Dynamics of a rotor partially filled with a viscous incompressible Fluid

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GAMM 2016 Section S05 – Nonlinear Oscillations
Rotors filled with liquids are of technical relevance for
- centrifuges,
- liquid-filled projectiles,
- drives and turbines with inner cooling.

E-drive with active cooling inside

https://incarplus.thyssenkrupp.com
Dynamics of a rotor partially filled with a viscous incompressible Fluid

Dominik Kern¹, Georg Jehle²

Introduction
Model
Discretization
Results
Summary

Scope

rotor partially filled with a liquid

goal: minimal model for simulation and control
Dynamics of a rotor partially filled with a viscous incompressible fluid

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Warning! This problem and its results (in some variations) are not new!

Kollmann 1962: *Experimentelle und theoretische*

Moiseev, Rumnyantsev 1968: *Dynamic Stability of Bodies*

Hendricks, Morton 1979: *Stability of a Rotor partially*

Ibrahim 2005: *Liquid Slooshing Dynamics*

Brommundt, Ostermeyer 1986: *Stabilität eines fliegend*

Keisenberg, Ostermeyer 2015: *Synchronization effects*

... 

However, the methodology developed here should be new (in some way) and be extendable to more complex models.
Model

planar model of

- rigid circular rotor, elastically mounted;
- partially filled with an incompressible, viscous liquid;
- rotating with prescribed angular velocity $\Omega(t) > \sqrt{\frac{g}{R_a}}$;
- and subjected to gravitational field.
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Rotor equations

\[
0 = m_R \ddot{u}_x + k_x u_x - l \int_0^{2\pi} p(r_a, \varphi) \cos \varphi \ r_a \ d\varphi
\]

\[
0 = m_R \ddot{u}_y + k_y u_y - l \int_0^{2\pi} p(r_a, \varphi) \sin \varphi \ r_a \ d\varphi
\]
Navier Stokes equation for incompressible fluids

\[
0 = \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) - \rho \mathbf{b} + \nabla p - \eta \Delta \mathbf{v}
\]

B.C.: \( v_r(t, r_a, \varphi) = 0, \quad v_\varphi(t, r_a, \varphi) = \Omega r_a, \quad p(t, r_i, \varphi) = p_{in} \)

I.C.: \( v_r(0, r, \varphi) = v_{r0}(r, \varphi), \quad v_\varphi(0, r, \varphi) = v_{\varphi0} \)
Evolution of the free boundary

\[
\frac{\partial r_i(t, \varphi)}{\partial t} + \mathbf{v} \cdot \nabla r_i = v_r(t, r_i, \varphi)
\]

I.C.: \( r_i(0, \varphi) = r_{i0}(\varphi) \)
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### Overview

<table>
<thead>
<tr>
<th>unknowns</th>
<th>equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotor $u_x(t), u_x(t), u_y(t), u_y(t)$</td>
<td>4 ode (state space)</td>
</tr>
<tr>
<td>fluid $v_r(t, r, \varphi), v_\varphi(t, r, \varphi)$</td>
<td>2 pde (N.S.eq. $r, \varphi$)</td>
</tr>
<tr>
<td>fluid $r_i(t, \varphi)$</td>
<td>1 pde (F.B.eq.)</td>
</tr>
</tbody>
</table>

7 7

Fluid \[\xrightarrow{\text{pressure}}\] rotor

Fluid \[\xrightarrow{\text{no-slip condition}}\] rotor

pressure
Approximation of the fluid velocity field

- Firstly, circumferential component $v_\varphi \approx \tilde{v}_\varphi$
- $\tilde{v}_r$ is solution of ode $\frac{\partial (r \tilde{v}_r)}{\partial r} = -\frac{\partial \tilde{v}_\varphi}{\partial \varphi}$ (incompressibility)

\begin{align*}
\tilde{v}_\varphi R(t, r, \varphi) &= \Omega r \\
\tilde{v}_\varphi 0(t, r, \varphi) &= (r_a - r) V_0(t) \\
\tilde{v}_\varphi 1(t, r, \varphi) &= (r_a - r) \left( V_{1s}(t) \sin \varphi + V_{1c}(t) \cos \varphi \right)
\end{align*}

\begin{align*}
\tilde{v}_r R(t, r, \varphi) &= 0 \\
\tilde{v}_r 0(t, r, \varphi) &= 0 \\
\tilde{v}_r 1(t, r, \varphi) &= \frac{1}{2r} (r - r_a)^2 \left( V_{1s}(t) \cos \varphi - V_{1c}(t) \sin \varphi \right)
\end{align*}
Approximation of the fluid velocity field

- Firstly, circumferential component \( v_\phi \approx \tilde{v}_\phi \)
- \( \tilde{v}_r \) is solution of ode \( \frac{\partial(r\tilde{v}_r)}{\partial r} = -\frac{\partial \tilde{v}_\phi}{\partial \phi} \) (incompressibility)

\[
\tilde{v}_\phi(t, r, \varphi) = \Omega r \\
\tilde{v}_\phi^0(t, r, \varphi) = (r_a - r)V_0(t) \\
\tilde{v}_\phi^1(t, r, \varphi) = (r_a - r)(V_{1s}(t) \sin \varphi + V_{1c}(t) \cos \varphi) \\
\tilde{v}_r^R(t, r, \varphi) = 0 \\
\tilde{v}_r^0(t, r, \varphi) = 0 \\
\tilde{v}_r^1(t, r, \varphi) = \frac{1}{2r}(r - r_a)^2(V_{1s}(t) \cos \varphi - V_{1c}(t) \sin \varphi)
\]
Approximation of the fluid velocity field

- Firstly, circumferential component \( \nu_\varphi \approx \tilde{\nu}_\varphi \)
- \( \tilde{\nu}_r \) is solution of ode \( \frac{\partial (r \tilde{\nu}_r)}{\partial r} = -\frac{\partial \tilde{\nu}_\varphi}{\partial \varphi} \) (incompressibility)

\[
\begin{align*}
\tilde{\nu}_\varphi R(t, r, \varphi) &= \Omega r \\
\tilde{\nu}_\varphi 0(t, r, \varphi) &= (r_a - r) V_0(t) \\
\tilde{\nu}_\varphi 1(t, r, \varphi) &= (r_a - r)(V_{1s}(t) \sin \varphi + V_{1c}(t) \cos \varphi) \\
\tilde{\nu}_r R(t, r, \varphi) &= 0 \\
\tilde{\nu}_r 0(t, r, \varphi) &= 0 \\
\tilde{\nu}_r 1(t, r, \varphi) &= \frac{1}{2r}(r - r_a)^2(V_{1s}(t) \cos \varphi - V_{1c}(t) \sin \varphi)
\end{align*}
\]
free boundary is approximated by an excentric circle

\( e(t) = \sqrt{e_x(t)^2 + e_y(t)^2} < R_a - R_i \) of radius \( R_i \)

\[
 r_i(t, \varphi) = e_x(t) \cos \varphi + e_y(t) \sin \varphi \\
+ \sqrt{R_i^2 - (e_x(t) \sin \varphi - e_y(t) \cos \varphi)^2}
\]
Approximation of the fluid pressure field

\[ \tilde{p}_R(t, r, \varphi) = p_{in} - \frac{1}{2} \varrho \Omega^2 (r^2 - R_i^2 + e(t)^2) \]
\[ + \varrho \Omega^2 r (e_x(t) \cos \varphi + e_y(t) \sin \varphi) \]

with \[ e_x = -\frac{\ddot{u}_x}{\Omega^2} \quad e_y = -\frac{\ddot{u}_y + g}{\Omega^2} \]

\[ \tilde{p}_0(t, r, \varphi) = (r - r_i) P_0(t) \]
\[ \tilde{p}_1(t, r, \varphi) = (r - r_i)(P_{1s}(t) \sin \varphi + P_{1c}(t) \cos \varphi) \]
Approximation of the fluid pressure field

\[ \tilde{p}_R(t, r, \varphi) = p_{in} - \frac{1}{2} \rho \Omega^2 (r^2 - R_i^2 + e(t)^2) + \rho \Omega^2 r (e_x(t) \cos \varphi + e_y(t) \sin \varphi) \]

with \( e_x = -\frac{\ddot{u}_x}{\Omega^2} \quad e_y = -\frac{\ddot{u}_y + g}{\Omega^2} \)

\[ \tilde{p}_0(t, r, \varphi) = (r - r_i) P_0(t) \]

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Approximation of the fluid pressure field

\[ \tilde{p}_R(t, r, \varphi) = p_{\text{in}} - \frac{1}{2} \rho \Omega^2 (r^2 - R_i^2 + e(t)^2) + \rho \Omega^2 r (e_x(t) \cos \varphi + e_y(t) \sin \varphi) \]

with \( e_x = -\frac{\ddot{u}_x}{\Omega^2} \)

\[ e_y = -\frac{\ddot{u}_y + g}{\Omega^2} \]

\[ \tilde{p}_0(t, r, \varphi) = (r - r_i) P_0(t) \]

\[ \tilde{p}_1(t, r, \varphi) = (r - r_i) (P_{1s}(t) \sin \varphi + P_{1c}(t) \cos \varphi) \]
Approximation of the PDE solution

Eight unknowns for the fluid field

\[ V_0(t), V_{1s}(t), V_{1c}(t), P_0(t), P_{1s}(t), P_{1c}(t), e_x(t), e_y(t). \]

Eight ODEs by method of weighted residuals

\[
\begin{align*}
0 &= \int \int_{r_i}^{r_a} NSE_r \left\{ \cdot 1 \cdot \sin \varphi \cdot \cos \varphi \cdot rd\varphi dr ight\} \\
0 &= \int \int_{r_i}^{r_a} NSE_\varphi \left\{ \cdot 1 \cdot \sin \varphi \cdot \cos \varphi \cdot rd\varphi dr \right\} \\
0 &= \int \int_{r_i}^{r_a} FBE \left\{ \cdot \sin \varphi \cdot \cos \varphi \cdot rd\varphi dr \right\}
\end{align*}
\]
Assuming $e/R_i \ll 1$

$$\int_0^{2\pi} \int_{r_i}^{r_a} \text{NSE} \ r \, d\phi \, dr \approx \int_0^{2\pi} \int_{R_i}^{r_a} \text{NSE} \ r \, d\phi \, dr$$

$$(r - r_i)P_0(t) \approx (r - R_i)P_0(t)$$

$$(r - r_i)(P_{1s}(t) \ldots) \approx (r - R_i)(P_{1s}(t) \ldots)$$

$$r_i(t, \varphi) \approx R_i + e_x(t) \cos \varphi + e_y(t) \sin \varphi$$
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**Introduction**

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**Results**

**Summary**

**Technische Mechanik/Dynamik**

**Total system**

DAE \( M\dot{\mathbf{z}} = L\mathbf{z} + n + c \)

\[ M\dot{\mathbf{z}} = \begin{bmatrix}
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
\dot{V}_0 \\
\dot{V}_{1s} \\
\dot{V}_{1c} \\
\dot{e}_x \\
\dot{e}_y \\
\dot{P}_0 \\
\dot{P}_{1s} \\
\dot{P}_{1c} \\
\dot{u}_x \\
\dot{u}_x \\
\dot{u}_y \\
\dot{u}_y \\
\end{bmatrix} \]
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Total system

\[
\text{DAE} \quad M\dot{z} = Lz + n + c
\]

\[
Lz = \begin{bmatrix}
\bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ \\
\circ & \circ & \bullet & \bullet & \bullet & \circ & \circ & \circ \\
\circ & \circ & \circ & \bullet & \circ & \circ & \circ & \circ \\
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\circ & \circ & \circ & \circ & \circ & \circ & \bullet & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \bullet \\
\end{bmatrix}
\begin{bmatrix}
V_0 \\
V_{1s} \\
V_{1c} \\
e_x \\
e_y \\
P_0 \\
P_{1s} \\
P_{1c} \\
u_x \\
u_y \\
\dot{u}_x \\
\dot{u}_y \\
\end{bmatrix}
\]
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Total system

\[
\text{DAE} \quad M\dot{z} = Lz + n + c
\]

\[
n + c = \begin{bmatrix}
\cdot V_0^2 + \cdot V_{1s}^2 + \cdot V_{1c}^2 \\
\cdot V_0 V_{1s} \\
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\end{bmatrix}
\]
DAE is solved numerically with the *IDA algorithm* from *SUNDIALS*. Here for a run-up prescribed by the angular velocity

\[ \Omega(t) = 2\omega_0(1 - e^{-t/\tau}). \]
Fluid and rotor motion

Excentricity

Rotor displacement ($t = 0 \ldots 0.3s$)
Velocity field

\[ v_\phi(t = 0 \ldots 0.03s, r, \varphi) \]

\[ v_r(t = 0 \ldots 0.03s, r, \varphi) \]
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Pressure field

\[ p(t = 0 \ldots 0.03s, r, \varphi) \]
The **minimum model** for a fluid filled and elastically mounted rotor, obtained in this contribution, is a **nonlinear DAE of dimension 12**, which is small enough for inclusion into simulations of composed systems and controller design.

**Future work is related with**

- verification (limit cases, literature results),
- further analysis types (modal, stability, bifurcations),
- algorithmic speed-up, enable large slooshing,
- 3D model (rotor as beam, bearing models).
### Input data

<table>
<thead>
<tr>
<th>model parameters</th>
<th>initial conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_a$ 0.02 m</td>
<td>$V_0(0)$ 0 (m/s)/m</td>
</tr>
<tr>
<td>$R_i$ 0.0175 m</td>
<td>$V_{1s}(0)$ 0 (m/s)/m</td>
</tr>
<tr>
<td>$p_{in}$ 10$^5$ Pa</td>
<td>$V_{1c}(0)$ 0 (m/s)/m</td>
</tr>
<tr>
<td>$\rho$ 1000 kg/m$^3$</td>
<td>$e_x(0)$ 0 m</td>
</tr>
<tr>
<td>$\eta$ 550 · 10$^{-6}$ Pa s</td>
<td>$e_y(0)$ 0 m</td>
</tr>
<tr>
<td>$g$ 9.81 m/s</td>
<td>$u_x(0)$ 0 m</td>
</tr>
<tr>
<td>$l$ 0.5 m</td>
<td>$\dot{u}_x(0)$ 0 m/s</td>
</tr>
<tr>
<td>$k_x$ 11192.9 N/m</td>
<td>$u_y(0)$ 0 m</td>
</tr>
<tr>
<td>$k_y$ 11192.9 N/m</td>
<td>$\dot{u}_y(0)$ 0 m/s</td>
</tr>
<tr>
<td>$m_R$ 3.93 kg</td>
<td></td>
</tr>
</tbody>
</table>
Navier Stokes equation in polar coordinates for an incompressible fluid

\[
0 = \rho \left( \frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial r} v_r + \frac{\partial v_r}{\partial \varphi} \frac{v_r}{r} - \frac{v_\varphi^2}{r} \right) - \rho b_r + \frac{\partial p}{\partial r} - \eta \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \varphi^2} - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\varphi}{\partial \varphi} \right)
\]

\[
0 = \rho \left( \frac{\partial v_\varphi}{\partial t} + \frac{\partial v_\varphi}{\partial r} v_r + \frac{\partial v_\varphi}{\partial \varphi} \frac{v_\varphi}{r} + \frac{v_r v_\varphi}{r} \right) - \rho b_\varphi + \frac{1}{r} \frac{\partial p}{\partial \varphi} - \eta \left( \frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_\varphi}{\partial \varphi^2} - \frac{v_\varphi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \varphi} \right)
\]