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Numerische Simulation auf massiv parallelen Rechnern

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Computation of 3D vertex singularities for linear elasticity: Error estimates for a finite element method on graded meshes

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Abstract   This paper is concerned with the computation of 3D vertex singularities of anisotropic elastic fields with Dirichlet boundary conditions, focusing on the derivation of error estimates for a finite element method on graded meshes. The singularities are described by eigenpairs of a corresponding operator pencil on spherical polygonal domains. The main idea is to introduce a modified quadratic variational boundary eigenvalue problem which consists of two self-adjoint, positive definite sesquilinear forms and a skew-Hermitean form. This eigenvalue problem is discretized by a finite element method on graded meshes. Based on regularity results for the eigensolutions estimates for the finite element error are derived both for the eigenvalues and the eigensolutions. Finally, some numerical results are presented.

Key Words   quadratic eigenvalue problems, linear elasticity, 3D vertex singularities, finite element methods, error estimates

AMS(MOS) subject classification   65N25; 65N30, 74G70 (old: 73B50)

Contents

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1 Introduction

The present paper is concerned with the study of the behaviour of three-dimensional elastic fields satisfying Dirichlet boundary conditions near the vertex of a polyhedral cone. It is well known, that stress singularities can arise in the neighborhood of the vertex [9, 33, 39]. The detailed knowledge of the singular terms of the elastic fields is of interest, e. g., in crack mechanics where the intersection of crack fronts or notches with the surface of the body generates vertices. Moreover, in computational mechanics, the lack of regularity near edges or corners demands modified discretization procedures. Our goals are to describe a mathematical method which leads to an efficient computation of the vertex singularities in very general situations, to derive error estimates for finite element solutions and to present numerical results.

Writing the boundary value problems in spherical coordinates (r, φ, θ) centered in the vertex of the cone and using the Mellin-transformation $(r\partial_r \to \alpha)$ we get a parameter-depending boundary value problem $\mathcal{A}_c(\alpha)u = \mathcal{F}$ in a curved polygon $\tilde{\Omega} \subset S^2$ on the unit sphere $S^2 \subset \mathbb{R}^3$. The eigenvalues α_i and the eigensolutions $u(\alpha_i; \cdot)$ of the operator pencil $\mathcal{A}_c(\cdot)$ generate the singular terms of the elastic fields near the vertex. They have the form (here written without logarithmic terms)

$$\sum_i c_i r^{\alpha_i} u(\alpha_i; \varphi, \theta).$$

In order to solve this generalized eigenvalue problem efficiently we formulate a modified quadratic eigenvalue problem by introducing the parameter $\lambda = \alpha + 1/2$ leading to an eigenvalue problem with a symmetric distribution of the eigenvalues. It reads in the weak formulation: Find $\lambda \in \mathbb{C}$ and $u \in V \setminus \{0\}$ such that

$$k(u, v) = \lambda\, g(u, v) + \lambda^2\, m(u, v) \quad \forall v \in V \tag{1}$$

with self-adjoint, positive definite forms $k(\cdot, \cdot)$ and $m(\cdot, \cdot)$ and a skew-Hermitean form $g(\cdot, \cdot)$. By V we denote a complex Hilbert space of vector functions defined on $\tilde{\Omega}$. For the numerical solution we construct a finite element subspace $V_h \subset V$ and look for the finite element solution of problem (1): Find $\lambda_h \in \mathbb{C}$ and $u_h \in V_h \setminus \{0\}$ such that

$$k(u_h, v_h) = \lambda_h\, g(u_h, v_h) + \lambda_h^2\, m(u_h, v_h) \quad \forall v_h \in V_h. \tag{2}$$

This approach is widely used in the engineering literature [27, 45]. Modern methods to solve the algebraic eigenvalue problem (2) by exploiting the structure can be found, for example, in [3, 7, 37].

The main interest of the current paper is to investigate the finite element errors $|\lambda - \lambda_h|$ and $\|u - u_h\|_V$. The particular difficulty is that the domain $\tilde{\Omega}$ has, in general, corners such that the eigenfunctions are not smooth. In Section 2, we formulate the quadratic boundary-eigenvalue problems at the sphere. They are related to the three-dimensional boundary value problems for the elasticity operator. Furthermore, we define Hilbert spaces on the spherical domain $\tilde{\Omega}$, sesquilinear forms and formulate the modified variational quadratic

eigenvalue problem with the parameter λ. We also describe basic properties of the sesquilin-
ear forms. In Section 3, we introduce more notation and derive regularity results for eigen-
solutions of two related operator pencils. These results are in principle well known but
here we need modified statements which are necessary for the proof of the approximation
results.

Section 4 is devoted to error estimates for the approximate eigenvalue problem. We
start with some general approximation results in Subsection 4.1. Then we introduce graded
meshes which are appropriate for the approximation of the eigenfunctions with piecewise
linear finite elements. In order to get approximation error estimates some non-standard
local interpolation error estimates are proved in Subsection 4.3. The main challenge lay in
the use of appropriate weighted Sobolev spaces due to the transformation of the eigenvalue
problem from the sphere into the plane parameter domain and due to the reduced regularity
of the eigenfunctions. Together with the abstract results of Subsection 4.1 we conclude
approximation error estimates for the eigenvalues and eigenfunctions. The estimates are
optimal only due to the use of the graded meshes. The main result is that an appropriately
calculated approximate eigenvalue $\hat{\lambda}_h$ is second order accurate, $|\lambda - \hat{\lambda}_h| \lesssim h^2$, where h is
the global mesh size which relates to the number N of degrees of freedom by $N \sim h^{-2}$.
Numerical results are discussed in Section 5. They confirm the theoretically predicted
convergence orders.

For simplicity we consider Dirichlet boundary conditions; other boundary conditions as
Neumann or mixed boundary conditions can be treated similarly [23].

We use the notation $a \lesssim b$ and $a \sim b$ which means the existence of positive constants
C_1 and C_2 (which are independent of the discretization parameter h and of the function
under consideration) such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively.

2   Formulation of the eigenvalue problem and review of known results

2.1   Description of corner singularities by an operator eigenvalue problem

The equilibrium equations for linear anisotropic fields in a polyhedral domain G read

$$-\sum_{j=1}^{3} \partial_j \sigma_{ij} = F_i \quad \text{in } G, \quad i = 1, 2, 3, \tag{3}$$

where F_i are given body forces, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor, $x = (x_1, x_2, x_3)$ are
Cartesian coordinates, and $\partial_j := \partial/\partial x_j$. Denote by $U = (U_1, U_2, U_3)^\top$ the displacement
field and by $\boldsymbol{\varepsilon}(U) = (\varepsilon_{ln}(U))_{l,n=1}^3$ the associated linearized strain tensor where $\varepsilon_{ln}(U) =$

$\frac{1}{2}(\partial_l U_n + \partial_n U_l)$. For small strains, Hooke's law yields

$$\sigma_{ij}(U) = \sum_{l,n=1}^{3} a_{ijmn}\varepsilon_{ln}(U), \quad i,j,l,n = 1,2,3. \tag{4}$$

The elastic moduli a_{ijmn} are real valued constants and satisfy the symmetry relations

$$a_{ijln} = a_{lnij} = a_{jiln} = a_{ijnl}. \tag{5}$$

The energy conservation law yields a strong ellipticity and boundedness condition for the corresponding quadratic form

$$M_1 \sum_{i,j=1}^{3} |\xi_{ij}|^2 \leq \sum_{i,j,l,n=1}^{3} a_{ijln}\xi_{ij}\xi_{ln} \leq M_2 \sum_{i,j=1}^{3} |\xi_{ij}|^2 \quad \forall \xi_{ij} \in \mathbb{R}, \ i,j = 1,2,3. \tag{6}$$

It follows that the elastic matrix $\mathbf{A}$ is symmetric and positive definite,

$$\mathbf{A} = \begin{pmatrix} a_{1111} & a_{1122} & a_{1133} & a_{1123} & a_{1131} & a_{1112} \\ \cdot & a_{2222} & a_{2233} & a_{2223} & a_{2231} & a_{2212} \\ \cdot & \cdot & a_{3333} & a_{3323} & a_{3331} & a_{3312} \\ \cdot & \cdot & \cdot & a_{2323} & a_{2331} & a_{2312} \\ \cdot & \cdot & \cdot & \cdot & a_{3131} & a_{3112} \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{1212} \end{pmatrix}. \tag{7}$$

Inserting (4) into (3), we get the linear elasticity equations $-LU = F$ in G, with $L = D^\top \mathbf{A} D$ and

$$D^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{8}$$

In the following we will study the homogeneous Dirichlet problem

$$-LU = F \quad \text{in } G, \qquad U = 0 \quad \text{on } \partial G.$$

The variational formulation of the problem is to find $U \in \overset{o}{W}{}^{1,2}(G)^3$ for given $F \in L^2(G)^3$ such that

$$\int_G \mathbf{A} DU \cdot DV \, \mathrm{d}x = \int_G F \cdot V \, \mathrm{d}x \quad \forall v \in \overset{o}{W}{}^{1,2}(G)^3, \tag{9}$$

with $\mathbf{A}$ and D from (7) and (8), respectively.

The behaviour of elastic fields near vertices of the polyhedron G can be locally investigated by means of a partition of unity. Therefore we can consider the problem in an infinite cone $K \subset R^3$ with vertex O at the origin, i. e.

$$K = \{x \in \mathbb{R}^3 : 0 < |x| < \infty, x/|x| \in \tilde{\Omega}\}, \tag{10}$$

where $\tilde{\Omega}$ is a subdomain of the unit sphere $S^2 \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\tilde{\Omega}$. We introduce spherical coordinates (r, φ, θ) centered in O. Vertex singularities are searched in the form

$$r^\alpha u(\alpha; \varphi, \theta). \tag{11}$$

The functions (11) have to be a solution of the homogeneous system

$$LU = 0 \quad \text{in } K, \qquad U = 0 \quad \text{on } \partial K.$$

In other words, $u(\alpha; \cdot)$ is solution of

$$\mathcal{A}_c(\alpha)u(\alpha; \cdot) = 0 \quad \text{in } \tilde{\Omega} \times \partial\tilde{\Omega}, \tag{12}$$

where $\mathcal{A}_c(\alpha)$ is the operator pencil which is obtained by inserting (11) into our homogeneous problem in the cone K and using the Mellin transform

$$u(\alpha; \varphi, \theta) := \int_0^\infty r^{-\alpha} U(r, \varphi, \theta) \frac{\mathrm{d}r}{r},$$

which maps $r\partial_r$ into the complex parameter α, $\partial_r := \partial/\partial r$. We will discuss this in more detail in Subsection 3.2. The problem (12) is a quadratic eigenvalue problem which has a finite number of eigenvalues in any strip $c_1 \leq \operatorname{Re}\alpha \leq c_2$. The set of these eigenvalues is discrete, see also Subsection 3.3 below.

2.2   Function spaces and weak formulation of the problem

For the weak formulation of the quadratic eigenvalue problem (12) we introduce now appropriate function spaces. We consider $\tilde{\Omega} \subset S^2$ in spherical coordinates (φ, θ) and Cartesian coordinates $x = (x_1, x_2, x_3) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$, and denote by $\Omega \subset \mathbb{R}^2$ the corresponding domain in the parameter plane,

$$\tilde{\Omega} = \{(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \in \mathbb{R}^3 : (\varphi, \theta) \in \Omega\}.$$

In this sense we write $v(\varphi, \theta) = \tilde{v}(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. We also use two different norm symbols to define the complex Lebesgue space L^2,

$$\|\tilde{v}\|_{0,\tilde{\Omega}}^2 := \int_{\tilde{\Omega}} |\tilde{v}|^2 \, \mathrm{d}S, \qquad |[v]|_{0,\Omega}^2 := \int_\Omega |v|^2 \, \mathrm{d}\omega, \quad \mathrm{d}\omega := \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

Moreover, let $W^{1,2}$ and $W^{2,2}$ be the complex Sobolev spaces endowed with the norms (expressed in the parameter domain Ω)

$$|[v]|_{k,\Omega}^2 \quad := \quad |[v]|_{0,\Omega}^2 + \sum_{j=1}^k [v]_{j,\Omega}^2, \quad k = 1, 2,$$

$$[v]_{1,\Omega}^2 \quad := \quad \int_\Omega \left(\left| \sin^{-1}\theta \, \partial_\varphi v \right|^2 + |\partial_\theta v|^2 \right) \, \mathrm{d}\omega,$$

$$[v]_{2,\Omega}^2 \quad := \quad \int_\Omega \left(\left| \sin^{-2}\theta \, \partial_{\varphi\varphi} v \right|^2 + \left| \sin^{-1}\theta \, \partial_{\varphi\theta} v \right|^2 + |\partial_{\theta\theta} v|^2 \right) \, \mathrm{d}\omega,$$

where $\partial_\theta := \partial/\partial\theta$, $\partial_\varphi := \partial/\partial\varphi$, $\partial_{\theta\theta} = \partial_\theta\partial_\theta$ etc. Note that in this setting $\|\tilde{v}\|_{0,\tilde{\Omega}} = |[v]|_{0,\Omega}$, $|\tilde{v}|_{1,\tilde{\Omega}} = [v]_{1,\Omega}$, but $|\tilde{v}|_{2,\tilde{\Omega}} \neq [v]_{2,\Omega}$, only $|\tilde{v}|_{2,\tilde{\Omega}} \gtrsim [v]_{2,\Omega}$. This is, however, sufficient for our purposes.

The usual norm symbols are used in connection with the parameter domain Ω in Subsection 4.3

$$\|v\|_{k,\Omega}^2 := \sum_{j=0}^k |v|_{j,\Omega}^2, \qquad |v|_{k,\Omega}^2 := \sum_{i+j=k} \int_\Omega \left|\partial_\varphi^i \partial_\theta^j v\right|^2 \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

The same symbols, $\|\cdot\|$ and $|[\,\cdot\,]|$, are used for the norms/seminorms of *vector functions* $v = (v_1, v_2, v_3)^\top$. The square of the (semi-)norm of a vector function is defined as the sum of the squares of the (semi-)norms of its components.

By $\overset{o}{W}{}^{1,2}(\tilde{\Omega})$ we denote the space of $W^{1,2}(\tilde{\Omega})$ functions vanishing on $\tilde{\Gamma} = \partial\tilde{\Omega}$. Furthermore, we define the spaces $H := L^2(\tilde{\Omega})^3$ and $V := \overset{o}{W}{}^{1,2}(\tilde{\Omega})^3$ of complex vector functions and identify them with the corresponding spaces $L^2(\Omega)^3$ and $\overset{o}{W}{}^{1,2}(\Omega)^3$, respectively. These spaces are equipped with the norms

$$\|v\|_H := |[v]|_{0,\Omega}, \qquad \|v\|_V := \left(|[v]|_{0,\Omega}^2 + \tfrac{1}{4}[v]_{1,\Omega}^2\right)^{1/2}.$$

The factor $\frac{1}{4}$ is used in order to underline that the constants M_1 and M_2 in Lemma 1 below are just the constants in relation (6).

The weak formulation of problem (12) is derived, e. g., in [26] by inserting the ansatz (11) into (9). For presenting the result, we introduce some abbreviating notation. We define for $u \in \overset{o}{W}{}^{1,2}(\Omega)$

$$e_j(u) := -\tfrac{1}{2}A_j u + B_j\partial_\theta u + C_j\partial_\varphi u, \quad s_j(u) := A_j u, \quad j = 1, 2, 3,$$

where

$$\begin{aligned}
A_1 &:= \cos\varphi\sin\theta, & B_1 &:= \cos\varphi\cos\theta, & C_1 &:= -\sin\varphi/\sin\theta, \\
A_2 &:= \sin\varphi\sin\theta, & B_2 &:= \sin\varphi\cos\theta, & C_2 &:= \cos\varphi/\sin\theta, \\
A_3 &:= \cos\theta, & B_3 &:= -\sin\theta, & C_3 &:= 0.
\end{aligned}$$

The corresponding vector functions A, B, and C arise in the representation of the nabla operator $(\partial_1, \partial_2, \partial_3)^\top$ in spherical coordinates after setting $r = 1$. Now we introduce the mappings $k : V \times V \to \mathbb{C}$, $m : H \times H \to \mathbb{C}$, $d : H \times V \to \mathbb{C}$, and $g : V \times V \to \mathbb{C}$,

$$k(u,v) := \sum_{i,j,l,n=1}^3 \int_\Omega a_{ijln} e_j(u_i) e_n(\overline{v}_l) \, \mathrm{d}\omega,$$

$$m(u,v) := \sum_{i,j,l,n=1}^3 \int_\Omega a_{ijln} s_j(u_i) s_n(\overline{v}_l) \, \mathrm{d}\omega,$$

$$d(u,v) := \sum_{i,j,l,n=1}^3 \int_\Omega a_{ijln} s_j(u_i) e_n(\overline{v}_l) \, \mathrm{d}\omega,$$

$$g(u,v) := \overline{d(v,u)} - d(u,v).$$

The eigenvalue problem reads: Find $\alpha \in \mathbb{C}$ and $u \in V \setminus \{0\}$ such that

$$\begin{aligned} 0 \; = \; & \alpha(\alpha+1)\, m(u,v) - (\alpha+1)\, d(u,v) + \alpha\, \overline{d(v,u)} + \\ & + \tfrac{1}{4}\, m(u,v) + \tfrac{1}{2}\, d(u,v) + \tfrac{1}{2}\, \overline{d(v,u)} \qquad \forall v \in V. \end{aligned}$$

Since the eigenvalues α of the generalized eigenvalue problem (12) are distributed symmetrically with respect to the line $\mathrm{Re}\,\alpha = -1/2$ (we will see this in Theorem 10, page 13), we introduce the new parameter

$$\lambda = \alpha + 1/2. \tag{13}$$

Then the weak formulation of the transformed eigenvalue problem is much simpler and reads: Find $\lambda \in \mathbb{C}$ and $u \in V \setminus \{0\}$ such that

$$k(u,v) = \lambda\, g(u,v) + \lambda^2\, m(u,v) \quad \forall v \in V. \tag{14}$$

The number λ is called *eigenvalue* of problem (14), and the vector function u is called *eigenelement* corresponding to λ.

Lemma 1 *The sesquilinear forms k and m are Hermitean, the form g is skew-Hermitean,*

$$\begin{aligned} k(u,v) &= \overline{k(v,u)} & \forall u,v \in V, \\ g(u,v) &= -\overline{g(v,u)} & \forall u,v \in V, \\ m(u,v) &= \overline{m(v,u)} & \forall u,v \in H. \end{aligned}$$

Moreover, the ellipticity and boundedness properties

$$\begin{aligned} M_1 \|u\|_V^2 &\leq & k(u,u) &\leq & M_2 \|u\|_V^2 & \qquad \forall u \in V, \\ M_1 \|u\|_H^2 &\leq & m(u,u) &\leq & M_2 \|u\|_H^2 & \qquad \forall u \in H, \\ & & |g(u,u)| &\leq & 2\sqrt{k(u,u)}\sqrt{m(u,u)} & \quad \forall u \in V, \\ & & |d(u,v)| &\leq & \sqrt{m(u,u)}\sqrt{k(v,v)} & \quad \forall u \in H, v \in V, \end{aligned}$$

hold.

Proof The properties follow from the definitions of the sesquilinear forms, the symmetry assumptions (5) and the ellipticity and boundedness assumptions (6) on the coefficients a_{ijln}, $i,j,l,n = 1,2,3$. $\qquad\square$

3   Two related eigenvalue problems in infinite dimensional spaces

3.1   Notation for operator eigenvalue problems

In this section we recall notation for general eigenvalue problems which will be used in two different cases in Subsections 3.2 and 3.3.

Let $\mathcal{A}(\alpha) : X \to Y$ be an operator pencil (operator function) mapping, for fixed $\alpha \in \mathbb{C}$, the Banach space X into the Banach space Y. Thus we have $\mathcal{A} : \mathbb{C} \to \mathcal{L}(X, Y)$. In this paper we specifically assume that $\mathcal{A}$ is quadratic in α. A set

$$\sigma(\mathcal{A}) \subset \mathbb{C}$$

is called the *spectrum* of $\mathcal{A}(\cdot)$ if for $\alpha \in \sigma(\mathcal{A})$ there is no bounded inverse operator $(\mathcal{A}(\alpha))^{-1} : Y \to X$. The number α_0 is called *eigenvalue* of $\mathcal{A}(\cdot)$, if there exists an element $u(\alpha_0; \cdot) \in X \setminus \{0\}$ such that $\mathcal{A}(\alpha_0)u(\alpha_0; \cdot) = 0$. This element $u(\alpha_0; \cdot)$ is called *eigenelement* of the operator pencil $\mathcal{A}(\cdot)$ corresponding to α_0. Shortly, we say $(\alpha_0, u(\alpha_0; \cdot))$ is an *eigenpair*. The set

$$U(\mathcal{A}, \alpha_0) := \{u(\alpha_0; \cdot) : \mathcal{A}(\alpha_0)u(\alpha_0; \cdot) = 0\} \cup \{0\}$$

is a closed subspace in X, which is called the *eigensubspace* of the operator pencil $\mathcal{A}(\cdot)$ corresponding to α_0. The dimension

$$n(\mathcal{A}, \alpha_0) := \dim U(\mathcal{A}, \alpha_0)$$

of this subspace is called the *geometric multiplicity* of the eigenvalue α_0.

A set of elements u^0, u^1, ..., u^{k-1}, is called *Jordan chain* of the *length k* of $\mathcal{A}(\cdot)$ at α_0 if the following relations hold:

$$
\begin{aligned}
\mathcal{A}(\alpha_0)\,u^0 &= 0, \\
\mathcal{A}(\alpha_0)\,u^1 + \mathcal{A}'(\alpha_0)\,u^0 &= 0, \\
\mathcal{A}(\alpha_0)\,u^i + \mathcal{A}'(\alpha_0)\,u^{i-1} + \tfrac{1}{2}\mathcal{A}''(\alpha_0)\,u^{i-2} &= 0, \quad i = 2, 3, \ldots, k-1.
\end{aligned}
\tag{15}
$$

The elements $u^0 = u^0(\alpha_0; \cdot)$, $u^1 = u^1(\alpha_0; \cdot)$, ..., $u^{k-1} = u^{k-1}(\alpha_0; \cdot)$, of any Jordan chain of the operator pencil $\mathcal{A}(\cdot)$ at α_0 are called *generalized eigenelements* corresponding to α_0. The closed linear hull of all the generalized eigenelements of $\mathcal{A}(\cdot)$ at α_0 is called *generalized eigensubspace*

$$W(\mathcal{A}, \alpha_0) \tag{16}$$

of $\mathcal{A}(\cdot)$ at α_0. The maximal length of Jordan chains beginning with $u^0 \in U(\mathcal{A}, \alpha_0)$ is called *order*

$$\nu(\mathcal{A}, u^0)$$

of the eigenelement u^0. Different eigenelements to the same eigenvalue α_0 may have different orders, therefore we introduce by

$$\varkappa(\mathcal{A}, \alpha_0) = \max_{u \in U(\mathcal{A}, \alpha_0) \setminus \{0\}} \nu(\mathcal{A}, u)$$

the maximal order of generalized eigenelements corresponding to α_0.

A system of eigenelements u^0_1, u^0_2, ..., u^0_l, from $U(\mathcal{A}, \alpha_0)$ is a *canonical basis* of eigenelements of $\mathcal{A}(\cdot)$ at α_0 if

$$\nu(\mathcal{A}, u^0_1) = \varkappa(\mathcal{A}, \alpha_0)$$

and u_i^0, $2 \leq i \leq l$, is an eigenelement of the maximal possible order belonging to some direct complement $\mathcal{M}_i$ in $U(\mathcal{A}, \alpha_0)$ to the linear hull $\mathcal{L}(u_1^0, \ldots, u_{i-1}^0)$ of the eigenelements $u_1^0, \ldots, u_{i-1}^0$, i. e.

$$\nu(\mathcal{A}, u_i^0) = \max_{u \in \mathcal{M}_i \setminus \{0\}} \nu(\mathcal{A}, u), \quad i = 2, 3, \ldots, l.$$

The vector $(\nu_1, \nu_2, \ldots, \nu_l)$ with $\nu_i = \nu(\mathcal{A}, u_i^0)$, $i = 1, 2, \ldots, l$, is the same for every canonical basis of eigenelements. The number

$$\mu(\mathcal{A}, \alpha_0) = \nu_1 + \nu_2 + \ldots + \nu_l$$

is called *algebraic multiplicity* of the eigenvalue α_0. If the algebraic multiplicity of an eigenvalue α_0 exceeds it geometric multiplicity, the eigenvalue α_0 is called *defective*. In the other case, where no generalized eigenelements exist, it is called *non-defective*.

Remark 2 Consider generalized eigenelements $u^0 = u^0(\alpha_0; \cdot)$, $u^1 = u^1(\alpha_0; \cdot)$, $\ldots$, $u^{k-1} = u^{k-1}(\alpha_0; \cdot)$ of $\mathcal{A}_c(.)$ at α_0. They are also generalized eigenelements of the weakly formulated eigenvalue problem (14) to some eigenvalue $\lambda_0 = \alpha_0 + \frac{1}{2}$ and satisfy the following variational equations,

$$\begin{aligned}
k(u^0, v) - \lambda_0\, g(u^0, v) - \lambda_0^2\, m(u^0, v) &= 0 && \forall v \in V, \\
k(u^1, v) - \lambda_0\, g(u^1, v) - \lambda_0^2\, m(u^1, v) &= g(u^0, v) + 2\lambda_0\, m(u^0, v) && \forall v \in V, \\
k(u^i, v) - \lambda_0\, g(u^i, v) - \lambda_0^2\, m(u^i, v) &= g(u^{i-1}, v) + 2\lambda_0\, m(u^{i-1}, v) + m(u^{i-2}, v) && \forall v \in V, \\
&&& i = 2, 3, \ldots, k-1.
\end{aligned}$$

3.2   Regularity of the eigenfunctions of the operator pencil $\mathcal{A}_c(\alpha)$

First, we consider the operator pencil

$$\mathcal{A}_c(\alpha) : \overset{o}{W}{}^{1,2}(\tilde{\Omega})^3 \to W^{-1,2}(\tilde{\Omega})^3 \tag{17}$$

defined by (12). Let us assume that $\tilde{\Omega} \subset S^2$ is a smooth domain. The function $\tilde{u}(\alpha; \cdot)$ is solution of the eigenvalue problem (12),

$$\mathcal{A}_c(\alpha)\tilde{u}(\alpha; \cdot) = 0 \quad \text{in } \tilde{\Omega} \times \partial\tilde{\Omega}.$$

For every fixed α this equation describes an elliptic boundary value problem [22, page 98]. Therefore one can conclude: if a nontrivial eigensolution $\tilde{u}^0(\alpha; \cdot) = \tilde{u}^0(\lambda - \frac{1}{2}; \cdot) \in X = \overset{o}{W}{}^{1,2}(\tilde{\Omega})^3$ of $\mathcal{A}_c(\cdot)$ exists, then it belongs to $W^{2,2}(\tilde{\Omega})^3$ and

$$[u(\alpha; \cdot)]_{2,\Omega} \lesssim |\tilde{u}(\alpha; \cdot)|_{2,\tilde{\Omega}} < \infty.$$

The regularity of generalized eigenfunctions, given by the relation (15) can be derived successively: Since $\tilde{u}^0 = \tilde{u}(\alpha; \cdot) \in W^{2,2}(\tilde{\Omega})^3$ it follows from the second equation $\mathcal{A}_c(\alpha)\tilde{u}^1 = -\mathcal{A}_c'(\alpha)\tilde{u}^0 \in L_2(\tilde{\Omega})^3$ and therefore $\tilde{u}^1 = \tilde{u}^1(\alpha, \cdot) \in \overset{o}{W}{}^{1,2}(\tilde{\Omega})^3$ belongs to $W^{2,2}(\tilde{\Omega})^3$. The third equation implies $\tilde{u}^2(\alpha, \cdot) \in W^{2,2}(\tilde{\Omega})^3$. Continuing this procedure we get:
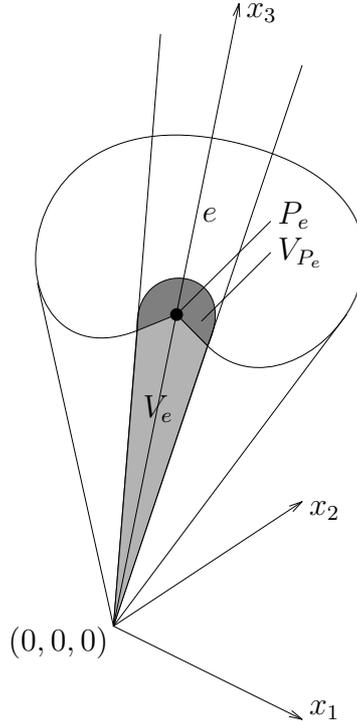
Figure 1: Vertex with one edge

Lemma 3 *If $\tilde{\Omega} \subset S^2$ is a smooth domain then the generalized eigenelements $\tilde{u}^i(\alpha, \cdot) \in \overset{o}{W}{}^{1,2}(\tilde{\Omega})^3$, $i = 0, 1, \ldots, k-1$, are contained in $W^{2,2}(\tilde{\Omega})^3$.*

In the case of a polyhedral vertex, the above operator pencil $\mathcal{A}_c(\alpha)$ is also defined on $\overset{o}{W}{}^{1,2}(\tilde{\Omega})^3$, see (17). Since $\tilde{\Omega}$ has corner points (generated by the edges of the domain G), we cannot conclude in general that the generalized eigenelements $u^i(\alpha; \cdot) \in W^{2,2}(\tilde{\Omega})^3$ and we have to analyze the influence of the corners on the asymptotic behaviour carefully. We follow here a paper of Dauge [11], where the corner-edge asymptotics is studied. The main idea is to introduce special fitted spherical coordinates which allow to couple the edge and corner singularities easily. We underline, that these special coordinates have auxiliary character and that the final results are independent of them.

Since singularities are of local nature we can assume, without loss of generality, that $\partial\tilde{\Omega}$ is smooth, except in one angular point $P_e = (0, 0, 1)$, Thus, the infinite cone K, see (10), with the vertex $c = (0, 0, 0)$ has one edge e only which coincides with the half-line $\{(0, 0, z), z > 0\}$. Besides the Cartesian coordinates $x = (x_1, x_2, x_3)$ we introduce cylindrical coordinates (ϱ, φ, z) with $\varrho = (x_1^2 + x_2^2)^{1/2}$. We assume that in a conical neighborhood V_e of the edge e the cone K coincides with a wedge W (see Figure 1),

$$W = \{(\varrho, \varphi, z) : \varrho > 0, \ 0 < \varphi < \omega_0, \ z \in \mathbb{R}\}.$$

Now, we introduce special spherical coordinates (r, X), where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and X denotes a coordinate system at the sphere S^2 such that $X = (X_1, X_2)$ with $X_1 = x_1/r = \cos\varphi \sin\theta$ and $X_2 = x_2/r = \sin\varphi\sin\theta$ locally in $V_{P_e} = V_e \cap S^2$. Note that in V_{P_e}

$$\text{dist}(P_e, (X_1, X_2)) = (X_1^2 + X_2^2)^{1/2} = \frac{\varrho}{r} = \sin\theta =: R_e. \tag{18}$$

We adapt the definition of the vertex-operator pencil (12) to the new spherical coordinates and introduce an edge-operator correspondingly. To this aim we write the linear elasticity operator L, introduced in Section 2, as

$$L = L(\partial_1, \partial_2, \partial_3) = r^{-2}\mathcal{L}_c(X_1, X_2, r\partial r, \partial_{X_1}, \partial_{X_2})$$

Applying the Mellin transformation with respect to r we get the vertex-operator pencil $\mathcal{A}_c(\alpha_c) : \overset{o}{W}{}^{1,2}(\tilde\Omega)^3 \to W^{-1,2}(\tilde\Omega)^3$,

$$\mathcal{A}_c(\alpha_c) := \mathcal{L}_c(X_1, X_2, \alpha_c, \partial_{X_1}, \partial_{X_2}).$$

We denote by $\mathcal{C}_{[-\frac{1}{2},\frac{1}{2})}$ the set of eigenvalues of $\mathcal{A}_c(\cdot)$ in the strip $-\frac{1}{2} \le \operatorname{Re}\alpha_c < \frac{1}{2}$ and by $u_c(\alpha_c; \cdot)$ the corresponding eigensolutions.

The edge singularities are generated by the non-tangential (non-tangential to the edge) part of the operator L. Thus we remove the derivatives ∂_3 in L and define

$$L_e(\partial_1, \partial_2) := L(\partial_1, \partial_2, 0).$$

Writing this operator in polar coordinates (ϱ, φ) we have

$$L_e(\partial_1, \partial_2) = \varrho^{-2}\mathcal{L}_e(\varphi, \varrho\partial_\varrho, \partial_\varphi)$$

and after the Mellin transform with respect to ϱ

$$\mathcal{A}_e(\alpha_e) := \mathcal{L}_e(\varphi, \alpha_e, \partial_\varphi).$$

Let be $\mathcal{E}_{[0,1)}$ the set of eigenvalues of $\mathcal{A}_e(\cdot)$ in the strip $0 \le \operatorname{Re}\alpha_e < 1$ and $\Phi_e(\alpha_e; \cdot)$ the corresponding eigensolutions.

Now we are in position to formulate the regularity results for the eigenfunctions $u_c(\alpha_c; \cdot)$.

Theorem 4 *[10, 35, 39, 40] Assume for simplicity that the eigenvalues in $\mathcal{C}_{[-\frac{1}{2},\frac{1}{2})}$ and $\mathcal{E}_{[0,1)}$ are non-defective. Then the weak solution $U \in \overset{o}{W}{}^{1,2}(G)^3$ of (9) admits the following decomposition in the vicinity of the vertex c:*

$$U = \sum_{\alpha_c \in \mathcal{C}_{[-\frac{1}{2},\frac{1}{2})}} C_{\alpha_c} r^{\alpha_c} u_c(\alpha_c; X_1, X_2) + u_{\text{rem},c}, \tag{19}$$

where C_{α_c} are constants,

$$u_c(\alpha_c; X_1, X_2) = \sum_{\alpha_e \in \mathcal{E}_{[0,1)}} C_{\alpha_e}(\alpha_c) R_e^{\alpha_e} \Phi_e(\alpha_e; \varphi) + u_{\text{rem},e}(\alpha_c), \tag{20}$$

and $u_{\text{rem},c}$ is a remainder depending on the edge singularities,

$$u_{\text{rem},c} = \sum_{\alpha_e \in \mathcal{E}_{[0,1)}} d_{\alpha_e}(x_3, r) \varrho^{\alpha_e} \Phi_e(\alpha_e; \varphi) + u_{\text{rem},c,e}(\alpha_c). \tag{21}$$

Here, $R_e := \sin\theta$, C_{α_c} and $C_{\alpha_e}(\alpha_c)$ are constants, d_{α_e} are functions, and $u_{\text{rem},e}(\alpha_c)$, $u_{\text{rem},c,e}(\alpha_c)$ are regular remainders.

Remark 5 If we admitted defective eigenvalues in $\mathcal{C}_{[-\frac{1}{2},\frac{1}{2})}$ and $\mathcal{E}_{[0,1)}$, then there would occur Jordan chains of $\mathcal{A}_c$ and $\mathcal{A}_e$ together with logarithmic terms of r, R_e, and ϱ in the expansions (19), (20), and (21) [8, 34, 36, 39].

Consider now a polyhedral cone K, see (10). We denote by $\mathcal{E}$ the set of edges for which the set of eigenvalues $\mathcal{E}_{[1,0)}$ is not empty. Since corner and edge singularities are of local nature we see from (20) that the eigenfunctions u_c behave like $R_e^{\alpha_e}$ in the neighborhood $\tilde{N}_e$ of a corner P_e of $\tilde{\Omega}$ for $e \in \mathcal{E}$. Here, α_e is the minimal element in the corresponding set $\mathcal{E}_{[0,1)}$. In the parameter domain we introduce the weighted Sobolev space $V_\beta^2(\Omega)$, $\beta = (\beta_e)_{e \in \mathcal{E}}$, via the norm $\|\!|\,\cdot\,|\!\|_{2,\beta,\Omega}$ which locally in N_e ($N_e \subset \Omega$ corresponds to $\tilde{N}_e \subset \tilde{\Omega}$) is given by

$$
\begin{aligned}
\|\!| v |\!\|_{2,\beta_e,N_e} &\sim [R_e^{\beta_e} v]_{2,N_e}, \quad \text{for all } e \in \mathcal{E}, \\
\|\!| v |\!\|_{2,\beta_e,\Omega\setminus\bigcup_{e\in\mathcal{E}} N_e} &\sim |[v]|_{2,\Omega\setminus\bigcup_{e\in\mathcal{E}} N_e},
\end{aligned}
\tag{22}
$$

where $\beta_e \in (1 - \operatorname{Re}\alpha_e, 1)$ is arbitrary.

Corollary 6 *The eigenfunction $u_c(\alpha_c; \cdot)$ with respect to some eigenvalue $\alpha_c \in \mathcal{C}_{[-\frac{1}{2},\frac{1}{2})}$ of $\mathcal{A}_c(\cdot)$ belongs to $V_\beta^2(\Omega)^3$, with entries $\beta_e \in (1 - \operatorname{Re}\alpha_e, 1)$ of β. That means, the corresponding norm is bounded, $\|\!| u_c |\!\|_{2,\beta,\Omega} < \infty$.*

Proof Let P_e be a corner which is described in the parameter plane by (φ_0, θ_0). Assume first that $\theta_0 \notin \{0, \pi\}$. From (20) we find that the exponent of R_e of the expansion of $R_e^{\beta_e} u_c$ is $\beta_e + \operatorname{Re}\alpha_e > 1$, therefore

$$[R_e^{\beta_e} u_c]_{2,N_e} \lesssim |R_e^{\beta_e} \tilde{u}_c|_{2,\tilde{N}_e} < \infty.$$

If $\theta_0 = 0$ then $R_e = \sin\theta \sim \theta$ and we obtain

$$[\theta^{\beta_e} u_c]_{2,N_e} < \infty$$

by direct calculation. The case $\theta_0 = \pi$ can be treated analogously by using $R_e = \sin(\pi - \theta)$. $\qquad\square$

We end this section by citing some results about the distribution of the eigenvalues which are of interest in the context of our work.

Lemma 7 *Let be* $K = \{(x_1, x_2, x_3) : x_3 > f(x_1, x_2)\}$ *where* f *is a continuous and positively homogeneous function of degree 1, i. e.,* $f(ax_1, ax_2) = af(x_1, x_2)$ *for all* $a > 0$.

(i) *Then the strip* $\left|\operatorname{Re}\alpha_c + \frac{1}{2}\right| < \frac{1}{2}$ *does not contain eigenvalues of the pencil* $\mathcal{A}_c(\alpha_c)$ *[23, Theorem 11.2.2, page 353].*

(ii) *If the restriction of* f *to the sphere* $S^1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ *belongs to the Sobolev space* $W^{1,2}(S^1)$, *then the strip* $\left|\operatorname{Re}\alpha_c + \frac{1}{2}\right| \leq \frac{1}{2}$ *has no eigenvalues of the pencil* $\mathcal{A}_c(\alpha_c)$ *[23, Theorem 11.3.3, page 364].*

(iii) *If* N *edges with the opening angles* φ_j, $j = 1 \ldots N$, *meet in the vertex* c, $\varphi_j \in (0, \pi)$ *for* $j = 2, \ldots, N$ *and* $\varphi_1 \in (0, 2\pi]$, *then the strip* $\left|\operatorname{Re}\alpha_c + \frac{1}{2}\right| \leq 1$ *is free of eigenvalues of the pencil* $\mathcal{A}_c(\alpha_c)$ *[23, Theorem 11.4.1, page 367].*

There are also results on the distribution of the eigenvalues of the pencil $\mathcal{A}_e(\alpha_e)$.

Lemma 8 *[23, Theorem 8.6.2, page 286, and Theorem 11.2.2, page 353] Let be*

$$K = \{(x_1, x_2) : 0 < (x_1^2 + x_2^2)^{1/2} < \infty, 0 < \arctan \frac{x_2}{x_1} < \omega_0\}.$$

(i) *If* $\omega_0 \in (0, \pi)$, *then the strip* $|\operatorname{Re}\alpha_e| \leq 1$ *does not contain eigenvalues of the pencil* $\mathcal{A}_e(\alpha_e)$ *[23, Theorem 8.6.2, page 286].*

(ii) *If* $\omega_0 \in (\pi, 2\pi)$, *then the strip* $0 < \operatorname{Re}\alpha_e \leq 1$ *contains exactly 2 eigenvalues (counting multiplicity) of the pencil* $\mathcal{A}_e(\alpha_e)$ *[23, Theorem 8.6.2, page 286].*

(iii) *If* $\omega_0 = \pi$ *or* $\omega_0 = 2\pi$, *then in the half plane* $\operatorname{Re}\alpha_e > 0$ *the only eigenvalues* $\alpha_{e,k} = k$ *or* $\alpha_{e,k} = \frac{k}{2}$, *respectively, occur with two linearly independent eigensolutions (no associated eigenfunctions exist) [23, Section 8.6.2, pages 284f.].*

(iv) *The strip* $|\operatorname{Re}\alpha_e| < \frac{1}{2}$ *is free of eigenvalues of the pencil* $\mathcal{A}_e(\alpha_e)$ *[23, Theorem 11.2.2, page 353].*

3.3   Regularity of the eigenfunctions of the operator pencil $\mathcal{B}(\lambda)$

In the previous section we reviewed regularity results for the eigenfunctions of the operator $\mathcal{A}_c(\alpha) : V \to V'$, $V = \overset{o}{W}{}^{1,2}(\Omega)^3$ which is the typical setting in the corresponding literature. For the investigation of our discretization, however, we will use the operator pencil

$$\mathcal{B}(\lambda) = J^{-1}\mathcal{A}_c(\lambda - 1/2),$$

where J is the isometry from V onto V' defined by

$$k(u, v) = (Ju, v)_H \quad \forall u, v \in V,$$

$H := L^2(\Omega)^3$ as above. In this section we describe properties of the operator pencil $\mathcal{B}(\lambda)$ and its adjoint which we will apply to formulate general spectral approximation results in the next section.

We come back to the weak eigenvalue problem (14) which is basic for the discretization. We introduce the operators $\mathcal{M} : V \to V$ and $\mathcal{D} : V \to V$ by the relations

$$\begin{aligned}
\mathcal{M}u \in V, \quad k(\mathcal{M}u, v) &= m(u, v) \quad \forall v \in V, \\
\mathcal{D}u \in V, \quad k(\mathcal{D}u, v) &= d(u, v) \quad \forall v \in V.
\end{aligned}$$

The adjoint operator $\mathcal{D}^* : V \to V$ is defined by the equality

$$k(\mathcal{D}u, v) = k(u, \mathcal{D}^*v) \quad \forall u, v \in V.$$

We set $\mathcal{G} = \mathcal{D}^* - \mathcal{D}$ and introduce the operator pencil $\mathcal{B}(\cdot) : \mathbb{C} \to \mathcal{L}(V, V)$ by

$$\mathcal{B}(\lambda) = I - \lambda \mathcal{G} - \lambda^2 \mathcal{M}, \quad \lambda \in \mathbb{C}.$$

The variational eigenvalue problem (14) is then equivalent to the eigenvalue problem for the operator pencil $\mathcal{B}(\cdot)$: Find $\lambda \in \mathbb{C}$, $u \in V \setminus \{0\}$, such that

$$\mathcal{B}(\lambda)u = 0.$$

Moreover, we define the operators $\mathcal{G}^*$ and $\mathcal{M}^*$ analogously to $\mathcal{D}^*$ and introduce the operator pencil $\mathcal{B}^*(\cdot)$ by the equality

$$\mathcal{B}^*(\lambda) = [\mathcal{B}(\lambda)]^*, \quad \lambda \in \mathbb{C}.$$

Now, we investigate the spectral properties of the operator pencil $\mathcal{B}(\cdot)$ on the basis of classical results from the spectral theory of operator pencils, see, for example, [24, 25, 30, 46]. First, from Lemma 1 and the compact embedding $V \stackrel{c}{\hookrightarrow} H$, where $H = L^2(\Omega)^3$, see Subsection 2.2, we obtain the following lemma which then implies Theorem 10.

Lemma 9 *The operators $\mathcal{G} : V \to V$, $\mathcal{M} : V \to V$, $\mathcal{D} : V \to V$, $\mathcal{D}^* : V \to V$, are compact and $\mathcal{M}^* = \mathcal{M}$, $\mathcal{G}^* = -\mathcal{G}$.*

Theorem 10 *The spectrum $\sigma(\mathcal{B})$ of problem (14) consists of isolated eigenvalues with finite algebraic multiplicities and with the only one possible accumulation point at infinity. If λ_0 is an eigenvalue of problem (14) then so are $-\lambda_0$, $\overline{\lambda}_0$ and $-\overline{\lambda}_0$.*

Proof Since $\lambda = 0$ is a regular value of the operator pencil $\mathcal{B}(\lambda) = I - \lambda \mathcal{G} - \lambda^2 \mathcal{M}$, $\lambda \in \mathbb{C}$, with compact operators $\mathcal{G}$ and $\mathcal{M}$, the assertions follow from [26, 30]. $\qquad\square$

Lemma 11 *$\mathcal{B}^*$ and $\mathcal{B}$ are also related by $\mathcal{B}^*(\lambda) = \mathcal{B}(-\overline{\lambda})$ for $\lambda \in \mathbb{C}$.*

Proof The assertion follows from the definition and the properties of $\mathcal{B}$, $\mathcal{G}$ and $\mathcal{M}$:

$$
\begin{aligned}
k(\mathcal{B}(\lambda)u, v) &= k((I - \lambda\mathcal{G} - \lambda^2\mathcal{M})u, v) = k(u, (I - \overline{\lambda}\mathcal{G}^* - \overline{\lambda}^2\mathcal{M}^*)v) \\
&= k(u, (I - (-\overline{\lambda})\mathcal{G} - (-\overline{\lambda})^2\mathcal{M})v) = k(u, \mathcal{B}(-\overline{\lambda})v) = k(u, \mathcal{B}^*(\lambda)v)
\end{aligned}
$$

for $u, v \in V$, $\lambda \in \mathbb{C}$. $\qquad\square$

Now, we consider the adjoint eigenvalue problem: Find $\lambda \in \mathbb{C}$, $u \in V \setminus \{0\}$, such that $\mathcal{B}^*(\lambda)u = 0$. By using Lemma 11, its variational form reads: Find $\lambda \in \mathbb{C}$, $u \in V \setminus \{0\}$, such that

$$
k(u, v) = -\overline{\lambda}\, g(u, v) + \overline{\lambda}^2\, m(u, v) \quad \forall v \in V.
$$

Theorem 12 *The spectra of $\mathcal{B}$ and $\mathcal{B}^*$ coincide,*

$$
\sigma(\mathcal{B}) = \sigma(\mathcal{B}^*),
$$

and for any eigenvalue $\lambda_0 \in \sigma(\mathcal{B})$ the maximal order of generalized eigenelements, $\varkappa$, the geometric multiplicity, n, and the algebraic multiplicity, μ, are finite and equal for $\mathcal{B}$ and $\mathcal{B}^*$:

$$
\begin{aligned}
\varkappa(\mathcal{B}, \lambda_0) &= \varkappa(\mathcal{B}^*, \lambda_0) < \infty, \\
n(\mathcal{B}, \lambda_0) &= n(\mathcal{B}^*, \lambda_0) < \infty, \\
\mu(\mathcal{B}, \lambda_0) &= \mu(\mathcal{B}^*, \lambda_0) < \infty.
\end{aligned}
$$

Proof These results follow, for example, from [16, 24, 25, 29, 31, 46]. $\qquad\square$

We use now the regularity results for the Jordan chains of the operator $\mathcal{A}_c$ in order to clarify the smoothness of the generalized eigenfunctions of the operators $\mathcal{B}$ and $\mathcal{B}^*$. The definition of these generalized eigenfunctions was already given in Remark 2 on page 8. Note that the operator pencils $\mathcal{A}_c(\alpha) : V \to V'$ and $\mathcal{B}(\lambda) : V \to V$ are closely connected via $\lambda = \alpha + \frac{1}{2}$ and the Riesz representation theorem.

Lemma 13 *The generalized eigenelements of the operator pencil $\mathcal{B}(\cdot)$ at λ and of the operator pencil $\mathcal{B}^*(\cdot)$ at $-\overline{\lambda}$, respectively, belong to $V_\beta^2(\Omega)^3$, with entries $\beta_e \in (1 - \operatorname{Re}\alpha_e, 1)$ of β. Moreover, for an eigenpair $(\lambda, u(\lambda; \cdot))$ of $\mathcal{B}(\cdot)$ or $\mathcal{B}^*(\cdot)$ the estimate*

$$
\| u(\lambda; \cdot) \|_{2,\beta,\Omega} \le C \left(|\lambda|^2 |[u(\lambda; \cdot)]|_{0,\Omega} + |\lambda| |[u(\lambda; \cdot)]|_{1,\Omega} \right) \tag{23}
$$

holds with a constant C independent of λ and $u(\lambda; \cdot)$.

Proof For simplicity, we consider again the case of non-defective eigenvalues $\alpha_c \in \mathcal{C}_{[-\frac{1}{2}, \frac{1}{2})}$ and $\alpha_e \in \mathcal{E}_{[0,1)}$. Setting $\alpha_c = \lambda_c - \frac{1}{2}$, formulae (19) and (20) describe the behaviour of the eigensolutions $u(\lambda; \cdot)$ of the operator pencil $\mathcal{B}$. As in the proof of Corollary 6 we find that $\| u(\lambda; \cdot) \|_{2,\beta,\Omega} < \infty$. Using the definition of the generalized eigenfunctions, see Remark 2, recurrently, we get analogous results for the generalized eigenfunctions.

Now we pass to the estimate (23). For a fixed eigenpair $(\lambda, u(\lambda; \cdot))$ we have

$$k(u(\lambda; \cdot), v(\cdot)) = \lambda g(u(\lambda; \cdot), v(\cdot)) + \lambda^2 m(u(\lambda; \cdot), v(\cdot)) \quad \forall v \in V.$$

Since $u(\lambda; \cdot) \in V_\beta^2(\Omega)^3$ for $\beta_e \in (1 - \operatorname{Re} \alpha_e, 1)$ this weakly formulated V-elliptic boundary value problem is equivalent to a classical one,

$$K_2 u(\lambda; \cdot) = \lambda \, K_1 u(\lambda; \cdot) + \lambda^2 \, K_0 u(\lambda; \cdot) \quad \text{in } \tilde{\Omega}, \qquad u(\lambda; \cdot) = 0 \quad \text{on } \partial \tilde{\Omega},$$

where K_i are differential operators of the order i. Due to the well known a-priori estimates for elliptic boundary value problems in weighted spaces [21] the estimate

$$
\begin{aligned}
\| u(\lambda; \cdot) \|_{2,\beta,\Omega} &\leq C \, \|[\lambda \, K_1 u(\lambda; \cdot) + \lambda^2 \, K_0 u(\lambda; \cdot)]\|_{0,\Omega} \\
&\leq C \left(|\lambda| |[u(\lambda; \cdot)]|_{1,\Omega} + |\lambda|^2 |[u(\lambda; \cdot)]|_{0,\Omega} \right)
\end{aligned}
$$

follows.

We have seen that the regularity of the eigenfunctions $u(\lambda; \cdot)$ of $\mathcal{B}(\cdot)$ is dominated by the geometry of the spherical cone, that means, by the eigenvalues $\alpha_e \in \mathcal{E}_{[0,1)}$. Therefore, the arguments for the derivation of the estimate (23) can be repeated for an arbitrary eigenvalue λ.

Since $\mathcal{B}^*(-\overline{\lambda}) = \mathcal{B}(\lambda) : V \to V$, the regularity results are valid for the generalized eigenelements of $\mathcal{B}^*$, too. $\square$

4   The approximate eigenvalue problem

4.1   General spectral approximation results

In this subsection we review approximation results taken from the papers of Karma [18, 19, 20].

Let $V_h \subset V$ be given finite-dimensional subspaces such that for any $v \in V$

$$\varepsilon_h(v) := \inf_{v_h \in V_h} \|v - v_h\|_V \to 0 \quad \text{for } h \to 0. \tag{24}$$

Problem (14) is approximated by the following finite-dimensional problem: Find $\lambda_h \in \mathbb{C}$, $u_h \in V_h \setminus \{0\}$, such that

$$k(u_h, v_h) = \lambda_h \, g(u_h, v_h) + \lambda_h^2 \, m(u_h, v_h) \quad \forall v_h \in V_h. \tag{25}$$

The number λ_h is called *approximate eigenvalue* and the vector function u_h is called *eigenelement* corresponding to λ_h.

We define the projection operator $\mathcal{P}_h : V \to V_h$ by

$$\mathcal{P}_h u \in V_h, \quad k(\mathcal{P}_h u, v_h) = k(u, v_h) \quad \forall v_h \in V_h, u \in V,$$

set $\mathcal{G}_h = \mathcal{P}_h\mathcal{G}$, $\mathcal{M}_h = \mathcal{P}_h\mathcal{M}$, $\mathcal{D}_h = \mathcal{P}_h\mathcal{D}$ and introduce in analogy to Subsection 3.3 the operator pencil $\mathcal{B}_h(\cdot)$ by the formula $\mathcal{B}_h(\lambda) = I - \lambda\,\mathcal{G}_h - \lambda^2\,\mathcal{M}_h$, $\lambda \in \mathbb{C}$. Then the variational eigenvalue problem (25) is equivalent to: Find $\lambda_h \in \mathbb{C}$, $u_h \in V_h \setminus \{0\}$, such that

$$\mathcal{B}_h(\lambda_h)u_h = 0.$$

Lemma 14 *The operator* $\mathcal{P}_h : V \to V_h$ *is a self-adjoint projector with*

$$\|u - \mathcal{P}_h u\|_V \le \sqrt{\frac{M_2}{M_1}}\,\varepsilon_h(u) \quad \forall u \in V,$$

where M_1 and M_2 are the constants from (6).

Proof These results follow from the projection property of $\mathcal{P}_h$ and the properties of the sesquilinear form $k(.,.)$ given in Lemma 1. $\qquad\square$

Now, let us formulate properties of the operators $\mathcal{G}_h$, $\mathcal{M}_h$, $\mathcal{D}_h$, $\mathcal{B}_h(\cdot)$ analogously to the properties described in Subsection 3.3 for the infinite-dimensional case.

Lemma 15 *The operators* $\mathcal{G}_h : V_h \to V_h$, $\mathcal{M}_h : V_h \to V_h$, $\mathcal{D}_h : V_h \to V_h$, $\mathcal{D}_h^* : V_h \to V_h$ *are compact and* $\mathcal{M}_h^* = \mathcal{M}_h$, $\mathcal{G}_h^* = -\mathcal{G}_h$.

Lemma 16 $\mathcal{B}_h^*$ *and* $\mathcal{B}_h$ *are related by* $\mathcal{B}_h^*(\lambda) = \mathcal{B}_h(-\overline{\lambda})$ *for* $\lambda \in \mathbb{C}$.

Theorem 17 *The spectrum $\sigma(\mathcal{B}_h)$ of problem (25) consists of isolated eigenvalues with finite algebraic multiplicities. If λ_h is an eigenvalue of problem (25) then so are $-\lambda_h$, $\overline{\lambda}_h$ and $-\overline{\lambda}_h$.*

For $\lambda_0 \in \sigma(\mathcal{B})$ we define

$$\varepsilon_h = \max_{v \in W(\mathcal{B},\lambda_0),\|v\|_V=1} \varepsilon_h(v), \quad \varepsilon_h^* = \max_{v \in W(\mathcal{B}^*,\lambda_0),\|v\|_V=1} \varepsilon_h(v), \tag{26}$$

where $W(\mathcal{B}, \lambda_0)$ is the generalized eigensubspace, see (16).

Theorem 18 *Let λ_0 be an eigenvalue of problem (14) with maximal order $\varkappa = \varkappa(\mathcal{B}, \lambda_0)$ of generalized eigenelements. Then there exists a sequence λ_h of eigenvalues of problem (25), $\lambda_h \in \sigma(\mathcal{B}_h)$, such that $\lambda_h \to \lambda_0$ as $h \to 0$ and for sufficiently small h the following error estimate is valid:*

$$|\lambda_h - \lambda_0| \lesssim (\varepsilon_h \varepsilon_h^*)^{1/\varkappa}.$$

Proof The error estimate follows from classical results [18] and Lemmata 9 and 14. $\quad\square$

In the case $\varkappa \ge 2$ the estimate can be improved when the arithmetic mean of the approximate eigenvalues is considered.
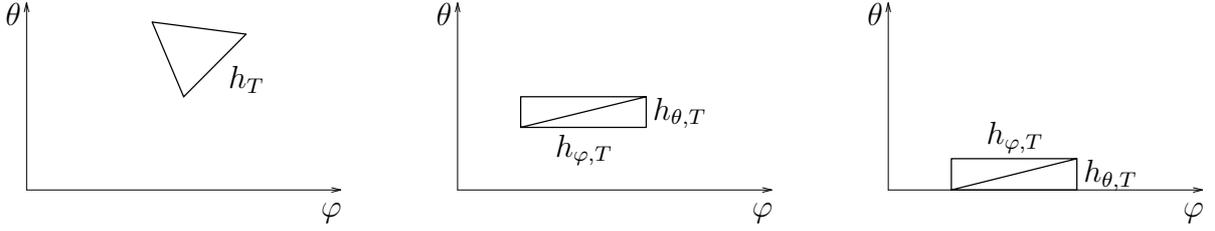
Figure 2: Illustration of different cases of triangles

Theorem 19 *Let λ_0 be an eigenvalue of problem (14) with algebraic multiplicity $\mu = \mu(\mathcal{B}, \lambda_0)$. Then there exist sequences $\lambda_i^h \in \sigma(\mathcal{B}_h)$, $i = 1, 2, \ldots, \mu$, of eigenvalues of problem (25), counted according to the algebraic multiplicity, such that $\lambda_i^h \to \lambda_0$ as $h \to 0$. For*

$$\hat{\lambda}_h = \frac{1}{\mu} \sum_{i=1}^{\mu} \lambda_i^h$$

and sufficiently small h the following error estimate is valid:

$$|\hat{\lambda}_h - \lambda_0| \lesssim \varepsilon_h \varepsilon_h^*.$$

Proof   With Lemmata 9 and 14 we have proved the assumptions to apply the recent results of Karma [19, 20]. □

Theorem 20 *Assume that λ_0 is an eigenvalue of problem (14) and (λ_h, u_h) are eigenpairs of problem (25), such that $\lambda_h \to \lambda_0$ as $h \to 0$. Then there exists an eigenelement $u^0 \in U(\mathcal{B}, \lambda_0)$ of problem (14) corresponding to λ_0 such that for sufficiently small h the following error estimate holds:*

$$\|u_h - u^0\|_V \lesssim \varepsilon_h + (\varepsilon_h \varepsilon_h^*)^{1/\varkappa}.$$

Proof   The error estimate follows from [17, 47], Lemmata 9 and 14 and Theorem 18.   □

4.2   Meshes

In order to describe the meshes we make the simplifying assumption that the domain $\tilde{\Omega} \subset S^2$ can be represented, by using a suitable choice of the north pole, by a polygonal domain Ω in the (φ, θ)-plane. Then, we define a family of meshes $\mathcal{T}_h = \{T\}$ with the following properties. For later use we define

$$\theta_{-,T} := \inf_{(\varphi,\theta) \in T} \sin\theta, \qquad \theta_{+,T} := \sup_{(\varphi,\theta) \in T} \sin\theta. \tag{27}$$

Admissibility   We assume that $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$ where the elements T are open triangles. Any side E of any triangle T is either part of the boundary $\partial\Omega$ or side of another triangle $T' \in \mathcal{T}_h$.

Aspect ratio For triangles T with $\theta_{-,T} \geq \theta_* = \text{const.} > 0$ we assume that T has bounded aspect ratio, without further constraints. The diameter of T is denoted by h_T. For an illustration see Figure 2, left hand side.

For $\theta_{-,T} < \theta_*$ we assume that two edges of T are parallel to the coordinate axes. Their lengths are denoted by $h_{\varphi,T}$ and $h_{\theta,T}$ which can be chosen independently, see also Figure 2, middle and right. That means the aspect ratio of T may not be bounded by a constant.

Refinement If $\tilde{\Omega}$ is a smooth domain then $h_T \sim h$ for all $T \in \mathcal{T}_h$, and the aspect ratio of all elements is bounded.

If $\tilde{\Omega}$ is not smooth the mesh is refined in the neighbourhoods of concave corners. (Due to Lemma 8, part (i), the eigenfunction is regular near corners $\tilde{P}$ with interior angle $\omega_0 \in (0, \pi)$.) For each of those corners $P = (\varphi_0, \theta_0)$ the refined mesh is defined as follows where the parameter β is the corresponding weight exponent as used in Corollary 6, this means $\beta_e \in (1 - \text{Re}\,\alpha_e, 1)$.

Case 1, $\theta_0 \notin \{0, \pi\}$: The aspect ratio of the elements is bounded and

$$h_T \sim \begin{cases} h^{1/(1-\beta)} & \text{if dist}\,(P, T) = 0, \\ h\,[\text{dist}\,(P, T)]^{\beta} & \text{if dist}\,(P, T) > 0. \end{cases} \tag{28}$$

This means that $h_T \sim h$ for $\text{dist}\,(P, T) > C_* = \text{const.}$

Case 2, $\theta_0 \in \{0, \pi\}$: The refinement zone is determined by $\sin\theta < \theta_*$. The elements might be anisotropic,

$$h_{\varphi,T} \quad \sim \quad h, \tag{29}$$

$$h_{\theta,T} \quad \sim \quad \begin{cases} h^{1/(1-\beta)} & \text{if } \theta_{-,T} = 0, \\ h\,\theta_{-,T}^{\beta} & \text{if } \theta_{-,T} > 0. \end{cases} \tag{30}$$

The space V_h is introduced by

$$V_h = \{v \in V : v|_T \in (\mathcal{P}_1)^3 \text{ for all } T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is, as usual, the space of polynomials of maximal degree one. For estimates of the approximation error of the eigenvalues and eigenfunctions via Theorems 18 – 20 we need estimates of ε_h and ε_h^* from (26). Since the regularity of the functions from $W(\mathcal{B}, \lambda_0)$ and $W(\mathcal{B}^*, \lambda_0)$ is the same, see Lemma 13, we need to estimate $\varepsilon_h(u)$ from (24) for functions $u \in V_\beta^2(\Omega)$. Therefore our aim is to bound $\|u - \text{I}_h u\|_V$ where $\text{I}_h : C(\overline{\Omega}) \to V_h$ is the interpolation operator with respect to the vertices of the triangulation. We start with local interpolation error estimates for scalar functions.

4.3 Local interpolation error estimates

The aim of this subsection is to estimate the interpolation error $u - I_h u$ for scalar u in the three cases which are indicated in Figure 2. Let us start with the simplest estimate.

Lemma 21 *Let T be an isotropic triangle, that means, T has bounded aspect ratio, and assume that*

$$\theta_{-,T} \geq h_T \tag{31}$$

Then the error estimate

$$[u - I_h u]_{m,T} \lesssim \theta_{-,T}^{-m} h_T^{2-m} [u]_{2,T}, \quad m = 0, 1,$$

holds when $|[u]|_{2,T} < \infty$.

Proof By using the results in standard norms we get for $m = 0, 1$

$$
\begin{aligned}
|[\partial_\theta^m (u - I_h u)]|_{0,T} &= \| \sin^{1/2}\theta \, \partial_\theta^m (u - I_h u) \|_{0,T} \\
&\leq \theta_{+,T}^{1/2} \| \partial_\theta^m (u - I_h u) \|_{0,T} \\
&\lesssim \theta_{+,T}^{1/2} h_T^{2-m} |u|_{2,T} \\
&\leq \theta_{+,T}^{1/2} \theta_{-,T}^{-1/2} h_T^{2-m} \left(|[\sin^{-2}\theta \, \partial_{\varphi\varphi} u]|_{0,T}^2 + |[\sin^{-1}\theta \, \partial_{\varphi\theta} u]|_{0,T}^2 + |[\partial_{\theta\theta} u]|_{0,T}^2 \right)^{1/2} \\
&\leq \theta_{+,T}^{1/2} \theta_{-,T}^{-1/2} h_T^{2-m} [u]_{2,T} \tag{32}
\end{aligned}
$$

We show now that $\theta_{+,T}/\theta_{-,T} \leq 2$. Indeed, by the definition of $\theta_{-,T}$ and $\theta_{+,T}$ there exist angles ϑ_- and ϑ_+ with $\theta_{-,T} = \sin\vartheta_-$, $\theta_{+,T} = \sin\vartheta_+$, and $|\vartheta_+ - \vartheta_-| \leq h_T$. Consequently, by using (31), we get

$$
\begin{aligned}
\theta_{+,T} &= \sin\vartheta_+ = \sin\vartheta_- \cos(\vartheta_+ - \vartheta_-) + \cos\vartheta_- \sin(\vartheta_+ - \vartheta_-) \\
&= \theta_{-,T} \left(\cos(\vartheta_+ - \vartheta_-) + \theta_{-,T}^{-1} \cos\vartheta_- \sin(\vartheta_+ - \vartheta_-) \right) \\
&\leq 2\theta_{-,T}. \tag{33}
\end{aligned}
$$

Analogously to (32) we show

$$
\begin{aligned}
|[\sin^{-1}\theta \, \partial_\varphi (u - I_h u)]|_{0,T} &= \| \sin^{-1/2}\theta \, \partial_\varphi (u - I_h u) \|_{0,T} \\
&\leq \theta_{-,T}^{-1/2} \| \partial_\varphi (u - I_h u) \|_{0,T} \\
&\lesssim \theta_{-,T}^{-1/2} h_T |u|_{2,T} \\
&\leq \theta_{-,T}^{-1} h_T [u]_{2,T}. \tag{34}
\end{aligned}
$$

With (32) – (34) we have proved the lemma.   □

We do not claim that this result is optimal with respect to the dependence of the right hand side on $\theta_{-,T}$. But this estimate is sufficient for our purposes since we use general isotropic triangles only in parts of the domain where $\theta_{-,T} \geq \theta_* = \text{const.} > 0$.

Lemma 22 *Let T be an anisotropic triangle with two edges parallel to the coordinate axes, compare Figure 2, middle. Assume that $|[u]|_{2,T} < \infty$. Under the assumption $\theta_{-,T} \gtrsim h_{\theta,T}$ the error estimates*

$$
\begin{aligned}
|[u - \mathrm{I}_h u]|_{0,T} &\lesssim& h_{\varphi,T}^2\,\theta_{-,T}^2 |[\sin^{-2}\theta\,\partial_{\varphi\varphi}u]|_{0,T} + \\
&& + h_{\varphi,T}\,h_{\theta,T}\,\theta_{-,T} |[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T} + h_{\theta,T}^2 |[\partial_{\theta\theta}u]|_{0,T},
\end{aligned}
$$

$$
|[\sin^{-1}\theta\,\partial_\varphi(u - \mathrm{I}_h u)]|_{0,T} \lesssim h_{\varphi,T}\theta_{-,T} |[\sin^{-2}\theta\,\partial_{\varphi\varphi}u]|_{0,T} + h_{\theta,T} |[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T},
$$

$$
|[\partial_\theta(u - \mathrm{I}_h u)]|_{0,T} \lesssim h_{\varphi,T}\theta_{-,T} |[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T} + h_{\theta,T} |[\partial_{\theta\theta}u]|_{0,T}
$$

hold.

Proof The proof is analogous to that of Lemma 21. Only, we use the anisotropic error estimates [2, 4]

$$
\|u - \mathrm{I}_h u\|_{0,T} \lesssim h_{\varphi,T}^2 \|\partial_{\varphi\varphi}u\|_{0,T} + h_{\varphi,T}\,h_{\theta,T}\|\partial_{\varphi\theta}u\|_{0,T} + h_{\theta,T}^2\|\partial_{\theta\theta}u\|_{0,T},
$$

$$
\|\partial_\varphi(u - \mathrm{I}_h u)\|_{0,T} \lesssim h_{\varphi,T}\|\partial_{\varphi\varphi}u\|_{0,T} + h_{\theta,T}\|\partial_{\varphi\theta}u\|_{0,T},
$$

$$
\|\partial_\theta(u - \mathrm{I}_h u)\|_{0,T} \lesssim h_{\varphi,T}\|\partial_{\varphi\theta}u\|_{0,T} + h_{\theta,T}\|\partial_{\theta\theta}u\|_{0,T}.
$$

For example, the last of the three estimates is obtained as follows:

$$
\begin{aligned}
|[\partial_\theta(u - \mathrm{I}_h u)]|_{0,T} &=& \|\sin^{1/2}\theta\,\partial_\theta(u - \mathrm{I}_h u)\|_{0,T} \\
&\le& \theta_{+,T}^{1/2}\|\partial_\theta(u - \mathrm{I}_h u)\|_{0,T} \\
&\lesssim& \theta_{+,T}^{1/2}\,(h_{\varphi,T}\|\partial_{\varphi\theta}u\|_{0,T} + h_{\theta,T}\|\partial_{\theta\theta}u\|_{0,T}) \\
&\lesssim& \theta_{+,T}^{1/2}\left(h_{\varphi,T}\theta_{+,T}^{1/2}|[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T} + h_{\theta,T}\theta_{-,T}^{-1/2}|[\partial_{\theta\theta}u]|_{0,T}\right) \\
&=& \theta_{+,T}^{1/2}\theta_{-,T}^{-1/2}\left(h_{\varphi,T}(\theta_{+,T}^{1/2}\theta_{-,T}^{-1/2})\theta_{-,T}|[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T} + h_{\theta,T}|[\partial_{\theta\theta}u]|_{0,T}\right).
\end{aligned}
$$

The arising factors $\theta_{+,T}^{1/2}\theta_{-,T}^{-1/2}$ is bounded by a constant as shown in (33). $\qquad\square$

Let S be the union of all elements T with $\theta_{-,T} = 0$. If S is not connected we treat all simply connected parts of S separately, such that we can assume that

$$
S = (\varphi_0, \varphi_1) \times (0, h_{\theta,S}). \tag{35}
$$

Lemma 23 *Assume that $|[u]|_{2,S} < \infty$. Then the interpolation operator I_h is well defined in S and the estimates*

$$
[u - \mathrm{I}_h u]_{m,S} \lesssim h_{\theta,S}^{2-m}[u]_{2,S}, \quad m = 0, 1,
$$

hold.

Proof By the mapping $\varphi = \hat{\varphi}$, $\theta = h_{\theta,S}\hat{\theta}$ we transform S to $\hat{S} = (\varphi_0, \varphi_1) \times (0, 1)$. On $\hat{S}$ we define the weighted Sobolev spaces

$$
\begin{aligned}
H^k_{1/2}(\hat{S}) &:= \{\hat{v} \in \mathcal{D}'(\hat{S}) : \|\hat{v}\|_{k;1/2,\hat{S}} < \infty\}, \\
\|\hat{v}\|^2_{k;1/2,\hat{S}} &:= \sum_{j=0}^{k} |\hat{v}|^2_{j;1/2,\hat{S}}, \\
|\hat{v}|^2_{k;1/2,\hat{S}} &:= \sum_{i+j=k} \|\hat{\theta}^{1/2} \partial_{\hat{\varphi}}^i \partial_{\hat{\theta}}^j \hat{v}\|^2_{0,\hat{S}}.
\end{aligned}
\tag{36}
$$

Such spaces were investigated by Mercier and Raugel [38] in the context of a transformation to polar/cylindrical coordinates. In particular, Theorem 4.7 of that paper states that $H^2_{1/2}(\hat{S}) \hookrightarrow C^0(\overline{\hat{S}})$, that means that

$$
\|\hat{u}\|_{C^0(\overline{\hat{S}})} \lesssim \|\hat{u}\|_{2;1/2,\hat{S}} \leq |[\hat{u}]|_{2,\hat{S}}.
$$

Consequently, I_h is well defined in S. Moreover,

$$
\|\partial_{\hat{\theta}}^m \hat{I}_h \hat{u}\|_{0;1/2,\hat{S}} \lesssim \|\hat{u}\|_{C^0(\overline{\hat{S}})} \|1\|_{0;1/2,\hat{S}} \lesssim \|\hat{u}\|_{2;1/2,\hat{S}} \quad m = 0, 1.
\tag{37}
$$

Let us now prove the interpolation error estimate. From [38, Theorem 4.6] we have the following Deny-Lions type argument,

$$
\inf_{\hat{w} \in \mathcal{P}_1} \|\hat{v} - \hat{w}\|_{2;1/2,\hat{S}} \lesssim |\hat{v}|_{2;1/2,\hat{S}}.
\tag{38}
$$

Therefore we can use the standard proof for the θ-derivative of the interpolation error. By transformation to $\hat{S}$ and using $\sin(h_{\theta,S}\hat{\theta}) \sim h_{\theta,S}\hat{\theta}$, (37), (38), we get for $m = 0, 1$,

$$
\begin{aligned}
|[\partial_\theta^m(u - I_h u)]|_{0,S} &\lesssim h_{\theta,S}^{1-m} \|\partial_{\hat{\theta}}^m(\hat{u} - \hat{I}_h \hat{u})\|_{0;1/2,\hat{S}} \\
&= h_{\theta,S}^{1-m} \inf_{\hat{w} \in \mathcal{P}_1} \|\partial_{\hat{\theta}}^m(\hat{u} - \hat{w} - \hat{I}_h(\hat{u} - \hat{w}))\|_{0;1/2,\hat{S}} \\
&\lesssim h_{\theta,S}^{1-m} \inf_{\hat{w} \in \mathcal{P}_1} \|\hat{u} - \hat{w}\|_{2;1/2,\hat{S}} \\
&\lesssim h_{\theta,S}^{1-m} |\hat{u}|_{2;1/2,\hat{S}} \\
&\lesssim h_{\theta,S}^{2-m} [u]_{2,S}.
\end{aligned}
\tag{39}
$$

For the estimate of $\partial_\varphi(u - I_h u)$ we distinguish in a first step two cases. First, if T is a triangle with one side E at the φ-axis then u is constant on E and $\partial_\varphi I_h u = 0$ in T, that means,

$$
|[\sin^{-1}\theta \, \partial_\varphi(u - I_h u)]|_{0,T} = |[\sin^{-1}\theta \, \partial_\varphi u]|_{0,T}.
$$

In the other case, T is a triangle with one vertex at the φ-axis and one edge E parallel to the φ-axis. The equivalence of norms in one-dimensional spaces, the interpolation property

and the trace theorem yield that

$$
\begin{aligned}
|[\sin^{-1}\theta\,\partial_\varphi \mathrm{I}_h u]|_{0,T} \quad &\sim \quad |[\hat{\theta}^{-1}\partial_{\hat{\varphi}}\hat{\mathrm{I}}_h\hat{u})]|_{0;1/2,\hat{T}} \\
&\sim \quad \left|\int_{\hat{E}} \partial_{\hat{\varphi}}\hat{\mathrm{I}}_h\hat{u}\right| = \left|\int_{\hat{E}} \partial_{\hat{\varphi}}\hat{u}\right| \lesssim \|\partial_{\hat{\varphi}}\hat{u}\|_{1,\hat{T}} \\
&\lesssim \quad \|\hat{\theta}^{-1}\partial_{\hat{\varphi}}\hat{u}\|_{0,\hat{T}} + \|\hat{\theta}^{-1}\partial_{\hat{\varphi}\hat{\theta}}\hat{u}\|_{0,\hat{T}} + \|\hat{\theta}^{-2}\partial_{\hat{\varphi}\hat{\varphi}}\hat{u}\|_{0,\hat{T}} \\
&\lesssim \quad |[\sin^{-1}\theta\,\partial_\varphi u]|_{0,T} + h_{\theta,S}\left(|[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,T} + |[\sin^{-2}\theta\,\partial_{\varphi\varphi}u]|_{0,T}\right).
\end{aligned}
$$

Combining the two cases we obtain

$$
\begin{aligned}
|[\sin^{-1}\theta\,\partial_\varphi(u - \mathrm{I}_h u)]|_{0,S} \quad \lesssim \quad &|[\sin^{-1}\theta\,\partial_\varphi u]|_{0,S} + \\
&+ h_{\theta,S}\left(|[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,S} + |[\sin^{-2}\theta\,\partial_{\varphi\varphi}u]|_{0,S}\right). \qquad (40)
\end{aligned}
$$

It remains to estimate the first term at the right hand side. Let $H_0^1(\hat{S}) := \{\hat{v} \in H^1(\hat{S}) : \hat{v}(\hat{\varphi},0) = 0\}$ and note that $\partial_{\hat{\varphi}}\hat{u} \in H_0^1(\hat{S})$. Using [38, Corollary 4.1], namely

$$
\|\hat{\theta}^{-1}\hat{v}\|_{0,\hat{S}} \lesssim |\hat{v}|_{1,\hat{S}} \quad \forall \hat{v} \in H_0^1(\hat{S}),
$$

we conclude

$$
\begin{aligned}
|[\sin^{-1}\theta\,\partial_\varphi u]|_{0,S} \quad &\sim \quad |[\hat{\theta}^{-1}\partial_{\hat{\varphi}}\hat{u}]|_{0,\hat{S}} \\
&\lesssim \quad \|\hat{\theta}^{-1}\partial_{\hat{\varphi}}\hat{u}\|_{0,\hat{S}} \\
&\lesssim \quad \|\partial_{\hat{\varphi}\hat{\theta}}\hat{u}\|_{0,\hat{S}} + \|\partial_{\hat{\varphi}\hat{\varphi}}\hat{u}\|_{0,\hat{S}} \\
&\lesssim \quad \|\hat{\theta}^{-1}\partial_{\hat{\varphi}\hat{\theta}}\hat{u}\|_{0,\hat{S}} + \|\hat{\theta}^{-2}\partial_{\hat{\varphi}\hat{\varphi}}\hat{u}\|_{0,\hat{S}} \\
&\lesssim \quad h_{\theta,S}\left(|[\sin^{-1}\theta\,\partial_{\varphi\theta}u]|_{0,S} + |[\sin^{-2}\theta\,\partial_{\varphi\varphi}u]|_{0,S}\right). \qquad (41)
\end{aligned}
$$

The desired estimate is obtained by combining (39) – (41).   □

Until now we assumed that $[u]_{2,T} < \infty$. In the case of polyhedral corners, however, the eigensolutions do in general not possess this regularity near the corners $\tilde{P}_e$ of $\tilde{\Omega}$. Let $\tilde{P}_e$ be represented by $P_e = (\varphi_0, \theta_0)$ and consider first the case $\theta_0 \notin \{0, \pi\}$. Then we can modify the proof of Lemma 21 by using the following result of Raugel [42], see also [14, Section 8.4.1],

$$
|u - \mathrm{I}_h u|_{m,T} \lesssim h_T^{2-m-\beta} \sum_{i+j=2} \|R_e^\beta \partial_\varphi^i \partial_\theta^j u\|_{0,T}, \quad m = 0, 1, \qquad (42)
$$

$R_e := R_e(\varphi, \theta) = [(\varphi - \varphi_0)^2 + (\theta - \theta_0)^2]^{1/2}$, $\beta \in [0, 1)$. This leads immediately to the following lemma.

Lemma 24 *Let T be an isotropic triangle and assume $\theta_{-,T} \gtrsim 1$. Then the error estimate*

$$
[u - \mathrm{I}_h u]_{m,T} \lesssim h_T^{2-m-\beta} \sum_{i+j=2} |[R_e^\beta \partial_\varphi^i \partial_\theta^j u]|_{0,T}, \quad m = 0, 1,
$$

holds for $\beta \in [0, 1)$.

Proof The error estimate follows from (42) and $\sin\theta \sim 1$ in T. $\qquad\square$

In the case that $\theta_0 = 0$ we have to modify Lemma 23.

Lemma 25 *Assume that $\|\!\|u\|\!\|_{2,\beta,S} < \infty$ and $1 - \operatorname{Re}\alpha_e < \beta_e < \frac{1}{2}$. Then the interpolation operator I_h is well defined in S, S from (35), and the estimates*

$$[u - \mathrm{I}_h u]_{m,S} \lesssim h_{\theta,S}^{2-m-\beta}[\theta^\beta u]_{2,S}, \quad m = 0, 1,$$

hold.

Proof We follow the lines of the proof of Lemma 23. It is shown in [32, Lemma 1.1] that $\|u\|_{C(\overline{S})} \lesssim \|u\|_{2,1/2+\beta,S}$ if $\beta_e < \frac{1}{2}$. Consequently, I_h is well defined for functions from $V_{1/2+\beta}^2$. Estimate (38) holds also when $\frac{1}{2}$ is replaced by $\frac{1}{2} + \beta$, $\beta \in [0,1)$. This can be proved in the same way as in [38]; we have to ensure only that the embedding $H_{1/2+\beta}^2(\hat{S}) \hookrightarrow H_{1/2+\beta}^1(\hat{S})$ is compact. (The definition of these weighted spaces is analogous to (36).) This is indeed true, see [21, Lemma 3.5].

For proving the estimate which is analogous to (39) we need another embedding theorem, namely $H_{1/2+\beta}^2(\hat{S}) \hookrightarrow H_{1/2}^1(\hat{S})$, which holds for $\beta \in [0,1)$, see also [21, Lemma 3.5]. Consequently,

$$
\begin{aligned}
|[\partial_\theta(u - \mathrm{I}_h u)]|_{0,S} &\lesssim \|\partial_{\hat\theta}^m(\hat{u} - \hat{\mathrm{I}}_h\hat{u})\|_{0;1/2,\hat{S}} \\
&= \inf_{\hat{w}\in\mathcal{P}_1} \|\partial_{\hat\theta}(\hat{u} - \hat{w} - \hat{\mathrm{I}}_h(\hat{u} - \hat{w}))\|_{0;1/2,\hat{S}} \\
&\lesssim \inf_{\hat{w}\in\mathcal{P}_1} \|\hat{u} - \hat{w}\|_{2;1/2+\beta,\hat{S}} \\
&\lesssim |\hat{v}|_{2;1/2+\beta,\hat{S}} \\
&\lesssim h_{\theta,S}^{1-\beta}[\theta^\beta u]_{2,S}.
\end{aligned}
$$

The derivation of (40), (41), is easily modified since all conclusions hold true also when $h_{\theta,T}$ is substituted by $h_{\theta,T}^{1-\beta}$, $\|\hat\theta^{-1}\partial_{\hat\varphi\hat\theta}\hat{u}\|_{0,\hat{T}}$ by $\|\hat\theta^{-1+\beta}\partial_{\hat\varphi\hat\theta}\hat{u}\|_{0,\hat{T}}$, $\|\hat\theta^{-2}\partial_{\hat\varphi\hat\varphi}\hat{u}\|_{0,\hat{T}}$ by $\|\hat\theta^{-2+\beta}\partial_{\hat\varphi\hat\varphi}\hat{u}\|_{0,\hat{T}}$ etc. $\qquad\square$

4.4   Approximation error estimates

The circular cone is the simplest case since Ω is a rectangle and the eigenfunctions are regular, see Subsection 3.2. Let $\{\mathcal{T}_h\}_h$ be a sequence of quasi-uniform triangular meshes. Each mesh is obtained from a rectangular mesh by dividing each rectangle into two triangles of diameter h, see Figure 3.

Theorem 26 *Assume that $u \in V$ is a vector function with $[u]_{2,\Omega} < \infty$. Then the interpolation error can be estimated on the mesh described above by*

$$\|u - \mathrm{I}_h u\|_V \lesssim h[u]_{2,\Omega}.$$
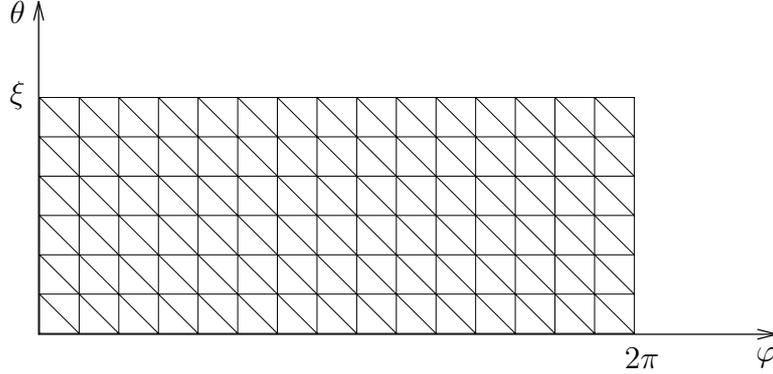
Figure 3: Illustration of the mesh for the conical domain.

Proof We apply the interpolation error estimates of Lemmata 21 – 23 and obtain

$$\|u - \mathrm{I}_h u\|_V^2 \sim \sum_{T \in \mathcal{T}_h} \sum_{m=0}^{1} [u - \mathrm{I}_h u]_{m,T}^2 \lesssim \sum_{m=0}^{1} (h^2 + h)[u]_{2,T}^2 \sim h[u]_{2,\Omega}^2$$

which is the desired result.   □

In the case of polyhedral vertices, the eigenfunctions have singularities itself, see Subsection 3.2. On uniform meshes we do not achieve the optimal convergence order, see Remark 28. Therefore, we use refined meshes as defined in Subsection 4.2.

Theorem 27 *Assume that $u \in V$ is a vector function with $\||u\||_{2,\beta,\Omega}$, $\beta \in [0,1)$, where $\|| \cdot \||_{2,\beta,\Omega}$ was introduced in (22). On the graded meshes introduced in Subsection 4.2, the interpolation error can be estimated by*

$$\|u - \mathrm{I}_h u\|_V \lesssim h \||u\||_{2,\beta,\Omega}.$$

Proof If a corner P_e is at position (φ_0, θ_0), $\theta_0 \notin \{0, \pi\}$, then the refined mesh in the neighbourhood of P_e consists of isotropic triangles only. For all elements T with $(\varphi_0, \theta_0) \in \overline{T}$ Lemma 24 is applied. By using $2 - m - \beta \geq 1 - \beta$ and (28) we obtain for $m = 0, 1$,

$$
\begin{aligned}
[u - \mathrm{I}_h u]_{m,T} &\lesssim& h_T^{1-\beta} \sum_{i+j=2} |[R_e^\beta \partial_\varphi^i \partial_\theta^j u]|_{0,T} \\
&\sim& h \sum_{i+j=2} |[R_e^\beta \partial_\varphi^i \partial_\theta^j u]|_{0,T} \\
&\lesssim& h \||u\||_{2,\beta,T}.
\end{aligned} \tag{43}
$$

For all other elements T in the refinement zone, $T \cap N_e \neq \emptyset$, we use Lemma 21, (28), and $\mathrm{dist}\,(T, P_e) < R_e$ for all points of T, and obtain for $m = 0, 1$,

$$
\begin{aligned}
[u - \mathrm{I}_h u]_{m,T} &\lesssim& h_T\,[u]_{2,T} \\
&\sim& h\,[\mathrm{dist}\,(P,T)]^\beta\,[u]_{2,T} \\
&\lesssim& h\,[R_e^\beta u]_{2,T} \\
&\lesssim& h\,\||u\||_{2,\beta,T}.
\end{aligned} \tag{44}
$$

Let now P_e be the north pole. Then for all elements T in a strip S as defined between Lemmata 22 and 23, Lemma 25 is applied. By using (29) and (30) we obtain, again for $m = 0, 1$,

$$
\begin{aligned}
[u - \mathrm{I}_h u]_{m,S} &\lesssim h_{\theta,S}^{1-\beta} \, [\theta^\beta u]_{2,S} \\
&\sim h \, [\theta^\beta u]_{2,S} \\
&\lesssim h \, \| u \|_{2,\beta,S}.
\end{aligned}
\tag{45}
$$

For all other elements T with $T \cap N_e \neq \emptyset$ we can use Lemma 22, (29), (30), $\theta_{-,T} \lesssim 1$, and $\theta_{-,T} \leq \theta$ for all points of T, and obtain for $m = 0, 1$,

$$
\begin{aligned}
[u - \mathrm{I}_h u]_{m,T} &\lesssim h_{\varphi,T} \theta_{-,T} \, |[\sin^{-2}\theta \, \partial_\varphi^2 u]|_{0,T} + \\
&\quad + (h_{\varphi,T} \theta_{-,T} + h_{\theta,T}) \, |[\sin^{-1}\theta \, \partial_\varphi \partial_\theta u]|_{0,T} + h_{\theta,T} \, |[\partial_\theta^2 u]|_{0,T} \\
&\lesssim h \, \theta_{-,T}^\beta \, [u]_{2,T} \\
&\lesssim h \, [\theta^\beta u]_{2,T} \\
&\lesssim h \, \| u \|_{2,\beta,T}.
\end{aligned}
\tag{46}
$$

If the south pole is also a singular corner of $\tilde{\Omega}$, then it is treated analogously. For elements T outside a refinement region N_e we use Lemma 21, 22 or 23, respectively, and obtain also

$$
[u - \mathrm{I}_h u]_{m,T} \lesssim h \, \| u \|_{2,\beta,T}.
$$

Combining all these estimates we obtain the desired result. $\square$

Remark 28 If the mesh is not refined, $h_T = h$ in (28) and $h_{\theta,T} = h$ in (30), then one can proceed analogously. However, we obtain in (43) – (46) only a local convergence order $h^{1-\beta}$ (using $h \leq h^{1-\beta} \, [\operatorname{dist}(P,T)]^\beta$ in (44) and (46)). That means that the global convergence order is

$$
1 - \max_e \beta_e = \min_e \operatorname{Re} \alpha_e - \varepsilon, \quad \varepsilon > 0 \text{ arbitrary,}
$$

since $\beta_e > 1 - \operatorname{Re} \alpha_e$ was assumed.

Corollary 29 *For the discretization as described in Subsection 4.2 the following error estimates for the eigenpairs $(\lambda_0, u(\lambda_0; \cdot))$ of (14) hold:*

$$
\begin{aligned}
|\lambda_h - \lambda_0| &\lesssim h^{2/\kappa} \, |\lambda_0|^{4/\kappa}, \\
|\hat{\lambda}_h - \lambda_0| &\lesssim h^2 \, |\lambda_0|^4, \\
\|u_h - u(\lambda_0; \cdot)\|_V &\lesssim h^\alpha \, |\lambda_0|^{2\alpha}, \quad \alpha = \min\{1, 2/\kappa\}.
\end{aligned}
$$

Proof Using Theorems 26 and 27 and defining

$$
\sigma(\lambda_0) := \max_{v \in W(\mathcal{B},\lambda_0), \|v\|_V = 1} \| v \|_{2,\beta,\Omega}, \qquad \sigma^*(\lambda_0) := \max_{v \in W(\mathcal{B}^*,\lambda_0), \|v\|_V = 1} \| v \|_{2,\beta,\Omega},
$$

we find that

$$
\begin{aligned}
\varepsilon_h &:= \max_{v\in W(\mathcal{B},\lambda_0),\|v\|_V=1} \varepsilon_h(v) \ \lesssim\ h\,\sigma(\lambda_0),\\
\varepsilon_h^* &:= \max_{v\in W(\mathcal{B}^*,\lambda_0),\|v\|_V=1} \varepsilon_h(v) \ \lesssim\ h\,\sigma^*(\lambda_0).
\end{aligned}
$$

There exist generalised eigenelements $w \in W(\mathcal{B},\lambda_0)$ and $w^* \in W(\mathcal{B}^*,\lambda_0)$, $\|w\|_V = \|w^*\|_V = 1$, such that

$$
\sigma(\lambda_0) = \vert\!\vert\!\vert\, w\,\vert\!\vert\!\vert_{2,\beta,\Omega}, \qquad \sigma^*(\lambda_0) = \vert\!\vert\!\vert\, w^*\,\vert\!\vert\!\vert_{2,\beta,\Omega}.
$$

Therefore we can conclude with Lemma 13 that

$$
\sigma(\lambda_0) \lesssim |\lambda_0|^2, \qquad \sigma^*(\lambda_0) \lesssim |\lambda_0|^2
$$

and, consequently,

$$
\varepsilon_h \lesssim h\,|\lambda_0|^2, \qquad \varepsilon_h^* \lesssim h\,|\lambda_0|^2.
$$

Using Theorems 18 – 20, we derive the desired estimates. $\square$

Remark 30 The number λ_0 does not depend on h and can thus be considered as a constant factor. But a large eigenvalue λ_0 diminishes the accuracy of its approximation.

5   Numerical tests

In order to confirm the theoretically predicted convergence order we test our method in a case where the the eigenvalues can be obtained by a simpler method to arbitrary precision, namely we consider the circular cone

$$
K = \{(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \in \mathbb{R}^3 : 0 \le \varphi < 2\pi, 0 < \theta < \xi\}
$$

and isotropic material. The values of $\lambda_{\min} = \lambda_{\min}(\xi)$ are calculated for the Poisson ratio $\nu = 0.3$ in [5, 6] for rotationally symmetric forces and in [43] for the general case. The eigenvalues satisfy a rather lengthy transcendental equation.

In the test we used a fifth order numerical integration rule with 7 points to generate the matrices. The eigenvalue problem is solved with SHIRA, the Skew-Hamiltonian Implicitly Restarted Arnoldi method [3, 37], with the shift value $\tau = 0.3$ and the convergence criterion $\epsilon = 10^{-12}$. The software was mainly written by Uwe Reichel and Cornelia Pester with the help of David S. Watkins. It makes use of various packages like ARPACK [28], Super-LU [12], LAPACK [1], as well as libraries of the Chemnitz SPC group for assembling matrices, graphics, memory management and other basic tasks [15, 41].

In Table 1 we display the smallest approximate eigenvalue and the error for various mesh sizes and angles $\xi \in \{90°, 120°, 150°\}$. From the results of two successive levels of refinement we compute the approximate convergence order.

We see that the relative error is nearly independent of the angle and the approximation order is surprisingly close to the predicted order two.

Angle	$(\pi h)^{-1}$	λ_{min}^h	error	approx. order
90	8	1.4917948929	8.20511 e−03	
	16	1.4978589231	2.14108 e−03	1.94
	32	1.4994719803	5.28020 e−04	2.02
	64	1.4998693746	1.30625 e−04	2.02
	128	1.4999674776	3.25224 e−05	2.01
	exact value	1.50000000		
120	8	1.0425845623 ± 0.0054051649 i	5.44107 e−03	
	16	1.0419631338 ± 0.0014806910 i	1.48069 e−03	1.88
	32	1.0419310513 ± 0.0003796528 i	3.80792 e−04	1.96
	64	1.0419564998 ± 0.0001091573 i	1.09230 e−04	1.80
	128	1.0419571416 ± 0.0000273475 i	2.75516 e−05	1.99
	exact value	1.0419604892		
150	8	0.8037192732 ± 0.0049080364 i	4.93810 e−03	
	16	0.8036384809 ± 0.0015122140 i	1.58161 e−03	1.64
	32	0.8032978815 ± 0.0003872941 i	4.06278 e−04	1.96
	64	0.8032079364 ± 0.0001010282 i	1.06218 e−04	1.94
	128	0.8031821396 ± 0.0000252988 i	2.62490 e−05	2.02
	exact value	0.80317514090		

Table 1: Example 1: Approximate eigenvalues, error and convergence order for various mesh sizes and angles.

As a second test we consider the Fichera corner. The domain can be described as the unit sphere where one octant is missing. A representation in the parameter plane is shown in Figure 4. This problem has been considered in the literature for isotropic material, for example in [44] for a single material and Dirichlet boundary conditions and in [13] for a bi-material joint and Neumann boundary conditions. We use also the bi-material joint but calculate the eigenvalues for the Dirichlet problem. The material is defined by

$$\nu = 0.3, \qquad E = \begin{cases} E_1 = 1.0 & \text{for } \theta < \frac{1}{2}\pi, \\ E_2 & \text{for } \theta > \frac{1}{2}\pi, \end{cases}$$

where E_2 is varied in the interval $[0.016, 64]$. The three smallest eigenvalues are plotted against E_2 in Figure 5.

The exact eigenvalues are not known in this case. Therefore we estimate the convergence order by calculating with three different meshes of mesh size h, $\frac{1}{2}h$, and $\frac{1}{4}h$. By assuming $\lambda_h - \lambda = C$, $\lambda_{h/2} - \lambda = C\,2^{-z}$ and $\lambda_{h/4} - \lambda = C\,2^{-2z}$ we obtain the estimated convergence order

$$z = \log_2 \frac{\lambda_h - \lambda_{h/2}}{\lambda_{h/2} - \lambda_{h/4}}.$$

Table 2 shows the results.
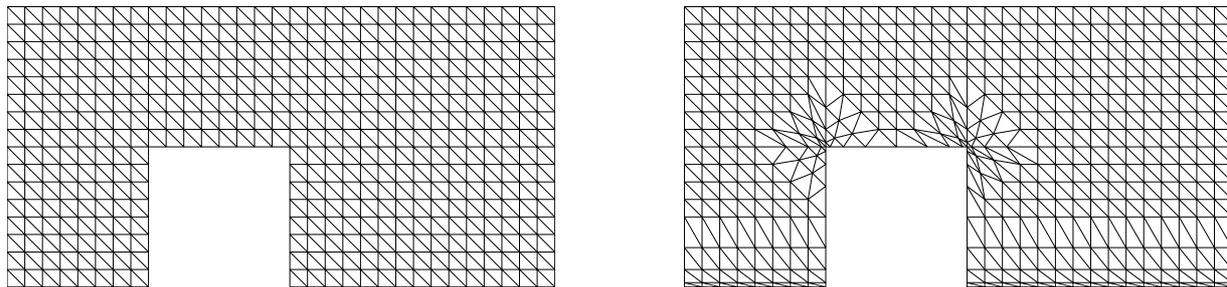
Figure 4: Fichera corner in the parameter plane: uniform mesh (left) and graded mesh (right)

We see that graded meshes deliver a more accurate eigenvalue approximation when the same number of unknowns is used. We see also that the estimated convergence order is significantly larger for graded meshes. The optimal convergence order is not yet reached. However, we are not able to do calculations with a still smaller mesh size.

The dependence of the constants on material parameters was not studied. Hence we cannot explain the deterioration when E_2 is increased. In further tests we observed the same behaviour for the second smallest eigenvalue but a slightly different dependence for the third eigenvalue. There, the maximum error order is attained for the single material case, $E_2 = 1$.
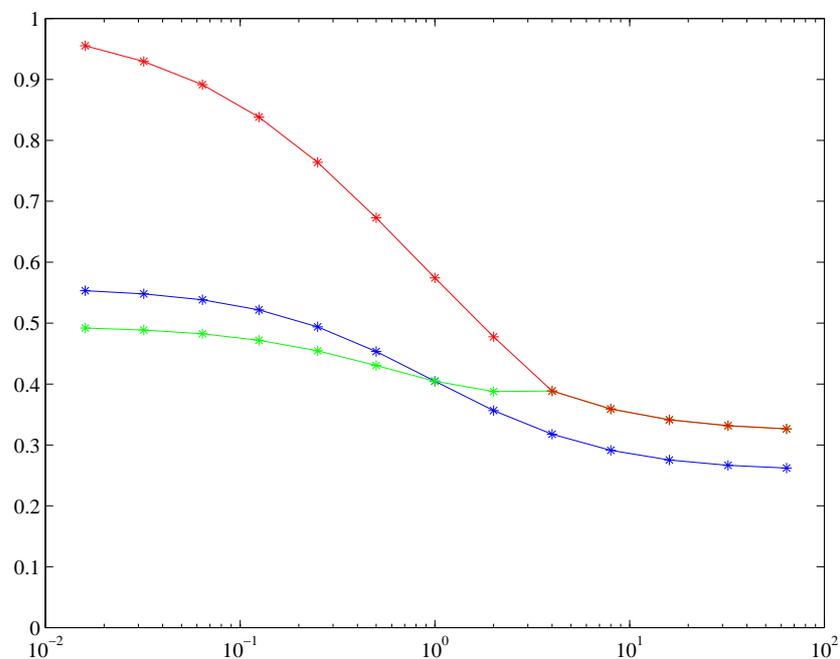
Figure 5: Three minimal eigenvalues for various material parameters

E_2	0.016	0.0625	0.025	1.0	4.0	16.0	64.0
$h = \pi/16$	0.5017	0.4920	0.4633	0.4123	0.3282	0.2880	0.2755
$h = \pi/32$	0.4957	0.4862	0.4581	0.4078	0.3223	0.2813	0.2685
$h = \pi/64$	0.4933	0.4839	0.4560	0.4059	0.3196	0.2781	0.2650
$h = \pi/128$	0.4924	0.4829	0.4551	0.4051	0.3184	0.2765	0.2633
est. order	1.3354	1.3274	1.2951	1.2396	1.1418	1.0420	1.0062
$h = \pi/16$	0.4974	0.4878	0.4598	0.4096	0.3262	0.2857	0.2729
$h = \pi/32$	0.4933	0.4838	0.4560	0.4060	0.3203	0.2788	0.2656
$h = \pi/64$	0.4922	0.4827	0.4549	0.4049	0.3183	0.2762	0.2629
$h = \pi/128$	0.4919	0.4824	0.4546	0.4046	0.3177	0.2754	0.2620
est. order	1.9343	1.9349	1.9062	1.8556	1.7092	1.6233	1.5828

Table 2: Approximate minimum eigenvalues for different mesh sizes and various material parameters, for both uniform (above) and graded meshes (below).

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