Reconstructing 3D compact sets– (Gentle) Introduction to Computational Morse Theory and Persistence Analysis

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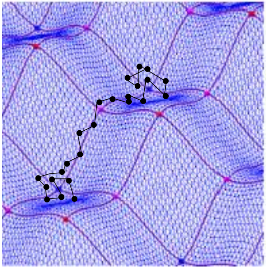
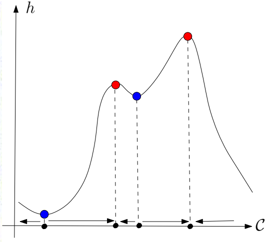
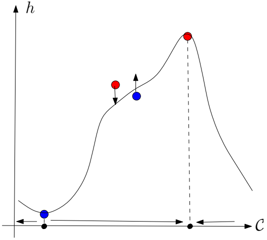
Motivations

- Stratification of point clouds
- Multi-scale representation of shapes





Original



Simplified

Outline

Reconstruction?

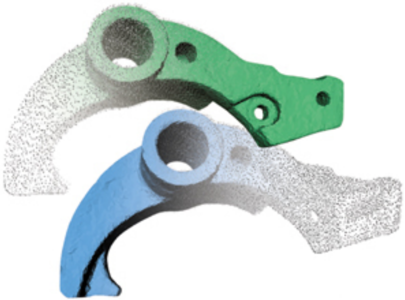
The Flow Complex (A Gentle Introduction)

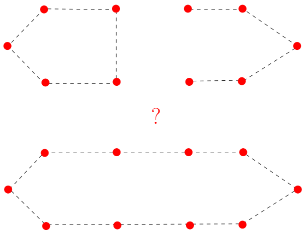
Flow Complex Based Reconstruction of Compact Sets

Conclusion - Outlook

Surface Reconstruction

▷Pb.: Compute a (piece-wise) surface from a point cloud
▷Context: Reverse Engineering, Medical data processing, Geology, Cultural Heritage





▷Sc. Challenge(s): Sampling models –minimal amount of information required to reconstruct *faithfully (geometry, topology)*
▷Previous work: heuristics before Amenta - Bern (98): *local Feature Size* —aka *Local Width*

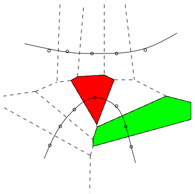
Reconstruction Difficulties

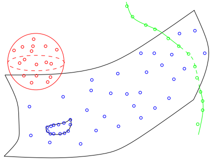
▷ Using an a priori model: classical surface-based reconstruction

- (Compact) Smooth surfaces versus sharps features, boundaries
- Samples may not comply with the model:
  - Unstable estimates for differential geometry based quantities:
    - normals, principal curvatures
  - The sampling may not be a surface at all
- Unique versus plausible reconstructions

▷ Calls for a general reconstruction strategy:

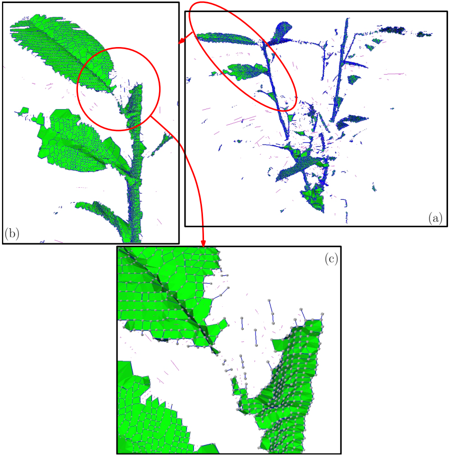
- No a priori: handling compact sets
- Providing multi-scale reconstruction i.e. plausible shapes





Practically: Reconstructing the Boundary of a Solid?

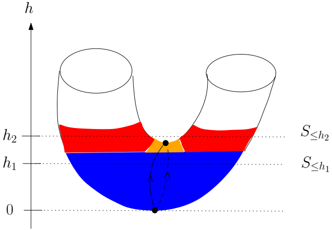
- ▷ Thin parts might just be too thin

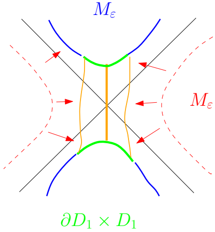


- ▷ Calls for a general reconstruction strategy

Beech tree reconstruction, with parameters $t_r = 2.2$, $t_p = 1.1$ (a) Overview of this noisy and under-sampled model (b,c) Zoom near a an under-sampled peduncle. The point cloud is courtesy of J-C. Chambelland et al, UMR 547 PIAF - INRA/UBP.

Morse theory - Critical points - Level sets





Theorem

Let p (assumed to be at the origin) be a c.p. of index λ, $M_h = \{p \in M \mid f(p) \le h\}$. For $\varepsilon >$ critical value c:

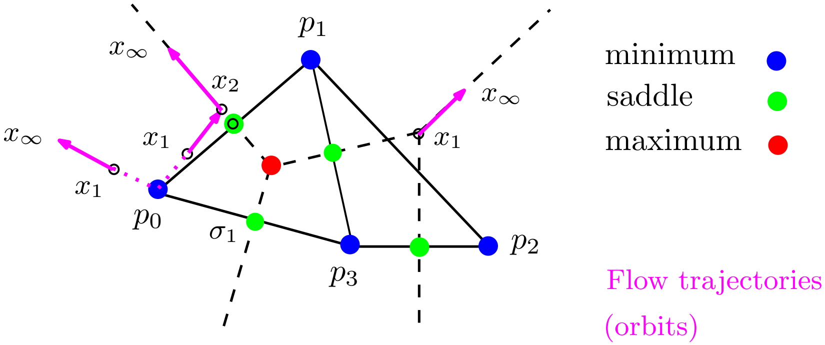
- *M_ε is homotopy equivalent to a CW complex obtained by attaching a λ-cell (D_λ) to $M_{-\varepsilon}$.*
- *M_ε and $M_{-\varepsilon} \cup D_\lambda \times D_{n-\lambda}$ are diffeomorphic*

Distance Function to Sample Points

- Given a collection of points $\{p_i\}_{i=1,\ldots,n}$, consider:

$$d(p) = \min_i \| pp_i \|$$

- Increasing the value of d:



The Morse Puzzle of the Distance Function

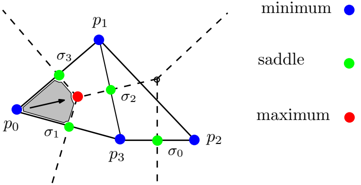
Flow Complex
- Critical points
- Stable manifolds
- Unstable manifolds

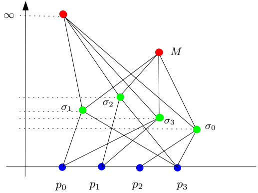
Morse-Smale diagram
- regions of homogeneous flow:
- $\bigcap$ of (un)stable manifolds

Hasse diagram
- orbits of the distance function
- through consecutive crit. pts

▷`Ref: Giesen, John; ACM SODA; 2003`

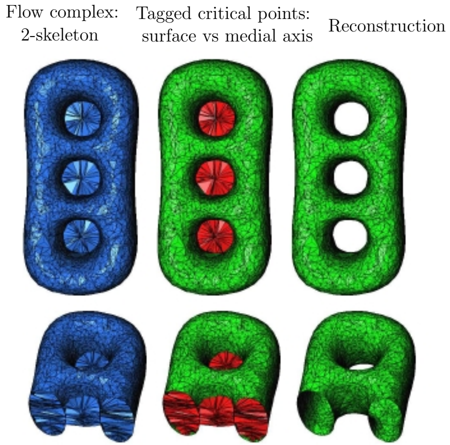
▷`Ref: Cazals, Pion; ACM SoCG; 2008`

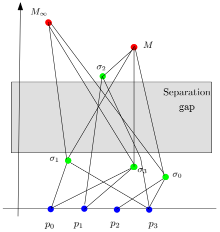




Flow Based Surface Reconstruction: Separating Critical Points

▷ ε-samples accommodate a separation of critical points:





Algo. based on the separation of c.p.:
- surface c.p.: angle criterion
- MA c.p.: distance criterion wrt poles

Under mild hypothesis on the sampling:
- Reconstruction $\hat{S}$:
  - $\hat{S} \subset$ tube around S
  - $\hat{S}$ isotopic to S

▷Ref: Dey, Giesen, Ramos,Sadri; ACM SoCG 2005

Generalization to Compact Sets: Finding Cuts from the Hasse Diagram

▷ Reconstruction:

- Collection of Stable Manifolds (SM) of all indices i.e. 0 to 3
- Governed by a parameter $t_r > 0$
- Abusing terminology: inserting a Hasse node $\sim$ inserting its SM

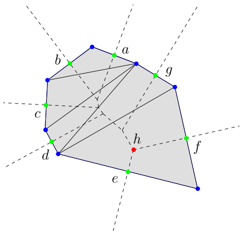
▷ Key ingredient: ratio $V(b)/V(a)$
between crit. values of incicdent crit. pts in the Hasse diagram

▷ Initialization: selected Gabriel edges

▷ Upflow extension:
a in reconstruction; $index(b) = index(a) + 1$
a sponsors b with priority $r_u = V(b)/V(a)$

▷ Regularization:
h in reconstruction
h triggers insertion of a, b, c, d, e, f, g; $r_r = 1$

▷ Horizontal extension:
a in reconstruction; $index(b) = index(a)$
a sponsors b with priority $r_h = \max(V(a)/V(b), V(b)/V(a))$

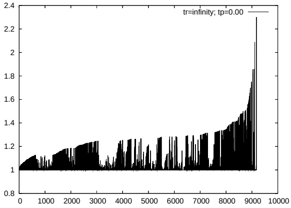


The Three Steps: Canonical Ordering of Insertions

Reconstruction Profiles

- ▷ Initialization: start from selected one-dimensional SM
  - – Gabriel edge $e = (v_0, v_1)$ is an init edge iff
    v_1 if the nearest neighbor of v_0 or vice-versa

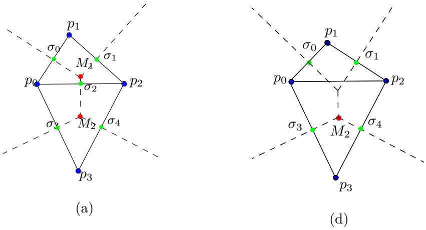
- ▷ Sponsoring as an iterative process:
  - – sponsored nodes placed into a priority queue Q
    - priority : least r_u, r_r, r_h ratio from any sponsor
    - requires insertion and/or updates of nodes already in Q
  - – induces a canonical insertion of nodes: take the easiest step
    i.e. pop node with least priority provided it is $< t_r$
- ▷ Reconstruction profile: $(r_i)_{i \geq 1}$ for $t_r = \infty$
  - – encompasses all possible reconstructions



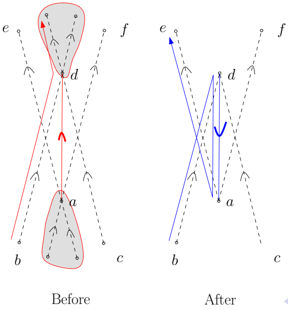
Example recons. profile (more later!)
Note: tail of diagram $\sim$ features
(significant maxima) of the model

Widening the Gaps: Persistence on the Hasse Diagram

- ▷ Insufficient sampling: widen the gaps
- ▷ Cancelling pairs of *nearby* critical points: reverting the flow



- ▷ Hasse simplification: multiplexing and redistribution of SMs

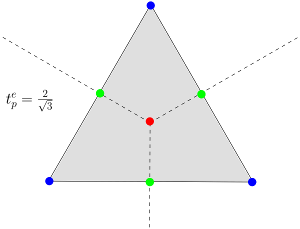


Hasse Simplification Cont'd

▷ Algorithm

- Iterative process based on the persistence priority $p(a, b) = V(b)/V(a)$
- Incremental cancellation of all pairs up to priority $\leq t_p$

▷ Persistence: the obtuse triangle



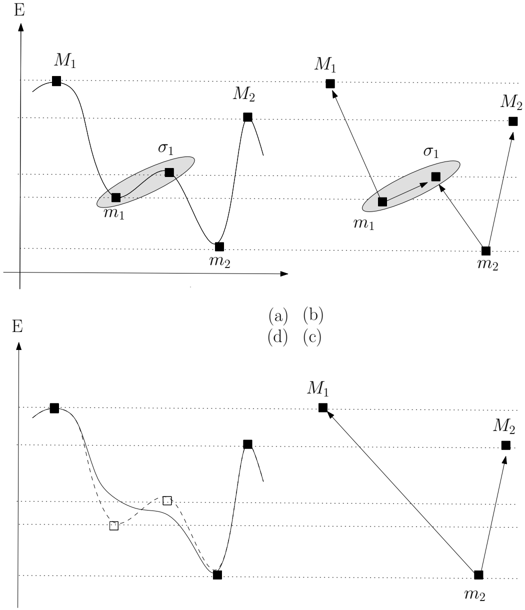
$t_p > t_p^e$ cancels edges-triangle

▷ Key steps

- Update of the Hasse diagram
  remove edges incident on nodes cancelled
  add edges of bipartite graph
  $\text{In}(b) \times \text{Out}(a)$
- Redistribution of SM

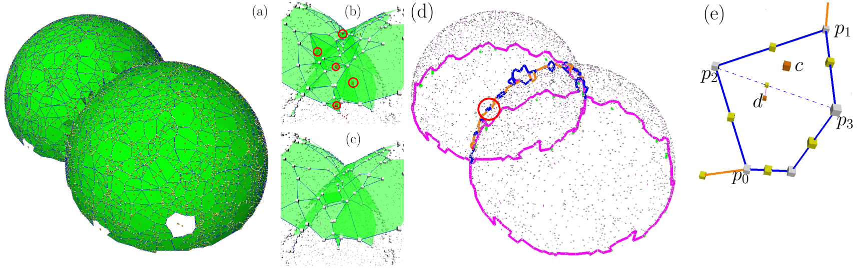
- surface with boundaries:
  may create leaks of SM
- but not systematic:
  SM rescued thanks to multiplexing

Application to (Sampled) Landscapes



Non Manifold Reconstruction

▷ Reconstructing two intersecting spheres

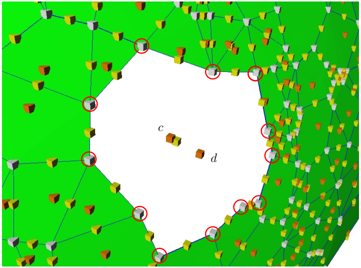


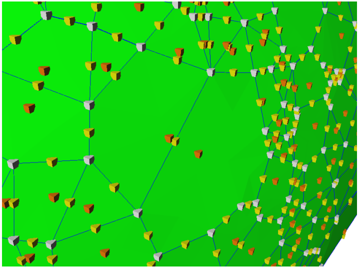
(a) $t_r = 1.9,\ t_p = 0$
(b) Transparent view of (a)
(c) $t_r = 2.5,\ t_p = 1.05$:
maxima cancelled

(d) $t_r = 2.5,\ t_p = 1.05$,
multiplicity of Gabriel edges:
zero one three four five
(e) Circled region of Fig. (d):
intersection curve stretched to a disk

Enumerating Plausible Reconstructions

- Do the circled points punch a hole or not?

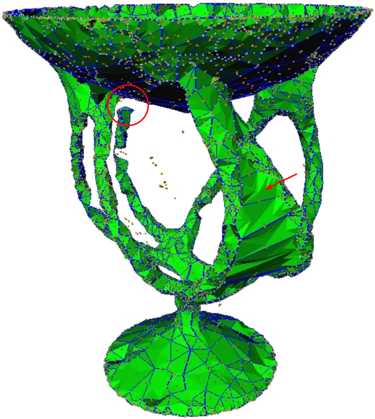




(a) At $t_r = 1.9, t_p = 0$: hole (b) At $t_r = 2.1, t_p = 0$: hole filled

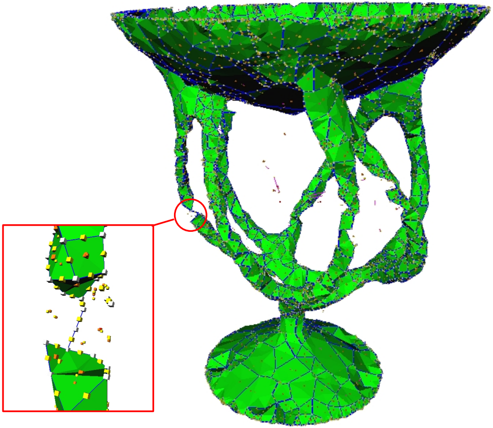
On the Importance of Persistence

▷ Undesirable extensions



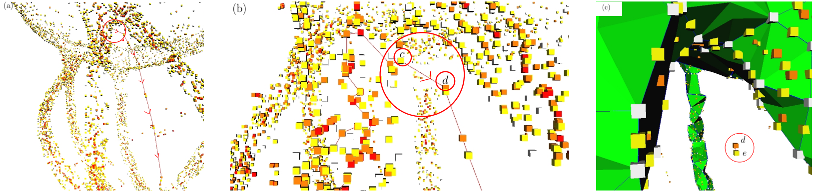
(a) At $t_r = 1.7$, $t_p = 0$: fin

▷ Fixed by persistence



(b) At $t_r = 2$, $t_p = 1.02$: fin is gone

Untangling the Role of Persistence

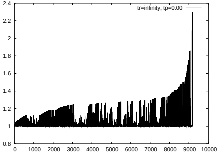


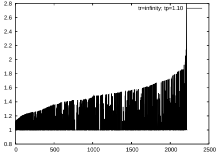
(a,b)The extension path followed by the algorithm–red arrows.
(c) Pairing by persistence: d paired to an index one critical point e

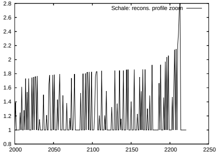
▷ Note: offers a local control so as to fix the sampling

Combining Parameters t_r and t_p: the Exple of Vase

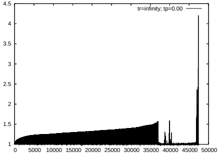
▷ Vase: (a) $t_p = 0, t_r = \infty$ (a) $t_p = 1.1, t_r = \infty$ (c) Tail of (b)

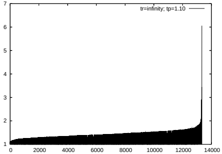


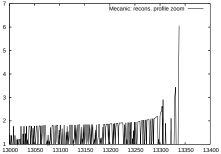




▷ Mechanic.: (a) $t_p = 0, t_r = \infty$ (b) $t_p = 1.1, t_r = \infty$ (c) Tail of (b)







▷ Recommendations

– values of $t_p \in [1.02, 1.2]$ and $t_r \in [1.9, 2.1]$ yield comparable results
– beyond $t_r \in [1.9, 2.1]$: number of <u>features</u> of the model (significant max.)

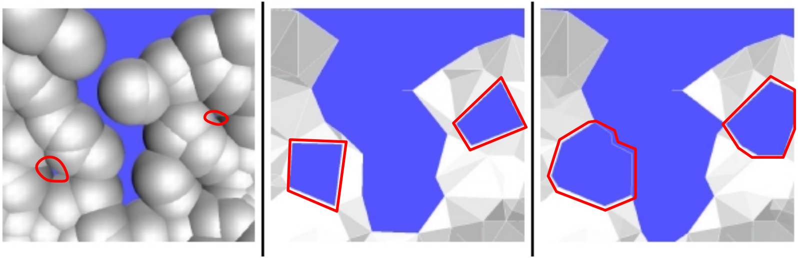
Union of balls, α-shapes, Flow-shapes

- Topology of a union of balls: union of balls $\sim$ flow shape $\sim$ α-shape

  union of balls and α-shapes are homotopy equivalent [Edelsbrunner, 92]
  α-shapes and flow shapes are homotopy equivalent [Dey, Giesen, John, 03]

- Key differences

  Events triggering addition in α-shape: Gabriel simplices
  critical points



Union of balls α-complex / α-shape Flow shape

Reconstruction Guarantees

▷ **Def.** Point cloud P is a ρ-*uniform* approximation of a compact set K if half the distance between the two closest sample points in P is at least ρ times the the Hausdorff distance between P and K, where $0 < \rho < 1$.

▷ **Thm.** Let K be a compact subset of $\mathbb{R}^3$;
Assume P is a ρ-uniform (κ, μ)-approximation of K. If

$$\frac{4}{\rho\mu^2} < t_r < \frac{\mu^2}{4\kappa} - 1,$$

then the reconstruction is homotopy equivalent to K^η for small enough η.

▷ Proof sketch:

Dey-Gisen-John'03 | Edelsbrunner'95 | Recons. thm.

$$\begin{tikzcd} F_\alpha\text{: flow shape} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & \alpha\text{-shape} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & P^\alpha\text{: union balls} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & K^\eta\text{: compact set} \end{tikzcd}$$

($\cup$ stable manifolds) | ($\cup$ of domains of simplices)

▷ **Note.** Thm uses uniform sampling; algorithm does not.

Conclusion

- Given samples on a landscape
  - Morse theory of the distance function to the samples
    - Multi-scale analysis
    - Stratification of the point cloud
    - Plausible reconstructions
    - For landscapes: is it worth it (paucity of the sampling)?
  - Morse theory of the height function (energy)
    - Detection of stable minima
    - Assignment of samples to the stable basin
    - Maintenance of connectivity $m - \sigma - m$
  - Extension to the dynamic setting—cf move operator

- Samples versus continuous Hamiltonian
  - Smooth case: compact representation - parametric model (possibly false)
  - Finite sampling: costly representation - no parametric model