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Numerische Simulation auf massiv parallelen Rechnern

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Strengthened Cauchy inequality in anisotropic meshes and application to an a-posteriori error estimator for the Stokes problem

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Abstract

In this document, we show an a-posteriori error estimator which is efficient and reliable even on highly stretched meshes for the Crouzeix-Raviart/$\mathcal{P}_0$ pair. It relies on hierarchical space splitting whose main ingredient is the strengthened Cauchy-Schwarz inequality. We demonstrate a method to enrich the Crouzeix-Raviart element so that the strengthened Cauchy constant is always bounded away from 1 independently of the aspect ratio. The performance of the a-posteriori error estimator is confirmed by our numerical results.

Keywords: anisotropic mesh, Stokes problem, strengthened Cauchy, error estimator,

AMS: 65N30, 65N15, 65N50

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1 Introduction

Anisotropic meshes have gained attention in practical as well as theoretical problems over the past twenty years (See among many others [Ape99], [ANSth], [Rac97], [KV00]) because they are very efficient in many practical situations such as problems presenting boundary or internal layers.

In contrast to several other velocity-pressure pairs ([Ran01]), the Crouzeix-Raviart/$\mathcal{P}_0$ is known ([ANSal], [AD99]) to be unconditionally stable on any anisotropic mesh. Unfortunately, this flexibility of the Crouzeix-Raviart/$\mathcal{P}_0$ pair cannot yet be fully exploited in adaptive FEM for the Stokes problem due to the lack of a-posteriori error estimator (APEE later) which functions in anisotropic grids. APEE's permit to evaluate the FE-errors without knowing the exact solution. That feature makes it possible to dynamically identify regions in the domain where one should have further refinement if the error there is too large and therefore adaptive refinements are mainly based on the quality of APEE's.

For isotropic grids, many different APEE's have been already proposed for the Stokes problem, see [Ver89], [Ver91], [Ain97],[JL00] and many others.

In the context of anisotropic meshes, there are already a variety of APEE's for Poisson and reaction-diffusion problems ([Kun97],[KV00],[DGP99], [Kun00], [Kun01]). But for the Stokes equation, APEE's have not yet been inspected in anisotropic meshes.

In section 2, we recall the definition of an anisotropic mesh and we introduce various definitions and notations. Section 3 treats exclusively the strengthened Cauchy-Schwarz inequality for the Crouzeix-Raviart element in stretched grids. We detail the APEE in section 4 and our theory is supported by numerical examples in the last section.

2 The model problem and notations

2.1 The Stokes problem

The Stokes problem consists of searching for the velocity $\mathbf{u} = (u_1, u_2) \in V := H_0^1(\Omega)^2$ and the pressure $p \in Q := L_0^2(\Omega)$ such that:

$$\left\{ \begin{array}{rcll} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) & = & (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in V \\ (\operatorname{div} \mathbf{u}, q) & = & 0 & \forall q \in Q \text{ , where} \end{array} \right. \tag{1}$$

$$H_0^1(\Omega) := \left\{ v \in H^1(\Omega) : \ v = 0 \,\text{on}\, \partial\Omega \right\} , \tag{2}$$

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \ \int_\Omega q = 0 \right\} . \tag{3}$$

2.2 Anisotropic mesh

Definition 1 An anisotropic mesh $\mathcal{T}_h$ is a set of disjoint triangles such that:

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}, \quad \text{and} \tag{4}$$

every edge of any element $T_i \in \mathcal{T}_h$ is *either* a part of the boundary $\partial\Omega$ *or* an edge of another element T_j of $\mathcal{T}_h$.

Remark 1 *For a triangle T, we denote*

$$\begin{aligned} h(T) &:= \textit{diam}(T) = \sup\{\|\mathbf{x} - \mathbf{y}\|_{\mathbf{R}^2},\, \mathbf{x}, \mathbf{y} \in T\} \\ \rho(T) &:= \textit{supremum of the diameters of all balls contained in } T \\ \sigma(T) &:= h(T)/\rho(T) = \textit{aspect ratio of } T, \end{aligned}$$

We require neither uniformity condition to $\mathcal{T}_h$ nor shape regularity for each triangle $T \in \mathcal{T}_h$. That means that all elements T of $\mathcal{T}_h$ are allowed to have an arbitrary aspect ratio $\sigma(T)$. We do not require that the aspect ratio $\sigma(T)$ is bounded. Since we do not put any angle requirement, very thin triangles are allowed to belong to $\mathcal{T}_h$. We will denote by $\partial\mathcal{T}_h$ the set of all edges of elements in the mesh $\mathcal{T}_h$ and

$$\sigma_h = \max_{T \in \mathcal{T}_h} \sigma(T).$$

2.3 Crouzeix-Raviart/$\mathcal{P}_0$ pair

We approximate the velocity and the pressure in the following discrete spaces:

$$\begin{aligned} V_h &:= \left\{ \mathbf{v}_h \in L^2(\Omega)^2 :\, \mathbf{v}_h|_T \in (P_1)^2\, \forall\, T \in \mathcal{T}_h, \text{ and} \int_F [\mathbf{v}_h] = 0\ \forall F \in \partial\mathcal{T}_h \right\}, \\ Q_h &:= \left\{ q_h \in L_0^2(\Omega) :\ q_h|_T \in P_0 \quad \forall\, T \in \mathcal{T}_h \right\}, \end{aligned}$$

where $[\mathbf{v}_h]$ stands for the jump of $\mathbf{v}_h$ across the edge F if F is an internal edge, and it is equal to $\mathbf{v}_h$ itself if F is a boundary edge. For all $\mathbf{u}, \mathbf{v} \in V_h$ and $q \in Q_h$, we define

$$a_T(\mathbf{u}, \mathbf{v}) := \sum_{j=1}^{2} \int_T \mathbf{grad}\, u_j \cdot \mathbf{grad}\, v_j\,, \quad b_T(\mathbf{v}, q) := \int_T q \operatorname{div} \mathbf{v}\,, \text{ and}$$

$$a_h(\mathbf{u}, \mathbf{v}) := \sum_{T \in \mathcal{T}_h} a_T(\mathbf{u}, \mathbf{v})\,, \quad b_h(\mathbf{v}, q) := \sum_{T \in \mathcal{T}_h} b_T(\mathbf{v}, q)\,.$$

The discrete problem deals with finding $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that:

$$\left\{ \begin{array}{rcll} a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) & = & (\mathbf{f}, \mathbf{v}_h), & \forall\, \mathbf{v}_h \in V_h \\ b_h(\mathbf{u}_h, q_h) & = & 0 & \forall\, q_h \in Q_h, \end{array} \right. \tag{5}$$

Let us introduce the broken Sobolev space:

$$\mathcal{H} := \left\{ u \in L^2(\Omega) : \ u|_T \in H^1(T) \ \ \forall T \in \mathcal{T}_h \right\} .$$

The exact velocity and the pressure errors are respectively:

$$\mathbf{u}_{\mathrm{err}} := \mathbf{u} - \mathbf{u}_h \in \mathcal{H}^2 \tag{6}$$
$$p_{\mathrm{err}} := p - p_h \in Q \ . \tag{7}$$

Later we will need the following scalar product and its corresponding energy norm:

$$\langle \mathbf{u}, \mathbf{v} \rangle := a_h(\mathbf{u}, \mathbf{v}), \qquad |||\mathbf{u}||| := \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} \tag{8}$$

The goal of APEE is to be able to evaluate $\mathbf{u}_{\mathrm{err}}$ and p_{err} without knowing $\mathbf{u}$ and p.

2.4 Simplification of the errors

Our idea is to avoid the evaluation of $\mathbf{u}_{\mathrm{err}}$ and p_{err} directly. Rather, we will first reduce these errors into a single error with a Poisson problem.

Lemma 1 Let $E \in \mathcal{H}^2$ be the solution of

$$a_h(E, \mathbf{v}) = a_h(\mathbf{u}_{\mathrm{err}}, \mathbf{v}) - b_h(\mathbf{v}, p_{\mathrm{err}}) \quad \forall\, \mathbf{v} \in \mathcal{H}^2 \, , \tag{9}$$

then

$$C_1 |||E|||^2 \leq |||\mathbf{u}_{\mathrm{err}}|||^2 + \|p_{\mathrm{err}}\|_0^2 \leq C_2 |||E|||^2 \, ,$$

where the constants C_1 and C_2 are independent of h and the aspect ratio σ_h of the mesh $\mathcal{T}_h$. C_1 and C_2 depend exclusively on Ω.

Proof

This is a particular case of Theorem 1.1. of [Ain97] to the Crouzeix-Raviart/$\mathcal{P}_0$ pair. Note that $C_1 = \frac{1}{2}$ and $C_2 = \frac{4}{\zeta^2} + 1$ where $\zeta = \zeta(\Omega)$ is the continuous infsup constant ([GR86]).

3 Enrichment of the Crouzeix-Raviart element

We propose here a way to enrich the Crouzeix-Raviart element in anisotropic meshes. We emphasize the fact that the strengthened Cauchy-Schwarz constant should be always strictly smaller than 1.

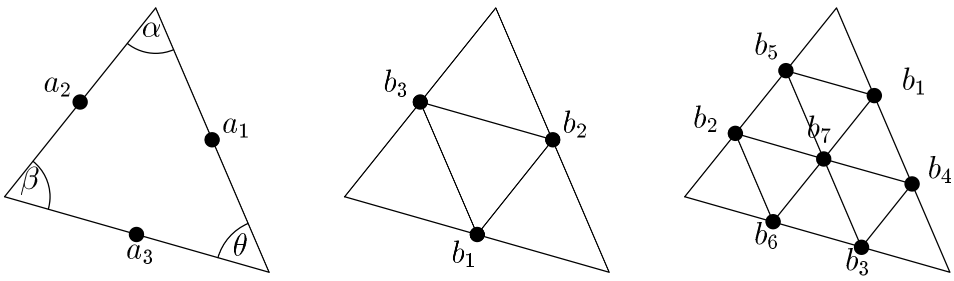

Figure 1: Graphical illustration of the notations.

Theorem 1 (Strengthened Cauchy-Schwarz inequality) Let T be an arbitrary triangle and $k = 2, 3$. Denote by $a_1,\ a_2, a_3$ the midpoints of its edges and by $\phi_i,\ i = 1, 2, 3$ the linear polynomials in T for which:

$$\phi_i(a_j) = \delta_{ij} \quad i,\ j = 1, 2, 3\,.$$

We refine T into k^2 similar triangles and denote the new nodes by b_j(see Fig. 1). Introduce the piecewise linear nodal basis functions at b_j by ψ_j ($j = 1, 2, 3$ for $k = 2$ and $j = 1, ..., 7$ for $k = 3$).

$$\begin{aligned} V(T) &:= \text{span}(\phi_i) \\ Z(T) &:= \text{span}(\psi_j)\,. \end{aligned}$$

There exists therefore a constant $\gamma \in [0, 1)$ which is independent of $\sigma(T)$, meas(T), $h(T)$ and $\rho(T)$ such that

$$a_T(u, v) \le \gamma |u|_{1,T}.|v|_{1,T} \quad \forall u \in\ V(T),\ \forall v \in\ Z(T).$$

Proof

The case $k = 2$ is already implicitly proved in [MM81] where $\gamma^2 = 3/4$, we only need to show it for $k = 3$. We should show that

$$\gamma := \sup_{u\in V(T)} \sup_{v\in Z(T)} \frac{a_T(u, v)}{|u|_{1,T}|v|_{1,T}} < 1\ . \tag{10}$$

By introducing the stiffness matrix corresponding to $(\phi_1, \phi_2, \phi_3, \psi_1, ..., \psi_7)$, which has a block structure:

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

one obtains:

$$\gamma = \sup_{\underline{u}\in\mathbf{R}^3} \sup_{\underline{v}\in\mathbf{R}^7} \frac{\underline{u}^T B\underline{v}}{\sqrt{\underline{u}^T A\underline{u}}\sqrt{\underline{v}^T C\underline{v}}}\ . \tag{11}$$

γ is therefore given by the square root of the largest eigenvalue of generalized eigenvalue problem:

$$(BC^{-1}B^T)\mathbf{v} = \lambda A\mathbf{v}\ . \tag{12}$$

Our aim is to express this largest eigenvalue in terms of the angles α and β(we can get rid of $\theta = \pi - \alpha - \beta$). We have:

$$A = 2\begin{bmatrix} c+a & -a & -c \\ -a & a+b & -b \\ -c & -b & b+c \end{bmatrix}, \quad B = \frac{1}{3}[A|A|0], \quad C = \begin{bmatrix} \zeta I & R & u_1 \\ R^T & \zeta I & u_2 \\ u_1^T & u_2^T & 2\zeta \end{bmatrix}$$

where $a := \cot\alpha$, $b := \cot\beta$, $c := \cot\theta$, $\zeta := a+b+c$, I is the identity matrix of order 3, and

$$R = \begin{bmatrix} -b/2 & -a & 0 \\ 0 & -c/2 & -b \\ -c & 0 & -a/2 \end{bmatrix}, \quad u_1 = \begin{bmatrix} -c \\ -a \\ -b \end{bmatrix}, \quad u_2 = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}.$$

$$K := \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} := \begin{bmatrix} \zeta I & R \\ R^T & \zeta I \end{bmatrix}^{-1} \tag{13}$$

is given by:

$$\begin{array}{rcl} K_{11} & := & (\zeta I - (1/\zeta)RR^T)^{-1} \\ K_{21} & := & (-1/\zeta)R^T E \\ K_{22} & := & (\zeta I - (1/\zeta)R^T R)^{-1} \\ K_{12} & := & (-1/\zeta)RD\ . \end{array} \tag{14}$$

Therefore C^{-1} has the form:

$$C^{-1} = \begin{bmatrix} K + \mu K U U^T K & -\mu K U \\ -\mu U^T K & \mu \end{bmatrix},$$

where:

$$U := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mu := \frac{1}{2\zeta - U^T K U}.$$

Because K can be expressed in block structure (see relations (13) and (14)), $K + \mu K U U^T K$ can also be written block-wise:

$$K + \mu K U U^T K =: \begin{bmatrix} P & Q \\ S & T \end{bmatrix},$$

so that matrix in the left hand side of (12) becomes:

$$\frac{1}{9}[A^T|A^T]\begin{bmatrix} P & Q \\ S & T \end{bmatrix}\begin{bmatrix} A \\ A \end{bmatrix}.$$

The eigenproblem (12) is therefore equivalent to:

$$\frac{1}{9}A(P+Q+S+T)\mathbf{w} = \lambda\mathbf{w} \tag{15}$$

Because we deal with 3×3 matrices in (15), simple (but long) computations yield that the three eigenvalues of (15) are:

$$\begin{aligned}
\lambda_1 &= \lambda_1(\alpha,\beta) = \frac{2}{D}(-35 + 17c_{42} - \sqrt{d}c_{44} - \sqrt{d}c_{55}s + 34c_{33}s + \\
&\quad \sqrt{d}c_{66} - 52c_{11}s + 17c_{24} + 35c_{22} - 34c_{44}) \qquad (16)\\
\lambda_2 &= \lambda_2(\alpha,\beta) = \frac{2}{D}(-35 + 17c_{42} + \sqrt{d}c_{44} + \sqrt{d}c_{55}s + 34c_{33}s - \\
&\quad \sqrt{d}c_{66} - 52c_{11}s + 17c_{24} + 35c_{22} - 34c_{44}) \qquad (17)\\
\lambda_3 &= 0\ , \qquad (18)
\end{aligned}$$

In all these relations, we used:

$$\begin{aligned}
c_{ij} &:= \cos^i\alpha\cos^j\beta\\
s &:= \sin\alpha\sin\beta\\
D &:= -13c_{42} + 138 + 4c_{55}s - 95c_{33}s + 7c_{35}s + 7c_{53}s - 2c_{51}s +\\
&\quad 20c_{02} - 20c_{04} - 2c_{15}s - 38c_{31}s - 38c_{13}s + 238c_{11}s - 20c_{40} +\\
&\quad 20c_{20} - 13c_{24} + 6c_{26} - 4c_{66} - 5c_{64} + 6c_{62} - 211c_{22} + 101c_{44} - 5c_{46}\\
d &:= (121 - 185c_{42} - 32c_{55}s + 182c_{33}s - 56c_{35}s - 56c_{53}s + 16c_{51}s -\\
&\quad 160c_{02} + 160c_{04} + 16c_{15}s + 304c_{31}s + 304c_{13}s - 714c_{11}s + 160c_{40} -\\
&\quad 160c_{20} - 185c_{24} - 48c_{26} + 32c_{66} + 40c_{64} - 48c_{62} + 787c_{22} - 230c_{44} +\\
&\quad 40c_{46})/c_{88}
\end{aligned}$$

Since λ_1 and λ_2 are functions of (α,β), it is very easy to compute the maximum taken value. For $(\alpha,\beta) \in (0,\pi)\times(0,\pi)$, it can be shown (see Fig. 2 and 3) that

$$\max(\lambda_1,\lambda_2) \le \frac{8}{9} \approx 0.888... \tag{19}$$

The theorem is then proved with $\gamma = \frac{2}{3}\sqrt{2}$.

Remark 2 The proof was simple but lengthy, a computation supported by Maple was helpful in performing all the elementary calculus. The maximum value of (19) is approached when one of the angle is tending to π(see Fig. 2 and 3). The result seems to be true for any $k \ge 2$ with

$$\gamma^2 = \frac{k^2-1}{k^2}\ ,$$

as the following numerical results shows. Readers are referred to [BG73], [Ban96], [EV91] for some ways to determine numerically the strengthened Cauchy-Schwarz constant. The test

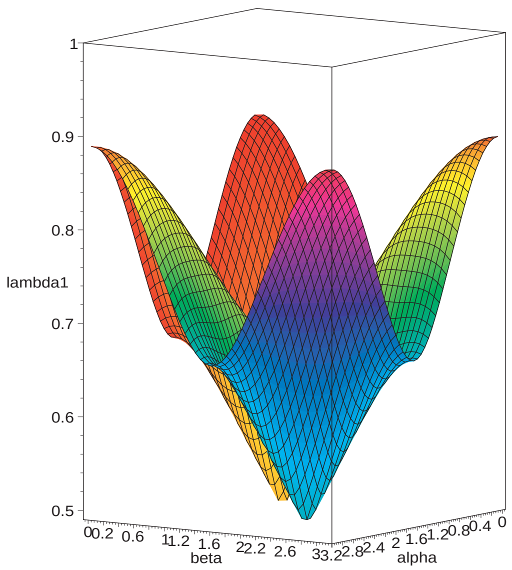

Figure 2: Plot of $\lambda_1(\alpha, \beta)$

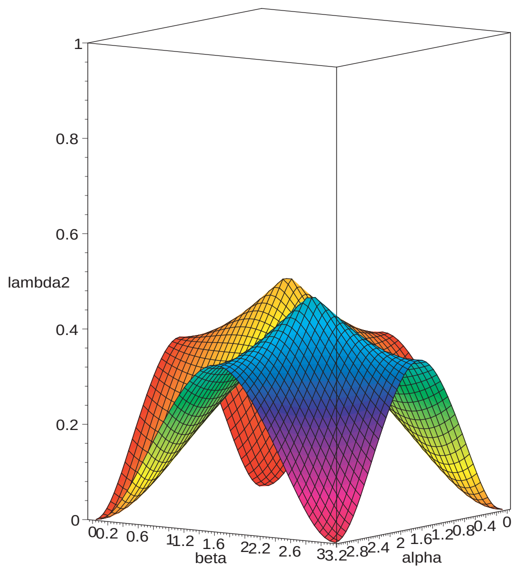

Figure 3: Plot of $\lambda_2(\alpha, \beta)$

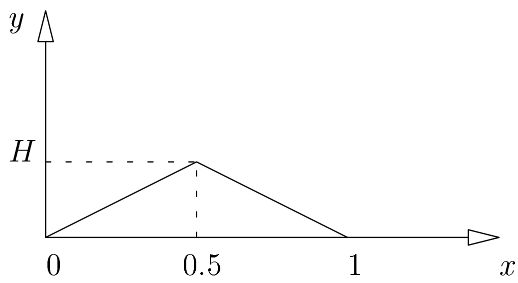

Figure 4: The investigated triangle.

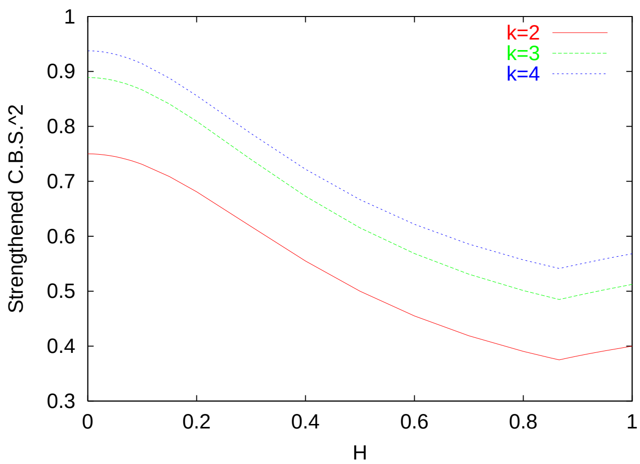

Figure 5: Strengthened Cauchy-Schwarz constant

consists of varying the value of the parameter H (see Fig. 4) and compute the corresponding strengthened Cauchy-Schwarz constant.

The results of the tests are presented in Figure 5. One can clearly see that when the triangle becomes anisotropic (H tends to 0) then the strengthened Cauchy-Schwarz Constant is tending to:

$$\begin{array}{lll} \gamma^2 = 0.75 & \text{for} & k=2 \\ \gamma^2 = 0.888... & \text{for} & k=3 \\ \gamma^2 = 0.9375 & \text{for} & k=4 \end{array}$$

The smallest value of the Cauchy-Schwarz constant is had for $H = 0.8660254$ where the triangle is equilateral. It is however worth noting that $k \geq 4$ is not of any interest because that leads to overly many nodes for each element and that results in too many costs for the APEE.

4 APEE for $|||E|||$

Since the a-posteriori errors $|||\mathbf{u}_{\text{err}}|||$ and $||p_{\text{err}}||_0$ are equivalent to $|||E|||$ (See Lemma 1), the work is left to the evaluation of $|||E|||$ without any a-priori knowledge of the exact solution $\mathbf{u}$ and p.

For each element $T \in \mathcal{T}_h$, we define

$$\begin{aligned} V(T) &:= \{v|_T : v \in V_h\}, \quad \mathbf{u}_T := \mathbf{u}_h|_T, \quad p_T := p_h|_T. & (20) \\ R(T) &:= \{\mathbf{v} \in H^1(T)^2 : \operatorname{div}\mathbf{v} = 0\}, & (21) \end{aligned}$$

We introduce also the spaces:

$$\begin{aligned} R &:= \{\mathbf{v} \in L_2(\Omega)^2 : \mathbf{v}|_T \in R(T)\ \forall T \in \mathcal{T}_h\} & (22) \\ Z_h &:= \{\mathbf{v} \in L_2(\Omega)^2 : \mathbf{v}|_T \in Z(T)^2\ \forall T \in \mathcal{T}_h\} & (23) \end{aligned}$$

(See Theorem 1 for the definition of $Z(T)$).

Now, we enlarge the space $V_h \cap R$ hierarchically into W_h:

$$W_h := (V_h \cap R) \oplus Z_h \tag{24}$$

Let us introduce the following problems (we do not need to solve any of them in practice):

$$\left\{ \begin{array}{l} \text{Find } E \in \mathcal{H}^2 : \\ a_h(E, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v}) \quad \forall\, \mathbf{v} \in V \cap R \end{array} \right. \tag{25}$$

$$\left\{ \begin{array}{l} \text{Find } \mathbf{v}_h \in V_h : \\ a_h(\mathbf{v}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v}) \quad \forall\, \mathbf{v} \in V_h \cap R \end{array} \right. \tag{26}$$

$$\begin{cases} \text{Find } \mathbf{w}_h \in W_h \ : \\ a_h(\mathbf{w}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v}) \quad \forall\, \mathbf{v} \in W_h \end{cases} \tag{27}$$

Note that the solution of (25) is nothing else than that of (9) since $b_h(\mathbf{v}, p_h) = 0$ for all $\mathbf{v} \in R$. Besides, (26) has an evident solution which is $\mathbf{v}_h = 0$.

We suppose the following two assumptions:

(A1) *Strengthened Cauchy-Schwarz inequality:*

$$\exists\, \gamma \in [0, 1): \quad \langle \mathbf{v}, \mathbf{z} \rangle \leq \gamma |||\mathbf{v}|||.|||\mathbf{z}||| \quad \forall \mathbf{v} \in V_h,\ \forall \mathbf{z} \in Z_h. \tag{28}$$

(A2) *Saturation assumption:*

$$\exists\, \beta < 1: \quad |||E - \mathbf{w}_h||| \leq \beta |||E - \mathbf{v}_h||| \tag{29}$$

Remark 3 *The saturation assumption (29) quantifies that the solution in the larger space W_h is more accurate than that in the smaller space V_h (It is a very natural assumption because each triangle $T \in \mathcal{T}_h$ is divided into 4 or 9 sub-triangles). In [Noc93], the author has shown in isotropic meshes that the saturation assumption is only an additional assumption which can be completely removed. In our anisotropic case, removing this assumption is still an open problem.*

Definition 2 For each element $T \in \mathcal{T}_h$, let $\mathbf{e}_T \in Z(T)^2$ be the solution of:

$$a_T(\mathbf{e}_T, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_T - a_T(\mathbf{u}_T, \mathbf{v}) \quad \forall\, \mathbf{v} \in Z(T)^2\,, \tag{30}$$

and our APEE will be:

$$\eta_T := |\mathbf{e}_T|_{1,T}\ .$$

Theorem 2 There exist two constants $\underline{C}$ and $\overline{C}$ which are independent of h and the aspect ratio σ_h of the mesh $\mathcal{T}_h$ such that

$$\underline{C} \sum_{T \in \mathcal{T}_h} \eta_T^2 \leq |||E|||^2 \leq \overline{C} \sum_{T \in \mathcal{T}_h} \eta_T^2 \tag{31}$$

Remark 4 *A theorem similar to this has been discussed in [JL00] where the authors have used extensively the shape regularity of the mesh as well as local quasi-uniformity to prove the theorem. Here, we present another proof which does not use any uniformity at all for the mesh. Our mesh $\mathcal{T}_h$ is allowed to have an arbitrary aspect ratio. We do not require shape regularity for elements. Our elements can be as thin as desired. Our proof use a similar idea as [AABM98].*

Proof

Part 1 (Efficiency):

Let us define $E_h \in Z_h$ by

$$E_h|_T := \mathbf{e}_T \quad \forall T \in \mathcal{T}_h \tag{32}$$

We note immediately that E_h is the solution of:

$$a_h(E_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v}) \qquad \forall \mathbf{v} \in Z_h \tag{33}$$

because equation (30) implies:

$$\sum_{T \in \mathcal{T}_h} a_T(\mathbf{e}_T, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} \left[(\mathbf{f}, \mathbf{v})_T - a_T(\mathbf{u}_T, \mathbf{v})\right] .$$

Therefore, we obtain:

$$\begin{aligned}
\left[\sum_{T \in \mathcal{T}_h} \eta_T^2\right]^{1/2} &= |||\mathbf{E}_h||| \leq \sup_{\substack{\mathbf{z} \in Z_h \\ |||\mathbf{z}|||=1}} a_h(\mathbf{E}_h, \mathbf{z}) = \sup_{\substack{\mathbf{z} \in Z_h \\ |||\mathbf{z}|||=1}} a_h(\mathbf{w}_h, \mathbf{z}) && (34) \\
&\leq \sup_{\substack{\mathbf{z} \in Z_h \\ |||\mathbf{z}|||=1}} |||\mathbf{w}_h|||.|||\mathbf{z}||| = |||\mathbf{w}_h||| && (35)
\end{aligned}$$

(we have the second equality in (34) because the right hand sides of (27) and (33) coincide for all $\mathbf{z} \in Z_h \subset W_h$).

On the other hand, we have:

$$|||\mathbf{w}_h||| \leq |||E - \mathbf{w}_h||| + |||E||| \leq \beta|||E||| + |||E||| = (1+\beta)|||E||| . \tag{36}$$

This last inequality with (35) yield:

$$\left[\sum_{T \in \mathcal{T}_h} \eta_T^2\right]^{1/2} \leq (1+\beta)\, |||E||| \tag{37}$$

Part 2 (Reliability):

Let $\mathbf{v} \in V_h \cap R$ and $\mathbf{z} \in Z_h$ be such that $|||\mathbf{v} + \mathbf{z}||| = 1$.

$$\begin{aligned}
1 &= |||\mathbf{v} + \mathbf{z}|||^2 = |||\mathbf{v}|||^2 + |||\mathbf{z}|||^2 + 2\langle \mathbf{v}, \mathbf{z}\rangle && (38) \\
&\geq |||\mathbf{v}|||^2 + |||\mathbf{z}|||^2 - 2\gamma|||\mathbf{v}|||.|||\mathbf{z}||| && (39) \\
&= (|||\mathbf{v}|||^2 - \gamma|||\mathbf{z}|||)^2 + (1-\gamma^2)|||\mathbf{z}|||^2 && (40)
\end{aligned}$$

Consequently,

$$1 \geq (1-\gamma^2)|||\mathbf{z}|||^2 . \tag{41}$$

We have on the other hand:

$$|||E||| \leq |||E - \mathbf{w}_h||| + |||\mathbf{w}_h||| \leq \beta|||E||| + |||\mathbf{w}_h|||\,,$$

which implies:

$$(1-\beta)|||E||| \leq |||\mathbf{w}_h|||\,. \tag{42}$$

Now, we use (41) to obtain

$$\begin{aligned}
|||\mathbf{w}_h||| &\leq \sup_{\substack{|||\mathbf{v}+\mathbf{z}|||=1\\(\mathbf{v},\mathbf{z})\in(V_h\cap R)\times Z_h}} a_h(\mathbf{w}_h, \mathbf{v}+\mathbf{z}) && (43)\\
&= \sup_{\substack{|||\mathbf{v}+\mathbf{z}|||=1\\(\mathbf{v},\mathbf{z})\in(V_h\cap R)\times Z_h}} (\mathbf{f}, \mathbf{v}+\mathbf{z}) - a_h(\mathbf{u}_h, \mathbf{v}+\mathbf{z}) && (44)\\
&= \sup_{\substack{|||\mathbf{v}+\mathbf{z}|||=1\\(\mathbf{v},\mathbf{z})\in(V_h\cap R)\times Z_h}} \underbrace{(\mathbf{f}, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v})}_{=0} + \underbrace{(\mathbf{f}, \mathbf{z}) - a_h(\mathbf{u}_h, \mathbf{z})}_{a_h(\mathbf{E}_h, \mathbf{z})}\\
&\leq \frac{1}{\sqrt{1-\gamma^2}}|||\mathbf{E}_h||| = \frac{1}{\sqrt{1-\gamma^2}}\left[\sum_{T\in\mathcal{T}_h}\eta_T^2\right]^{1/2}\,. && (45)
\end{aligned}$$

According to (42) and this last inequality,

$$|||E||| \leq \frac{1}{1-\beta}|||\mathbf{w}_h||| \leq \frac{1}{(1-\beta)\sqrt{1-\gamma^2}}\left[\sum_{T\in\mathcal{T}_h}\eta_T^2\right]^{1/2}\,.$$

Finally, the theorem is proved and:

$$\frac{1}{(1+\beta)^2}\sum_{T\in\mathcal{T}_h}\eta_T^2 \leq |||E|||^2 \leq \frac{1}{(1-\beta)^2(1-\gamma^2)}\sum_{T\in\mathcal{T}_h}\eta_T^2\,. \tag{46}$$

5 Numerical Results

Our numerical results are not done in order to replace theoretical proofs but rather to support them. They consist of two tests: the first one are performed on ordinary isotropic meshes and the second on anisotropic meshes. In both cases, the domain Ω is the unit square. The right hand side of (1) is chosen in such a way that the exact solutions are:

$$\begin{aligned}
u_1(x,y) &= (x/10)^2(x-1)^2(y/10)(y-1)(2y-1) \in H_0^1(\Omega) && (47)\\
u_2(x,y) &= -(y/10)^2(y-1)^2(x/10)(x-1)(2x-1) \in H_0^1(\Omega) && (48)\\
p(x,y) &= (x-0.5)(y-0.5) \in L_0^2(\Omega) && (49)
\end{aligned}$$

Let M denote the number of subintervals along the x-axis and N along the y-axis. In the case of Fig. 6, we have $M=4$ and $N=3$. The performance of our APEE is demonstrated numerically in Table 1 and Table 2. In both tables, the last column is the ratio between the error computed with APEE and the exact error. In the second table

$$\text{A.R.} := \max\left\{\frac{M}{N}, \frac{N}{M}\right\} \ .$$

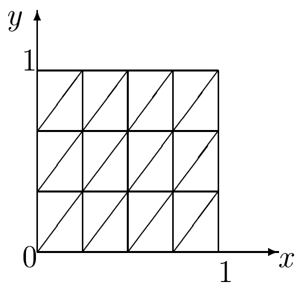

Figure 6: $M = 4$ and $N = 3$

M	N	$\lvert\lvert\lvert\mathbf{u}_{\text{err}}\rvert\rvert\rvert^2 + \lVert p_{\text{err}}\rVert_0^2$	$\sum_{T\in\mathcal{T}_h} \eta_T^2$	ratio
5	5	0.001105	0.001542	1.3954
10	10	0.000293	0.000409	1.3959
20	20	6.69 10^{-5}	10.3 10^{-5}	1.5396
40	40	1.56 10^{-5}	2.56 10^{-5}	1.6410
80	80	3.75 10^{-6}	6.38 10^{-6}	1.7013

Table 1: Results for isotropic grids

M	N	A.R.	$\lvert\lvert\lvert\mathbf{u}_{\text{err}}\rvert\rvert\rvert^2 + \lVert p_{\text{err}}\rVert_0^2$	$\sum_{T\in\mathcal{T}_h} \eta_T^2$	ratio
128	2	64.0	0.002624	0.003259	1.2420
128	4	32.0	9.89 10^{-4}	11.6 10^{-4}	1.1729
128	8	16.0	3.18 10^{-4}	3.77 10^{-4}	1.1855
128	16	8.00	8.02 10^{-5}	10.1 10^{-5}	1.2593
128	32	4.00	1.59 10^{-5}	2.42 10^{-5}	1.5220
128	64	2.00	4.13 10^{-6}	7.04 10^{-6}	1.7046
128	128	1.00	1.46 10^{-6}	2.49 10^{-6}	1.7054

Table 2: Results for anisotropic grids

The linear systems are solved by means of the Bramble-Pasciak conjugate gradient(see [BP88]) which has been improved in [MS01].

6 Conclusion and future work

We have shown an APEE having the following features:

- It can be computed element-wise,
- Efficient and reliable on any meshes (isotropic and anisotropic),
- Solve a local Poisson problem for each element,
- The linear system to solve for each element is small.

Making a rigorous analysis about which to choose $k = 2$ or $k = 3$ is difficult because the value of β in the saturation assumption is not known exactly. Depending on the particular mesh that is used and the solution of the problem, we may find a better β in practice than in any theoretical estimation. Since our APEE works for both $k = 2$ and $k = 3$, we can think of using acceleration techniques such as Richardson extrapolation to obtain a more accurate APEE.

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