Energy-entropy-consistent time integration of nonlinear thermo-viscoelastic continua

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- Summary & outlook

Goto strong forms
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Overview of structure-preserving time integrators

1 Stuart & Humphries [1998], 2 Hairer et al. [2006], 3 Gonzalez [1996], 4 Armero & Romero [2001], 5 Marsden & West [2001], 6 Romero [2010], 7 Betsch & Steinmann [2000], 8 Mohr et al. [2008], 9 Bauchau & Bottasso [1999], 10 Ober-Blöbaum & Saake [2013]

Characteristics

1. Preserve physical structures of (constraints on) solution spaces of time evolution equations (ODE or PDE or DAE)

2. Examples: Geometric constraints, conservation laws (e.g. energy or linear and angular momentum), balance laws (e.g. entropy, Lyapunov function).

Finite difference methods in time

1. Symplectic methods

2. Energy-momentum time stepping schemes

3. Energy dissipative schemes

4. Variational integrators (VI)

5. Energy-entropy consistent (TC) integrators for thermoelastic closed systems

Finite element methods in time

1. Higher-order accurate symplectic methods

2. GALERKIN-based energy-momentum methods

3. GALERKIN-based energy dissipative methods

4. GALERKIN-based variational integrators (GVI)
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Motion of a single isotropic continuum

\[
\begin{align*}
T & = F_i^T F_i \\
N & = \nabla \varphi \\
Q & = B_0 \psi \\
\rho & = J \rho_c \circ \varphi \\
\rho_c & = J \rho_0 \circ \varphi \\
F_i & = \nabla \varphi \\
C & = F_i^T F_i \\
\rho & = J \rho_c \circ \varphi \\
\Theta & = \theta \circ \varphi \\
P & = J (\rho_t v) \circ \varphi \\
\Theta & = \theta \circ \varphi
\end{align*}
\]

State functionals

1. **Total energy, total entropy & Lyapunov function**
   \[
   H = K + E \\
   S = \int_{B_0} s \, dV \\
   V = H - \Theta_\infty S
   \]

2. **Kinetic energy & internal energy**
   \[
   K = \int_{B_0} k(p) \, dV = \int_{B_0} \frac{1}{2} \rho \cdot p \cdot dV \\
   E = \int_{B_0} e(C, s, C_i) \, dV
   \]
Constitutive laws of the considered continuum


Material motion and heat conduction

1. **First Piola-Kirchhoff stress tensor**

\[ P = 2 F \frac{\delta E}{\delta C} \quad e = e^{\text{ ela}}(C) + e^{\text{ the}}(C, s) + e^{\text{ vis}}(C, C_i) \]

2. **Fourier’s law of isotropic heat conduction**

\[ Q = -\kappa J C^{-1} \nabla \Theta \]

Dissipative behaviour

1. **Total dissipation**

\[ D^{\text{tot}} = D^{\text{cdu}} + D^{\text{vis}} = -\frac{1}{\Theta} Q \cdot \nabla \Theta - \frac{\delta E}{\delta C_i} : \frac{\partial C_i}{\partial t} \geq 0 \]

2. **Viscous evolution equation**

\[ \frac{\partial C_i}{\partial t} = -\tilde{V}^{-1}(C_i) : \frac{\delta E}{\delta C_i} \quad \tilde{V}^{-1}(C_i) = C_i V^{-1} C_i \]

\[ V^{-1} = \frac{1}{2 V^{\text{dev}}} I^{\text{dev}^T} + \frac{1}{n_{\text{dim}}} V^{\text{vol}} I^{\text{vol}} \]
Equations of motion

1. Balance of total linear momentum

\[ \int_{\mathcal{B}_0} \frac{\partial p}{\partial t} \, dV = \int_{\mathcal{B}_0} B \, dV + \int_{\partial \mathcal{B}_0} T \, dA \quad \text{with} \quad T = P N \]

2. Equations of motion

\[ \frac{\partial p}{\partial t} = \text{Div} P + B \]
\[ \frac{\partial \varphi}{\partial t} = 1 \rho \frac{p}{p} = \frac{\delta H}{\delta p} \]

Heat conduction equations

1. Balance of total entropy

\[ \int_{\mathcal{B}_0} \frac{D^{\text{tot}}}{\Theta} \, dV = \int_{\mathcal{B}_0} \left[ \frac{\partial s}{\partial t} - \frac{R}{\Theta} \right] dV + \int_{\partial \mathcal{B}_0} \frac{1}{\Theta} Q \cdot N \, dA \geq 0 \]

2. Heat conduction equations

\[ \frac{\partial s}{\partial t} = \frac{1}{\Theta} \left[ D^{\text{vis}} + R - \text{Div} Q \right] \]
\[ \Theta = \frac{\partial e}{\partial s} \equiv \frac{\delta H}{\delta s} \]
Strong forms and the energy balances

Technische Mechanik/Dynamik

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Balance of total energy \[ \frac{\partial H}{\partial t} = P_{\text{mec}} + P_{\text{the}} \]

1. Balance of mechanical energy
\[
\int_{B_0} \left[ \frac{\delta H}{\delta p} \cdot \frac{\partial p}{\partial t} + \frac{\delta H}{\delta F} \cdot \frac{\partial F}{\partial t} \right] \, dV = \int_{B_0} \frac{\partial \varphi}{\partial t} \cdot B \, dV + \int_{\partial B_0} \frac{\partial \varphi}{\partial t} \cdot T \, dA
\]

2. Balance of thermal energy
\[
\int_{B_0} \left[ \frac{\delta H}{\delta s} \cdot \frac{\partial s}{\partial t} + \frac{\delta H}{\delta C_i} \cdot \frac{\partial C_i}{\partial t} \right] \, dV = \int_{B_0} R \, dV - \int_{\partial B_0} Q \cdot N \, dA
\]

Balance of Lyapunov function \[ V = H - \Theta_\infty S \]

\[
\frac{\partial V}{\partial t} = -\int_{B_0} \frac{\Theta_\infty}{\Theta} D^{\text{tot}} \, dV + \int_{B_0} \frac{\partial \varphi}{\partial t} \cdot B \, dV + \int_{\partial B_0} \frac{\partial \varphi}{\partial t} \cdot T \, dA
\]
\[
+ \int_{B_0} \frac{\Theta - \Theta_\infty}{\Theta} R \, dV - \int_{\partial B_0} \frac{\Theta - \Theta_\infty}{\Theta} Q \cdot N \, dA \quad (\leq 0)
\]
Weak forms derived from the strong forms


Weak equations of motion (integration by parts)

\[ \int_{B_0} \left[ \delta \varphi \cdot \frac{\partial \mathbf{p}}{\partial t} + \nabla (\delta \varphi) : \frac{\partial H}{\partial \mathbf{F}} \right] \mathrm{d}V = \int_{B_0} \delta \varphi \cdot \mathbf{B} \mathrm{d}V + \int_{\partial B_0} \delta \varphi \cdot \mathbf{T} \mathrm{d}A \]

\[ \int_{B_0} \delta \mathbf{p} \cdot \left[ \frac{\partial \varphi}{\partial t} - \frac{\partial H}{\partial \mathbf{p}} \right] \mathrm{d}V = 0 \]

Weak heat conduction equations

\[ - \int_{B_0} \frac{\delta \Theta}{\Theta} \text{Div} \mathbf{Q} \mathrm{d}V = \int_{B_0} \nabla \left( \frac{\delta \Theta}{\Theta} \right) \cdot \mathbf{Q} \mathrm{d}V - \int_{\partial B_0} \frac{\delta \Theta}{\Theta} \mathbf{Q} \cdot \mathbf{N} \mathrm{d}A \]

Integration by parts

\[ \int_{B_0} \left[ \delta \Theta \frac{\partial s}{\partial t} + \frac{\delta \Theta}{\Theta} \frac{\partial H}{\partial \mathbf{C}_i} : \frac{\partial \mathbf{C}_i}{\partial t} \right] \mathrm{d}V = \int_{B_0} \left[ \frac{\delta \Theta}{\Theta} \mathbf{R} + \nabla \left( \frac{\delta \Theta}{\Theta} \right) \cdot \mathbf{Q} \right] \mathrm{d}V - \int_{\partial B_0} \frac{\delta \Theta}{\Theta} \mathbf{Q} \cdot \mathbf{N} \mathrm{d}A \]

\[ \int_{B_0} \delta s \left[ \Theta - \frac{\delta H}{\delta s} \right] \mathrm{d}V = 0 \]
Weak forms fulfill all balance laws (I)

Balance of total linear momentum

Choose \( \delta \varphi(X, t) = c = \text{const.} \)

\[
c \cdot \left[ \int_{B_0} \left( \frac{\partial p}{\partial t} - B \right) \, dV - \int_{\partial B_0} T \, dA \right] = 0
\]

Balance of linear momentum

Balance of total entropy

Choose \( \delta \Theta(X, t) = \Theta_\infty = \text{const.} \)

\[
\Theta_\infty \left[ \int_{B_0} \left( \frac{\partial s}{\partial t} - \frac{D^{\text{vis}}}{\Theta} + \frac{R}{\Theta} - \nabla \left( \frac{1}{\Theta} \right) \cdot Q \right) \, dV + \int_{\partial B_0} \frac{1}{\Theta} Q \cdot N \, dA \right] = 0
\]

Entropy inequality principle with \( D^{\text{int}} \geq 0 \)

Balance of mechanical energy

Choose \( \delta \varphi(X, t) = \frac{\partial \varphi(X,t)}{\partial t} \) and \( \delta p(X, t) = \frac{\partial p(X,t)}{\partial t} \)

\[
\int_{B_0} \left[ \frac{\delta H}{\delta p} \cdot \frac{\partial p}{\partial t} + \frac{\delta H}{\delta F} : \frac{\partial F}{\partial t} \right] \, dV = \int_{B_0} \frac{\partial \varphi}{\partial t} \cdot B \, dV + \int_{\partial B_0} \frac{\partial \varphi}{\partial t} \cdot T \, dA
\]

Balance of mechanical energy
Weak forms fulfill all balance laws (II)

Balance of thermal energy

Choose \( \delta \Theta(X, t) = \Theta(X, t) \) and \( \delta s(X, t) = \frac{\partial s(X, t)}{\partial t} \)

\[
\int_{B_0} \left[ \frac{\delta H}{\delta s} \frac{\partial s}{\partial t} + \frac{\delta H}{\delta C_i} : \frac{\partial C_i}{\partial t} \right] dV = \int_{B_0} p^{\text{ent}} dV - \int_{\partial B_0} Q \cdot N dA
\]

Balance of thermal energy

Balance of total energy

Add balance of mechanical and thermal energy

\[
\frac{\partial H}{\partial t} = \int_{B_0} \left[ \frac{\partial \varphi}{\partial t} \cdot B + R \right] dV + \int_{\partial B_0} \left[ \frac{\partial \varphi}{\partial t} \cdot T - Q \cdot N \right] dA
\]

Balance of total energy

Balance of Lyapunov function

1. Choose \( \delta \Theta(X, t) = \Theta(X, t) - \Theta_\infty \), \( \delta s(X, t) = \frac{\partial s(X, t)}{\partial t} \) and add the balance of mechanical energy (without entropy and total energy consistency possible, cp. ehG method).

or 2. \( \frac{\partial V}{\partial t} = \frac{\partial H}{\partial t} - \Theta_\infty \int_{B_0} \frac{\partial s}{\partial t} dV \)
Boundary conditions in the weak forms

**Mechanical boundary**

\[ \int_{\partial B_0} \delta \varphi \cdot \mathbf{T} \, dA = \int_{\partial \varphi B_0} \delta \varphi \cdot \mathbf{T} \, dA + \int_{\partial T B_0} \delta \varphi \cdot \mathbf{T}(t) \, dA \]

**Thermal boundary (with LAGRANGE multiplier technique)**

\[ \int_{\partial B_0} \frac{\delta \Theta}{\Theta} \mathbf{Q} \cdot \mathbf{N} \, dA = \int_{\partial \Theta B_0} \delta \Theta \lambda_{\Theta} \, dA - \int_{\partial \mathcal{Q} B_0} \frac{\delta \Theta}{\Theta} \mathbf{Q}(t) \, dA \]

with

\[ \int_{\partial \Theta B_0} \delta \lambda_{\Theta} [\Theta - \Theta_\infty] \, dA = 0 \quad \lambda_{\Theta} : \text{LAGRANGE multiplier (outward normal entropy flux)} \]

**Boundary conditions**

- Mechanical boundaries
- Thermal boundaries

**Mechanical boundary**

- \( \partial_T B_0 \)
- \( \mathbf{T}(t) \)

**Thermal boundary**

- \( \partial_Q B_0 \)
- \( \mathbf{Q}(t) \)

**Boundary conditions in the weak forms**

### Equations of motion

\[
\int_{B_0} \delta p_{n+1} \cdot \left[ \frac{\Delta \varphi}{\Delta t} - \frac{\Delta P H}{\Delta p} \right] \, dV = 0
\]

\[
\int_{B_0} \left[ \delta \varphi_{n+1} \cdot \frac{\Delta p}{\Delta t} + \nabla (\delta \varphi_{n+1}) : \frac{\Delta P H}{\Delta F} \right] \, dV = \int_{B_0} \delta \varphi_{n+1} \cdot B_{\frac{1}{2}} \, dV + \int_{\partial T B_0} \delta \varphi_{n+1} \cdot T_{\frac{1}{2}} \, dA
\]

### Heat conduction equations

\[
\int_{B_0} \delta s_{n+1} \cdot \left[ \Theta_{n+1} - \frac{\Delta P H}{\Delta s} \right] \, dV = 0
\]

\[
\int_{B_0} \left[ \delta \Theta_{n+1} \frac{\Delta s}{\Delta t} + \frac{\delta \Theta_{n+1}}{\Theta_{n+1}} \frac{\Delta P H}{\Delta C_i} : \frac{\Delta C_i}{\Delta t} \right] \, dV = \int_{B_0} \left[ \nabla \left( \frac{\delta \Theta_{n+1}}{\Theta_{n+1}} \right) \cdot Q_{\frac{1}{2}} + \frac{\delta \Theta_{n+1}}{\Theta_{n+1}} R_{\frac{1}{2}} \right] \, dV
\]

\[+ \int_{\partial \Theta B_0} \delta \Theta_{n+1} \lambda_{n+1} \, dA + \int_{\partial \Theta B_0} \frac{\delta \Theta_{n+1}}{\Theta_{n+1}} \bar{Q}_{\frac{1}{2}} \, dA
\]

### Local and boundary time evolutions

1. Viscous evolution

\[
\frac{\Delta C_i}{\Delta t} + \bar{V}^{-1} (C_{i,n+\frac{1}{2}}) : \frac{\Delta P H}{\Delta C_i} = 0
\]

2. Boundary condition

\[
\int_{\partial \Theta B_0} \delta \lambda_{n+1} [\Theta_{n+1} - \Theta_{\infty}] \, dA = 0
\]
Time discrete operators ($2^{nd}$ order accurate)


Single-variable function discrete derivative of $f(z)$

\[
\frac{\Delta f}{\Delta z} = \frac{\partial f(z_{n+\frac{1}{2}})}{\partial z} + \frac{f(z_{n+1}) - f(z_n) - \frac{\partial f(z_{n+\frac{1}{2}})}{\partial z} \circ [z_{n+1} - z_n]}{[z_{n+1} - z_n] \circ [z_{n+1} - z_n]} (z_{n+1} - z_n)
\]

Special case 1: Scalar-valued $z$

\[
\frac{\Delta f}{\Delta t} = \frac{f(z_{n+1}) - f(z_n)}{t_{n+1} - t_n}
\]

Special case 2: Quadratic $f$

\[
\frac{\Delta k}{\Delta p} = \frac{\partial k(p_{n+\frac{1}{2}})}{\partial p} \equiv \frac{1}{\rho} p_{n+\frac{1}{2}}
\]

Multi-variable functional (partitioned) discrete derivatives

\[
\frac{\Delta^P H}{\Delta p} = \frac{\Delta k}{\Delta p} \quad \text{with} \quad k(p) \quad \text{and} \quad e(C, s, C_i)
\]

\[
\frac{\Delta^P H}{\Delta F} = F_{n+\frac{1}{2}} \begin{bmatrix}
\Delta e \\ \Delta C
\end{bmatrix}_{s_n, C_{in}} + \begin{bmatrix}
\Delta e \\ \Delta C
\end{bmatrix}_{s_{n+1}, C_{in+1}}
\]

\[
\frac{\Delta^P H}{\Delta s} = \frac{1}{2} \begin{bmatrix}
\Delta e \\ \Delta s
\end{bmatrix}_{C_n, C_{in+1}} + \begin{bmatrix}
\Delta e \\ \Delta s
\end{bmatrix}_{C_{n+1}, C_{in}}
\]

\[
\frac{\Delta^P H}{\Delta C_i} = \frac{1}{2} \begin{bmatrix}
\Delta e \\ \Delta C_i
\end{bmatrix}_{C_n, s_n} + \begin{bmatrix}
\Delta e \\ \Delta C_i
\end{bmatrix}_{C_{n+1}, s_{n+1}}
\]
Time discrete balance laws (I)

Balance of total linear momentum

Choose \( \delta \varphi_{n+1}(X) = c = \text{const.} \)

\[
c \cdot \left[ \int_{B_0} \left[ \frac{\Delta p}{\Delta t} - B_{\frac{1}{2}} \right] dV - \int_{\partial T B_0} \overline{T}_{\frac{1}{2}} dA \right] = 0
\]

Time discrete balance of linear momentum

Balance of total entropy

Choose \( \delta \Theta_{n+1}(X) = \Theta_\infty = \text{const.} \)

\[
\Theta_\infty \left[ \int_{B_0} \frac{\Delta s}{\Delta t} - \frac{D_{\text{tot}}^{\frac{1}{2}} + R_{\frac{1}{2}}}{\Theta_{n+1}} dV + \int_{\partial_\Theta B_0} \lambda_{n+1} dA - \int_{\partial_\Theta B_0} \frac{\bar{Q}_{\frac{1}{2}}}{\Theta_{n+1}} dA \right] = 0
\]

Time discrete entropy inequality principle with \( D_{\text{tot}}^{\frac{1}{2}} \geq 0 \)

Balance of mechanical energy

Choose \( \delta \varphi_{n+1}(X) = \frac{\Delta \varphi(X)}{\Delta t} \) and \( \delta p_{n+1}(X) = \frac{\Delta p(X)}{\Delta t} \)

\[
\frac{\Delta K}{\Delta t} + \frac{1}{2} \left[ \frac{\Delta E}{\Delta t} \bigg|_{s_n, C_{in}} + \frac{\Delta E}{\Delta t} \bigg|_{s_{n+1}, C_{in+1}} \right] = \int_{B_0} \frac{\Delta \varphi}{\Delta t} \cdot B_{\frac{1}{2}} dV + \int_{\partial T B_0} \frac{\Delta \varphi}{\Delta t} \cdot \overline{T}_{\frac{1}{2}} dA
\]

Time discrete balance of mechanical energy
**Balance of thermal energy**

Choose \( \delta \Theta_{n+1}(X) = \Theta_{n+1}(X) - \Theta_\infty \), \( \delta s_{n+1}(X) = \frac{\Delta s(X)}{\Delta t} \) and \( \delta \lambda_{n+1}(X) = \lambda_{n+1}(X) \)

\[
\frac{1}{2} \left[ \frac{\Delta E}{\Delta t} \bigg|_{C_n,C_{n+1}} + \frac{\Delta E}{\Delta t} \bigg|_{C_{n+1},C_{in}} + \frac{\Delta E}{\Delta t} \bigg|_{C_{n},s_n} + \frac{\Delta E}{\Delta t} \bigg|_{C_{n+1},s_{n+1}} \right]
= \int_{B_0} R_\frac{1}{2} \, dV - \Theta_\infty \int_{\partial_\Theta B_0} \lambda_{n+1} \, dA + \int_{\partial Q B_0} \bar{Q}_{\frac{1}{2}} \, dA
\]

**Balance of total energy**

Add time discrete balance of mechanical and thermal energy

\[
\frac{\Delta H}{\Delta t} = \int_{B_0} \left[ \frac{\Delta \varphi}{\Delta t} \cdot B_\frac{1}{2} + R_\frac{1}{2} \right] \, dV + \int_{\partial TB_0} \frac{\Delta \varphi}{\Delta t} \cdot \bar{T}_\frac{1}{2} \, dA - \Theta_\infty \int_{\partial_\Theta B_0} \lambda_{n+1} \, dA + \int_{\partial Q B_0} \bar{Q}_{\frac{1}{2}} \, dA
\]

**Balance of Lyapunov function (Lagrange multiplier will be eliminated)**

1. Choose \( \delta \Theta_{n+1}(X) = \Theta_{n+1}(X) - \Theta_\infty \), \( \delta s_{n+1}(X) = \frac{\Delta s(X)}{\Delta t} \), \( \delta \lambda_{n+1}(X) = \lambda_{n+1}(X) \) and add the time discrete balance of mechanical energy (cp. ehG method)

or 2. \( \frac{\Delta V}{\Delta t} = \sqrt{\frac{\Delta H}{\Delta t}} - \Theta_\infty \int_{B_0} \frac{\Delta s}{\Delta t} \, dV \)
Free rotating insulated thermo-viscoelastic disc

Boundary conditions

- $\omega_z = 0.33 \text{ s}^{-1}$
- $\Theta_\infty = 300 \text{ K}$
- No b.c.

ETC integrator with $\Delta t = 0.02 \text{ s}$

- $t = 5 \text{ s}$
- $t = 10 \text{ s}$
- $t = 15 \text{ s}$
- $t = 20 \text{ s}$
- $t = 25 \text{ s}$
- $t = 30 \text{ s}$

Reference configuration

- $t = 0 \text{ s}$
- $v_0$
- $v_0$

Temperature distribution over time:

- $0 \text{ s}$ to $30 \text{ s}$
- Temperature range: 310 K to 380 K
- Color scale: Red to Yellow
Free rotating insulated thermo-viscoelastic disc

**Midpoint rule ($\Delta t = 0.1$ s)**

$t = 25$ s

**ETC integrator ($\Delta t = 0.1$ s)**

$t = 25$ s

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Rail-bound uninsulated thermo-viscoelastic disc

Midpoint rule (\(\Delta t = 0.14\) s)

ETC integrator (\(\Delta t = 0.14\) s)
Thermo-viscoelastic disc with heat exchange

Boundary conditions

\[ \Theta_{\infty} = 300 \text{ K} \]

Load history

MPR \((\Delta t = 0.02 \text{ s})\)

ETC \((\Delta t = 0.02 \text{ s})\)

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Midpoint rule ($\Delta t = 0.05$ s)

ETC integrator ($\Delta t = 0.20$ s)

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Thermo-viscoelastic compression buffer

Boundary conditions

- Bottom supported
- Top loaded
- \( v = 0 \text{s}^{-1} \)
- \( \Theta_0 = \Theta_\infty = 300 \text{K} \)

ETC (\( \Delta t = 0.1 \text{s} \))

- \( t = 20 \text{s} \)
- \( t = 25 \text{s} \)
- \( t = 80 \text{s} \)
- \( t = 85 \text{s} \)
- \( t = 95 \text{s} \)
- \( t = 100 \text{s} \)

Load history

- Time history of load Q (W/m²)
- Time history of temperature T (N/m²)

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Midpoint rule

ETC integrator

Total energy balance [J] vs. time [s] for different time steps.

Total energy balance [J] vs. time [s] for different time steps.

Total energy balance [J] vs. time [s] for different time steps.

Midpoint rule

Total energy balance [J] vs. time [s] for different time steps.

ETC integrator

Total energy balance [J] vs. time [s] for different time steps.

Total energy balance [J] vs. time [s] for different time steps.

Total energy balance [J] vs. time [s] for different time steps.
Conclusions

Summary

1 Goal: The ETC integrator fulfills
   ▶ the balance of total linear/angular momentum AND total entropy
   ▶ the balance of total energy AND Lyapunov function
   also with Dirichlet and Neumann boundary conditions.

2 Algorithmic basis: A spatially weak formulation, which fulfills
   ▶ ALL balance laws for STANDARD finite element spaces

3 Algorithmic key: A time discretisation with
   ▶ time discrete differential operators which satisfy the
equation version of the fundamental theorem of calculus.

4 Benefit: A transient simulation with an improved
   ▶ numerical stability for large time steps, and
   ▶ physically consistent solutions (NO hour-glassing, NO waves)

Outlook

5 Introduction of a (mixed) variational formulation of the time evolution
   \( \rightsquigarrow \) (mixed) variational integrator \( \rightsquigarrow \) physical consistency with time adaptivity

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1 Maugin & Kalpakides [2002], Bargmann [2008]
2 Zielonka [2006], Zielonka, Ortiz & Marsden [2008], Mata & Lew [2011]
3 Ibrahimbegovic, Chorfi & Gharzeddine [2001], Hartmann, Quint & Arnold [2008], Birken et al. [2010], Moore [2011], Gleim & Kuhl [2013]