Disorder Phenomena in Chaotic Systems

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Abstract: The influence of static disorder on chaotic systems is investigated for two different classes. In the first case disorder is present in spatial inhomogeneities of low-dimensional extended systems. We show that chaotic diffusion is suppressed by the disorder and that such systems are simple systems exhibiting the phenomenon of aging. In the second class the randomness lies in the coupling parameters of many degrees of freedom. We demonstrate that for a system of randomly coupled limit cycle oscillators the occurrence of a first-order phase transition into a spin-glass like phase. We discuss, how these phenomena are reflected in Lyapunov spectra and other characteristic quantities of dynamical systems.

1 Introduction

Chaos in dynamical systems is nowadays a well established research field [1,2] with applications in all branches of physics, e.g. atomic physics, hydrodynamics, quantum optics, or solid state physics. The key concepts for the characterization of such systems are Lyapunov exponents, various dynamical entropies, and fractal dimensions of strange attractors in dissipative systems. Many investigations considered systems with few effective degrees of freedom living in a compact phase space. Cases where phase space is infinite were mostly treated in the presence of a discrete translational symmetry, which by reduction to a unit cell again lead to compact phase spaces, typically a high-dimensional torus. Clearly these are limiting cases, and one wonders what happens for systems with broken translational invariance, or for systems with many non-equivalent degrees of freedom. In the following we treat examples, where order in this sense is not present, but the other extreme is prevailing, namely full disorder. Disordered systems are usually considered as a branch of statistical physics or solid state physics [3] with concepts and methods very different from the ones in dynamical systems theory. The two model classes considered below, suggest that spin-glass order, aging, or disorder induced anomalous transport are phenomena that arise very frequently also in disordered dynamical systems. We will also see how the above mentioned characteristics of dynamical systems reflect these disorder phenomena.

2 Few Degrees of Freedom in Disordered Environments

The simplest disordered dynamical systems are those with one or two degrees of freedom with disorder in the environment. An example is the Hamiltonian motion of a point particle in a two-dimensional disordered potential consisting of hard discs (Lorentz gas [10]) or smoothed version thereof. The periodic counterparts of these models have recently received much experimental and theoretical attention in connection with mesoscopic transport in quantum dot lattices [11,12]. There the importance of classical phase space structures and chaotic diffusion was realized [12]. Here we will concentrate first on dissipative systems which may also show chaotic transport. A simple much studied example is the damped motion of a periodically driven particle in a periodic potential. This provides e.g. a model for superionic conductors in an external field and currently finds renewed interest in the context of ratchet physics. Simplified versions which are assumed to capture essential aspects of these systems are one-dimensional iterated maps [5–7]. While the periodicity in the equations of motion allows for the application of advanced methods such as periodic orbit theory, the thermodynamic formalism, or Levy flight statistics [9], it is clearly of much greater importance to understand the effects of static disorder in such systems. From the physics of disordered systems, it is known that static or quenched randomness may drastically alter macroscopic quantities such as transport coefficients. In the following we will first report on such an effect for dynamical systems, namely the total *suppression of normal or anomalous chaotic diffusion by quenched randomness* in the equations of motion. We will show that this can occur in both, dissipative and Hamiltonian systems. This turns out to be a non-trivial effect, since the mean-square displacement will remain finite, although chaotic transport is *not* inhibited locally. Secondly we will point out that in the presence of a global bias these systems show various *phase transitions* characterized by *anomalous chaotic transport* properties. Finally we argue that these systems can also show the phenomenon of *aging*.

2.1 Suppression of Chaotic Diffusion

Dissipative Systems

We will first concentrate on one-dimensional non-invertible maps of the type studied in [5–9]. They have the general form $x_{t+1} = f(x_t) = x_t + F(x_t)$, with $F(x)$ periodic in x. The periodicity interval, which we set equal to one, i.e. $F(x) = F(x+1)$, defines cells or half open intervals $A_i = [i, i+1), i \in \mathbb{Z}$, on the real axis. We will modify these dynamical systems by randomly changing $F(x)$ in each cell A_i to a function $F^{(i)}(x)$ resulting in

$$x_{t+1} = x_t + F^{(i)}(x_t) \tag{2.1}$$

for $x_t \in A_i$. This corresponds to a random variation in space of the driving force felt by the particle. A natural choice for $F^{(i)}(x)$ consists of random shifts of F

$$F^{(i)}(x) = F(x) + \varepsilon(i). \tag{2.2}$$

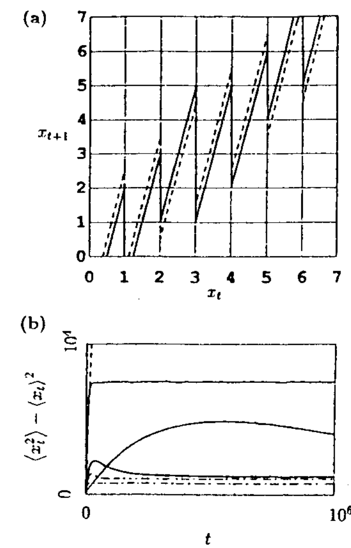

Figure 1
(a) Simple piecewise linear maps corresponding to a periodic (dashed) and a random driving 'force' (bold). The mean-square displacement (b) increases linearly for the former (dashed line) and saturates in the latter case as shown for several disorder realizations (full lines: $\sigma^2(t)$ for $\varepsilon(i) = \pm 1/2$, dot-dashed graphs: $\varepsilon(i)$ equally distributed in $(-1/2, +1/2)$). There exist environments where a first constant level is observed only after more than $t = 10^6$ iterations ($0.5 \ldots 1.0 \times 10^7$ for the second graph from the top).

In order to avoid complications connected with a global bias we assume for the moment the symmetry $F(-x) = -F(x)$, and further that the $\varepsilon(i)$ are independent, identically distributed random variables with a symmetric distribution function $p(\varepsilon) = p(-\varepsilon)$ implying $\overline{\varepsilon(i)\varepsilon(j)} \propto \delta_{ij}$ and $\overline{\varepsilon(i)} = 0$. Through the cell index i, defined as $i = [x]$, the largest integer smaller than x, the term $\varepsilon(i)$ is recognized as piecewise constant random function of x. In contrast to previous studies [5], where time-dependent noise was added to the deterministic dynamics, the random term $\varepsilon(i)$ remains constant in time, (2.1) is still deterministic, it describes a dynamical system with *quenched* randomness.

Let us now investigate the effect of this static randomness first for the simplest maps which in the absence of disorder ($\varepsilon(i) = 0$) exhibit chaotic diffusion. These are systems where $F(x)$ varies linearly in each cell, i.e. $F(x) = a\{x\} - a/2$ with $\{x\} = x - [x]$. Since the slope of $f(x)$ is $a+1$ these maps are chaotic for $a > 0$ and show chaotic diffusion for $a > 1$. The dashed graph in Fig. 1a is an example with $a = 3$. The diffusive motion for this ordered case is verified by the linear increase of the mean-square displacement $\sigma^2(t) = \langle x_t^2 \rangle - \langle x_t \rangle^2 = 2Dt$ with the correct diffusion constant $D = 1/4$ [6] (dashed line in Fig. 1b). This and the following results for $\sigma^2(t)$ were obtained numerically by iterating ensembles of 2×10^4 points (initially distributed homogeneously or inhomogeneously in one cell) for 10^6 (occasionally 10^7) time steps. An example of a map with binary disorder, $\varepsilon(i) = \pm 1/2$ in (2.2), is shown as full line in Fig. 1a. Now, with disorder $\varepsilon(i) \neq 0$, a very different behavior is observed: $\sigma^2(t)$ saturates and remains bounded for large times. As is seen from Fig. 1b this is true for discrete random variations as well as for continuously distributed random variables $\varepsilon(i)$. We emphasize that for both cases there exists no obvious reason why the spreading of the distribution $\rho_t(x)$ should be limited because the *a priori* probability for reaching one of the neighboring cells is always finite. More explicitly, *independent* of the chosen sequence $\varepsilon(i)$, a fraction $p = 1/4$ of a homogeneous distribution in some cell A_i is always transferred to the right neighboring cell A_{i+1}, the same fraction to the left cell A_{i-1}, and one quarter remains within the cell A_i. From this point of view there is no difference between the homogeneous situation ($\varepsilon(i) = 0$) and

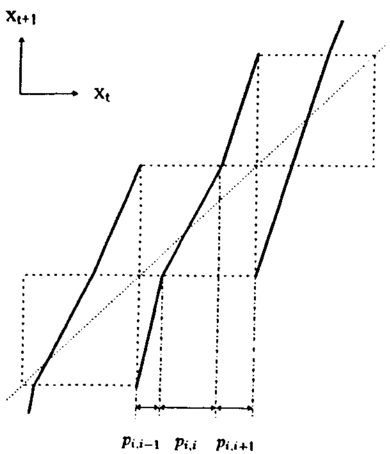

Figure 2
An example from the class of iterated maps for which the asymptotically finite mean-square displacement follows rigorously due the work of Sinai and Golosov [14][15]. Also shown by dashed lines are the unit squares of the integer grid (see Fig. 1a) along the bisectrix. The indicated intervals p_{ii} and $p_{i,i\pm1}$ mediate the transitions from the i-th cell A_i to itself and its neighbors respectively.

the inhomogeneous case ($\varepsilon(i) \neq 0$)! The randomness affects only the last quarter, which is mapped into one or both of the next-nearest cells $A_{i\pm2}$. Note also that the degree of chaoticity as measured by the Lyapunov exponent (Fig. 1: $\lambda = \ln 4$) is not altered by the random shifts.

The explanation of this localization effect in the case of quenched randomness follows from the following connection. For the map $f(x)$ with discrete random shifts as in Fig. 1 the cells A_i define a (generating) Markov partition [13]. This implies that the evolution of piecewise constant distributions $\rho_t(x)$ (constant in the cells A_i) is fully equivalent to a Markov process, i.e. the content $\pi_i(t) = \int_i^{i+1} \rho_t(x)\,dx$ of cell A_i at time t is iterated according to

$$\pi_j(t+1) = \sum_i \pi_i(t) p_{ij}. \tag{2.3}$$

For the above piecewise linear map with $\varepsilon(i) = \pm 1/2$ (Fig. 1) the only non-zero transition probabilities p_{ij} are given by $p_{ii} = p_{i,i\pm1} = 1/4$ and $p_{i,i\pm2} = (1/2 \pm \varepsilon(i))/4$. The results of Fig. 1 were also checked by iterating (2.3) with these transition probabilities. Such a model defines a discrete random walk in a locally asymmetric random environment. The above localization effect, i.e. $\sigma^2(t)$ remaining finite for $t \to \infty$, is known as *Golosov phenomenon* in the random walk literature [4]. Inspired by Sinai's work [14] it was proven rigorously for systems with only nearest neighbor transitions by Golosov [15]. Reversing the above arguments which led us from iterated maps to random walks, it is obvious that also for the latter systems there exist realizations in terms of dynamical systems. These consist of piecewise linear chaotic maps of the form (2.1), with a typical example shown in Fig. 2. Again the cells A_i provide a Markov partition for this system. The segments of length p_{ii} and $p_{i,i\pm1}$ in each unit cell, where the map $f(x)$ is linear, correspond to the non-zero transition probabilities p_{ii} and $p_{i,i\pm1}$ of the associated Markov chain.

The asymptotically finite mean-square displacement was proven in [15] for independent random sequences p_{ii} and $p_{i,i-1}/p_{i,i+1}$ with $\overline{\ln(p_{i,i-1}/p_{i,i+1})} = 0$. The latter condition means that there is no global bias in the system and that one observes recurrent behavior with probability one.

So far we have seen that the dynamical systems defined in Fig. 1 and Fig. 2 both can be mapped to random walk models with (locally) asymmetric random transitions probabilities to next-nearest respectively nearest neighbors. Because of the short ranged correlations in the quenched disorder they belong to the same universality class [4].

An intuitive picture for the relevant physical processes is obtained from the continuum limit of these discrete random walk models, which is Brownian motion in a spatially random force field $\tilde{F}(x)$ [4]. In this limit the dynamics is governed by the Langevin equation

$$\dot{x}(t) = -\frac{\partial \tilde{V}}{\partial x}(x(t)) + \xi(t) \tag{2.4}$$

with Gaussian white noise $\xi(t)$. The important point is that the associated potential $\tilde{V}(x) = -\int^x \tilde{F}(x')dx'$ itself can be thought of as a spatial realization of a Brownian path. The resulting statistical self-similarity of the potential $\tilde{V}(Lx) \simeq L^{1/2}\tilde{V}(x)$ implies the occurrence of deeper and deeper potential wells as the particle proceeds. The work of Sinai and Golosov shows that an ensemble of initially close particles moves in a coherent fashion from one deep minimum to the next deeper potential well. In this stepwise process it is typically one minimum which dominates and therefore determines the (finite) width $\sigma^2(t)$ of the ensemble. Since the random environment in the neighborhood of these minima is the same only in a statistical sense one observes for a fixed environment still fluctuations in $\sigma^2(t)$. These fluctuations become extremely rare for large times t as follows from an Arrhenius argument [4] which says that the typical time to overcome the ever increasing relevant potential barriers increases exponentially with the barrier height, i.e. it takes a time of the order $\exp(\sqrt{x})$ for the state to travel a distance x. Solving this relation for x says that the typical distance reached in time t increases only as $\ln^2 t$. Indeed it has been shown rigorously that the mean displacement grows anomalously as $\langle x(t) \rangle = \xi(t) \ln^2 t$ with $\xi(t)$ a random function of $O(1)$ [4,14,15].

Applying this picture of a thermally activated process in a random Brownian landscape to dynamical systems presupposes the existence of a Markov partition. The results of Fig. 1 for continuous distributions of shifts $\varepsilon(i)$, however, show that the observed localization phenomenon is not bound to the existence of a Markov partition. One may also expect that the feature of piecewise linearity is not necessary for the occurrence of this effect. Indeed we have shown in [16] that even anomalously enhanced diffusion generated by certain nonlinear maps [8] is totally suppressed by the introduction of disorder of the above type.

Area Preserving Systems

One may wonder whether this sort of dynamical localization can be observed also in Hamiltonian systems or area preserving maps. This question is answered affirmatively by an explicit construction of area preserving maps, which show the same behavior. Our goal is achieved by extending the concept of homogeneous multibaker maps [17] to inhomogeneous chains by an appropriate use of the fact that every finite Markov chain can be realized by area preserving maps [18]. The construction is shown in Fig. 3 below.

It consists of an array of L rectangles called cells $A_i, i = 1, \ldots, L$, of unit width in the x-direction and heights π_i. During one iteration step the inscribed rectangles within one cell A_i are mapped to the neighboring cells and into the present cell as follows: Take e.g. the black rectangle of height π_i and width $p_{i,i-1}$, which is labeled by its area $\pi_i p_{i,i-1}$ in Fig. 3. This rectangle is in one iteration (from top to bottom in Fig. 3) squeezed in the y-direction,

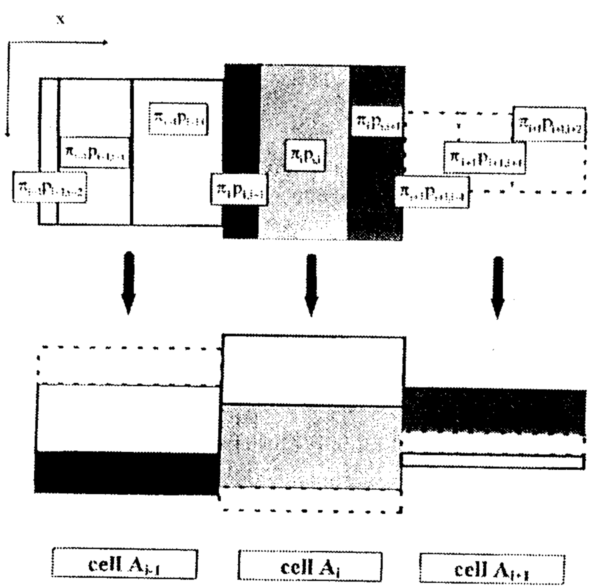

Figure 3
The construction of an area preserving inhomogeneous chain of baker maps, which in the projection on the x-axis reveals exactly the dynamics of the one-dimensional iterated map of Fig.2.

stretched in the x-direction, and transferred to the left neighboring cell A_{i-1}in an area preserving manner. In the same way the grey-shaded rectangle $\pi_i p_{i,i+1}$ is transferred to the right, and the light-grey rectangle gets squeezed and stretched, but remains in the cell A_i as shown in the figure. To make this scheme consistent we impose periodic boundary conditions $A_{i+L} = A_i$ ($L = 3$ in Fig. 3). Alternatively we could confine transport within the array by appropriate modifications of the boundary maps ($p_{1,0} = p_{L,L+1} = 0$). One observes that the x-coordinate x_t of a point (x_t,y_t) in this two-dimensional area preserving map is iterated exactly as a point x_t of the one-dimensional map of Fig. 2 (with slopes $p_{i,j}^{-1}$). This implies that the dynamical localization phenomenon of the previous section iş found also in these area preserving maps!

Note, however, that in order to get globally an area preserving map, the heights π_i cannot be chosen arbitrary. This follows from the condition that in one iteration the outflow of area of a given cell, say A_i, has to be equal to its inflow,

$$\pi_{i-1}p_{i-1,i} + \pi_{i+1}p_{i+1,i} = \pi_i p_{i,i-1} + \pi_i p_{i,i+1}. \tag{2.5}$$

Adding to both sides of this equation the term $\pi_i p_{i,i}$ and using the normalization $\sum_j p_{i,j} = 1$, one finds that the π_i have to be proportional to the stationary probability distribution of a Markov chain with transition probabilities $p_{i,j}$. This is also clear from the fact that the projection of the two-dimensional stationary distribution of the inhomogeneous chain of baker maps as constructed above (which is constant in its area of definition), should yield the stationary distribution of the one-dimensional map of Fig. 2. For reflecting boundary conditions the stationary distribution can be found exactly as

$$\pi_i = \pi_1 \prod_{k=1}^{i-1} \frac{p_{k,k+1}}{p_{k+1,k}}. \tag{2.6}$$

For quenched random transition probabilities, as in the models treated here, this means that the π_i follow a random multiplicative process, which occur naturally in many branches of physics. For periodic boundary conditions a similar result is obtained [19]. A characterization of such stationary distributions in terms of its cumulants was given only recently in [20]. Of course it would be of interest if for some class of area preserving maps or corresponding continuous time Hamiltonian systems, such a distribution of chaotic areas occurs naturally. In such cases one expects that one finds the same anomalous transport properties as in the simple maps introduced here. This will be observable until the boundaries of the system are reached, which for extended systems occurs only at exponentially large times, i.e. at times of the order $O(\exp\sqrt{L})$.

2.2 Anomalous Drift and Phase Transitions

Releasing the conditions $\overline{\varepsilon(i)} = 0$ for the system of Fig. 1 or $\overline{\ln(p_{i,i-1}/p_{i,i+1})} = 0$ for that of Fig. 2 leads us from systems without global bias to such with a bias. The latter may simulate an external static field or may be attributed to systematic asymmetries of the underlying potential in more realistic models, such as driven damped particles in some potential landscape. Such aspects are clearly of great importance in the context of transport in ratchets [21,22]. Although the connection between the simplified one-dimensional maps and more realistic models is complicated and not fully understood, the investigation of the former with bias will provide some insight into the possible scenarios in the latter.

Exploiting again the connection between maps and discrete time Markov chains, allows us to apply known results from the literature to disordered iterated maps. Analytical results by Derrida and Pomeau [19] for the biased case $\overline{\ln(p_{i,i-1}/p_{i,i+1})} \neq 0$ lead to transport properties for the corresponding dynamical systems, which we summarize in Fig. 4 for a special example of the type of Fig. 2.

The example consists of setting $p_{ii} = 0$ and choosing a binary distribution for the transition probabilities $p_{i,i\pm1}$ of Fig. 2. More explicitly, we construct a chain of maps consisting of two sorts of cells (see Fig. 2), which we denote as A_+ and A_-. For the type A_+ we choose $p_{i,i+1} = a$ and correspondingly $p_{i,i-1} = 1-a$, while for type A_- we reverse the assignments, i.e. we set $p_{i,i+1} = 1-a$ and $p_{i,i-1} = a$. These cells are concatenated randomly and independently so that type A_+ is present in a concentration c and correspondingly a fraction $1-c$ of cells are of type A_-. For Fig. 4 we have chosen $a = 1/3$ so that for $c = 0$ we have an ordered system consisting only of cells A_-, which map points with larger probability, namely with probability 2/3, to the right, and with probability 1/3 to the left. In this limit one gets chaotic diffusion with a mean drift to the right, where diffusion constant D and drift velocity V are given by $D = D_0 = 2a(1-a)$ and $V = V_0 = 1-2a$, respectively. The behavior of the transport coefficients, normalized by its bare values D_0

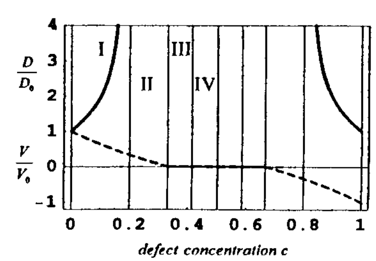

Figure 4
The dependence of the normalized transport coefficients D/D_0 (bold) and V/V_0 (dashed) is shown as fuction of the defect concentration c. In the regimes where these are finite they are self-averaging quantities and therefore these values are observed with probability one independent of the disorder realization.

and V_0, is shown in Fig. 4 as function of the concentration c. We need to discuss only the case $0 \le c \le 1/2$, since the rest follows by symmetry. For the other extreme of full disorder $c = 1/2$ we get the unbiased situation $\overline{\ln(p_{i,i-1}/p_{i,i+1})} = 0$, where the anomalous Sinai and Golosov results of section 2.1 hold. This means we get in this case $D = 0$ and $V = 0$. Between these extremes various transitions between dynamically different phases occur. In Fig. 4 these different regimes are numbered as I-IV. In phase I both D and V are finite and non-zero, i.e. one has normal chaotic transport as in the ordered limit. The transition to phase II occurs at concentrations where conditions $\overline{(p_{i,i-1}/p_{i,i+1})^{\pm 2}} = 1$ are fulfilled (two symmetric solutions in c) . This transition is accompanied by D becoming infinite, which holds in regimes II an III. In these phases one has therefore anomalously enhanced diffusion, i.e. the mean-square displacement grows superlinear. The transition between II an III is signaled by a vanishing drift velocity, i.e. the mean displacement grows slower than linearly in time. The transition points are given by $\overline{(p_{i,i-1}/p_{i,i+1})^{\pm 1}} = 1$. The anomalously slow growth of the displacement holds up to the value $c = 1/2$, i.e. also in regime IV. The transition from III to the latter is characterized by a crossover from superdiffusive to subdiffusive chaotic transport implying that in addition to V also D vanishes. The last transition is observed for $\overline{(p_{i,i-1}/p_{i,i+1})^{\pm 1/2}} = 1$. The fact that the qualitative changes in the drift and diffusion properties occur at different values of the concentration is quite common in disordered systems and is expected to hold also more generally. A more extensive description and a discussion of the different regimes in terms of activated Brownian motion in tilted Brownian potentials can be found in the review article [4]. Interestingly the same conclusions were reached very recently in the context of disordered ratchets by a totally different reasoning within a continuous time model [22].

2.3 Aging

The phenomenon of aging is a well-known experimental fact from glasses, spin glasses, and other complex materials [23,24]. Theoretical investigations are also mainly concerned with these systems [25], but recently aging was found also in much simpler model systems [26,27]. Aging can be defined as an anomalous behavior of response and correlation functions. Consider e.g. the correlation function $C_{AB}(t,t_w) \equiv \langle A(t_w)B(t+t_w)\rangle$ of two variables A and B, where t_w is the waiting time after the preparation of the initial state at time $t = 0$. For $t \ll t_w$ the correlation function $C_{AB}(t,t_w)$ is independent of t_w and a fluctuation-dissipation theorem is supposed to hold. Aging is present if $C_{AB}(t,t_w)$ depends strongly on t_w when t is of the same order as t_w. Furthermore for t and t_w large on expects a scaling behavior of the form

$$C_{AB}(t,t_w) = t^{-\nu} f(\frac{t}{t_w}). \tag{2.7}$$

Interestingly such a scaling behavior with $\nu = 1$ has been found numerically for various correlation functions by Marinari and Parisi [26] in the case of a random walk in a random environment of Sinai type. As pointed out above such systems are realized by dynamical systems as introduced in Fig. 2. This implies that aging occurs also in these simple disordered dynamical systems and its generalizations treated in Section 2.1. This fact has not been realized previously.

3 Many Degrees of Freedom with Random Couplings

In this Section we consider the other extreme of disordered dynamical systems, namely that of many degrees of freedom, each confined to a compact set (a circle) and coupled in a random manner. The results below are obtained for a model defined by N randomly interacting phase oscillators. Interest in such models comes from biology [28–30], e.g. from brain research [31,32], from laser physics [33,34], and other branches of physics [35]. They are a good approximation for many dynamical systems where limit cycle oscillators are the basic entities [30,35]. The model is defined by the equations of motion

$$\dot{\phi}_j(t) = \omega_j + \sum_{i=1}^{N} J_{ij} \sin(\phi_i(t) - \phi_j(t)), \tag{3.8}$$

where ϕ_i are angles, $\phi_i \in [0,2\pi)$, the ω_i are the unperturbed oscillator frequencies, and the J_{ij} are the couplings between the different limit cycle oscillators. This model has mostly been investigated for the special case of random frequencies ω_i and uniform all-to-all interactions $J_{ij} = K/N$ [35,36]. In this case, which is usually referred to as the Kuramoto model, it is well known that one observes a transition from completely incoherent motion of the oscillators for small interaction strength $K < K_c$, to partially coherent motion (phase locking) above a critical interaction strength K_c. This behavior is captured in the thermodynamic limit $N \to \infty$ by the order parameter

$$m = \lim_{t\to\infty} \langle \exp(i\phi(t)) \rangle, \tag{3.9}$$

where the brackets $\langle \dots \rangle := \lim_{N\to\infty} 1/N \sum_{j=1}^{N} \cdots$ define an ensemble average over all oscillators. The transition is then described by a vanishing order parameter for $K < K_c$, and an increase of the "magnetization" $|m|$ as $|m| \sim \sqrt{(K-K_c)/K_c}$ above K_c. It is known that this system is not chaotic as follows from the maximal Lyapunov exponent, which is always zero, but we note that the phase locking is reflected in the full Lyapunov spectrum, which can also be calculated exactly [37].

Here we present results for the more interesting and general case, where in addition to the frequencies also the couplings are quenched random variables. They are assumed to be gaussian distributed variables with mean zero $\overline{J_{ij}} = \overline{\omega_i} = 0$ and covariances

$$\overline{\omega_i \omega_j} = \tau^{-2} \delta_{ij} \tag{3.10}$$

$$\overline{J_{ij} J_{kl}} = \frac{J^2}{N} (\delta_{ik}\delta_{jl} + \eta \delta_{il}\delta_{jk}). \tag{3.11}$$

The assumption $\overline{\omega_i} = 0$ is actually no restriction, since one can always transform to a rotating frame where this condition is fulfilled. The condition $\overline{J_{ij}} = 0$ means that the couplings are on the average neither ferromagnetic nor antiferromagnetic, but since the J_{ij} are random with positive and negative values one gets competing interactions and possibly frustration for a given oscillator. (3.10) means that the oscillator frequencies are uncorrelated and their variance can be used for fixing the time scale. In other words, setting $\tau = 1$, means that we measure the J_{ij} and J in units of τ^{-1} and time in units of τ. This leaves us with two independent parameters characterizing the system (in the limit

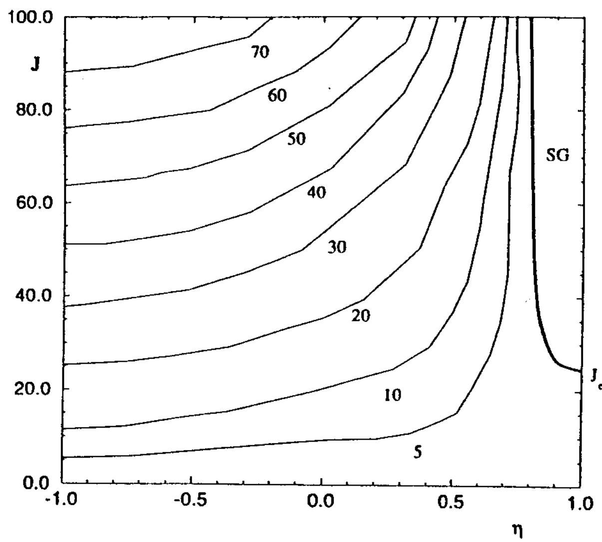

Figure 5
Value of the largest Lyapunov exponent λ_{max} in dependence on symmetry parameter η and interaction strength J represented by contour lines at levels $\lambda_{max} = 5,10,20,30,40,50,60,70$ ($N = 100$). The part of the parameter space (η, J) above $J_c(\eta)$ (bold line) where the oscillators freeze in random positions, is denoted by "SG" In this regime $\lambda_{max} = 0$. For $\eta < 0.8$ there is no spin glass order. The largest Lyapunov exponent decreases monotonically with η and for $\eta < 0.8$ grows monotonically with J.

$N \to \infty$): J with $0 < J < \infty$ is a measure for the interaction strength and $-1 \leq \eta \leq 1$ is the so-called symmetry parameter. From (3.11) follows that for $\eta = 0$ all coupling parameters are uncorrelated, while $\eta = 1$ imposes the constraint $J_{ij} = J_{ji}$ (symmetric interactions) and $\eta = -1$ implies $J_{ij} = -J_{ji}$ (antisymmetric interactions). The case of symmetric interactions has been treated previously [38].Our main results for this system are summarized in the phase diagram of Fig. 5.

As function of the parameters η and J one finds two different phases which can be regarded as a paramagnetic phase and a spin glass phase (SG in Fig. 5). The latter is characterized by a non-vanishing Edwards-Anderson (EA) parameter [3], which in our case may be defined as

$$q = \lim_{t \to \infty} \mathrm{Re} \left\langle \exp(i\phi_j(t_0) \exp(-i\phi_j(t+t_0)) \right\rangle . \qquad (3.12)$$

The spin glass phase, which exists above the full, bold line $J_c(\eta)$ in the region $0.8 \lesssim \eta \leq 1$ and for $J \geq J_c \equiv J_c(\eta = 1) \simeq 24.3$, is characterized by a vanishing "magnetization" $m = 0$ and an EA parameter $q = 1$. In addition we find in this phase that the maximal Lyapunov exponent is zero and that all other Lyapunov exponents are negative. This means that all oscillators are frozen into random directions, which is indeed a characteristic feature of a spin glass state at temperature $T = 0$. A similar freezing transition at a symmetry parameter $\eta_c \simeq 0.83$ has also been found in an asymmetric random neural network model [39], which may be related to our model at $J = \infty$. Note that in the symmetric case $\eta = 1$ the maximal Lyapunov exponent is zero for all values of J. Below $J_c \simeq 24.3$, however, not only one, but a finite fraction of zero Lyapunov exponents appears to exist which indicates that in this case a non-chaotic motion of the system on a lower dimensional manifold in phase space is still present.

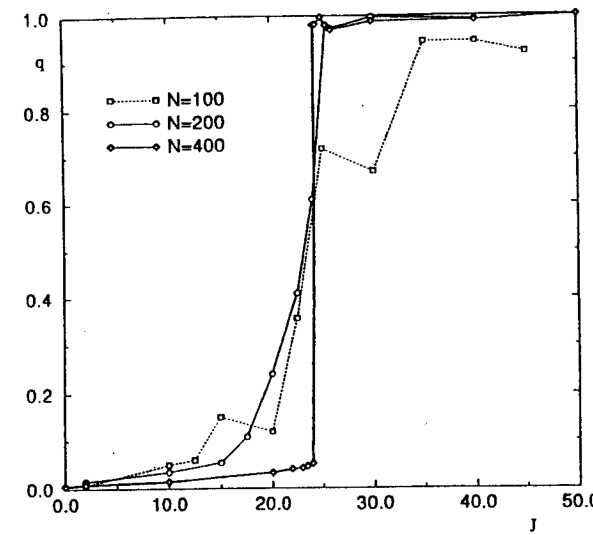

Figure 6
Order parameter q in dependence on interaction strength J for $N = 100,200,400$ oscillators. The transition from $q \approx 0$ for $J < J_c$ to $q \approx 1$ for $J > J_c$ becomes sharper with growing system size N. Between $J = 24$ and $J = 24.3$ the order parameter q changes from 0.046 to 0.982.

The situation is very different in the "paramagnetic" phase. There in addition to m also q vanishes, meaning that we are in a situation of dynamical disorder. This is also reflected in a positive maximal Lyapunov exponent implying chaotic motion, as can be seen from the level lines for the maximal Lyapunov exponent in Fig. 5. Thus we have here a certain correspondence between order parameters for disordered systems and characteristic quantities for dynamical systems.

A final point of interest, which shall be presented here concerns the nature of the phase transition at the transition line $J_c(\eta)$. Other interesting aspects of the Lyapunov spectra, behavior of correlation functions, the time-dependence of the magnetization, etc. can be found in [40]. The phase transition appears to be of first order as is demonstrated for $\eta = 1$ in Fig. 6. There the dependence of q on the coupling strength J is plotted for increasing system size. These simulations clearly suggests that the spin glass order parameter q jumps from zero to one in the thermodynamic limit $N \to \infty$ as J crosses $J_c \simeq 24.3$. The first order character of the transition is also confirmed for $0.8 \lesssim \eta \leq 1$, where interestingly we find for finite N strong intermittency phenomena in the neighborhood of the transition line [40]. A last speculative remark on the spin glass phase of this model is that one expects also aging phenomena to occur here.

4 Discussion

We have presented results for simple deterministic dynamical systems with quenched disorder. It turned out that disorder plays an extremely strong role for the macroscopic behavior of such systems. We have explicitly shown that normal and anomalous chaotic diffusion can be suppressed by disorder in dissipative and Hamiltonian systems. This may happen without any sign in Lyapunov exponents or related quantities. Interestingly, these systems show the phenomenon of aging. For biased systems one finds various phases with further anomalous transport properties. These results were obtained numerically and also rigorously. Furthermore, the underlying mechanisms appear to be universal and also robust against noise. For a high-dimensional randomly coupled dynamical system we found the occurrence of a first order phase transition into a spin-glass phase, which is also reflected in the nature of the Lyapunov spectrum. In summary the field of disordered dynamical systems provides us with many interesting problems with surprising effects to be expected also in the future.

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