)^{1/2}\Big(\sum_{|k|>N} k^4\|u_k\|^2_{L_{2,1/2}(\Omega_a)}\Big)^{1/2} \\ &\leq CN^{-2}\|u-u_{hN}\|_{X^0_{1/2}(\Omega)}\|f\|_{X^0_{1/2}(\Omega)}. \end{aligned} \tag{88}$$

Finally, collecting (84), (87), and (88) yields the assertion of Theorem 4. ■

6 Numerical results

For verifying the convergence rate of the Fourier-finite-element method with Nitsche mortaring on graded meshes, we consider the BVP $-\Delta\hat{u} = \hat{f}$ in $\widehat{\Omega}$, $\hat{u} = \hat{g}$ on $\partial\widehat{\Omega}$. The meridian domain Ω_a generating the axisymmetric domain $\widehat{\Omega}$ is a pentagon with the vertices $(0,0)$, $(2,0)$, $(1,1)$, $(2,2)$, and $(0,2)$. The subdomains of Ω_a are given by: $\Omega_a^1 = \{(r,z) \in \Omega_a : z > 1\}$ and $\Omega_a^2 = \{(r,z) \in \Omega_a : z < 1\}$, cf. also Figure 5. With the notation from Section 2 we deduce that the non-convex corner E_a has the coordinates $r_{E_a} = 1$, $z_{E_a} = 1$ and that the angle of the reentrant edge of $\widehat{\Omega}$ is $\theta_0 = \frac{3\pi}{2}$.
The data $\hat{f}$ and $\hat{g}$ are chosen so that the solution of the BVP is:

$$\begin{aligned} \hat{u} \;&=\; r^{1.1}R^\lambda \sin(\lambda\theta)\Psi(\varphi,R), \\ \Psi(\varphi,R) \;&=\; R - \ln\Big\{4\sinh^2\Big(\frac{R}{2}\Big) + 4\sin^2\Big(\frac{\varphi}{2}\Big)\Big\} = \sum_{k=1}^{\infty} \frac{2}{k}\,\mathrm{e}^{-kR}\cos k\varphi, \end{aligned} \tag{89}$$

where R, θ are local polar coordinates with respect to E_a (see Section 2) and $\lambda = \frac{\pi}{\theta_0} = \frac{2}{3}$. The right-hand side $\hat{g}$ of the boundary condition satisfies $\hat{g} = 0$ on that part of the boundary where $\theta = 0$ or $\theta = \theta_0$ holds. A complete homogenization of the boundary condition could be done by applying a suitable cut-off function to $\hat{u}$.

Near the reentrant edge the solution $\hat{u}$ from (89) is equal to a non-tensor product singularity function of the type (13), where the function $\Psi(\varphi, R)$ is explicitly given by the limit of the corresponding Fourier series. Using the complex form of the series at (13), the setting $\delta_0 = 0$, $\delta_k = |k|^{-1}$ for $k \in \mathbb{Z}\backslash\{0\}$ leads to the series and its limit at (89). In [12, Section 7], the properties of the function $\Psi(\varphi, R)$ are described in more detail.
For the experiments, meshes with the grading parameters $\mu_1 = 1$ (i.e. quasi-uniform meshes) as well as $\mu_2 = 0.8\lambda \approx 0.533$ according to Section 3 are used. Figure 5(a) shows the initial mesh for $\mu = \mu_1$. This mesh is refined globally by dividing each triangle into four equal triangles such that the mesh parameters form a sequence $\{h_1, h_2, \ldots\}$, here for five levels with $h_{i+1} = 0.5\, h_i$, $i = 1, 2, 3, 4$. The ratio of the number of mesh segments on the mortar interface is given by 2 : 3. For the locally graded meshes we also employ five levels h_i, $i = 1, \ldots, 5$, of triangulation. The mesh with $\mu = \mu_2$ on the level $h = h_1$ is represented in Figure 5(b).
For both types of meshes (i.e. $\mu = \mu_1$ and $\mu = \mu_2$), the trace $\mathcal{E}_h^1$ of the triangulation $\mathcal{T}_h^1$ of Ω_a^1 on the interface Γ is taken to form the partition $\mathcal{E}_h$. The mortar parameters (cf. Section 3) are chosen as follows: $\alpha_1 = 1$, $\alpha_2 = 0$, and $\gamma = 10$.
Furthermore, for the discretization with respect to N (the number of Fourier coefficients for the approximate solution), we employ five levels N_i, where $N_1 = 8$ and $N_{i+1} = 2\, N_i$ for $i = 1, 2, 3, 4$ holds.

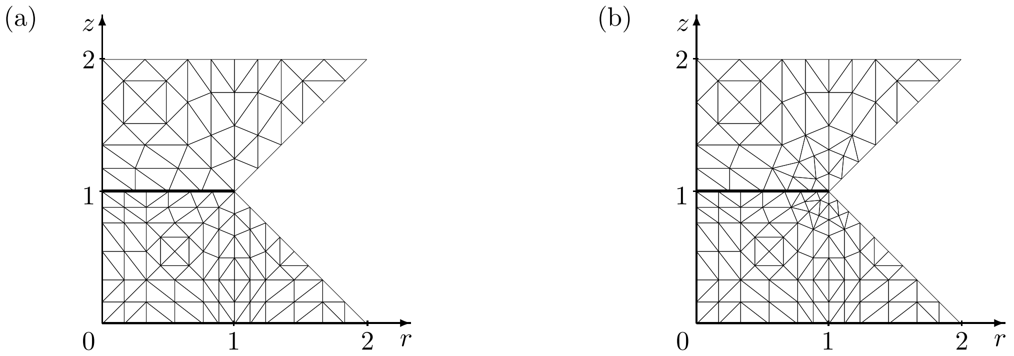

Figure 5: Triangulation with grading parameters $\mu = 1$ and $\mu = 0.8\lambda$

For the approximate measuring of the convergence rates stated in (64) and (75), the hypothesis for the tests is that

$$\|u - u_{hN}\|_{X^0_{1/2}(\Omega)} \approx C_1^{(0)} h^{\sigma_0} + C_2^{(0)} N^{-\tau_0}, \quad \|u - u_{hN}\|_{1,h,\Omega} \approx C_1^{(1)} h^{\sigma_1} + C_2^{(1)} N^{-\tau_1}, \qquad (90)$$

where u is associated with the solution $\hat{u}$ from (89) by relation (2), and u_{hN} is its approximate solution defined by (60). The parameters $C_1^{(i)}$ and $C_2^{(i)}$ $(i = 0, 1)$ are assumed to be approximately constant for two consecutive levels of h and N.
First we investigate the convergence order with respect to the discretization parameter h. Table 1 shows the observed σ-values $\sigma_{obs,0}(\mu)$ and $\sigma_{obs,1}(\mu)$ of the convergence orders σ_0 and σ_1 on meshes of the levels $h_2, \ldots, h_5$ with grading parameters $\mu = \mu_i$ $(i = 1, 2)$, for fixed $N = 64$. According to Theorems 3 and 4, the expected convergence orders are $\sigma_{exp,0}(\mu_1) = 2\lambda \approx 1.33$, $\sigma_{exp,0}(\mu_2) = 2$, $\sigma_{exp,1}(\mu_1) = \lambda \approx 0.67$, and $\sigma_{exp,1}(\mu_2) = 1$. We can state that for $\mu = \mu_1$ the observed rates are slightly better than the expected ones, and for $\mu = \mu_2$ the observed rates are very close to the expected ones.

level	$\sigma_{obs,0}(\mu_1)$	$\sigma_{obs,0}(\mu_2)$	$\sigma_{obs,1}(\mu_1)$	$\sigma_{obs,1}(\mu_2)$
h_2	1.51	2.05	0.74	1.06
h_3	1.50	1.99	0.74	1.01
h_4	1.47	1.96	0.73	1.00
h_5	1.43	1.95	0.72	1.00

Table 1: Convergence orders on the levels $h = h_2, \dots h_5$ for $\mu_1 = 1$, $\mu_2 = 0.8\lambda$, and $N = 64$

For the purpose of testing the convergence order with respect to N, some computations on the meshes of the level $h = h_5$ with the grading parameters $\mu = \mu_1$, $\mu = \mu_2$ and N varying from N_1 to N_5 are carried out. As predicted by theory (estimates (64) and (75)), the observed values $\tau_{obs,0}(\mu)$, $\tau_{obs,1}(\mu)$ of the convergence orders τ_0, τ_1 are nearly equal (leading digits coincide) for different values $\mu = \mu_1$, $\mu = \mu_2$ of the mesh grading parameter. Therefore, in Table 2 we represent the observed values $\tau_{obs,0}$, $\tau_{obs,1}$ without any dependence on the grading parameter μ. Comparing $\tau_{obs,0}$, $\tau_{obs,1}$ with the expected values $\tau_{exp,0} =$

level	$\tau_{obs,0}$	$\tau_{obs,1}$
N_2	1.95	1.08
N_3	2.05	1.13
N_4	2.11	1.16
N_5	2.14	1.21

Table 2: Convergence orders on the levels $N = N_2, \dots, N_5$ for $h = h_5$

2, $\tau_{exp,1} = 1$ we establish that the observed convergence rates are slightly better than the expected ones. This could be explained by the fact that the function $\Psi(\varphi, R)$ (and, consequently, the solution $\hat{u}$) is more regular with respect to φ than in Theorems 1 and 3 required.

Thus, the numerical example illustrates that local refinement of the mesh with an appropriate grading parameter is suited for improving the convergence order of the Fourier-finite-element method combined with Nitsche mortaring when the solution of the BVP has singularities. Especially, using meshes with a grading parameter $\mu < \lambda$ (here, $\mu = 0.8\lambda$), we get the same convergence order as in case of a regular solution (see [15]).

References

[1] D.N. Arnold, F. Brezzi, B. Cockburn, D. Marini. Discontinuous Galerkin methods for elliptic problems. *Discontinuous Galerkin methods: Theory, Computation and Applications* (B. Cockburn, G. Karniadakis & C.-W. Shu, eds.), pp. 89-101, Berlin: Springer Verlag, 1999.

[2] R. Becker, P. Hansbo, R. Stenberg. A finite element method for domain decomposition with non-matching grids. *M2AN Math. Model. Numer. Anal.*, 37(2):209-225, 2003.

[3] F.B. Belgacem. The Mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84:173-197, 1999.

[4] C. Bernardi, Y. Maday, A.T. Patera. A new nonconforming approach to domain decomposition: The mortar element method. *Nonlinear partial differential equations and their applications* (H. Brezis et al., Eds.), pp. 13-51, Paris, 1994.

[5] C. Bernardi, M. Dauge, Y. Maday. *Spectral methods for axisymmetric domains.* Series in Applied Mathematics, Gauthiers-Villars, Paris and North Holland, Amsterdam, 1999.

[6] H. Blum. Numerical treatment of corner and crack singularities. *Finite Element and Boundary Element Techniques from Mathematical and Engineering Point of View* (E. Stein & W. Wendland, eds.), pp. 172-212, Wien-New York: Springer Verlag, 1988.

[7] P. Butzer, R. Nessel. *Fourier Analysis and Approximation, Vol. I.* Birkhäuser Verlag, Basel, 1971.

[8] P. Clément. Approximation by finite element functions using local regularization. *R.A.I.R.O.*, 9(2):77-84, 1975.

[9] A. Fritz, S. Hüeber, B. Wohlmuth. A comparison of Mortar and Nitsche Techniques for linear elasticity. *Calcolo*, 41:115-137, 2004.

[10] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston-London-Melbourne, 1985.

[11] B. Heinrich. Singularity Functions at Axisymmetric Edges and their Representation by Fourier Series. *Math. Meth. Appl. Sci.*, 16:837-854, 1993.

[12] B. Heinrich. The Fourier-finite-element-method for Poisson's equation in axisymmetric domains with edges. *SIAM J. Numer. Anal.*, 33:1885-1911, 1996.

[13] B. Heinrich, K. Pietsch. Nitsche Type Mortaring for some Elliptic Problem with Corner Singularities. *Computing*, 68:217-238, 2002.

[14] B. Heinrich, S. Nicaise. The Nitsche mortar finite-element method for transmission problems with Singularities. *IMA J. Numer. Anal.*, 23:331-358, 2003.

[15] B. Heinrich, B. Jung. The Fourier finite-element method with Nitsche mortaring. *Preprint SFB393/04-11, Technische Universität Chemnitz*, 2004. (to appear in: *IMA J. Numer. Anal.*, 2006).

[16] B. Heinrich, K. Pönitz. Nitsche type mortaring for singularly perturbed reaction-diffusion problems. *Computing*, 75(4):257-279, 2005.

[17] B. Mercier, G. Raugel. Résolution d'un problème aux limites dans un ouvert axisymétrique par éléments finis en r, z et séries de Fourier en ϕ. *R.A.I.R.O. Analyse numérique*, 16(4):405-461, 1982.

[18] B. Nkemzi, B. Heinrich. Partial Fourier approximation of the Lamé equations in axisymmetric domains. *Math. Methods Appl. Sci.*, 22(12):1017-1041, 1999.

[19] L. A. Oganesjan und L. A. Ruchovec. *Variacionno-raznostnye metody rešenija elliptičeskich uravnenij.* Izd. Akad. Nauk Armjanskoj SSR, Erevan, 1979.

[20] G. Raugel. Résolution numérique par une méthode d'éléments finis du probléme Dirichlet pour le Laplacien dans un polygone. *C. R. Acad. Sci. Paris, Sér. A*, 286(18):A791-A794, 1978.

[21] R. Stenberg. Mortaring by a method of J. A. Nitsche. In: *Computational mechanics. New trends and applications.*, Barcelona: Centro Internac. Métodos Numér. Ing., 1998.

[22] G. Strang und G. Fix. *An analysis of the finite element method.* Prentice-Hall Inc., Englewood Cliffs, 1973.

[23] B. Weber. *Die Fourier-Finite-Elemente-Methode für elliptische Interfaceprobleme in axialsymmetrischen Gebieten.* PhD thesis, Technische Universität Chemnitz-Zwickau, 1994.

[24] B. I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition.* Lecture Notes in Computational Science and Engineering 17, Springer-Verlag, Berlin Heidelberg, 2001.

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