Computational Science 1

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Seminar
Exercises

Prof. M. Schreiber
schreiber@physik.tu-chemnitz.de
Room 2/P302, Phone 21910

Dr. P. Cain
cain@physik.tu-chemnitz.de
Room 2/P310, Phone 33144

Exercise 5 (5.12.2019):

Statistical test of randomness

from ***An Introduction to Computer Simulation Methods***,
Chapter 7, Problem 7.35

A rather simple but very fast random number generator can be constructed with the linear congruential method $x_n = (ax_{n-1} + c) \bmod m$, where a, c, m, and x_n are integers. The random numbers take values between 0 to m and should be divided by m to fit into the interval $[0, 1)$.

a) **Uniformity**. A random number sequence should contain numbers distributed in the unit interval with equal probability. The simplest test of uniformity is to divide this interval into M equal size bins. For example, consider the first $N = 10^4$ numbers generated with $a = 106$, $c = 1283$, and $m = 6075$. Place each number into one of $M = 100$ bins. Is the number n_i of entries in each bin approximately equal? What happens if you increase N? Plot the distribution of the n_i's for different N and M.

b) **Filling sites**. Although a random number sequence might be distributed in the unit interval with equal probability, the consecutive numbers might be correlated in some way. One test of this correlation is to fill a square lattice of L^2 sites at random. Consider an array $n(x, y)$ that is initially empty, where $1 \leq x_i, y_i \leq L$. A site is selected randomly by choosing its two coordinates x_i and y_i from two consecutive numbers in the sequence. If the site is empty, it is filled and $n(x_i, y_i) = 1$; otherwise it is not changed. This procedure is repeated t times, where t is the number of Monte Carlo steps per site. That is, the time is increased by $1/L^2$ each time a pair of random numbers is generated. Because this process is analogous to the decay of radioactive nuclei, we expect that the fraction of empty lattice sites should decay as e^{-t}. Determine the fraction of unfilled sites using the random number generator that you have been using for $L = 10$, 15, and 20. Are your results consistent with the expected fraction? Repeat the same test with $a = 65539$, $c = 0$, and $m = 2^{31}$.

c) **Hidden correlations**. Another way of checking for correlations is to plot x_{i+k} versus x_i. If there are any obvious patterns in the plot, then there is something wrong with the generator. Use $a = 16807$, $c = 0$, and $m = 2^{31} - 1$. Can you detect any structure in the plotted points for $k = 1$ to $k = 5$?

d) **Short-term correlations**. Another measure of short term correlations is the autocorrelation function

$$C(k) = \frac{\langle x_{i+k} x_i \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2}, \tag{1}$$

where x_i is the ith term in the sequence. If x_{i+k} and x_i are not correlated, then $\langle x_{i+k} x_i \rangle = \langle x_{i+k} \rangle \langle x_i \rangle$ and $C(k) = 0$. Is $C(k)$ identically zero for any finite sequence? Compute $C(k)$ for $a = 106$, $c = 1283$, and $m = 6075$.