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Numerische Simulation auf massiv parallelen Rechnern

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Vibrations of plates with masses

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Abstract This paper presents the investigation of the nonlinear eigenvalue problem describing free vibrations of plates with elastically attached masses. We study properties of eigenvalues and eigenfunctions and prove the existence theorem. Theoretical results are illustrated by numerical experiments.

Key Words nonlinear eigenvalue problem, vibrations of plates

AMS(MOS) subject classification 74H20, 74H45, 49R50, 65N25, 47J10, 47A75, 35P05, 35P30

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1 Introduction

Problems on eigenvibrations of mechanical structures with elastically attached loads have important applications [1]. A new general approach to investigate and solve these problems was proposed in [2]. In the present paper, we apply this approach for problems on eigenvibrations of plates with elastically attached loads.

Let us start with describing eigenvibrations of the plate-springs-loads system. We shall investigate the flexural vibrations of an isotropic thin clamped plate with the middle surface occupying the plane domain Ω with the boundary Γ. Denote by $\rho = \rho(x)$ the volume mass density, $E = E(x)$ the Young modulus, $\nu = \nu(x)$ the Poisson ratio, $d = d(x)$ the thickness of the plate, and $D = D(x) = Ed^3/12(1-\nu^2)$ the flexural rigidity of the plate at a point $x \in \Omega$, $0 < \nu < 1/2$. Assume that loads of mass M_i are joined by an elastic springs with the stiffness coefficients K_i at the points $x^{(i)} \in \Omega$, $i = 1, 2, \ldots, m$. Then the vertical deflection $w(x,t)$ of the plate at a point $x \in \Omega$ at time $t > 0$ and the vertical displacements $\eta_i(t)$ of the loads of mass M_i at time $t > 0$ satisfy the following equations (see, for example, [1]):

$$\begin{aligned} &Lw(x,t) + \rho(x)d(x)w_{tt}(x,t) = \sum_{i=1}^{m} M_i(\eta_i(t))_{tt}\delta(x - x^{(i)}), \quad x \in \Omega, \\ &w(x,t) = \partial_n w(x,t) = 0, \quad x \in \Gamma, \\ &M_i(\eta_i(t))_{tt} + K_i(\eta_i(t) - w(x^{(i)},t)) = 0, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{1}$$

where $\delta(x)$ is the delta function of Dirac, ∂_n is the outward normal derivative on Γ, L is the differential operator defined by the relation

$$Lw = \partial_{11}D(\partial_{11}w + \nu\partial_{22}w) + \partial_{22}D(\partial_{22}w + \nu\partial_{11}w) + 2\partial_{12}D(1-\nu)\partial_{12}w,$$

$\partial_{ij} = \partial_i\partial_j$, $\partial_i = \partial/\partial x_i$, $i, j = 1, 2$, $(\psi(t))_t = d\psi(t)/dt$.

The eigenvibrations of the plate-springs-loads system are characterized by the functions $w(x,t)$ and $\eta_i(t)$ of the form

$$w(x,t) = u(x)v(t), \quad x \in \Omega, \quad \eta_i(t) = c_i u(x^{(i)})v(t), \quad t > 0,$$

where $v(t) = a_0 \cos\sqrt{\lambda}t + b_0 \sin\sqrt{\lambda}t$ for $t > 0$, a_0, b_0, c_i, and λ are constants, $i = 1, 2, \ldots, m$.

From the third equation of (1), we conclude that $c_i = \sigma_i/(\lambda - \sigma_i)$, $\sigma_i = K_i/M_i$, $i = 1, 2, \ldots, m$. The first two equations of (1) lead to the following nonlinear eigenvalue problem: find values λ and nontrivial functions $u(x)$, $x \in \Omega$ such that

$$\begin{aligned} &Lu + \sum_{i=1}^{m} \frac{\lambda\sigma_i}{\lambda - \sigma_i} M_i\delta(x - x^{(i)})u = \lambda\,\rho d\,u, \quad x \in \Omega, \\ &u = \partial_n u = 0, \quad x \in \Gamma. \end{aligned} \tag{2}$$

The present paper is devoted to the investigation of nonlinear eigenvalue problem (2). In Section 2, we state the variational formulation for differential eigenvalue problem (2). In

Section 3, we introduce parameter linear eigenvalue problems and study their properties. These parameter eigenvalue problems are used for proving the existence theorem in Section 4. In Section 5, we consider the nonlinear biharmonic eigenvalue problem and demonstrate numerical experiments.

2 Variational statement of the problem

By $\mathbb{R}$ denote the set of real numbers. Let Ω be a plane domain with a Lipschitz-continuous boundary Γ. As usual, let $L_2(\Omega)$ and $W_2^2(\Omega)$ denote the real Lebesgue and Sobolev spaces equipped with the norms

$$|u|_0 = \left(\int\limits_\Omega u^2\, dx \right)^{1/2}, \quad \|u\|_2 = \left(\sum_{i=0}^{2} |u|_i^2 \right)^{1/2},$$

respectively, where

$$|u|_1 = \left(\sum_{i=1}^{2} |\partial_i u|_0^2 \right)^{1/2}, \quad |u|_2 = \left(\sum_{ij=1}^{2} |\partial_{ij} u|_0^2 \right)^{1/2},$$

$\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial_i \partial_j$, $i, j = 1, 2$. Denote by $\overset{\circ}{W}{}_2^2(\Omega)$ the space of functions u from $W_2^2(\Omega)$ such that $u = \partial_n u = 0$ on Γ, $\partial_n u$ is the outer normal derivative of u along the boundary Γ.

Put $\Lambda = (0, \infty)$, $H = L_2(\Omega)$, $V = \overset{\circ}{W}{}_2^2(\Omega)$. Note that the space V is compactly embedded into the space H, any function from V is continuous on $\overline{\Omega}$. The semi-norm $|.|_2$ is a norm over the space V, which is equivalent to the norm $\|.\|_2$.

Assume that $L_\infty(\Omega)$ is the space of measurable real functions u bounded almost everywhere on Ω with the norm

$$|u|_{0,\infty} = \operatorname*{ess.\,sup}_{x \in \Omega} |u(x)|.$$

Note that there exists c_0 such that

$$|v|_{0,\infty} \le c_0 |v|_2 \quad \forall v \in W_2^2(\Omega).$$

Introduce the numbers $K_i > 0$, $M_i > 0$, $\sigma_i = K_i/M_i$, $i = 1, 2, \ldots, m$. Define functions E, ν, ρ, and d from $L_\infty(\Omega)$, for which there exist positive numbers E_1, E_2, ρ_1, ρ_2, d_1, d_2 such that

$$\begin{aligned} E_1 &\le E(x) \le E_2, & 0 &< \nu(x) < 1/2, \\ \rho_1 &\le \rho(x) \le \rho_2, & d_1 &\le d(x) \le d_2, \end{aligned}$$

for almost all $x \in \Omega$. Set

$$D = \frac{Ed^3}{12(1 - \nu^2)}.$$

Define the bilinear forms $a : V \times V \to \mathbb{R}$, $b : H \times H \to \mathbb{R}$, $c_i : V \times V \to \mathbb{R}$, and the functions $\xi_i(\mu)$, $\mu \in \Lambda$, $\zeta_i(\mu)$, $\mu \in \Lambda$ by the formulae

$$\begin{aligned}
a(u,v) &= \int\limits_\Omega D[(\partial_{11}u + \partial_{22}u)(\partial_{11}v + \partial_{22}v) + \\
&\quad +(1-\nu)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]\,dx, \\
b(u,v) &= \int\limits_\Omega \rho d\,uv\,dx, \\
c_i(u,v) &= M_i u(x^{(i)})v(x^{(i)}), \quad u,v \in V, \\
\zeta_i(\mu) &= \frac{\sigma_i}{\sigma_i - \mu}, \quad \mu \in \Lambda, \\
\xi_i(\mu) &= \frac{\mu\sigma_i}{\mu - \sigma_i}, \quad \mu \in \Lambda,
\end{aligned}$$

where $x^{(i)}$ are fixed points on Ω, $i = 1, 2, \ldots, m$.

Consider the following differential eigenvalue problem: find $\lambda \in \Lambda$, $u \in V \setminus \{0\}$, $Lu \in H$ such that

$$Lu + \sum_{i=1}^{m} \xi_i(\lambda) M_i \delta(x - x^{(i)})u = \lambda\,\rho d\,u.$$

This differential problem is equivalent to the following variational eigenvalue problem: find $\lambda \in \Lambda$ and $u \in V \setminus \{0\}$ such that

$$a(u,v) + \sum_{i=1}^{m} \xi_i(\lambda)\,c_i(u,v) = \lambda\,b(u,v) \quad \forall v \in V. \tag{3}$$

The number λ that satisfies (3) is called an eigenvalue, and the element u is called an eigenelement of problem (3) corresponding to λ. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue λ and the zero element is a closed subspace in V, which is called the eigensubspace corresponding to the eigenvalue λ. The dimension of this subspace is called the multiplicity of the eigenvalue λ. The pair λ and u that satisfies (3) is called an eigensolution or eigenpair of problem (3).

3 Parameter eigenvalue problems

Set $\sigma_0 = 0$ and $\sigma_{m+1} = \infty$. For $\sigma_{k-1} < \sigma_k$, $1 \le k \le m+1$, we denote $\Lambda_k = (\sigma_{k-1}, \sigma_k)$. Let us write problem (3) for $\lambda \in \Lambda_k$ in the form:

Find $\lambda \in \Lambda_k$ and $u \in V \setminus \{0\}$ such that

$$a(u,v) + \sum_{i=1}^{k-1} \xi_i(\lambda)\,c_i(u,v) = \lambda(b(u,v) + \sum_{i=k}^{m} \zeta_i(\lambda)\,c_i(u,v)) \quad \forall v \in V. \tag{4}$$

Here we assume that

$$\sum_{i=n_1}^{n_2} a_i = 0$$

when $n_2 < n_1$.

For problem (4) we introduce the following parameter linear eigenvalue problem for fixed parameter μ:

Find $\varphi^{(k)}(\mu) \in \mathbb{R}$ and $u \in V \setminus \{0\}$ such that

$$a(u,v) + \sum_{i=1}^{k-1} \xi_i(\mu)\, c_i(u,v) = \varphi^{(k)}(\mu)(b(u,v) + \sum_{i=k}^{m} \zeta_i(\mu)\, c_i(u,v)) \tag{5}$$

for all $v \in V$ and fixed $\mu \in \Lambda_k$.

Assume that

$$\sigma_{k-1} < \sigma_k = \sigma_{k+1} = \ldots = \sigma_{k+r_k-1} < \sigma_{k+r_k}$$

for $r_k \geq 1$, $k + r_k \leq m+1$ and define the subspace $V_k = \{v : v \in V, v(x^{(i)}) = 0, i = k, k+1, \ldots, k+r_k-1\}$ of the space V. Let us consider the following linear eigenvalue problem:

Find $\lambda^{(k)} \in \mathbb{R}$ and $u \in V_k \setminus \{0\}$ such that

$$a(u,v) + \sum_{i=1}^{k-1} \xi_i(\sigma_k)\, c_i(u,v) = \lambda^{(k)}(b(u,v) + \sum_{i=k+r_k}^{m} \zeta_i(\sigma_k)\, c_i(u,v)) \tag{6}$$

for all $v \in V_k$.

The following lemma formulates the properties for bilinear forms of eigenvalue problems (3), (4)–(6).

Lemma 1 *The following inequalities hold:*

$$\begin{array}{rcccll} \alpha_1 |v|_2^2 & \leq & a(v,v) & \leq & \alpha_2 |v|_2^2 & \forall v \in V, \\ \beta_1 |v|_0^2 & \leq & b(v,v) & \leq & \beta_2 |v|_0^2 & \forall v \in H, \\ 0 & \leq & c_i(v,v) & \leq & \gamma_2^{(i)} |v|_2^2 & \forall v \in V, \end{array}$$

where $\alpha_j = D_j$, $\beta_j = \rho_j d_j$, $j = 1,2$, $D_1 = E_1 d_1^3/12$, $D_2 = E_2 d_2^3/9$, $\gamma_2^{(i)} = c_0^2 M_i$, $i = 1, 2, \ldots, m$.

Proof The inequalities follow from the definitions of the bilinear forms and the assumptions on the coefficients. □

For fixed $\mu \in \Lambda_k$ problem (5) has a countable set of real eigenvalues of finite multiplicity $\varphi_i^{(k)}(\mu)$, $i = 1, 2, \ldots$, which are repeated according to their multiplicity:

$$0 < \varphi_1^{(k)}(\mu) \leq \varphi_2^{(k)}(\mu) \leq \ldots \leq \varphi_i^{(k)}(\mu) \leq \ldots, \quad \lim_{i\to\infty} \varphi_i^{(k)}(\mu) = \infty.$$

The corresponding eigenelements $u_i^{(k)}$, $i = 1, 2, \ldots$, form a complete system in V.

Problem (6) has a countable set of real eigenvalues of finite multiplicity $\lambda_i^{(k)}$, $i = 1, 2, \ldots$, which are repeated according to their multiplicity:

$$0 < \lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \ldots \leq \lambda_i^{(k)} \leq \ldots, \quad \lim_{i \to \infty} \lambda_i^{(k)} = \infty.$$

The corresponding eigenelements $u_i^{(k)}$, $i = 1, 2, \ldots$, form a complete system in V_k.

By analogy with [2], we derive the following results.

Lemma 2 *The functions $\varphi_i^{(k)}(\mu)$, $\mu \in \Lambda_k$ are continuous nonincreasing functions such that*

(a) $\varphi_i^{(k)}(\mu) \to 0$ *as* $\mu \to \sigma_k$, $\mu \in \Lambda_k$, $i = 1, 2, \ldots, r_k$, $\varphi_{i+r_k}^{(k)}(\mu) \to \lambda_i^{(k)}$ *as* $\mu \to \sigma_k$, $\mu \in \Lambda_k$, $i = 1, 2, \ldots$ *for* $1 \leq k \leq m$,

(b) $\varphi_i^{(k)}(\mu) \to \lambda_i^{(k-1)}$ *as* $\mu \to \sigma_{k-1}$, $\mu \in \Lambda_k$, $i = 1, 2, \ldots$ *for* $2 \leq k \leq m+1$.

Lemma 3 *A number $\lambda \in \Lambda_k$ is an eigenvalue of problem (4) if and only if the number $\lambda \in \Lambda_k$ is a solutions of an equation from the set*

$$\mu - \varphi_i^{(k)}(\mu) = 0, \quad \mu \in \Lambda_k, \quad i = 1, 2, \ldots$$

4 Existence of eigensolutions

Define the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1-m, 2-m, \ldots$, by the formulae

$$\gamma_{k-m+i}(\mu) = \varphi_i^{(k)}(\mu), \quad \mu \in \Lambda_k, \quad i = 1, 2, \ldots,$$

$$\gamma_j(\mu) = 0, \quad \mu \in \Lambda_k, \quad j = 1-m, 2-m, \ldots, k-m$$

for $1 \leq k \leq m+1$.

Lemma 4 *The functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1-m, 2-m, \ldots$, are continuous nonincreasing functions.*

Proof The assertion of this lemma follows from Lemmata 2, 3, and the definition of the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1-m, 2-m, \ldots$ □

Lemma 5 *A number $\lambda \in \Lambda$ is an eigenvalue of problem (3) if and only if the number $\lambda \in \Lambda$ is a solution of an equation from the set*

$$\mu - \gamma_i(\mu) = 0, \quad \mu \in \Lambda, \quad i = 1-m, 2-m, \ldots$$

Proof The assertion of this lemma follows from Lemmata 2 and 3. □

As the generalization of the existence theorem from [2], we obtain the following existence result.

Theorem 6 *Assume that* $N_k = \max\{i : \lambda_i^{(k)} \leq \sigma_k, i \geq 1\}$, *where* $\lambda_i^{(k)}$, $i = 1, 2, \ldots$ *are eigenvalues of eigenvalue problem* (6), *and* r_k *is defined by*

$$\sigma_{k-1} < \sigma_k = \sigma_{k+1} = \ldots = \sigma_{k+r_k-1} < \sigma_{k+r_k}$$

for $k \geq 1$, $r_k \geq 1$, $k + r_k \leq m+1$. *Then eigenvalue problem* (3) *has a countable set of real eigenvalues of finite multiplicity* λ_i, $i = 1-m, 2-m, \ldots$, *which are repeated according to their multiplicity:*

$$0 < \lambda_{1-m} \leq \lambda_{2-m} \leq \ldots \leq \lambda_i \leq \ldots, \quad \lim_{i\to\infty} \lambda_i = \infty.$$

Each eigenvalue λ_i, $i \geq 1-m$ *is a unique root of the equation*

$$\mu - \gamma_i(\mu) = 0, \quad \mu \in \Lambda, \quad i \geq 1-m.$$

Moreover, it holds

$$\sigma_{k-1} < \lambda_{i_1} \leq \ldots \leq \lambda_{i_2} \leq \sigma_k,$$

where $i_1 = k - m + N_{k-1}$, $i_2 = k - m + N_k + r_k - 1$. *The relations*

$$\lambda_{n-r_k-1} < \lambda_{n-r_k} = \ldots = \lambda_n = \sigma_k < \lambda_{n+1}$$

are valid if and only if it holds

$$\lambda_{j-r_k-1}^{(k)} < \lambda_{j-r_k}^{(k)} = \ldots = \lambda_j^{(k)} = \sigma_k < \lambda_{j+1}^{(k)}$$

for $j = m - k + n - r_k$. *The eigensubspace* $U(\lambda_i)$ *of nonlinear eigenvalue problem* (3) *is*

(a) *the eigensubspace corresponding to the eigenvalue* $\varphi_j^{(k)}(\mu)$ *of linear eigenvalue problem* (5) *for* $\mu = \lambda_i$ *if* $\lambda_i \in \Lambda_k$, $j = m - k + i$,

(b) *the eigensubspace corresponding to the eigenvalue* $\lambda_j^{(k)}$ *of linear eigenvalue problem* (6) *if* $\lambda_j^{(k)} = \lambda_i = \sigma_k$, $j = m - k + i - r_k$.

5 Nonlinear biharmonic eigenvalue problem

Let us consider problem (2) for $D = 1$ and $\rho d = 1$. For this case we obtain the following nonlinear biharmonic eigenvalue problem: find values λ and nontrivial functions $u(x)$, $x \in \Omega$ such that

$$\begin{aligned} &\Delta^2 u + \sum_{i=1}^{m} \frac{\lambda \sigma_i}{\lambda - \sigma_i} M_i \delta(x - x^{(i)}) u = \lambda\, u, \quad x \in \Omega, \\ &u = \partial_n u = 0, \quad x \in \Gamma, \end{aligned} \tag{7}$$

where $\Delta^2 = \partial_1^4 + \partial_2^4 + 2\partial_1^2\partial_2^2$ denotes the biharmonic operator. Here we set $\Omega = (0,1)^2$, $m = 2$, $x^{(1)} = (9/26, 9/26)^\top$, $x^{(2)} = (9/26, 19/26)^\top$, $M_1 = M_2 = M$, $K_1 = K_2 = K$,

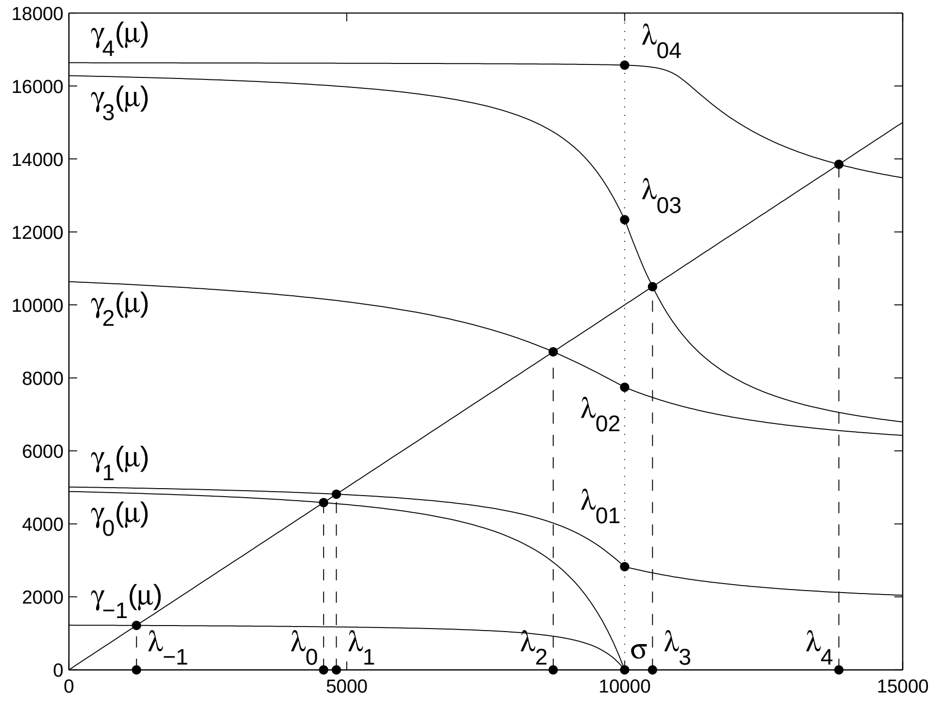

Figure 1: The six smallest eigenvalues of nonlinear biharmonic eigenvalue problem

$\sigma_1 = \sigma_2 = \sigma$, $M = 0.01$, $K = 100$, $\sigma = 10000$. This problem has been solved numerically by applying the finite difference method. We use the standard thirteen-point finite difference approximation of the biharmonic operator on the uniform mesh as in [2]. Results of numerical experiments are demonstrated by Figure 1. We show the functions $\gamma_i(\mu)$, $i = -1, 0, 1, 2, 3, 4$, eigenvalues $\lambda_{0i} = \lambda_i^{(1)}$, $i = 1, 2, 3, 4$ of problem (6), and eigenvalues λ_i, $i = -1, 0, 1, 2, 3, 4$ of nonlinear biharmonic eigenvalue problem (7). Thus, Figure 1 illustrates the existence result of Theorem 6 for nonlinear biharmonic eigenvalue problem (7).

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