Computational Science 1

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Seminar
Exercises

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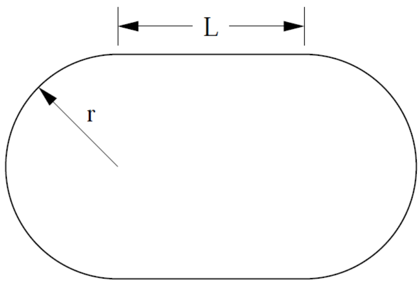
Exercise 5 (15.11.2012):

Billiard models

from ***An Introduction to Computer Simulation Methods***,
Chapter 6, Problems 6.26

Consider a two-dimensional planar geometry in which a particle moves with constant velocity along straight line orbits until it elastically reflects off the boundary.



Suppose that we divide the circle into two equal parts and connect them by straight lines of length L as shown in the Figure. This geometry is called a stadium billiard. How does the motion of a particle in the stadium compare to the motion in the circle? In both cases we can find the trajectory of the particle by geometrical considerations.

a) Write a program to simulate the stadium billiard model. Use the radius r of the semicircles as the unit of length. The algorithm for determining the path of the particle is as follows:

   (i) Begin with an initial position (x_0, y_0) and momentum (p_{x0}, p_{y0}) of the particle such that $|p_0| = 1$.

   (ii) Determine which of the four sides the particle will hit. The possibilities are the top and bottom line segments and the right and left semicircles.

   (iii) Determine the next position of the particle from the intersection of the straight line defined by the current position and momentum, and the equation for the segment where the next reflection occurs.

   (iv) Determine the new momentum, (p_x, p_y), of the particle after reflection such that the angle of incidence equals the angle of reflection. For reflection off the line segments we have $(p'_x, p'_y) = (p_x, -p_y)$. For reflection off a circle we have

$$p'_x = \left[y^2 - (x - x_c)^2\right] p_x - 2(x - x_c) y p_y \tag{1}$$
$$p'_y = -2(x - x_c) y p_x + \left[(x - x_c)^2 - y^2\right] p_y \tag{2}$$

where $(x_c, 0)$ is the center of the circle. Remember that all lengths are scaled by the radius of the circle.

(v) Repeat steps (ii)-(iv).

b) Determine if the particle dynamics is chaotic by estimating the largest Lyapunov exponent. One way to do so is to start two particles with almost identical positions and/or momenta (varying by say 10^{-5}). Compute the difference Δs of the two phase space trajectories as a function of the number of reflections n, where Δs is defined by

$$\Delta s = \sqrt{|\mathbf{r_1} - \mathbf{r_2}|^2 + |\mathbf{p_1} - \mathbf{p_2}|^2}. \tag{3}$$

Choose $L = 1$ and $r = 1$. The Lyapunov exponent can be found from a semilog plot of Δs versus n. Repeat your calculation for different initial conditions and average your values of Δs before plotting. Repeat the calculation for $L = 0.5$ and 2.0 and determine if your results depend on L.

c) Another test for the existence of chaos is the reversibility of the motion. Reverse the momentum after the particle has made n reflections, and let the drawing color equal the background color so that the path can be erased. What limitation does roundoff error place on your results? Repeat this simulation for $L = 1$ and $L = 0$.

d) Place a small hole of diameter d in one of the circular sections of the stadium so that the particle can escape. Choose $L = 1$ and set $d = 0.02$. Give the particle a random position and momentum, and record the time when the particle escapes through the hole. Repeat for at least 10^4 particles and compute the fraction of particles $S(n)$ remaining after a given number of reflections n. The function $S(n)$ will decay with n. Determine the functional dependence of S on n, and calculate the characteristic decay time if $S(n)$ decays exponentially. Repeat for $L = 0.1$, 0.5, and 2.0. Is the decay time a function of L? Does $S(n)$ decays exponentially for the circular billiard model ($L = 0$)?

e) Choose an arbitrary initial position for the particle in a stadium with $L = 1$, and a small hole as in part (d). Choose at least 5000 values of the initial value p_{x0} uniformly distributed between 0 and 1. Choose p_{y0} so that $|p| = 1$. Plot the escape time versus p_{x0}, and describe the visual pattern of the trajectories. Then choose 5000 values of p_{x0} in a smaller interval centered about the value of p_{x0} for which the escape time was greatest. Plot these values of the escape time versus p_{x0}. Do you see any evidence of self-similarity?