Computational Science 1

`http://www.tu-chemnitz.de/physik/THUS/de/lehre/CSM_WS1617.php`

Seminar
Exercises

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Exercise 4 (24.11.2016):

The fixed points of the logistic map

from ***An Introduction to Computer Simulation Methods***,
Chapter 6, Problem 6.4

a) Use `GraphicalSolutionApp` (see sample codes of Chapter 6) to show graphically that there is a single stable fixed point of $f(x)$ for $r < 3/4$. It would be instructive to modify the program so that the value of the slope $(df/dx)|_{x=x_n}$ is shown as you step each iteration. At what value of r does the absolute value of this slope exceed unity? Let b_1 denote the value of r at which the fixed point of $f(x)$ bifurcates and becomes unstable. Verify that $b_1 = 0.75$.

b) Describe the trajectory of $f(x)$ for $r = 0.785$. Is the fixed point given by $x^* = 1 - 1/4r$ stable or unstable? What is the nature of the trajectory if $x_0 = 1 - 1/4r$? What is the period of $f(x)$ for all other choices of x_0? What are the values of the two-point attractor?

c) The function $f(x)$ is symmetrical about $x = 1/2$ where $f(x)$ is a maximum. What are the qualitative features of the second iterate $f^{(2)}(x)$ for $r = 0.785$? Is $f^{(2)}(x)$ symmetrical about $x = 1/2$? For what value of x does $f^{(2)}(x)$ have a minimum? Iterate $x_{n+1} = f^{(2)}(x_n)$ for $r = 0.785$ and find its two fixed points x_1^* and x_2^*. (Try $x_0 = 0.1$ and $x_0 = 0.3$.) Are the fixed points of $f^{(2)}(x)$ stable or unstable for this value of r? How do these values of x_1^* and x_2^* compare with the values of the two-point attractor of $f(x)$? Verify that the slopes of $f^{(2)}(x)$ at x_1^* and x_2^* are equal.

d) Verify the following properties of the fixed points of $f^{(2)}(x)$. As r is increased, the fixed points of $f^{(2)}(x)$ move apart and the slope of $f^{(2)}(x)$ at its fixed points decreases. What is the value of $r = s_2$ at which one of the two fixed points of $f^{(2)}$ equals $1/2$? What is the value of the other fixed point? What is the slope of $f^{(2)}(x)$ at $x = 1/2$? What is the slope at the other fixed point? As r is further increased, the slopes at the fixed points become negative. Finally at $r = b_2 \approx 0.8623$, the slopes at the two fixed points of $f^{(2)}(x)$ equal -1, and the two fixed points of $f^{(2)}$ become unstable. (The exact value of b_2 is $b_2 = (1 + \sqrt{6})/4$.)

e) Show that for r slightly greater than b_2, for example, $r = 0.87$, there are four stable fixed points of $f^{(4)}(x)$. What is the value of $r = s_3$ when one of the fixed points equals $1/2$? What are the values of the three other fixed points at $r = s_3$?

f) Determine the value of $r = b_3$ at which the four fixed points of $f^{(4)}$ become unstable.

g) Choose $r = s_3$ and determine the number of iterations that are necessary for the trajectory to converge to period 4 behavior. How does this number of iterations change when neighboring values of r are considered? Choose several values of x_0 so that your results do not depend on the initial conditions.