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Galerkin-based energy-momentum time-stepping schemes for anisotropic hyperelastic materials

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Motivation, goals and strategy

Motivation

1. Dynamic simulations of fiber-reinforced materials in light-weight structures

energy-momentum schemes for nonlinear anisotropic materials

1. Galerkin approximations in space and time,
2. assumed 'strain' approximation in time for the matrix,
3. superimposed algorithmic stress field for the matrix

variationally consistent design of energy-momentum schemes

1. differential variational principles (Jourdain's, Gauss's etc.)
2. continuous assumed 'strain' approximation in time
3. discontinuous stress approximation in time

Strategy

(compare Betsch & Janz [2016], Schlögl & Leyendecker [2016])

1. Formulation of a mixed variational principle for continua
2. Space and time discretization of this variational principle
3. Energy-momentum schemes as discrete Euler-Lagrange equations

Continuum model for fiber-reinforced materials

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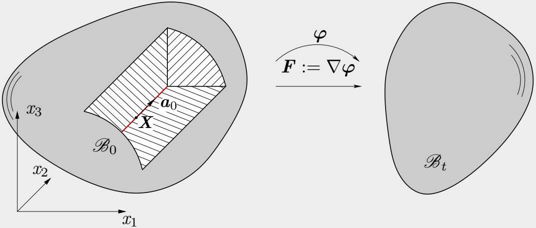
Transversely isotropic continuum

Schröder, Neff & Balzani [2005]



Matrix and fiber deformation

Klinkel, Sansour & Wagner [2005]

1. Deformation gradient of the fiber

$$\boldsymbol{a} = \boldsymbol{F}\boldsymbol{a}_0 \qquad \boldsymbol{F}_F := \boldsymbol{a}\otimes\boldsymbol{a}_0 = \boldsymbol{F}\boldsymbol{A}_0 \qquad \boldsymbol{A}_0 := \boldsymbol{a}_0\otimes\boldsymbol{a}_0$$

2. Right Cauchy-Green tensors

$$\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F} \qquad \boldsymbol{C}_F := \boldsymbol{F}_F^T\boldsymbol{F}_F := C_F\boldsymbol{A}_0 \qquad C_F := \boldsymbol{C} : \boldsymbol{A}_0 \equiv I_4^C$$

3. Second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} := 2\sum_{i=1}^{3}\frac{\partial\hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial I_i^C}\,\frac{\partial I_i^C}{\partial\boldsymbol{C}} + 2\,\frac{\partial\hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial C_F}\,\boldsymbol{A}_0$$

Functional form of the total energy balance

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Total energy balance $\dot{\mathcal{H}} = 0$

$$\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}; \rho_0) + \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{C}}}, \boldsymbol{S}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \tilde{\boldsymbol{S}}) + \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}; \rho_0, \boldsymbol{b}, \boldsymbol{t}, \bar{\boldsymbol{u}}) = 0$$

Kinetic power $\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}; \rho_0)$

$$\dot{\mathcal{T}} := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \dot{\boldsymbol{v}} \, \mathrm{d}V - \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \, \mathrm{d}V + \int_{\mathscr{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{u}} \, \mathrm{d}V$$

Stress power $\dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{C}}}, \boldsymbol{S}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \tilde{\boldsymbol{S}})$ with assumed 'strain' tensor

$$\dot{\Pi}^{\text{int}} := \frac{1}{2} \int_{\mathscr{B}_0} \left[2 \frac{\partial W(\tilde{\boldsymbol{C}}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} - \boldsymbol{S} \right] : \dot{\tilde{\boldsymbol{C}}} \, \mathrm{d}V - \frac{1}{2} \int_{\mathscr{B}_0} \dot{\boldsymbol{S}} : [\tilde{\boldsymbol{C}} - \boldsymbol{C}(\boldsymbol{u})] \, \mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{S} : \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}) \, \mathrm{d}V$$

External power $\dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}; \rho_0, \boldsymbol{b}, \boldsymbol{t}, \dot{\bar{\boldsymbol{u}}})$

$$\dot{\Pi}^{\text{ext}} := - \int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \boldsymbol{h} \cdot (\dot{\boldsymbol{u}} - \dot{\bar{\boldsymbol{u}}}) \, \mathrm{d}A$$

Mixed Jourdain principle

(cp. Vujanovic [1986], Klinkel & Wagner [1997], Hackl [1997])

Principle of virtual power $\delta_* \dot{\mathcal{H}} = 0$ in mixed form

$$\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}; \rho_0) + \delta_* \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{C}}}, \boldsymbol{S}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \tilde{\boldsymbol{S}}) + \delta_* \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}; \rho_0, \boldsymbol{b}, \boldsymbol{t}, \bar{\boldsymbol{u}}) = 0$$

Virtual kinetic power $\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}; \rho_0)$

$$\delta_* \dot{\mathcal{T}} := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \delta_* \dot{\boldsymbol{v}} \,\mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \,\mathrm{d}V + \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}V$$

Virtual stress power $\delta_* \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{C}}}, \boldsymbol{S}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \tilde{\boldsymbol{S}})$

$$\begin{aligned}\delta_* \dot{\Pi}^{\text{int}} := \quad & \frac{1}{2} \int_{\mathscr{B}_0} \left[2 \, \frac{\partial W(\tilde{\boldsymbol{C}}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} - \boldsymbol{S} \right] : \delta_* \dot{\tilde{\boldsymbol{C}}} \,\mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S} : \left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}) \right] \mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{S} : \delta_* \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}) \,\mathrm{d}V\end{aligned}$$

Virtual external power $\delta_* \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}; \rho_0, \boldsymbol{b}, \boldsymbol{t}, \dot{\bar{\boldsymbol{u}}})$

$$\delta_* \dot{\Pi}^{\text{ext}} := - \int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \boldsymbol{h} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}A$$

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Euler-Lagrange equations

Constitutive equations

$$\rho_0 \boldsymbol{v} = \boldsymbol{p} \qquad \forall t \geqslant t_0$$

$$2\,\frac{\partial W(\tilde{\boldsymbol{C}}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} = \boldsymbol{S} \qquad \forall t \geqslant t_0$$

$$\dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}) = \dot{\tilde{\boldsymbol{C}}} \quad \text{with} \quad \tilde{\boldsymbol{C}}(t_0) = \boldsymbol{C}(\boldsymbol{u}_0)$$

Boundary conditions

$$\boldsymbol{F}\boldsymbol{S}\boldsymbol{N} = \boldsymbol{t} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_t \mathscr{B}_0$$

$$\delta_* \dot{\boldsymbol{u}} = \boldsymbol{0} \Longleftrightarrow \dot{\boldsymbol{u}} = \dot{\bar{\boldsymbol{u}}} \text{ with } \boldsymbol{u}(t_0) = \bar{\boldsymbol{u}}(t_0) \quad \text{on} \quad \partial_u \mathscr{B}_0$$

Equations of motion

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{with} \quad \boldsymbol{u}(t_0) = \boldsymbol{u}_0$$

$$\mathrm{DIV}[\boldsymbol{F}\boldsymbol{S}] + \rho_0 \boldsymbol{b} = \dot{\boldsymbol{p}} \quad \text{with} \quad \boldsymbol{p}(t_0) = \boldsymbol{p}_0 \equiv \rho_0 \boldsymbol{v}_0$$

Time evolution characteristics

1. continuous time evolutions of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{p}$ as well as $\tilde{\boldsymbol{C}}$
2. discontinuous time evolution of the stress $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}} = \boldsymbol{O}$

Discrete variational principle (2nd-order accu.)

Discrete principle of virtual power in mixed form at $\xi_1 = 0.5$

$$\int_{t_0}^{t_N} \delta_* \dot{\mathcal{H}}(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{v}}(t), \dot{\boldsymbol{p}}(t), \dot{\tilde{\boldsymbol{C}}}(t), \boldsymbol{S}(t); \rho_0, \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \boldsymbol{b}(t), \boldsymbol{t}(t), \dot{\bar{\boldsymbol{u}}}(t), \tilde{\boldsymbol{S}}(t))\,\mathrm{d}t \approx$$

evaluation at collocation points ξ_i

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_1), \dot{\boldsymbol{v}}_h^n(\xi_1), \dot{\boldsymbol{p}}_h^n(\xi_1), \dot{\tilde{\boldsymbol{C}}}{}_h^n(\xi_1), \boldsymbol{S}_h^n(\xi_1); \rho_0, \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \boldsymbol{b}_h^n(\xi_1), \boldsymbol{t}_h^n(\xi_1), \dot{\bar{\boldsymbol{u}}}{}_h^n(\xi_1), \tilde{\boldsymbol{S}}{}_h^n(\xi_1))\, h_n =$$

$$\sum_{n=0}^{N-1} \delta_* \mathcal{H}_d(\boldsymbol{u}_{n+1}, \boldsymbol{v}_{n+1}, \boldsymbol{p}_{n+1}, \boldsymbol{C}_{n+1}, \boldsymbol{S}_{n+\frac{1}{2}}; \rho_0, \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \boldsymbol{b}_{n+\frac{1}{2}}, \boldsymbol{t}_{n+\frac{1}{2}}, \bar{\boldsymbol{u}}_{n+1}, \tilde{\boldsymbol{S}}_{n+\frac{1}{2}})\, h_n = 0$$

Galerkin-based approximations

$$\boldsymbol{u}_h^n(\alpha) := \boldsymbol{u}_n + \alpha\,(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) \qquad \boldsymbol{v}_h^n(\alpha) := \boldsymbol{v}_n + \alpha\,(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n)$$

$$\tilde{\boldsymbol{C}}{}_h^n(\alpha) := \boldsymbol{C}_n + \alpha\,(\boldsymbol{C}_{n+1} - \boldsymbol{C}_n) \qquad \boldsymbol{p}_h^n(\alpha) := \boldsymbol{p}_n + \alpha\,(\boldsymbol{p}_{n+1} - \boldsymbol{p}_n)$$

Semidiscrete variational forms

$$\int_{\mathscr{B}_0} \left[\rho_0 \boldsymbol{v}_{n+\frac{1}{2}} - \boldsymbol{p}_{n+\frac{1}{2}}\right] \cdot \delta_* \boldsymbol{v}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \left[\boldsymbol{S}_{n+\frac{1}{2}} - 2\,\frac{\partial W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}; \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \boldsymbol{C}} - \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : \delta_* \boldsymbol{C}_{n+1}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \delta_* \boldsymbol{p}_{n+1} \cdot \left[\boldsymbol{v}_{n+\frac{1}{2}} - \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{h_n}\right] \mathrm{d}V = 0 = \int_{\mathscr{B}_0} \left[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n - (\boldsymbol{F}_{n+1} + \boldsymbol{F}_n)^T (\boldsymbol{F}_{n+1} - \boldsymbol{F}_n)\right] : \delta_* \boldsymbol{S}_{n+\frac{1}{2}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \left[\frac{\boldsymbol{p}_{n+1} - \boldsymbol{p}_n}{h_n} + \boldsymbol{B}_{n+\frac{1}{2}}^T \boldsymbol{S}_{n+\frac{1}{2}}\right] \cdot \delta_* \boldsymbol{u}_{n+1}\,\mathrm{d}V = \int_{\mathscr{B}_0} \rho_0 \boldsymbol{b}_{n+\frac{1}{2}}\,\mathrm{d}V + \int_{\partial_t \mathscr{B}_0} \boldsymbol{t}_{n+\frac{1}{2}} \cdot \delta_* \boldsymbol{u}_{n+1}\,\mathrm{d}A + \int_{\partial_u \mathscr{B}_0} \boldsymbol{h}_{n+\frac{1}{2}} \cdot \delta_* \boldsymbol{u}_{n+1}\,\mathrm{d}A$$

Option 1 (this presentation)

1. local and weak forms

Option 2 Schröder, Wriggers & Balzani [2011]

1. discrete multifield form

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Fully discrete equations in option 1 (2nd-order accu.)

Initial conditions

$$\boldsymbol{u}(t_0)=\boldsymbol{u}_0 \qquad \boldsymbol{p}(t_0)=\rho_0\boldsymbol{v}_0 \qquad \boldsymbol{v}(t_0)=\boldsymbol{v}_0 \qquad \tilde{\boldsymbol{C}}(t_0)=(\nabla\boldsymbol{u}_0+\boldsymbol{I})^T(\nabla\boldsymbol{u}_0+\boldsymbol{I})$$

Discrete strong forms

(compare Betsch & Janz [2016])

$$\boldsymbol{p}_n=\rho_0\boldsymbol{v}_n \implies \rho_0\left[\boldsymbol{v}_n+\boldsymbol{v}_{n+1}\right]=\boldsymbol{p}_n+\boldsymbol{p}_{n+1} \implies \boldsymbol{p}_{n+1}=\rho_0\boldsymbol{v}_{n+1}$$

$$\boldsymbol{C}_{n+1}=\boldsymbol{F}_{n+1}^T\boldsymbol{F}_{n+1} \implies \boldsymbol{C}_{n+1}-\boldsymbol{C}_n=\boldsymbol{F}_{n+1}^T\boldsymbol{F}_{n+1}-\boldsymbol{F}_n^T\boldsymbol{F}_n \implies \boldsymbol{C}_{n+1}=\boldsymbol{F}_{n+1}^T\boldsymbol{F}_{n+1}$$

$$\boldsymbol{u}_n,\boldsymbol{v}_n \implies \boldsymbol{v}_{n+\frac{1}{2}}=\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{h_n} \implies \boldsymbol{v}_{n+1}$$

$$2\,\frac{\partial W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}};\boldsymbol{A}_0,\boldsymbol{\kappa}_0)}{\partial\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}}+\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}=\boldsymbol{S}_{n+\frac{1}{2}}$$

(parameter field)

Discrete weak momentum balance

$$\mathbf{M}\,\frac{\mathbf{v}_{n+1}-\mathbf{v}_n}{h_n}+\int_{\mathscr{B}_0}\mathbf{B}_{n+\frac{1}{2}}^T\mathbf{S}_{n+\frac{1}{2}}=\mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}}+\mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}}+\mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}}$$

$$\mathbf{M}:=\int_{\mathscr{B}_0}\rho_0\,\mathbf{N}^T\mathbf{N}\,\mathrm{d}V \qquad \mathbf{H}_t:=\int_{\partial_t\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V \qquad \mathbf{H}_u:=\int_{\partial_u\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V$$

$$2\,\boldsymbol{B}_{n+\frac{1}{2}}\left[\boldsymbol{u}_{n+1}-\boldsymbol{u}_n\right]:=\boldsymbol{C}_{n+1}-\boldsymbol{C}_n$$

Discrete total energy balance

(by multiplication with $\mathbf{v}_h^n(\xi_1)=\mathbf{v}_{n+\frac{1}{2}}$)

$$\frac{\mathcal{T}_{n+1}-\mathcal{T}_n}{h_n}+\frac{\Pi_{n+1}^{\mathrm{ext}}+\Pi_n^{\mathrm{ext}}}{h_n}=-\int_{\mathscr{B}_0}\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{h_n}\cdot\boldsymbol{B}_{n+\frac{1}{2}}^T\left[2\,\frac{\partial W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}};\boldsymbol{A}_0,\boldsymbol{\kappa}_0)}{\partial\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}}+\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right]\mathrm{d}V$$

Discrete superimposed stress tensor (2nd-order accu.)

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Algorithmic claim

(one equation for 6 unknowns)

$$\int_{\mathscr{B}_0} \frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{h_n}\cdot\boldsymbol{B}^T_{n+\frac{1}{2}}\left[2\,\frac{\partial W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}};\boldsymbol{A}_0,\boldsymbol{\kappa}_0)}{\partial\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}}+\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right]\mathrm{d}V=\frac{\Pi^{\mathrm{int}}_{n+1}-\Pi^{\mathrm{int}}_n}{h_n}$$

Constrained variational problem

(compare Gauss's principle in Ramm [2011])

$$\delta_*\mathcal{L}(\mu,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}})=0 \qquad \mathcal{L}(\mu,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}):=\frac{1}{2}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}:\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}+\mu\,\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}})$$

Local constraint

G., Betsch & Steinmann [2005]

$$\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}):=W_{n+1}-W_n-\left[\frac{\partial W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}};\boldsymbol{A}_0,\boldsymbol{\kappa}_0)}{\partial\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}}+\frac{1}{2}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right]:[\boldsymbol{C}_{n+1}-\boldsymbol{C}_n]=0$$

Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}}\equiv\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}-\frac{\mu}{2}\,[\boldsymbol{C}_{n+1}-\boldsymbol{C}_n]=\boldsymbol{O} \qquad \frac{\partial\mathcal{L}}{\partial\mu}\equiv\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}})=0$$

Discrete superimposed stress tensor

Armero & Zambrana-Rojas [2007]

$$\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}=2\,\frac{\mathcal{G}(\boldsymbol{O})}{\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}\,[\boldsymbol{C}_{n+1}-\boldsymbol{C}_n]:[\boldsymbol{C}_{n+1}-\boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}\,[\boldsymbol{C}_{n+1}-\boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}$$

Higher-order approximation (I) - $(k > 1)$

Discrete variation principle

$$\sum_{n=0}^{N-1} \sum_{i=1}^{k} \delta_* \dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_i), \dot{\boldsymbol{v}}_h^n(\xi_i), \dot{\boldsymbol{p}}_h^n(\xi_i), \dot{\tilde{\boldsymbol{C}}}_h^n(\xi_i), \boldsymbol{S}_h^n(\xi_i); \rho_0, \boldsymbol{A}_0, \boldsymbol{\kappa}_0, \boldsymbol{b}_h^n(\xi_i), \boldsymbol{t}_h^n(\xi_i), \bar{\boldsymbol{u}}_h^n(\xi_i))\, w_i\, h_n = 0$$

Discrete weak assumed strain equation

$$\sum_{i=1}^{k} \int_{\mathcal{B}_0} \delta_* \boldsymbol{S}_h^n(\xi_i) : \left[\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha} - \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n(\xi_i)) \right] w_i \,\mathrm{d}V = 0$$

Discrete local assumed strain equation (Euler-Lagrange equation)

$$\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha} - \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n(\xi_i)) = \boldsymbol{O} \qquad i = 1, \ldots, k$$

with

$$\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\alpha)}{\mathrm{d}\alpha} = \sum_{j=1}^{k+1} \overset{\circ}{M}_{j+1}(\alpha)\, \boldsymbol{C}_j^n \equiv \sum_{i=1}^{k} \tilde{M}_i(\alpha)\, \tilde{\boldsymbol{C}}_i^n$$

Unknown nodal values C_l^n, $l = 2, \ldots, k$

$$\boldsymbol{C}_l^n := \sum_{i=1}^{k} m_{li}\, \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n(\xi_i)) + \boldsymbol{C}_1^n \qquad \text{with} \qquad \boldsymbol{m} = \begin{bmatrix} \overset{\circ}{M}_2(\xi_1) & \ldots & \overset{\circ}{M}_{k+1}(\xi_1) \\ \vdots & \ldots & \vdots \\ \overset{\circ}{M}_2(\xi_k) & \ldots & \overset{\circ}{M}_{k+1}(\xi_k) \end{bmatrix}^{-1}$$

Higher-order approximation (II) - $(k > 1)$

Nodal values for $k = 1$

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_2^n := (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

Nodal values for $k = 2$

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_2^n := \frac{1}{3}\left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right]^T \left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right] + (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_3^n := (\boldsymbol{F}_3^n)^T \boldsymbol{F}_3^n$$

Old superimposed stress with mixed 'strain' approximation in time

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l}\; \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)$$

(energy consistent, **but not** variationally consistent approximation)

G., Betsch & Steinmann [2005]

with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, [\boldsymbol{F}_j^n]^T \boldsymbol{F}_j^n$$

New superimposed stress with uniform 'strain' approximation in t

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l)\, [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, w_l}\, [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)\, [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}$$

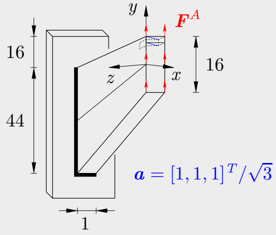
with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, \boldsymbol{C}_j^n$$

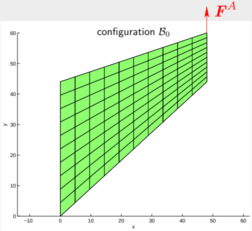
Stiff Cook-type cantilever 3D-beam (static load)

Beam geometry

Erler & G. [2015]

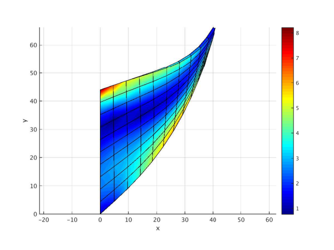


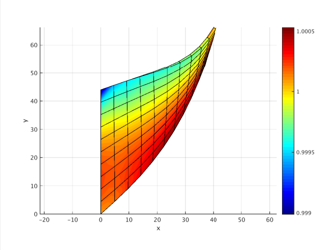
Reference configuration



Deformed configurations

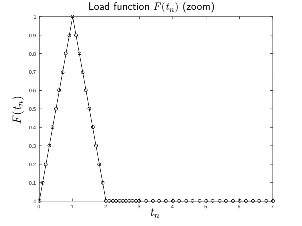
($\hat{F} = 3000$, 120 load steps, left: von Mises stress, right: I_3)



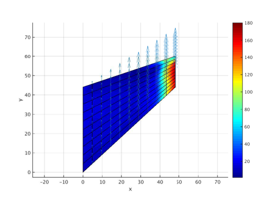


Stiff Cook-type cantilever 3D-beam (dynamic load)

Load function ($\hat{F} = 100,\ h_n = [0.1, 0.2]$)

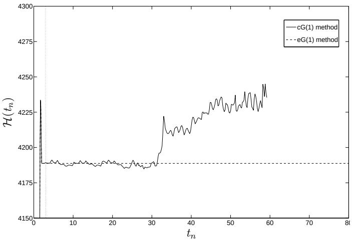


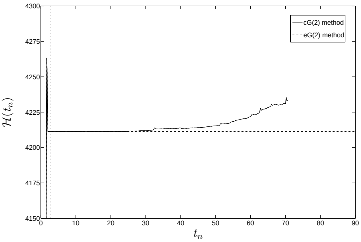
Initial configuration $\mathbf{v}^A = [0, v_{\max} \cdot \frac{x^A}{48}, 0]^T$



Total energy versus time

(compare G., Betsch & Steinmann [2005])

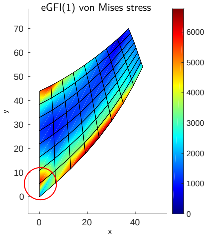




Stiff Cook membrane (I) - eG(1) method (dynamic load)

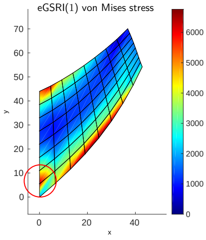
Q1(FI)-element

Erler & G. [2015]



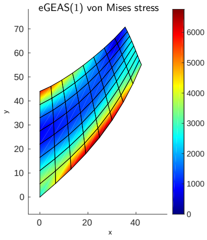
SRI method

Doll et al. [2000]

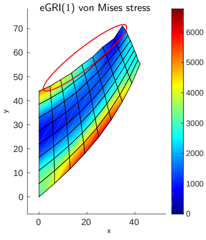


EAS method

Müller & Betsch [2007]



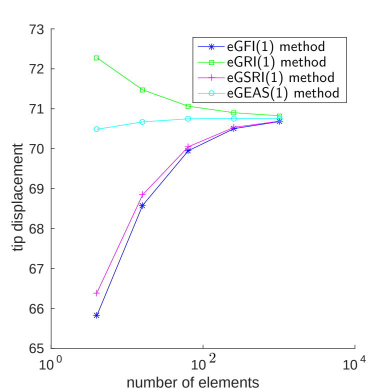
RI method (without Hourglass stabilization)

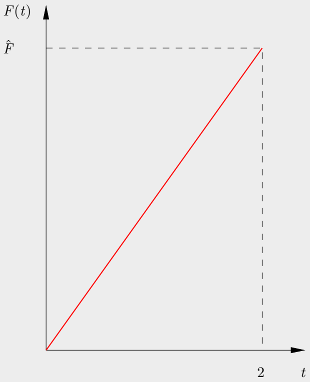


Stiff Cook membrane (II) - eG(1) method (dynamic load)

Convergence study

$(F(t) = \hat{F}\,\frac{t}{2}, \hat{F} = 18000, t \in [0,2])$





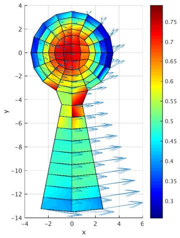
First concluding remarks

(compare Betsch & Janz [2016])

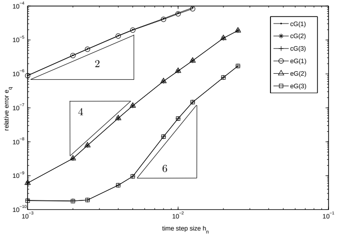
1. The EAS method applied to the eG(k) schemes is already recommendable,
2. but we going to complete the comparison with a multifield formulation.

Free flying 3D-propeller (I) (100 Q1-elements)

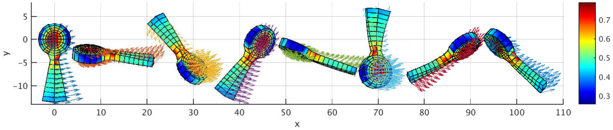
Initial configuration (von Mises stress)



Convergence study (non-stiff case)

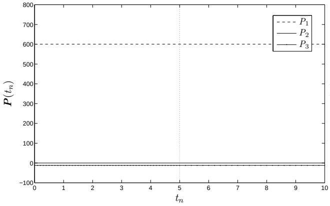


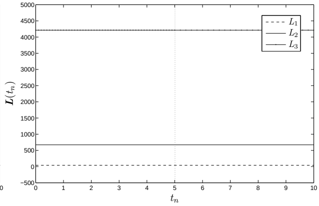
Typical motion (cG(1) method, non-stiff case, von Mises stress)



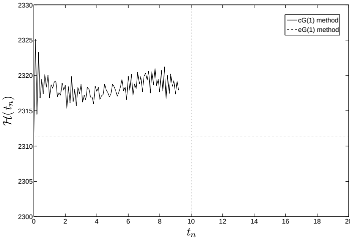
Free flying 3D-propeller (II) (100 Q1-elements)

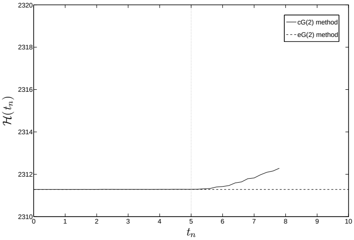
Total linear and angular momenta of the eG(2) method (stiff case)





Total energy versus time (stiff case)





Summary

Introduction
Variational setting
Continuum model
Total energy balance
Variational principle
Euler-Lagrange equations
Discrete setting
Discrete variation
Discrete E-L-equations
Superimposed stress
Higher-order approx.
Numerical studies
Summary

1. Motivation:
   - Dynamic simulation methods for fiber-reinforced structures
   - related with improved time and space approximations
2. Goals:
   - energy-momentum consistent time-stepping schemes
   - with variationally consistent Galerkin-based approximations
3. Strategy:
   - Formulation of a mixed variational principle
   - Introduction of Galerkin-based approximations for state variables
   - Time/space discretization at the time/space collocation points
   - Discrete variation at the time/space mesh points
4. Important results:
   - Variationally consistent assumed strain approximation in time
   - and a new energy-consistent higher-order stress approximation