On the Dynamics of Nonlinear Hysteretic Systems

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Summary. We show that the coupling of simple dynamical systems with a typical model for hysteresis, the Preisach model, leads to interesting new scenarios in nonlinear dynamics. The Preisach model is known from many branches of science to be a universal model for complex, non-local hysteresis. We provide results for three cases, for two, where the Preisach model simply acts as hysteretic transducer for stochastic and deterministically chaotic input, respectively, and thirdly for a model with hysteretic feedback into a nonlinear system. We find, for instance, new kinds of attractors and a fractal dependence of the asymptotic behavior as function of the starting values.

1 Introduction

Hysteresis is a well-known phenomenon in many branches of science [1]. It refers to situations, where for a given external input parameter or field multiple internal system states are possible. Which one of these states is assumed, depends on previous parameter variations and therefore on the history of the system. Classical examples are magnetic materials, where magnetization and external magnetic field are hysteretically related [2]. Sometimes one encounters bistable situations corresponding to only one single hysteresis loop describing the input-output or field vs. state relation. We are, however, interested in complex hysteretic systems with arbitrary many internal states, which correspond to a given value of the external field or input. Correspondingly, these systems in addition to a major hysteresis loop, show sub-loops, sub-sub-loops, etc. as the input is varied. Apart from magnetic materials such behavior is ubiquitously found in all kinds of systems, e.g. in piezoelectric materials [3], shape memory alloys [4], superconducting systems, porous materials such as soils, and also in economic systems [5,6]. Complex hysteresis and the model introduced below, provide a prominent example, how methodological approaches interconnect seemingly disparate disciplines and systems. The complex behavior of hysteretic systems is often difficult to access from
first principles. A phenomenological model which is very successfully applied to all of the mentioned systems is the so-called Preisach model. It was introduced in the context of magnetic systems by Weiss and de Freudenreich almost a century ago [7] and became popular through the work of Preisach [8]. The universal properties of this model and its limitations were elaborated in detail in [12, 13]. In the mathematical literature the corresponding operator, the so-called Preisach operator, which maps input time series to output time series, also found much interest [9, 10]. While the hysteretic behavior of these systems by itself has been investigated quite extensively, not much is known about scenarios that may arise if hysteretic subsystems are coupled dynamically to its environment. This amounts to considering dynamical systems with hysteretic nonlinearity. In addition there do not exist many rigorous results on the characteristic properties of the output time series generated by the Preisach model. Especially from an experimental or practical point of view one is interested, for instance, in correlations within the output time series and the corresponding spectral density, respectively. We indicate some exact results in this respect.

2 Models

We treat this problem by adopting a well-known model for the hysteretic subsystem, the so-called Preisach model, which is known to have universal properties and has been used by physicists and engineers in modeling all of the above mentioned systems [1, 12]. It is defined by the following operator, which maps an input time series \( x(t) \) to an output \( y(t) \) by superimposing with weights \( \mu(\alpha, \beta) \) the outputs \( s_{\alpha\beta}[x(t)] \) of infinitely many elementary hysteresis loops

\[
\int \int d\alpha d\beta \mu(\alpha, \beta) s_{\alpha\beta}[x(t)]
\]

The elementary hysteresis loops are rectangular loops with outputs \( s_{\alpha\beta}[x(t)] \in \{-1, +1\} \) and switching values \( \alpha \) (upper) and \( \beta \) (lower) (delayed relay or Schmitt trigger) as depicted in Fig. 1. Formally the output \( s_{\alpha\beta}[x(t)] \) of such a non-ideal relay with initial state \( s_{\alpha\beta}(t_0) = s_0 \) for a given input time series \( x(t), t \geq t_0 \), can be written as

\[
s_{\alpha\beta}[x(t)] = \begin{cases} 
+1 & \text{if there exists } t_1 \in [t_0, t] \text{ s.t. } x(t_1) \geq \alpha \text{ and } x(\tau) > \beta \text{ for all } \tau \in [t_1, t] \\
-1 & \text{if there exists } t_1 \in [t_0, t] \text{ s.t. } x(t_1) \leq \beta \text{ and } x(\tau) < \alpha \text{ for all } \tau \in [t_1, t] \\
s_0 & \text{if } \beta < x(\tau) < \alpha \text{ for all } \tau \in [t_0, t]
\end{cases}
\]

The weight function \( \mu(\alpha, \beta) \) in Eq.(1) is typically taken as a 2-dimensional Gaussian restricted to the region \( \alpha > \beta \) in the so-called Preisach plane, or a constant in the triangle \(-1 < \beta < \alpha < +1\). In the following we assume the latter. We consider input and output time series, \( \{x(t)\} \) and \( \{y(t)\} \),
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![Diagram](image)

**Fig. 1.** The elementary rectangular hysteresis loop switches to the state \( s_{\alpha\beta} = +1 \) if the input rises beyond the upper switching value \( \alpha \), and back to \( s_{\alpha\beta} = -1 \) if it afterwards decreases below the lower threshold \( \beta \).

respectively, which may be discrete or continuous in time. An important property of the Preisach model is that it can store certain extremal values of the previous input time series by remembering which of the elementary relays have switched and which have not. This behavior is visualized in Fig. 2: Each point \((\beta, \alpha)\) in the triangle of Fig. 2b represents an elementary hysteresis loop. The points or loops \((\beta, \alpha)\), which are in the state \( s_{\alpha\beta} = +1 \) lie in the area \( S^+ \) below the staircase-like separation line, the remaining elementary hysteresis loops in the area \( S^- \) are in the state \( s_{\alpha\beta} = -1 \). The appearance of the staircase-like separation line can be explained as follows. Rising the input from some strongly negative value to a first maximum \( M_1 \) as in Fig. 2a corresponds to a straight horizontal separation line wandering upwards in Fig. 2b thereby switching all loops \((\beta, \alpha)\) below it to values \( s_{\alpha\beta} = +1 \) because the individual upper threshold values \( \alpha \) were surpassed. Hereby it was assumed that in the initial state of the Preisach model all loops \((\beta, \alpha)\) were in the state \( s_{\alpha\beta}(x(0)) = -1 \). A subsequent decrease of \( x(t) \) to the minimum \( m_1 \) corresponds to a vertical line moving to the left in Fig. 2b, thereby causing a corner in the separation line at the value \((\beta, \alpha) = (m_1, M_1)\). The next increase to \( M_2 \) and decrease to \( m_2 \) causes in the same way the appearance of an additional corner at \((\beta, \alpha) = (m_2, M_2)\), and so on. Note that the current input always corresponds to the intersection of the separation line with the bisectrix \( \beta = \alpha \). In this way the Preisach model develops memory, which in principle can become infinitely long, e.g. for a damped sinusoidal input. The above construction also shows that some of the previously created corners may be deleted again, for instance, if the current input increases beyond some of the previously stored corner coordinates \( M_i \). Therefore the memory, and also its length, the number of corner coordinates, is typically fluctuating in time with the fluctuations depending on the input time series in a non-trivial way. One should emphasize, that the currently stored memory sequence determines, in addition to the current input, the actual output of the Preisach model.

We couple the Preisach model, as defined above, with ordinary dynamical systems in the following way. In model A and B we use the Preisach model simply as a transducer: We generate the input for the Preisach model from
deterministic or stochastic dynamical systems and observe the output of the Preisach model. We use for model A as input time series those generated by the logistic map

\[ x(t + 1) = f(x(t)) = 1 - ax^2(t) \]  

whereas for model B we generate the input by the stochastic Ornstein-Uhlenbeck process

\[ \frac{d}{dt} x(t) = -\frac{1}{\tau_{in}} x(t) + \xi(t) \]  

where \( \xi(t) \) denotes Gaussian white noise with zero mean and unit variance. \( \tau_{in} \) is the correlation time for this process. We will see that despite the simplicity of these input processes, the output shows quite complicated behavior. Model C provides a simple example for a nonlinear hysteretic feedback system: We apply to the output of the Preisach model the nonlinear logistic map, the result is used as new input for the Preisach model, the output of the latter is again iterated by the logistic map, then fed into the Preisach model, etc. Thus we repeatedly iterate the logistic map with Preisach nonlinearity consisting of one application of the Preisach transducer followed by one application of the logistic map

\[ x(t + 1) = P \circ f(x(t)) \]  

Note that the action of \( P \) depends in principle on all previous input values.

3 Results

The complexity of the output of a hysteretic system as described by the Preisach model when subject to a chaotic input (Model A) is demonstrated by the following results. We use as input the fully chaotic time series generated by the logistic map for the parameter value \( a = 2 \). In Fig. 3 we show
for reference the return map for the input $x(t+1)$ vs. $x(t)$, which is just the logistic parabola, and the subject of interest, the return map $y(t+1)$ vs. $y(t)$ of the corresponding output of the Preisach model. The many branches in the

![Graph](image.png)

**Fig. 3.** The return map for the output of the Preisach model (red for even, blue for odd memory length) is shown in comparison to the return map of the chaotic input, the logistic parabola (green). The structure in the output return map is attributed to the fluctuating memory of the Preisach model.

latter reflect the high-dimensionality of the system. Although a thorough understanding of the details of this structure needs further investigations, we see that the branches can be classified according to the parity (odd or even) of the memory length, the number of values needed to specify in the Preisach plane ($\alpha$-$\beta$ plane) the piecewise constant separation line (see Fig. 2). Since this line, and correspondingly the system memory and its length, changes dynamically depending on the input, we deal here with a class of dynamical systems which are currently not well understood: The memory length can be regarded as an effective dimensionality of the dynamical system, and therefore this is an example of a system with a time-dependent effective dimension (the "true" system dimension is infinite). The distribution of this effective dimension or memory length is shown in Fig. 4 for input data stemming from the logistic map with three different parameters. There is a slight difference in the distribution of odd (circles) and even values (crosses), but as the lines in this figure indicates, they are approximately Gauss distributed with mean value e.g. for $a = 2.00$ around $d = 18$. The last result means that Fig. 3 is the projection of an attractor with "mean dimension" $d = 18$. Since the properties of the memory are input dependent one finds a quite different behavior e.g. for the fractal input generated for the parameter $a$ corresponding to the Feigenbaum accumulation point of the logistic map. More details on this system and the behavior of the memory can be found in [14].
Fig. 4. Memory length distribution $p(L)$ of the Preisach model with input from logistic maps with parameters $a = 1.40$ (blue), $a = 1.89$ (green), and $a = 2.00$ (red).

Instead of a deterministic chaotic input one may be interested also in the system behavior under stochastic input (Model B). Some aspects of this problem were treated for the Ornstein-Uhlenbeck (OU) process as input and for the special case of symmetric hysteresis loops in [18]. Here we are interested in the dependence of the output properties as the correlation time $\tau_{in}$ of the input is varied. The output can be characterized by the two-point correlation functions which in the stationary case is given by

$$C(\tau) = \langle y(0) y(\tau) \rangle - \langle y \rangle^2$$

(6)

or its Fourier transform, the spectral density $S(\omega) = \int C(\tau) e^{i\omega \tau} d\tau$. In Fig. 5 the spectral density of the output is shown for four different values of $\tau_{in}$. While the input spectrum, i.e. the spectrum of the OU process, is Lorentzian

$$S(\omega) = \frac{1}{\omega^2 + \tau_{in}^{-2}},$$

(7)

the output spectrum and correspondingly the structure of its autocorrelation function is more complicated. Two features become apparent. For increasing input correlation time $\tau_{in}$, the output spectrum develops side peaks and structures hinting to a hitherto unexplained resonance phenomenon. Secondly, the wings of the output spectra deviate clearly from the Lorentzian line shape. Another interesting aspect is that the width of the spectrum becomes very narrow for small input correlation times $\tau_{in}$, implying that the output correlation time $\tau_{out}$ becomes longer for decreasing $\tau_{in}$. This is shown in Fig. 6, where the output correlation times $\tau_{out}$ were determined from the width of a Lorentzian fitted to the output spectrum. Actually, $\tau_{out}$ seems to diverge for $\tau_{in} \rightarrow 0$, a result which is supported also by some recent analytic results [11]: For uncorrelated discrete time input we can show that the output correlations
show quite generally a power law decay $C(t) \sim t^{-\eta}$ with $0 < \eta < \infty$. The actual value of $\eta$ depends on the form of the tails of the input density and the form of the Preisach density. This result means that depending on $\eta$ one may even observe $1/f$-noise in the output spectrum: if $0 < \eta < 1$, one obtains for the spectrum at small frequencies $S(\omega) \sim \omega^{-1+\eta}$. All these results are mathematical manifestations of the internal memory of hysteretic systems, which can become arbitrarily long.

For increasing $\tau$ the spectrum develops more structure with clear deviations from a Lorentzian profile. These deviations become even stronger for systems with a Gaussian Preisach density $\mu(\alpha, \beta)$. More details also on the related variation of the mean memory length with $\tau_{in}$ can be found in [15]. The origin of these structures, however, is not yet clear and needs further investigations.

The following results for model C seem to be the most complicated to understand. Some aspects of dynamical systems with Preisach nonlinearity have been published recently for a driven iron pendulum [16]. In contrast to that our discrete time hysteretic dynamical system is much simpler and allows e.g. for a systematic investigation of the dependence on initial conditions. In Fig. 7 we show the asymptotic state of the system as the initial value $x_0$ is
changed (as initial state of the Preisach model all elementary relays are set to 
\(s_{\alpha\beta}(t_0) = -1\), the parameter of the logistic map is \(a = 2\)). The asymptotic

![Graph showing correlation time dependence](image1)

**Fig. 6.** The dependence of the output correlation time \(\tau_{\text{out}}\) on the input correlation 
time \(\tau_{\text{in}}\). The three curves were obtained from time series of different lengths.

The asymptotic state may be simply a fixed point or a chaotic or other complicated orbit. 
The lines in Fig. 7 (e.g. for \(x_0 < -0.5\)) correspond to lines of fixed points, 
i.e. the attracting fixed point varies continuously as the initial condition is 
changed: The system exhibits infinitely many attractors, in this case point 
attractors. One observes, however, also more complicated asymptotic objects, 
which are visible as scattered points in the \(x^*\)-direction. Probably these 
are chaotic objects, but this needs to be investigated in more detail, too.

![Graph showing asymptotic state dependence](image2)

**Fig. 7.** The dependence of the asymptotic state of model C in dependence of the 
initial condition is shown. The right figure shows an enlargement of the region 
between \(x_0 = 0.24\) and \(x_0 = 0.25\) of the left figure indicating a self-similar structure.
Another interesting aspect of Fig. 7 is the apparent approximate self-similarity of the asymptotic structures in dependence on the initial conditions. E.g. the region of initial conditions between \( x_0 = 0.2454 \) and \( x_0 = 0.25 \) (from cusp to cusp in right panel of Fig. 7) seems to be similar to the region from \( x_0 = -0.187 \) to \( x_0 = 0.675 \) (left of Fig. 7). For other parameters of the logistic map the asymptotic state of the map with Preisach non-linearity shows also similar structures. In general the dissipative nature of the hysteretic Preisach transducer tends to suppress chaotic motion. For fixed initial condition \( x_0 \) one finds complicated bifurcation scenarios as the parameter \( a \) is varied. More results can be found in [17]. Some of them, e.g. the line of fixed points for \( x_0 < -0.5 \) are easily understood. A better understanding of the self-similar structures of Fig. 7 and the mentioned bifurcation scenarios, however, needs further investigations.

4 Summary

We have demonstrated that the coupling of simple dynamical systems with a typical hysteresis model leads to interesting new scenarios in nonlinear dynamics. This is a result of the ability of hysteretic systems to store information about previous input dynamically. The time-dependent memory, or fluctuating dimension of the system provides on one hand a challenge for the analysis of such systems. On the other hand such systems are of great practical importance and therefore are a rewarding subject of future research in nonlinear dynamics.

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References

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