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Numerische Simulation auf massiv parallelen Rechnern

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Stability of discretizations of the Stokes problem on anisotropic meshes

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Abstract Anisotropic features of the solution of flow problems are usually approximated on anisotropic (large aspect ratio) meshes. This paper reviews stability results of several velocity-pressure pairs with respect to growing aspect ratio of the elements in the mesh. For further pairs numerical tests are described. Related results are mentioned.

Key Words Stokes problem, edge singularity, anisotropic mesh, stable discretization

AMS(MOS) subject classification 65N30; 65N12

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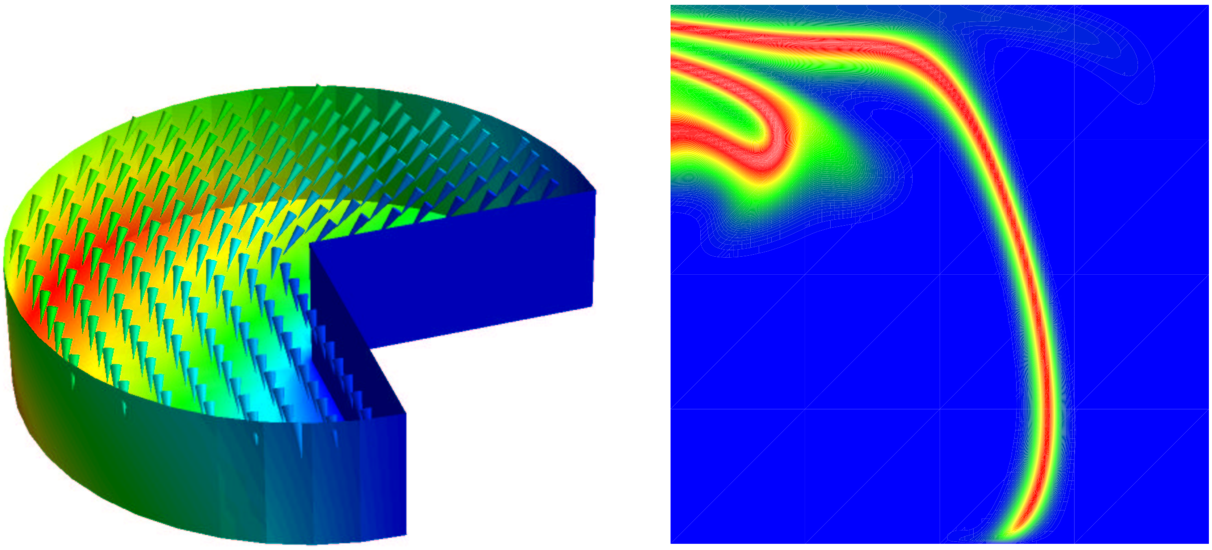

Figure 1: Illustration of a flow around an edge and of interior layers

1 Introduction

In the simulation of viscous flow problems we encounter anisotropic phenomena in the solution near edges or in layers, see Figure 1 for an illustration. Anisotropic solutions should be approximated on appropriate meshes which are usually also anisotropic. This means that the meshes contain elements with huge aspect ratio. Using isotropic (shape regular) meshes instead would lead to overrefinement.

In order to guarantee the stability of mixed methods, the approximating spaces must satisfy the well-known inf-sup condition. In our context, it is important that the inf-sup constant does not tend to zero when the meshes become anisotropic.

Consider the Stokes problem which is to find $u \in X = H_0^1(\Omega)^d$ and $p \in M = L_0^2(\Omega)$

$$a(u, v) + b(v, p) = (f, v) \quad \forall v \in X, \qquad b(u, q) = 0 \quad \forall q \in M, \tag{1}$$

with $a(u, v) = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i$ and $b(u, q) = -\int_\Omega q \operatorname{div} u$. Let $\mathcal{T}_h$ be an admissible mesh of elements, and X_h, M_h, finite element spaces corresponding to $\mathcal{T}_h$. Define by

$$\gamma_h := \inf_{0 \neq p_h \in M_h} \sup_{0 \neq u_h \in X_h} \frac{b(u_h, p_h)}{|u_h|_1 \|p_h\|_0}$$

the inf-sup constant with the usual modification for non-conforming methods. The inf-sup condition requires this constant to be bounded uniformly away from zero,

$$\exists \gamma > 0 : \quad \gamma_h > \gamma \quad \forall h > 0. \tag{2}$$

It is important for the stability of the method. Several pairs of finite element spaces (X_h, M_h) satisfying this condition are known for isotropic meshes, see, e. g., Brezzi and Fortin (1991) for an overview.

The aim of this paper is to review results for anisotropic meshes and to add new ones. We split the exposition into two parts and consider first non-conforming methods in Section 2, and then conforming methods in Section 3. In a short fourth section we point to some results concerning the difficulties with adaptive mesh generation and resolving the system of finite element equations when anisotropic meshes are used.

The notation $a \lesssim b$ means the existence of a positive constant C (which is independent of $\mathcal{T}_h$) such that $a \leq Cb$.

2 Nonconforming methods

We start with a review of results of Apel, Nicaise, and Schöberl (2001b, 2000) for triangular and tetrahedral meshes, namely for the pair of the Crouzeix-Raviart element for the velocity,

$$X_h := \{v_h \in L^2(\Omega)^3 : v_h|_K \in (\mathcal{P}_1)^3 \ \forall K, \int_F [v_h] = 0 \ \forall F\},$$

and the piecewise constants for the pressure,

$$M_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_0 \ \forall K, \int_\Omega q_h = 0\}.$$

Here, K and F denote finite elements and faces of them, respectively. At the end of the section we comment on rectangular and prismatic elements.

Since $X_h \not\subset X$ we use the weaker bilinear forms

$$a_h(u,v) := \sum_K \sum_{i=1}^{3} \int_K \nabla u_i \cdot \nabla v_i, \quad b_h(u,q) := -\sum_K \int_K q \operatorname{div} u,$$

to define the approximate solution $(u_h \times p_h) \in X_h \times M_h$ of (1) by

$$\begin{aligned} a_h(u_h,v_h) + b_h(v_h,p_h) &= (f,v_h) \quad \forall v_h \in X_h, \\ b_h(u_h,q_h) &= 0 \qquad\quad \forall q_h \in M_h. \end{aligned}$$

The bilinear form $a_h(.,.)$ defines a broken H^1-norm via $\|.\|_{1,h}^2 := a_h(.,.)$.

For the analysis of this method the interpolant $\mathrm{I}_h : X \to X_h$ is used which is defined elementwise by

$$\int_F u = \int_F \mathrm{I}_h u \qquad \forall F \subset \partial K, \forall K \in \mathcal{T}_h.$$

This interpolant has nice local properties. To describe them consider a finite element T with element sizes as illustrated in Figure 2. Then the estimates

$$\int_T \nabla u = \int_T \nabla \mathrm{I}_h u \qquad \forall u \in H^1(T), \tag{3}$$

$$|\mathrm{I}_h u|_{1,T} \leq |u|_{1,T} \qquad \forall u \in H^1(T), \tag{4}$$

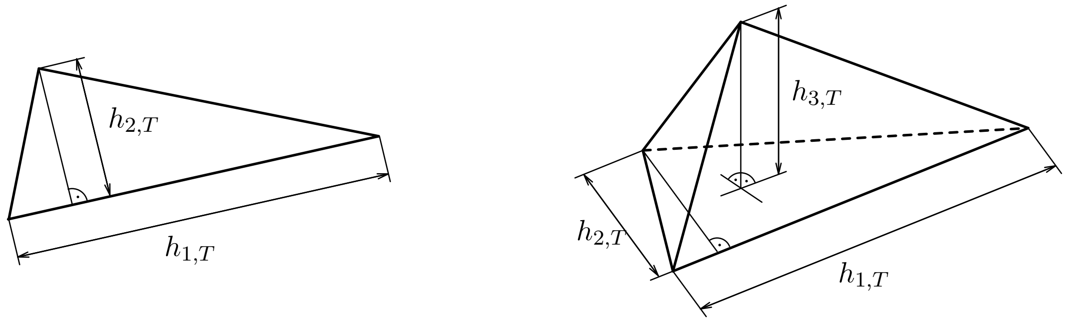

Figure 2: Illustration of the element sizes of an anisotropic finite element

hold for general elements (Apel, Nicaise, and Schöberl, 2001b), without demanding a maximal angle condition (e.g. Apel, 1999). The approximation error estimate

$$|u - \mathrm{I}_h u|_{1,T} \lesssim \sum_{i=1}^{d} h_{i,T} |\partial_i u|_{1,T} \qquad \forall u \in H^2(T), \tag{5}$$

$\partial_i := \partial/\partial x_i$, proved also in (Apel, Nicaise, and Schöberl, 2001b), holds under the assumption of the maximal angle condition and the coordinate system condition (e.g. Apel, 1999). It is worth mentioning that this estimate does not hold for the conforming $\mathcal{P}_1$ element in three dimensions, see Apel and Dobrowolski (1992) and Apel (1999).

Due to (3), (4), I_h can be used as the Fortin operator to show the inf-sup condition (2) by the standard proof (e.g. Brezzi and Fortin, 1991). Again, this estimate holds on general meshes, there is no bound on the aspect ratio, no maximal angle condition. This was independently pointed out by Acosta and Durán (1999) and Apel et al. (2000).

Apel, Nicaise, and Schöberl (2000) investigated carefully the flow around one concave edge of a prismatic domain $\Omega = G \times Z$, $G \subset I\!R^2$, $Z \subset I\!R^1$, see Figure 1 for an illustration. The Cartesian coordinate system is such that the edge is part of the x_3-axis. First, the regularity of the solution was investigated resulting in the estimate

$$\|u\|_{2;\beta} + \|\partial_3 u\|_{1;0} + \|p\|_{1;\beta} + \|\partial_3 p\|_0 \lesssim \|f\|_0, \tag{6}$$

where $\|v\|^2_{\ell;\beta} := \sum_{i+j+k\le\ell} \|r^{\beta-\ell+i+j+k}\partial_1^i\partial_2^j\partial_3^k u\|_0$, $r = r(x) := (x_1^2 + x_2^2)^{1/2}$, $\beta \in (1-\lambda, 1)$ is arbitrary, $\lambda \in (1/2, \pi/\omega)$ is the smallest positive solution of the transcendental equation $\sin(\lambda\omega) + \lambda \sin\omega = 0$, and ω is the interior angle of the edge. A closer look on (6) reveals that the derivatives of u and p in x_3-direction are regular. Other derivatives need to be weighted by a power of r in order to be square integrable.

Appropriate anisotropic finite element meshes for the treatment of elliptic problems in domains with edges are known from previous work, see, e. g., Apel and Dobrowolski (1992) and Apel (1999). For an illustration see Figure 3. For proving an approximation result, further local estimates were derived. For the local L^2-projector $\mathrm{M}_h : L^2(\Omega) \to M_h$ defined elementwise by $\mathrm{M}_h v|_T := (\mathrm{meas}_3 T)^{-1} \int_T v$ there holds

$$\|p - \mathrm{M}_h p\|_{0,T} \lesssim \sum_{i=1}^{3} h_{i,T} \|\partial_i p\|_{0,T} \qquad \forall p \in H^1(T).$$

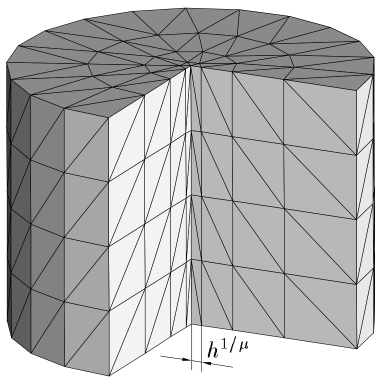

Figure 3: Anisotropic mesh near an edge

Moreover, for elements T touching the edge the estimates

$$\begin{aligned}\|\partial_j(u - \mathrm{I}_h u)\|_{0,T} &\lesssim \sum_{i=1}^{2} h_{i,T}^{1-\beta} \|\partial u\|_{1;\beta,T} + h_{3,T} |\partial_3 u|_{0;\beta,T}, \\ \|p - \mathrm{M}_h p\|_{0,T} &\lesssim h_{1,T}^{1-\beta} \|p\|_{1;\beta,T},\end{aligned}$$

are valid. From these local estimates and by using (6), the global estimate

$$\|u - \mathrm{I}_h u\|_{1,h} + \|p - \mathrm{M}_h p\|_0 \lesssim h\|f\|_0 \tag{7}$$

is derived, where h is the global mesh size being of order $N^{-1/3}$ where N is the global number of unknowns.

The most difficult part is the estimation of the consistency error. Apel et al. (2000) finally show that

$$\sup_{v_h \in V_h} \frac{|a_h(u, v_h) + b_h(v_h, p) - (f, v_h)|}{\|v_h\|_{1,h}} \lesssim h\,\|f\|_0, \tag{8}$$

where $V_h := \{v_h \in X_h : \operatorname{div} v_h|_K = 0\ \forall K\}$. So we can conclude the approximation error estimate

$$\|u - u_h\|_{1,h} + \|p - p_h\|_0 \lesssim h\,\|f\|_0.$$

Until now, we considered prismatic domains $\Omega = G \times Z$. This simplifies the structure of the solution enormously, since no additional corner singularities appear. In a general polyhedral domain, however, we have to treat both edge and corner singularities. The mesh should be refined towards corners and, anisotropically, towards edges. Apel and Nicaise (1998) investigate such meshes and derive error estimates for the Poisson problem. For the Stokes problem with Crouzeix-Raviart elements for the velocity and piecewise constants for the pressureX_h and M_h as defined above, the inf-sup condition (2) holds for general meshes. All the ideas for proving the approximation error estimate (7) are contained in the papers Apel and Nicaise (1998) and Apel et al. (2000). The critical point is the consistency error. The proof of (8) by Apel et al. (2001b; 2000) heavily relies on the tensor product structure of the mesh. The generalization is an open problem.

The rectangular pendant to the Crouzeix-Raviart element is the rotated Q_1 element (e. g. Rannacher and Turek, 1992). For solving the Stokes problem we can use this element for the velocity and again piecewise constants for the pressure. To be specific, consider a rectangle $(0,H) \times (0,h)$ with $h = o(H)$ in the Cartesian coordinate system (x_1, x_2) and the reference element $(0,1)^2$ in the coordinate system $(\hat{x}_1, \hat{x}_2)$. The polynomial spaces of the parametric and the non-parametric rotated Q_1 elements are

$$\begin{aligned} Q_{1,\mathrm{rot}}^{\mathrm{par}} &= \mathcal{P}_1 \oplus \operatorname{span}\{\hat{x}_1^2 - \hat{x}_2^2\} = \mathcal{P}_1 \oplus \operatorname{span}\{(h/H)^2 x_1^2 - x_2^2\}, \\ Q_{1,\mathrm{rot}}^{\mathrm{non}} &= \mathcal{P}_1 \oplus \operatorname{span}\{x_1^2 - x_2^2\} = \mathcal{P}_1 \oplus \operatorname{span}\{\hat{x}_1^2 - (h/H)^2 \hat{x}_2^2\}, \end{aligned}$$

respectively. Rannacher and Turek (1992) mention that the parametric variant of this element is not stable on anisotropic grids. Becker and Rannacher (1995) prove that the non-parametric variant is stable. Consistency error estimates, however, were not derived for anisotropic meshes.

Apel et al. (2000) suggest a modification, namely

$$Q_{1,\mathrm{mod}} = \mathcal{P}_1 \oplus \operatorname{span}\{x_1^2\} = \mathcal{P}_1 \oplus \operatorname{span}\{\hat{x}_1^2\}.$$

The analogon for prismatic elements $T = D \times (0,H)$, where D is an isotropic (shape regular) triangle of diameter h, is $\mathcal{P}_1 \oplus \operatorname{span}\{x_3^2\}$. These elements are also stable, allow to prove the interpolation error estimates (4), (5), and also the consistency error estimate.

3 Conforming methods

Becker (1995) analyzes stabilized $\mathcal{Q}_1 - \mathcal{P}_0$ and $\mathcal{Q}_1 - \mathcal{Q}_1$ elements on anisotropic rectangular meshes and proves the inf-sup condition with a constant independent of the aspect ratio.

Schötzau, Schwab, and Stenberg (1998; 1999) consider quadrilateral and triangular elements for the *hp*-version of the finite element method, in particular combinations $\mathcal{Q}_k - \mathcal{Q}_{k-2}$ and $\mathcal{P}_k - \mathcal{P}_{k-2}$, $k \geq 2$. For layer patches as in Figure 4 the inf-sup constant does not depend on the aspect ratio, but slightly on k^{-1} ($k^{-1/2}$ for the quadrilaterals and k^{-3} for the triangles). This is compensated by the exponentially good approximation. Near corners the meshes must be treated carefully, see the discussion below on the special case of the $\mathcal{Q}_2 - \mathcal{Q}_0$ pair. Schötzau, Schwab, and Stenberg (1999) advice to use geometric tensor product meshes near corners and prove stability for this.

Ainsworth and Coggins (2000) refine these results by considering smaller velocity spaces $\mathcal{Q}_{k+\max\{\mu k,1\},k}$ where $\mu \in [0,1]$ is a fixed constant. The pressure space is $\mathcal{P}_{k-1}$. They have shown that on the macroelement $(-1,1)^2$ discretized with any layer mesh $\mathcal{T}_x \times (0,1)$, the stability constant has a lower bound

$$\gamma = \begin{cases} Ck^{-1/2} & \text{for } \mu = 0, \\ C(\mu)(1 + \ln^{1/2} k)^{-1} & \text{for } \mu > 0, \end{cases}$$

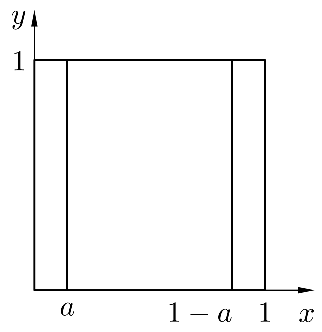

Figure 4: Illustration of a layer mesh

where C is independent of k and the aspect ratio. Near corners, they do not consider a geometric tensor product mesh like in the paper of Schwab. Instead, they use the macroelement $(1, 1+\rho)^2$ divided into 4 elements T_i, $i = 1, \dots, 4$, and they assume the polynomial degree distributions:

$$(k_x; k_y) = \begin{cases} (k; k) & \text{in } T_1 = (0,\rho)^2, \\ (k + \max(\mu k, 1); k) & \text{in } T_2 = (\rho, 1+\rho) \times (0,\rho), \\ (k + \max(\mu k, 1); k + \max(\mu k, 1)) & \text{in } T_3 = (\rho, 1+\rho)^2, \\ (k; k + \max(\mu k, 1)) & \text{in } T_4 = (0,\rho) \times (\rho, 1+\rho). \end{cases}$$

For this configuration, They found that the inf-sup constant for $\mathcal{Q}_{k_x,k_y} - \mathcal{P}_{k-1}$ has a lower bound

$$\gamma = \begin{cases} Ck^{-1/2} \min(1, k\sqrt{\rho}) & \text{for } \mu = 0, \\ C(\mu)(1 + \log^{1/2} k)^{-1} \min(1, k\sqrt{\rho}) & \text{for } \mu > 0. \end{cases}$$

For a general polygonal domain Ω, the mesh can therefore be constructed by using layer and corner macroelements.

Toselli and Schwab (2001) generalize the 2D results for the $\mathcal{Q}_k - \mathcal{Q}_{k-2}$ pair to the 3D case. First, they consider meshes which are geometrically refined towards the faces (geometric boundary layer meshes). Second, they treat edge singularities with special geometric edge meshes where also hanging nodes may occur. They proved for both cases a lower bound of the inf-sup constant in the form Ck^{-1} where C depends on the geometric grading factor but not on the aspect ratio.

As a particular case of $\mathcal{Q}_k - \mathcal{Q}_{k-2}$, Schötzau, Schwab, and Stenberg (1999) consider the $\mathcal{Q}_2 - \mathcal{P}_0$ rectangular element. Though this is stable on layered meshes (as in Figure 4) it is instable for certain corner grids (Figure 5, left hand side). We reproduce in Figure 5, right hand side, the inf-sup constants γ_h when the parameter a is varying in the interval $(0, 0.5)$. This example shows that stability cannot be proved in the general case and that one has to treat anisotropic meshes carefully.

The experience with the Taylor-Hood pair $\mathcal{P}_2 - \mathcal{P}_1$ is similar to $\mathcal{Q}_2 - \mathcal{P}_0$. On a layered mesh we found stability, see Figure 6. Negative examples are shown in Figures 7 and 8. If we keep some geometrical properties of the meshes but enlarge the number of elements, the pair may become stable, see Figure 9, or it may not, see Figure 10.

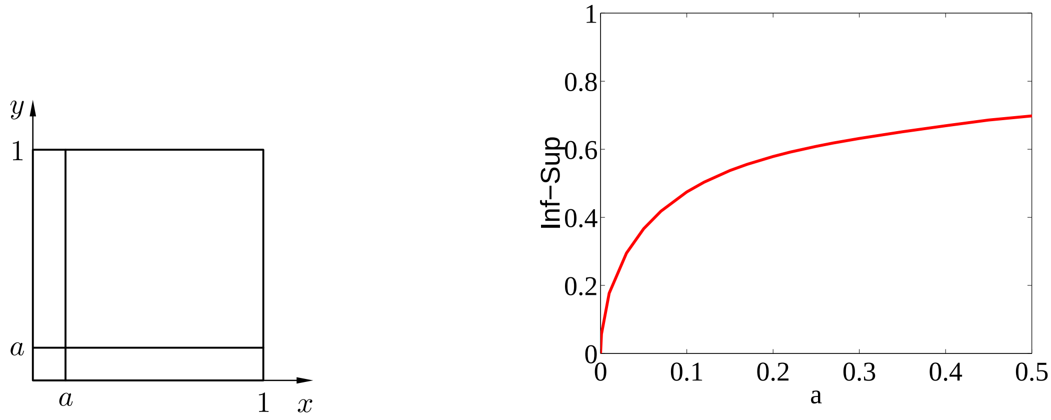

Figure 5: Corner mesh and the corresponding inf-sup constants γ_h for $\mathcal{Q}_2 - \mathcal{P}_0$ as a function of a

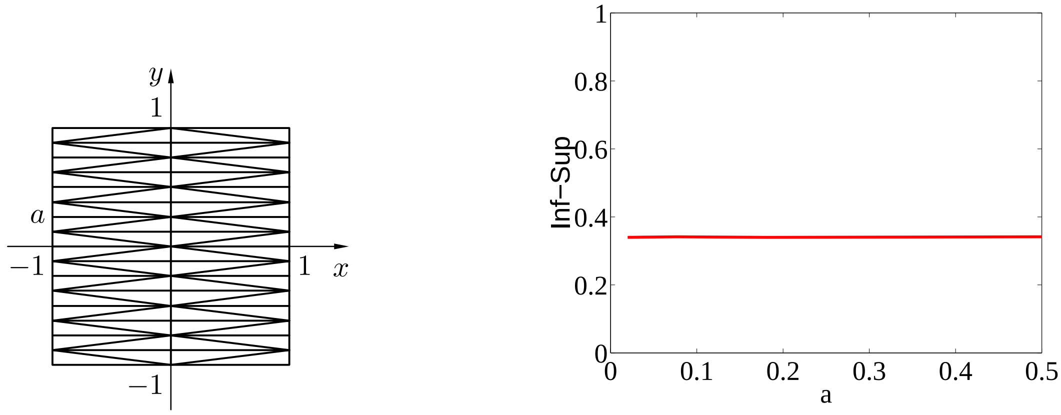

Figure 6: Layer mesh and the corresponding inf-sup constants γ_h for $\mathcal{P}_2 - \mathcal{P}_1$ as a function of a

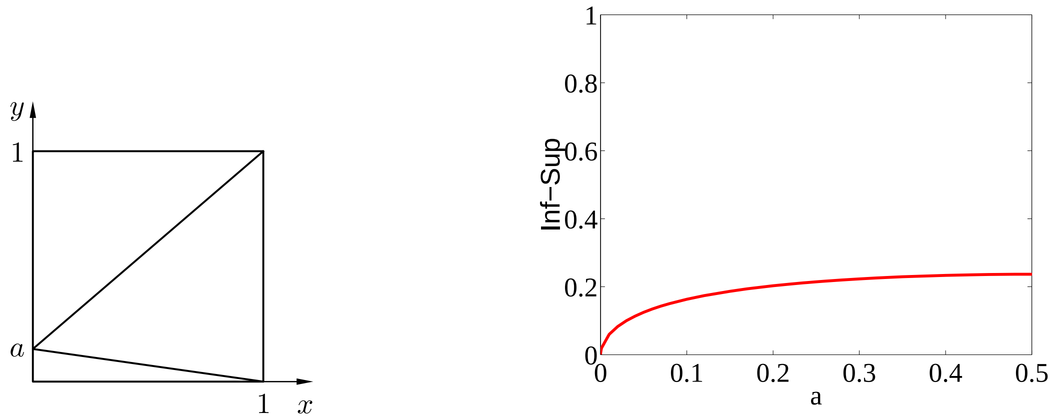

Figure 7: Simple mesh and the corresponding inf-sup constants γ_h for $\mathcal{P}_2 - \mathcal{P}_1$ as a function of a

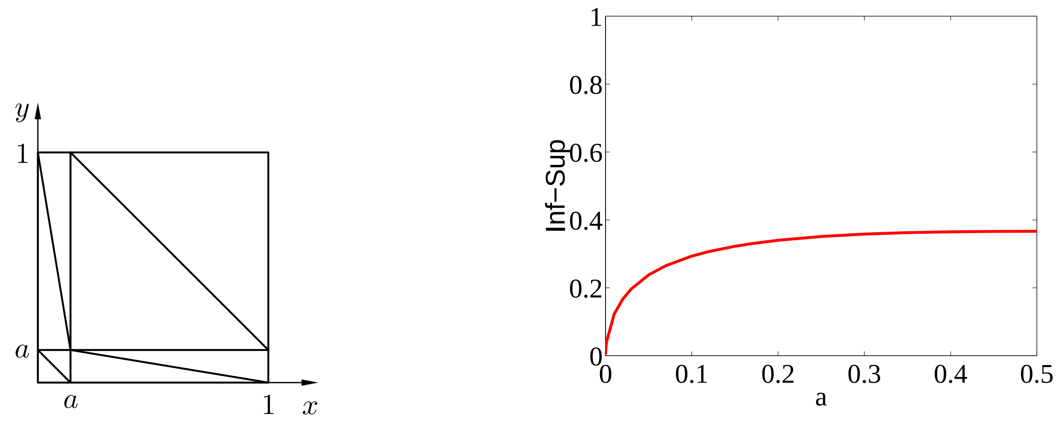

Figure 8: Corner mesh and the corresponding inf-sup constants γ_h for $\mathcal{P}_2 - \mathcal{P}_1$ as a function of a

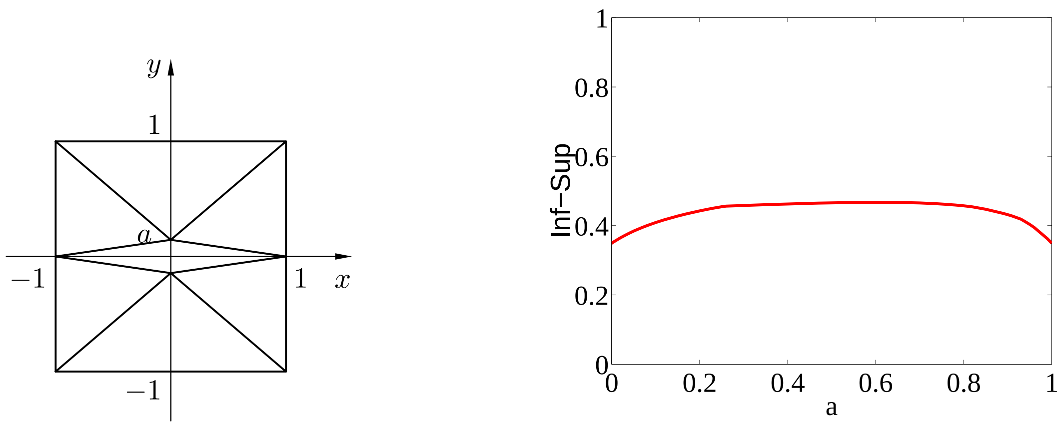

Figure 9: Extended simple mesh and the corresponding inf-sup constants γ_h for $\mathcal{P}_2 - \mathcal{P}_1$ as a function of a

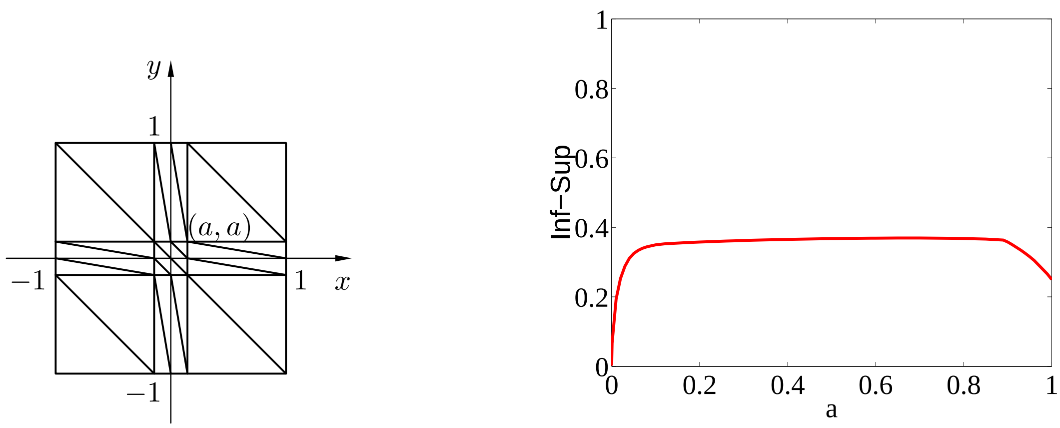

Figure 10: Extended corner mesh and the corresponding inf-sup constants γ_h for $\mathcal{P}_2 - \mathcal{P}_1$ as a function of a

If we enlarge the velocity space by a bubble functions per element and consider the pair $\mathcal{P}_2^+ - \mathcal{P}_1$ then we find of course stability in all cases where already the pair $\mathcal{P}_2 - \mathcal{P}_1$ is stable. Additionally, stability is obtained for some meshes where the pair $\mathcal{P}_2 - \mathcal{P}_1$ is unstable. For instance, the family of simple meshes from Figure 7 and the corner meshes from Figure 8 led to inf-sup constants $\gamma_h > 0.5$ uniformly in the parameter a.

Finally, Acosta and Durán (1999) write that Russo reported instabilities of the mini element on anisotropic meshes. This was supported also by our tests. On layer meshes as in Figure 6 we found $\gamma_h \sim a$.

4 Further results

In the previous sections we reviewed stability results for the Stokes problem which are the basis also for an effective discretization of the Navier-Stokes equations. For reliable computations, however, we need also an appropriate error estimator. Local a-posteriori error estimators for discretizations of the Stokes problem on isotropic meshes are known from the literature, see, for example, papers by Verfürth (1989, 1991, 1996), Ainsworth and Oden (1997), and Kay and Silvester (1999). These estimators have been investigated under the assumption of a bounded aspect ratio of the elements and it is not clear yet, how they depend on the aspect ratio. On the other hand, Kunert (1999, 2000a, 2001a,b) and Kunert and Verfürth (2000) investigate error estimators which work also on anisotropic meshes efficiently and reliably. But these investigations are still restricted to scalar equations with self-adjoint elliptic operators (Poisson problem, reaction-diffusion problem). First results for the Stokes problem on anisotropic meshes have been derived by Randrianarivony (2001).

An adaptive discretization needs not only an efficient and reliable error estimator but also the derivation of local directions in which the elements should be stretched, and of the optimal aspect ratio. There is one main approach known from the literature, namely the use of eigenvalues and eigenvectors of an approximated Hessian (the matrix of second derivatives) of the solution or one component of a vector valued solution. This goes back to Peraire et al. (1987) and has subsequently been refined by many authors including D'Azevedo and Simpson (1989, 1991), Zienkiewicz and Wu (1994), Castro-Diaz et al. (1995), Ait-Ali-Yahia et al. (1996), Dolejší (1998, 2001), and Kunert (2000b). In other applications the stretching direction is determined from the data, for example from the streamlines in convection-diffusion problems, see Skalický and Roos (1999). One can also try to detect internal layers or shocks by analyzing the gradient (or gradient jump) of some values, see Zienkiewicz and Wu (1994). In a further approach, Apel, Grosman, Jimack, and Meyer (2001a) introduce a new strategy for controlling the use of anisotropic mesh refinement based upon the gradients of an a posteriori approximation of the error in a computed finite element solution.

The discretization of the Stokes problem leads to a symmetric indefinite system of equations. For the resolution we suggest to use a modification of the conjugate gradient method. Bramble and Pasciak (1988) show that this system of equations can be viewed as a positive definite system in an appropriate scalar product, see also Meyer and Steidten

(2001) for further discussions. This methods is efficient when an optimal preconditioner for the Laplace operator on the mesh under consideration is available.

Multi-grid methods have been suggested and analyzed for anisotropic problems with tensor product structure. One approach is to take care of the strong connections by properly designed line or plane smoothers (e. g. Wittum, 1989; Hackbusch, 1989; Stevenson, 1993; Bramble and Zhang, 2001), another is to build up the hierarchy of triangulations by semi-coarsening (e. g. Zhang, 1995; Griebel and Oswald, 1995; Margenov et al., 1995). For conforming discretizations near edges, Apel and Schöberl (2000) suggest to use a multi-grid method with semi-coarsening perpendicularly to the edge combined with a line smoother in the orthogonal direction. A similar idea was proposed by Börm and Hiptmair (1999). The difference is that these authors consider a certain class of singularly perturbed problems, and suggest to use semi-coarsening with respect to the "harmless" coordinate and line relaxation in the direction of the singular perturbation.

Reviewing all these papers one can state that optimized meshes are often not hierarchical and can therefore not be used for the most efficient solver techniques. It is one of the main challenges to bridge this gap.

5 Summary

This work focuses on the stability of velocity-pressure pairs on anisotropic meshes for both non-conforming and conforming methods. First, we reviewed results on stability, consistency and approximation for the Crouzeix-Raviart $-\mathcal{P}_0$ pair and some related results for non-conforming quadrilateral elements. Next, we recapitulated some stability results of some conforming pairs that are known so far, again with special emphasis on anisotropic meshes. We provided a example for the instability of the mini element. Although the Taylor-Hood pair is not always stable on anisotropic grids we point out that its behaviour is satisfactory in some cases. Special features are seen for the $\mathcal{P}_2^+ - \mathcal{P}_1$ pair which was in all tests stable. Theoretical investigations of these pairs are still to be done.

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