Nonlinear dynamics with dissipative delays

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Abstract. Recently, it was shown that there is a hitherto unknown dichotomy for systems with time-varying delay [Phys. Rev. Lett. 118, 044104 (2017)]. There are systems with conservative delays, which are equivalent to time delay systems with constant delay. On the other hand, there are dissipative delays leading to completely new dynamics, which are not known from systems with constant delay. For example, a new type of chaos called laminar chaos was discovered in scalar delay differential equations (DDEs) [Phys. Rev. Lett. 120, 084102 (2018)]. In this contribution, we study the effects of dissipative delays on the dynamics of a harmonic oscillator with nonlinear delayed feedback. We show that laminar dynamics including laminar chaos is possible in higher dimensional DDEs.

Introduction

Time delay systems are dynamical systems, where the instantaneous dynamics at time t depends on the history of the system at the retarded time $R(t) < t$. Since the history is relevant for the future dynamics of time delay systems, they are infinite dimensional by nature, and in combination with nonlinearities, they exhibit a rich variety of dynamics including multistability and high dimensional chaos. Such systems can be found in various fields, ranging from engineering and control theory to life science or neuroscience [1]. They are described by delay differential equations (DDEs). In the literature often DDEs with constant time delay τ_0 are studied, that is $R(t) = t - \tau_0$. However, fluctuating time delays with $R(t) = t - \tau(t)$ and an invertible argument argument $R(t)$ are more relevant in practice, and therefore, we study such systems in this contribution.
Recently, it was shown that there are two classes of variable delays, namely *conservative* and *dissipative* delays [2]. Whereas systems with conservative delay can be transformed to systems with constant delay, systems with dissipative delay are qualitatively different from systems with constant delay [2, 3]. As a consequence, for example, a new type of chaos was observed in scalar systems with large dissipative delay, which is called *laminar chaos* [4]. In this contribution, we answer the question whether a similar dynamics is possible in higher dimensional systems with dissipative delay. In particular, we choose the prototypical example of a harmonic oscillator with nonlinear delayed feedback, which is given by the DDE

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = f\left(x(t-\bar{\tau}(t)), \dot{x}(t-\bar{\tau}(t))\right), \tag{1}$$

where $\gamma > 0$ and ω_0 is the damping and the eigenfrequency of the oscillator, respectively.

Solution of DDEs for large time-varying delay

From [4] we know that the effects from dissipative delays become more important for large delays. Therefore, we study Eq. (1) with large periodic delays $\bar{\tau}(t+T) = \bar{\tau}(t)$. Since we demand for a strictly increasing retarded argument $t - \bar{\tau}(t)$, a large time-varying delay automatically means a delay variation with large amplitude and large period T. Nevertheless, whether a delay is 'large', depends on the internal time scales of the system. In our case, it means that the delay is larger than the asymptotic relaxation time λ^{-1} of the system, where $\lambda = \gamma$, if $\gamma \leq \omega_0$, or $\lambda = \gamma - \sqrt{\gamma^2 - \omega_0^2}$ otherwise. For the following analysis we use the state variables $\mathbf{y}(t/T) = \mathrm{col}(x(t), \dot{x}(t))$, whose time argument is rescaled by the period T. In this case, Eq. (1) reads

$$\frac{1}{T}\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \begin{pmatrix} 0 \\ f\left(y_1(R(t)), y_2(R(t))\right) \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix}, \tag{2}$$

$R(t) = t - \tau(t)$ and $\tau(t) = \bar{\tau}(tT)/T$. Thus, the parameter T is an indicator for the largeness of the delay in Eq. (2), where time is rescaled such that $\tau(t+1) = \tau(t)$ and $R(t+1) = R(t) + 1$. In general, the method of steps can be used to solve Eq. (2) stepwise in the intervals $I_n := (t_{n-1}, t_n]$ with $R(t_n) = t_{n-1}$ and $n \geq 0$. In particular, the initial function of Eq. (2) is given by $\mathbf{y}(t)$ with $t \in I_0$ and the solution in the interval $t \in I_{n+1}$ can be constructed from the solution in the previous interval I_n by

$$\mathbf{y}(t) = \mathbf{M}(t - t_n)\mathbf{y}(t_n) + \int_{t_n}^{t} \mathbf{M}(t - t')T \begin{pmatrix} 0 \\ f\left(y_1(R(t')), y_2(R(t'))\right) \end{pmatrix} dt', \tag{3}$$

where $\mathbf{M}(\theta) = \exp(T\mathbf{A}\theta)$ is the fundamental matrix solution of the ODE part of the DDE, i.e. Eq. (2) with $f = 0$. In other words $\mathbf{M}(\theta)$ describes the dynamics of a damped harmonic oscillator which converges exponentially with the relaxation rate λT to the equilibrium $\mathbf{y} = \mathbf{0}$. For large T, we have $\mathbf{M}(t - t_n) \approx \mathbf{0}$ for $t > t_n$ and

$\mathbf{M}(t-t')T\ \Theta(t-t') \approx -\mathbf{A}^{-1}\delta(t-t')$, where Θ is the Heaviside step function. This means that in the limit $T\to\infty$, the solution can be constructed via the map

$$y_1(t) = \frac{1}{\omega_0^2} f\left(y_1(R(t)), 0\right), \quad y_2(t) = 0. \tag{4}$$

Eq. (4) can be also obtained directly from the DDE because the left hand side of Eq. (2) vanishes for $T\to\infty$. Eq. (4) means that the time arguments and the function values from the solution in two consecutive intervals are connected via the maps $t' = R(t)$ and $y' = f(y,0)/\omega_0^2$, respectively. Following [4], we call Eq. (4) limit map and the map $t' = R(t)$ access map, because it describes the retarded access to the history. Eq. (4) holds for $T\to\infty$. For large but finite T the integral in Eq. (3) with the exponentially decreasing kernel $\mathbf{M}(t-t')T$ leads to an additional smoothing of the solution of the limit map.

Conservative vs. dissipative delays - Turbulent chaos vs. laminar chaos

From the limit map Eq. (4) it follows that the access map R significantly affects the dynamics of systems with time-varying delay. A classical example are sinusoidally time-varying delays with access maps of the form

$$t' = t - \tau_0 - A\sin(2\pi t)/(2\pi), \tag{5}$$

where τ_0 and $0 \le A < 1$ are the mean and the amplitude of the delay variation in the rescaled time. Eq. (5) is also known as circle map. Depending on the choice of the parameters τ_0 and A two qualitatively different dynamics can be observed. White regions in Fig. 1a) correspond to quasiperiodic dynamics, where the distance between nearby trajectories remains constant on average. We call these delays conservative delays. Systems with conservative delay can be transformed to systems with constant delay [2, 3]. On the other hand, the black regions in Fig. 1a) correspond to attracting periodic dynamics, which is called mode-locking. These regions are the Arnold tongues of the circle map and the corresponding delays are called dissipative delays.

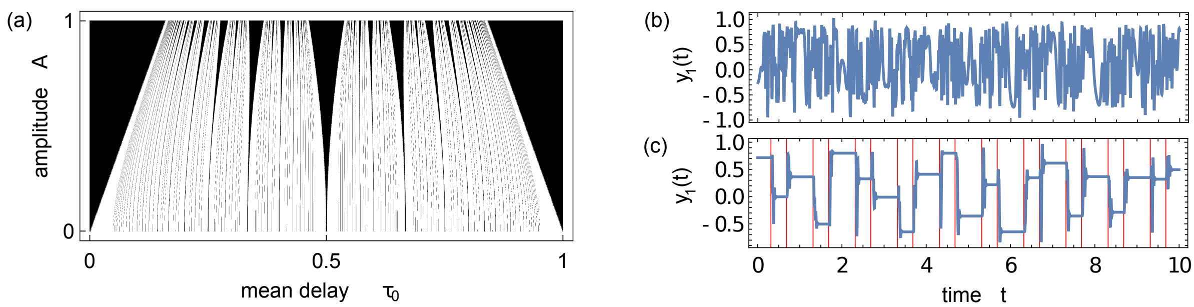

Figure 1: (a) Parameter space of the sinusoidally varying delay $\tau(t) = \tau_0 + A\sin(2\pi t)/(2\pi)$. (b,c) Exemplary trajectories $y_1(t)$ for Eq. 2, with $f(y_1, y_2) = 0.8\cos(\pi y_1)$ and $\tau(t)$ as in (a), showing (b) turbulent chaos and (c) laminar chaos, where $\tau_0 = 1.54$ and $\tau_0 = 1.5$, respectively. The remaining parameters are given by $A = 0.9$, $\omega_0 = 1.0$, $\gamma = 0.5$, and $T = 150$

For large T, the quasiperiodic dynamics of the access map for conservative delays means that the infinitely many function values of $y_1(t)$ in the interval I_n are iterated rather independently via the map $y' = f(y,0)/\omega_0^2$. If this map exhibits chaos, strong chaotic fluctuations are observed in the solution of the DDE as can be seen in Fig. 1b). However, if there is a dissipative delay, the access map has infinitely many stable and unstable periodic points $t_l^* = R^q(t_l^*) + p$, where R^q denotes the qth iterate of the access map. Under certain conditions, the attracting dynamics of the access map leads to laminar phases in the DDE, which are separated by the unstable periodic points of the map R [4]. An exemplary solution for that can be found in Fig. 1c), where the map $y' = f(y,0)/\omega_0^2$ is chaotic. This so-called laminar chaos is characterized by laminar phases with nearly constant displacement $y_1(t)$, separated by burstlike transitions near the unstable periodic points (red vertical lines) of the access map.

Summary

Based on the prototypical example of a harmonic oscillator with nonlinear delayed feedback, we have shown that laminar chaos can appear also for multi-component DDEs with large time-varying delay. If the delay is large compared to the relaxation time λ^{-1} of the harmonic oscillator, the solution of the DDE can be approximated via the two dimensional limit map Eq. (4). Similar to the existing results for scalar DDEs [4], in the delayed oscillator a dissipative delay with its attracting dynamics of the access map also leads to solutions with laminar phases.

References

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