Energy and Momentum conserving variational based time integration of anisotropic hyperelastic continua

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Advances in time integration for solid, fluid and coupled systems

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Motivation, goals and strategy

Motivation

1. Dynamic simulations of fiber-reinforced materials in light-weight structures

Design of energy-momentum schemes for anisotropic materials

1. Higher-order approximations
2. Enhanced displacement gradients
3. Assumed 'strain' approximations in time for matrix and fiber,
4. Superimposed algorithmic stress fields for matrix and fiber

Variationally consistent design of energy-momentum schemes

1. differential variational principles (Jourdain's, Gauss's etc.)
2. continuous assumed 'strain' approximation in time
3. discontinuous stress approximation in time

Strategy

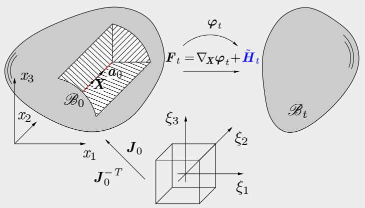
(compare Betsch & Janz [2016], Schlögl & Leyendecker [2016])

1. Formulation of a mixed variational principle for continua
2. Space and time discretization of this variational principle
3. Energy-momentum schemes as discrete Euler-Lagrange equations

Continuum model for fiber-reinforced materials

Transversely isotropic material

Schröder, Neff & Balzani [2005]



Enhanced displacement gradient Q1E9

Simo, Armero & Taylor [1993]
Müller & Betsch [2007]

$$[\tilde{\boldsymbol{H}}_\square(\boldsymbol{\xi})]^\alpha{}_\beta := \sum_{I=1}^{n_{\text{dim}}} [\boldsymbol{\Gamma}^I]^\alpha \, [\nabla_{\boldsymbol{\xi}} \tilde{N}_I(\boldsymbol{\xi})]_\beta$$

$$[\tilde{\boldsymbol{H}}_0(\boldsymbol{\xi})]^B{}_A := \frac{j_0}{j(\boldsymbol{\xi})} \, [\boldsymbol{J}_0]^B{}_\alpha [\tilde{\boldsymbol{H}}_\square(\boldsymbol{\xi})]^\alpha{}_\beta [\boldsymbol{J}_0^{-1}]^\beta{}_A$$

$$[\tilde{\boldsymbol{H}}_t(\boldsymbol{\xi})]^a{}_A := [\nabla_{\boldsymbol{X}} \boldsymbol{\varphi}_t^0]^a{}_B [\tilde{\boldsymbol{H}}_0(\boldsymbol{\xi})]^B{}_A$$

$$\boxed{[\tilde{\boldsymbol{H}}_t(\boldsymbol{\xi})]^a{}_A = \sum_{I=1}^{n_{\text{dim}}} [\boldsymbol{\alpha}_t^I]^a \, [\tilde{\nabla}_{\boldsymbol{X}} \tilde{N}_I(\boldsymbol{\xi})]_A}$$

Matrix and fiber deformation

Klinkel, Sansour & Wagner [2005]

1. Deformation gradient of the fiber

$$\boldsymbol{F}_F := \boldsymbol{a} \otimes \boldsymbol{a}_0 = \boldsymbol{F}\boldsymbol{A}_0 \qquad \boldsymbol{a} = \boldsymbol{F}\boldsymbol{a}_0 \qquad \boldsymbol{A}_0 := \boldsymbol{a}_0 \otimes \boldsymbol{a}_0$$

2. Right Cauchy-Green tensors

$$\boldsymbol{C}_F := \boldsymbol{F}_F^T \boldsymbol{F}_F := C_F \boldsymbol{A}_0 \qquad C_F := \boldsymbol{C} : \boldsymbol{A}_0 \equiv I_4^C \qquad \boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$$

3. Second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} := 2 \sum_{i=1}^{3} \frac{\partial \hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial I_i^C} \frac{\partial I_i^C}{\partial \boldsymbol{C}} + 2 \frac{\partial \hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial C_F} \boldsymbol{A}_0 + 2 \frac{\partial W_F(C_F)}{\partial C_F} \boldsymbol{A}_0$$

Functional form of the total energy balance

Total energy balance $\dot{\mathcal{H}} = 0$

$$\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F) + \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}}) = 0$$

Kinetic power functional

$$\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \dot{\boldsymbol{v}} \, \mathrm{d}V - \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \, \mathrm{d}V + \int_{\mathscr{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{u}} \, \mathrm{d}V$$

Stress power functional $\dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F)$

$$\begin{aligned}
\dot{\Pi}^{\text{int}} := \; & \frac{1}{2}\int_{\mathscr{B}_0} \Big\{ \big[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} - \boldsymbol{S}\big] : \dot{\tilde{\boldsymbol{C}}} + \big[2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F - \boldsymbol{S}_F : \boldsymbol{A}_0\big] : \dot{\tilde{C}}_F \Big\} \, \mathrm{d}V \\
& - \frac{1}{2}\int_{\mathscr{B}_0} \dot{\boldsymbol{S}} : \big[\tilde{\boldsymbol{C}} - \boldsymbol{C}(\boldsymbol{u}, \tilde{\boldsymbol{H}})\big] \, \mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0} \boldsymbol{S} : \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \, \mathrm{d}V \\
& - \frac{1}{2}\int_{\mathscr{B}_0} \dot{\boldsymbol{S}}_F : \big[\tilde{C}_F \, \boldsymbol{A}_0 - \boldsymbol{C}_F(\boldsymbol{u}, \tilde{\boldsymbol{H}})\big] \, \mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0} \boldsymbol{S}_F : \dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \, \mathrm{d}V
\end{aligned}$$

External power functional $\dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}})$

$$\dot{\Pi}^{\text{ext}} := -\int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \{\boldsymbol{h} \cdot (\dot{\boldsymbol{u}} - \dot{\bar{\boldsymbol{u}}}) + \dot{\boldsymbol{h}} \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}})\} \, \mathrm{d}A$$

Mixed Jourdain principle

(cp. Schröder & Kuhl [2015], Klinkel & Wagner [1997], Hackl [1997])

Principle of virtual power $\delta_* \dot{\mathcal{H}} = 0$ in mixed form

$$\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \delta_* \dot{\Pi}^{\mathrm{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F) + \delta_* \dot{\Pi}^{\mathrm{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}}) = 0$$

Virtual kinetic power $\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}})$

$$\delta_* \dot{\mathcal{T}} := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \delta_* \dot{\boldsymbol{v}} \, \mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \, \mathrm{d}V + \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot \delta_* \dot{\boldsymbol{u}} \, \mathrm{d}V$$

Virtual stress power $\delta_* \dot{\Pi}^{\mathrm{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F)$

$$\begin{aligned} \delta_* \dot{\Pi}^{\mathrm{int}} := & \frac{1}{2} \int_{\mathscr{B}_0} \left\{ \left[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} - \boldsymbol{S} \right] : \delta_* \dot{\tilde{\boldsymbol{C}}} + \left[2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F - \boldsymbol{S}_F : \boldsymbol{A}_0 \right] : \delta_* \dot{\tilde{C}}_F \right\} \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S} : \left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \right] \mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{F}\,\boldsymbol{S} : \left[\nabla_{\boldsymbol{X}}(\delta_* \dot{\boldsymbol{u}}) + \delta_* \dot{\tilde{\boldsymbol{H}}} \right] \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S}_F : \left[\dot{\tilde{C}}_F \boldsymbol{A}_0 - \dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \right] \mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{F}_F\,\boldsymbol{S}_F : \left[\nabla_{\boldsymbol{X}}(\delta_* \dot{\boldsymbol{u}}) + \delta_* \dot{\tilde{\boldsymbol{H}}} \right] \boldsymbol{A}_0 \, \mathrm{d}V \end{aligned}$$

Virtual external power $\delta_* \dot{\Pi}^{\mathrm{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}})$

$$\delta_* \dot{\Pi}^{\mathrm{ext}} := - \int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \delta_* \dot{\boldsymbol{u}} \, \mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \delta_* \dot{\boldsymbol{u}} \, \mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \left\{ \boldsymbol{h} \cdot \delta_* \dot{\boldsymbol{u}} + \delta_* \dot{\boldsymbol{h}} \cdot [\boldsymbol{u} - \bar{\boldsymbol{u}}] \right\} \mathrm{d}A$$

Euler-Lagrange equations

Constitutive equations

$$\rho_0 \boldsymbol{v} = \boldsymbol{p} \qquad \forall t \geqslant t_0 \qquad \delta_* \dot{\tilde{\boldsymbol{H}}} = \boldsymbol{O} \text{ and } \tilde{\boldsymbol{H}}(t_0) = \boldsymbol{O}$$

$$2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} = \boldsymbol{S} \qquad \forall t \geqslant t_0 \qquad \boldsymbol{S}_F = \left[2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F\right] \boldsymbol{A}_0$$

$$\dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) = \dot{\tilde{\boldsymbol{C}}} \quad \text{with} \quad \tilde{\boldsymbol{C}}(t_0) = \boldsymbol{C}(\boldsymbol{u}_0, \tilde{\boldsymbol{H}}(t_0))$$

$$\dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) : \boldsymbol{A}_0 = \dot{\tilde{C}}_F \quad \text{with} \quad \tilde{C}_F(t_0) = \boldsymbol{C}_F(\boldsymbol{u}_0, \tilde{\boldsymbol{H}}(t_0)) : \boldsymbol{A}_0$$

Neumann and Dirichlet boundary conditions

$$[\boldsymbol{F}\boldsymbol{S} + \boldsymbol{F}_F\,(\boldsymbol{S}_F : \boldsymbol{A}_0)]\,\boldsymbol{N} = \boldsymbol{t} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_t \mathscr{B}_0$$

$$\delta_* \dot{\boldsymbol{u}} = \boldsymbol{0} \text{ and } \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ with } \boldsymbol{u}(t_0) = \bar{\boldsymbol{u}}(t_0) \quad \text{on} \quad \partial_u \mathscr{B}_0$$

Equations of motion in first order form

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{with} \quad \boldsymbol{u}(t_0) = \boldsymbol{u}_0$$

$$\mathrm{DIV}[\boldsymbol{F}\boldsymbol{S} + \boldsymbol{F}_F\,(\boldsymbol{S}_F : \boldsymbol{A}_0)] + \rho_0 \boldsymbol{b} = \dot{\boldsymbol{p}} \quad \text{with} \quad \boldsymbol{p}(t_0) = \boldsymbol{p}_0 \equiv \rho_0 \boldsymbol{v}_0$$

Time evolution characteristics

1. continuous time evolutions of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{p}$ as well as $\tilde{\boldsymbol{H}}$, $\tilde{\boldsymbol{C}}$, $\tilde{C}_F$
2. discontinuous time evolution of the stresses $\boldsymbol{S}$, $\boldsymbol{S}_F$ and $\tilde{\boldsymbol{S}} = \boldsymbol{O}$, $\tilde{S}_F = O$

Discrete variational principle (2nd-order accurate)

Discrete principle of virtual power at $\xi_1 = 0.5$

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_1), \dot{\boldsymbol{v}}_h^n(\xi_1), \dot{\boldsymbol{p}}_h^n(\xi_1), \dot{\tilde{\boldsymbol{H}}}(\xi_1), \dot{\tilde{\boldsymbol{C}}}_h^n(\xi_1), \dot{\tilde{C}}_{F_h}^n(\xi_1), \boldsymbol{S}_h^n(\xi_1), \boldsymbol{S}_{F_h}^n(\xi_1); \tilde{\boldsymbol{S}}_h^n(\xi_1), \tilde{S}_{F_h}^n(\xi_1))\, h_n =$$

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}_d(\boldsymbol{u}_{n+1}, \boldsymbol{v}_{n+1}, \boldsymbol{p}_{n+1}, \boldsymbol{H}_{n+1}, \boldsymbol{C}_{n+1}, C_{F_{n+1}}, \boldsymbol{S}_{n+\frac{1}{2}}, \boldsymbol{S}_{F_{n+\frac{1}{2}}}; \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}, \tilde{S}_{F_{n+\frac{1}{2}}})\, h_n = 0$$

Galerkin approximations

$$\boldsymbol{u}_h^n(\alpha) := \boldsymbol{u}_n + \alpha\,(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) \qquad \boldsymbol{v}_h^n(\alpha) := \boldsymbol{v}_n + \alpha\,(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) \qquad \boldsymbol{p}_h^n(\alpha) := \boldsymbol{p}_n + \alpha\,(\boldsymbol{p}_{n+1} - \boldsymbol{p}_n)$$

$$\tilde{\boldsymbol{H}}_h^n(\alpha) := \boldsymbol{H}_n + \alpha\,(\boldsymbol{H}_{n+1} - \boldsymbol{H}_n) \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) := \boldsymbol{C}_n + \alpha\,(\boldsymbol{C}_{n+1} - \boldsymbol{C}_n) \qquad \tilde{C}_{F_h}^n(\alpha) := C_{F_n} + \alpha\,(C_{F_{n+1}} - C_{F_n})$$

Semidiscrete variational forms

$$\int_{\mathscr{B}_0} \left[\boldsymbol{S}_{n+\frac{1}{2}} - 2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) - \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : \delta \boldsymbol{C}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \left[\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0 - 2\,\mathrm{D}W_F(\tilde{C}_{Fn+\frac{1}{2}}) - \tilde{S}_{F_{n+\frac{1}{2}}}\right] : \delta C_{F_{n+1}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \left[\rho_0 \boldsymbol{v}_{n+\frac{1}{2}} - \boldsymbol{p}_{n+\frac{1}{2}}\right] \cdot \delta \boldsymbol{v}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \left[(C_{F_{n+1}} - C_{F_n})\,\boldsymbol{A}_0 - \left(\boldsymbol{F}_{F_{n+1}}^T + \boldsymbol{F}_{F_n}^T\right)(\boldsymbol{F}_{F_{n+1}} - \boldsymbol{F}_{F_n})\right] : \delta \boldsymbol{S}_{F_{n+\frac{1}{2}}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \left[\boldsymbol{v}_{n+\frac{1}{2}} - \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{h_n}\right] \cdot \delta \boldsymbol{p}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \left[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n - \left(\boldsymbol{F}_{n+1}^T + \boldsymbol{F}_n^T\right)(\boldsymbol{F}_{n+1} - \boldsymbol{F}_n)\right] : \delta \boldsymbol{S}_{n+\frac{1}{2}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \delta \boldsymbol{h}_{n+1} \cdot [\boldsymbol{u}_{n+1} - \bar{\boldsymbol{u}}_{n+1}]\,\mathrm{d}A = 0 = \int_{\mathscr{B}_0} \boldsymbol{F}_{n+\frac{1}{2}} \left[\boldsymbol{S}_{n+\frac{1}{2}} + \left(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\right)\boldsymbol{A}_0\right] : \delta \boldsymbol{H}_{n+1}\,\mathrm{d}V$$

$$\int_{\partial_t \mathscr{B}_0} \boldsymbol{t}_{n+\frac{1}{2}} \cdot \delta \boldsymbol{u}_{n+1}\,\mathrm{d}A + \int_{\partial_u \mathscr{B}_0} \boldsymbol{h}_{n+\frac{1}{2}} \cdot \delta \boldsymbol{u}_{n+1}\,\mathrm{d}A$$

$$= \int_{\mathscr{B}_0} \left\{\frac{\boldsymbol{p}_{n+1} - \boldsymbol{p}_n}{h_n} + \boldsymbol{B}_{n+\frac{1}{2}}^T \left[\boldsymbol{S}_{n+\frac{1}{2}} + \left(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\right)\boldsymbol{A}_0\right] - \rho_0 \boldsymbol{b}_{n+\frac{1}{2}}\right\} \cdot \delta \boldsymbol{u}_{n+1}\,\mathrm{d}V$$

Discrete Euler-Lagrange equations (2nd-order accurate)

Initial conditions

$$\boldsymbol{u}(t_0)=\boldsymbol{u}_0 \qquad \boldsymbol{p}(t_0)=\rho_0\boldsymbol{v}_0 \qquad \boldsymbol{v}(t_0)=\boldsymbol{v}_0 \qquad \tilde{\boldsymbol{C}}(t_0)=(\nabla\boldsymbol{u}_0+\boldsymbol{I})^T(\nabla\boldsymbol{u}_0+\boldsymbol{I}) \qquad \tilde{\boldsymbol{H}}(t_0)=\boldsymbol{O}$$

Discrete strong forms

(compare Betsch & Janz [2016])

$$\rho_0\left[\boldsymbol{v}_n+\boldsymbol{v}_{n+1}\right]=\boldsymbol{p}_n+\boldsymbol{p}_{n+1} \qquad h_n\left[\boldsymbol{v}_n+\boldsymbol{v}_{n+1}\right]=2\left(\boldsymbol{u}_{n+1}-\boldsymbol{u}_n\right)$$

$$2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}})+\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}=\boldsymbol{S}_{n+\frac{1}{2}} \qquad \boldsymbol{S}_{F_{n+\frac{1}{2}}}:\boldsymbol{A}_0=2\,\mathrm{D}W_F(\tilde{C}_{F_{n+\frac{1}{2}}})+\tilde{S}_{F_{n+\frac{1}{2}}}$$

$$2\,\mathbb{I}^{\mathrm{sym}}:\boldsymbol{F}^T_{n+\frac{1}{2}}\left(\boldsymbol{F}_{n+1}-\boldsymbol{F}_n\right)=\boldsymbol{C}_{n+1}-\boldsymbol{C}_n \qquad \left(C_{F_{n+1}}-C_{F_n}\right)\boldsymbol{A}_0=2\,\mathbb{I}^{\mathrm{sym}}:\boldsymbol{F}^T_{F_{n+\frac{1}{2}}}\left(\boldsymbol{F}_{F_{n+1}}-\boldsymbol{F}_{F_n}\right)$$

Discrete weak forms due to Q1E9

compare Simo, Armero & Taylor [1993]
Müller & Betsch [2007]

$$\frac{2}{h_n}\mathbf{M}\left[\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{h_n}-\mathbf{v}_n\right]+\int_{\mathscr{B}_0}\mathbf{B}^T_{n+\frac{1}{2}}\left[\mathbf{S}_{n+\frac{1}{2}}+\left(\boldsymbol{S}_{F_{n+\frac{1}{2}}}:\boldsymbol{A}_0\right)\mathbf{A}_0\right]\mathrm{d}V=\mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}}+\mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}}+\mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}}$$

$$\mathbf{M}:=\int_{\mathscr{B}_0}\rho_0\,\mathbf{N}^T\mathbf{N}\,\mathrm{d}V \qquad \mathbf{H}_t:=\int_{\partial_t\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V \qquad \mathbf{H}_u:=\int_{\partial_u\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V$$

$$\boldsymbol{B}_{n+\frac{1}{2}}\left[\boldsymbol{u}_{n+1}-\boldsymbol{u}_n\right]:=\mathbb{I}^{\mathrm{sym}}:\boldsymbol{F}^T_{n+\frac{1}{2}}\left[\nabla\boldsymbol{u}_{n+1}-\nabla\boldsymbol{u}_n\right] \qquad \boldsymbol{G}_{n+\frac{1}{2}}\left[\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n\right]:=\mathbb{I}^{\mathrm{sym}}:\boldsymbol{F}^T_{n+\frac{1}{2}}\left[\boldsymbol{H}_{n+1}-\boldsymbol{H}_n\right]$$

$$\int_{\mathscr{B}_0}\mathbf{G}^T_{n+\frac{1}{2}}\left[\mathbf{S}_{n+\frac{1}{2}}+\left(\boldsymbol{S}_{F_{n+\frac{1}{2}}}:\boldsymbol{A}_0\right)\mathbf{A}_0\right]\mathrm{d}V=\mathbf{0} \qquad \boldsymbol{H}_{n+1}=\sum_{A=1}^{n_{\mathrm{dim}}}\boldsymbol{\alpha}^A_{n+1}\otimes\tilde{\nabla}\tilde{N}_A$$

Discrete total energy balance

$$\frac{\mathcal{T}_{n+1}-\mathcal{T}_n}{h_n}+\frac{\Pi^{\mathrm{ext}}_{n+1}+\Pi^{\mathrm{ext}}_n}{h_n}=-\frac{1}{2}\int_{\mathscr{B}_0}\left[\boldsymbol{S}_{n+\frac{1}{2}}+\left(\boldsymbol{S}_{F_{n+\frac{1}{2}}}:\boldsymbol{A}_0\right)\boldsymbol{A}_0\right]:\underbrace{2\left[\boldsymbol{B}_{n+\frac{1}{2}}\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{h_n}+\boldsymbol{G}_{n+\frac{1}{2}}\frac{\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n}{h_n}\right]}_{(\boldsymbol{C}_{n+1}-\boldsymbol{C}_n)/h_n}\mathrm{d}V$$

Discrete superimposed fiber stress (2nd-order accurate)

Algorithmic claim for the fiber (one equation for 1 unknown)

$$\frac{1}{2}\int_{\mathscr{B}_0}\left[2\,\mathrm{D}W_F(\tilde{C}_{F_{n+\frac{1}{2}}}) + \tilde{S}_{F_{n+\frac{1}{2}}}\right]\underbrace{\boldsymbol{A}_0 : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]}_{C_{F_{n+1}} - C_{F_n}}\,\mathrm{d}V = \int_{\mathscr{B}_0}\left[W_F(C_{F_{n+1}}) - W_F(C_{F_n})\right]\mathrm{d}V$$

Constrained variational problem (compare Gauss's principle in Ramm [2011])

$$\mathcal{L}(\mu, \tilde{S}_{F_{n+\frac{1}{2}}}) := \frac{1}{2}\left(\tilde{S}_{F_{n+\frac{1}{2}}}\right)^2 + \mu\,\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) \qquad \delta_*\mathcal{L}(\mu, \tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

Local constraint G., Betsch & Steinmann [2005]

$$\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) := W_{F_{n+1}} - W_{F_n} - \frac{1}{2}\left[2\,\mathrm{D}W_F(\tilde{C}_{n+\frac{1}{2}}) + \tilde{S}_{F_{n+\frac{1}{2}}}\right]\left[C_{F_{n+1}} - C_{F_n}\right] = 0$$

Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{S}_{F_{n+\frac{1}{2}}}} \equiv \tilde{S}_{F_{n+\frac{1}{2}}} - \frac{\mu}{2}\left[C_{F_{n+1}} - C_{F_n}\right] = 0 \qquad \frac{\partial\mathcal{L}}{\partial\mu} \equiv \mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

Discrete superimposed fiber stress Gonzalez [2000]

$$\tilde{S}_{F_{n+\frac{1}{2}}} = 2\,\frac{\mathcal{G}(O)}{\left[C_{F_{n+1}} - C_{F_n}\right]\cancel{\left[C_{F_{n+1}} - C_{F_n}\right]}}\cancel{\left[C_{F_{n+1}} - C_{F_n}\right]}$$

Discrete superimposed stress tensor (2nd-order accurate)

Algorithmic claim for the matrix (one equation for 6 unknowns)

$$\frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) + \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\Big] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\mathrm{d}V = \int_{\mathscr{B}_0}[W(\boldsymbol{C}_{n+1}) - W(\boldsymbol{C}_n)]\,\mathrm{d}V$$

Constrained variational problem

$$\mathcal{L}(\mu, \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := \frac{1}{2}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} : \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} + \mu\,\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) \qquad \delta_*\mathcal{L}(\mu, \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

Local constraint

$$\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := W_{n+1} - W_n - \frac{1}{2}\Big[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) + \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\Big] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] = 0$$

Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}} \equiv \tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} - \frac{\mu}{2}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] = \boldsymbol{O} \qquad \frac{\partial\mathcal{L}}{\partial\mu} \equiv \mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

Discrete superimposed stress tensor Armero & Zambrana-Rojas [2007]

$$\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} = 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}$$

Higher-order accurate approximation

Discrete variation principle

$$\sum_{n=0}^{N-1}\sum_{i=1}^{k}\delta_*\dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_i),\dot{\boldsymbol{v}}_h^n(\xi_i),\dot{\boldsymbol{p}}_h^n(\xi_i),\dot{\tilde{\boldsymbol{C}}}_h^n(\xi_i),\boldsymbol{S}_h^n(\xi_i);\rho_0,\boldsymbol{A}_0,\boldsymbol{\kappa}_0,\boldsymbol{b}_h^n(\xi_i),\boldsymbol{t}_h^n(\xi_i),\bar{\boldsymbol{u}}_h^n(\xi_i))\,w_i\,h_n=0$$

Galerkin-based approximations

$$\boldsymbol{u}_h^n(\alpha):=\sum_{j=1}^{k+1}M_j(\alpha)\,\boldsymbol{u}_j^n \qquad \boldsymbol{v}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{v}_j^n \qquad \boldsymbol{p}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{p}_j^n$$

$$\tilde{\boldsymbol{H}}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{H}_j^n \qquad \tilde{\boldsymbol{C}}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{C}_j^n \qquad \tilde{C}_{F_h}^n(\alpha):=\sum M_j(\alpha)\,C_{F_j}^n$$

Discrete weak assumed strain equation

$$\sum_{i=1}^{k}\int_{\mathcal{B}_0}\delta_*\boldsymbol{S}_h^n(\xi_i):\left[\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha}-\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))\right]w_i\,\mathrm{d}V=0 \qquad i=1,\dots,k$$

Discrete local assumed strain equation (Euler-Lagrange equation)

$$\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha}-\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))=\boldsymbol{O} \qquad i=1,\dots,k \qquad \frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\alpha)}{\mathrm{d}\alpha}=\sum_{j=1}^{k+1}\mathring{M}_j(\alpha)\,\boldsymbol{C}_j^n\equiv\sum_{i=1}^{k}\tilde{M}_i(\alpha)\,\tilde{\boldsymbol{C}}_i^n$$

Unknown nodal values C_l^n, $l=2,\dots,k$

$$\boldsymbol{C}_l^n:=\sum_{i=1}^{k}m_{li}\,\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))+\boldsymbol{C}_1^n \qquad \text{with} \qquad \boldsymbol{m}=\begin{bmatrix}\mathring{M}_2(\xi_1) & \dots & \mathring{M}_{k+1}(\xi_1)\\ \vdots & \dots & \vdots\\ \mathring{M}_2(\xi_k) & \dots & \mathring{M}_{k+1}(\xi_k)\end{bmatrix}^{-1}$$

Higher-order accurate stress approximation

Nodal values for $k = 1$

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_2^n := (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

Nodal values for $k = 2$

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_2^n := \frac{1}{3}\left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right]^T \left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right] + (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_3^n := (\boldsymbol{F}_3^n)^T \boldsymbol{F}_3^n$$

Old superimposed stress with mixed 'strain' approximation in time

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2 \frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_l) : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l} \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_i)$$

(energy consistent, **but not** variationally consistent approximation)

G., Betsch & Steinmann [2005]

with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, [\boldsymbol{F}_j^n]^T \boldsymbol{F}_j^n$$

New superimposed stress with uniform 'strain' approximation in t

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2 \frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1} \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_l) : \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_l)\, [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, w_l} [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1} \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_i)\, [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}$$

with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\tilde{\boldsymbol{C}}_h^n}(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, \boldsymbol{C}_j^n$$

Rotating pipe with inner taper (I) (Q1-element)
(influence of reinforcing fibers; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) \;:=\;& \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
&+ \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
&+ \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
&+ \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
&+ \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi \in [0, \tfrac{\pi}{2}[
$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

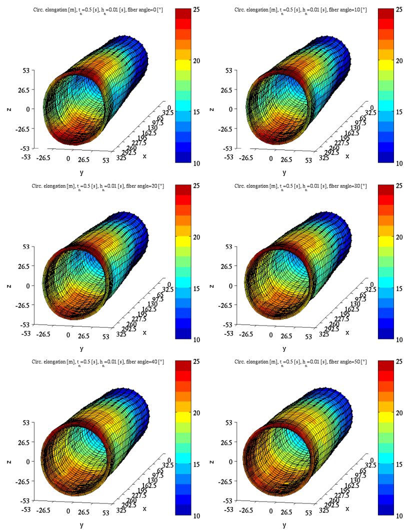
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1
$$

no external forces

Current configurations at $T = 0.5$

Colour indicates circumferential elongation ($h_n = 0.01$)



Rotating pipe with inner taper (II) (Q1-element)

(influence of reinforcing fibers; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

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$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi \in [0, \tfrac{\pi}{2}[$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

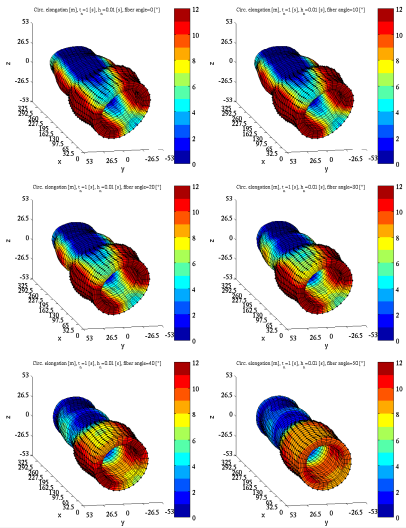
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1$$

no external forces

Current configurations at $T = 1.0$

Colour indicates circumferential elongation ($h_n = 0.01$)



Rotating pipe with inner taper (III) (Q1-element)
(influence of reinforcing fibers; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := {} & \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
& + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
& + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
& + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
& + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

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W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
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W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c + 1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

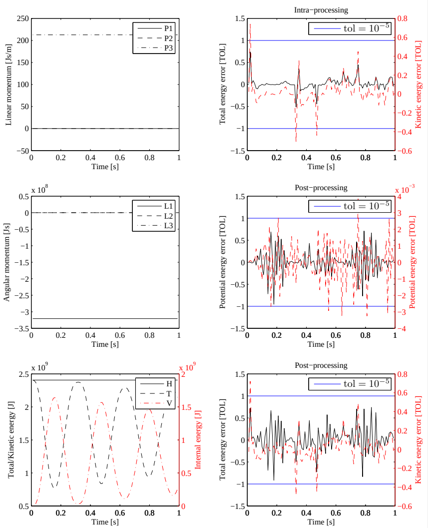
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1
$$

no external forces

Conservation properties

time step size $h_n = 0.01$ and Newton tolerance 10^{-5}



Rotating pipe with inner taper (IV) (Q1-element)
(influence of reinforcing fibers; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

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Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\mathrm{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\mathrm{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\mathrm{iso}}))^{-1/2} - 4\right]$$

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$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi \in [0, \tfrac{\pi}{2}[$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

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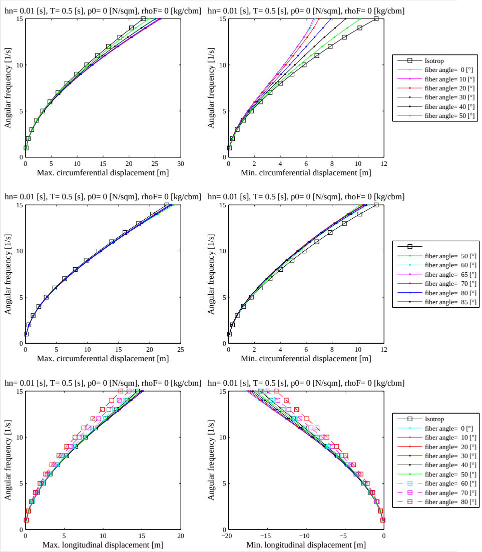
$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -15\,\boldsymbol{e}_1$$

no external forces

Pipe elongations at $T = 0.5$ $(h_n = 0.01)$



Rotating pipe under pressure (I) (Q1-element)
(influence of internal pressure; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

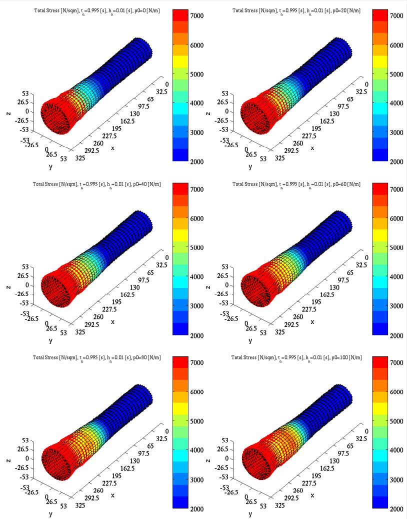
$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -7\, \boldsymbol{e}_1$$

Fluid pressure in the pipe

$$p(r) = p_0 + \tfrac{1}{2}\,\rho_F\,\omega_0^2\, r^2$$

Pipe configurations at $T = 1.0$ $(h_n = 0.01)$

Colours indicate von-Mises stress and red arrows the pressure load



Rotating pipe under pressure (II) (Q1-element)

(influence of internal pressure; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) \;:=\; & \mu_{10}\,[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})] \\
& + \mu_{20}\,[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})]^2 \\
& + \mu_{30}\,[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})]^3 \\
& + \mu_{01}\,[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})] \\
& + \frac{Y_1}{Y_2}\,\{1 - \exp[-Y_2\,(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I}))]\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) \;:=\; \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) \;:=\; \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) \;:=\; \frac{g_0}{g_c + 1}\,[I_4(\boldsymbol{C})]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 \;:=\; \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

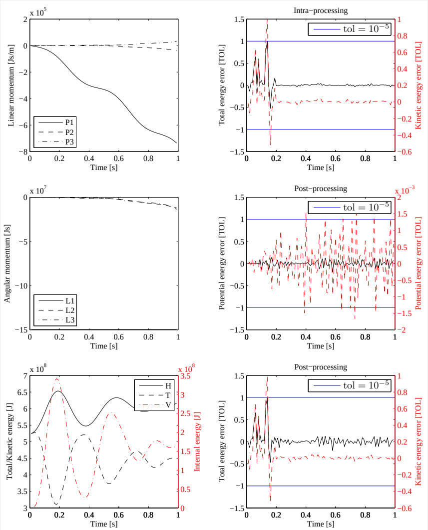
$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -7\,\boldsymbol{e}_1$$

Fluid pressure in the pipe

$$p(r) = p_0 + \tfrac{1}{2}\,\rho_F\,\omega_0^2\,r^2$$

Conservation properties

time step size $h_n = 0.01$ and pressure $p_0 = 120\,[\text{N/sqm}]$



Rotating pipe under pressure (III) (Q1-element)
(influence of higher-order finite elements in time)

Michael Groß, Rajesh Ramesh and Julian Dietzsch

Pipe material

Isotropic strain energy functions

$$W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\mathrm{vol}}(\boldsymbol{C}) := \frac{\kappa^{\mathrm{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\mathrm{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\mathrm{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50°, 60°$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

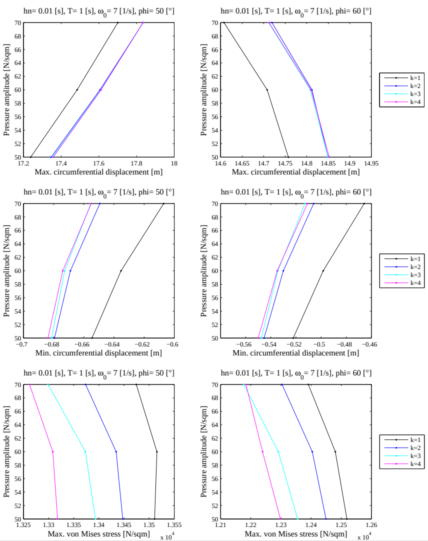
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -7\, \boldsymbol{e}_1$$

Fluid pressure in the pipe

$$p(r) = p_0 + \tfrac{1}{2}\,\rho_F\,\omega_0^2\, r^2$$

Deformation/stress for higher-order FeT



Rotating pipe under pressure (IV) (Q1-element)
(influence of higher-order finite elements in time)

Pipe material

Isotropic strain energy functions

$$\begin{aligned} W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := {} & \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\ & + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\ & + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\ & + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\ & + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\} \end{aligned}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

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Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 60^\circ$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

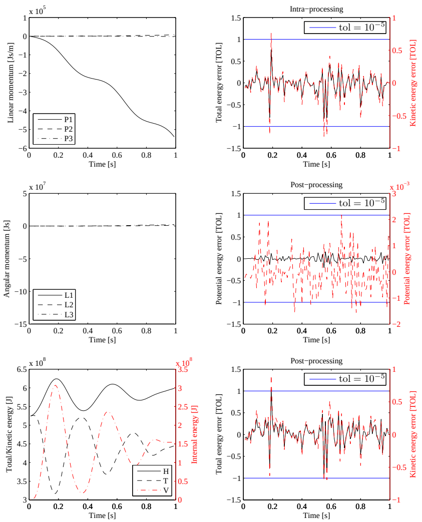
$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -7\,\boldsymbol{e}_1$$

Fluid pressure in the pipe

$$p(r) = p_0 + \frac{1}{2}\,\rho_F\,\omega_0^2\,r^2$$

Conservation properties for quartic FeT

time step size $h_n = 0.01$ and pressure $p_0 = 70\,[\text{N/sqm}]$



Rotating pipe under pressure (Q1-element)

(rotor instability without bearings; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\,[I_4(\boldsymbol{C})]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 70^\circ$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

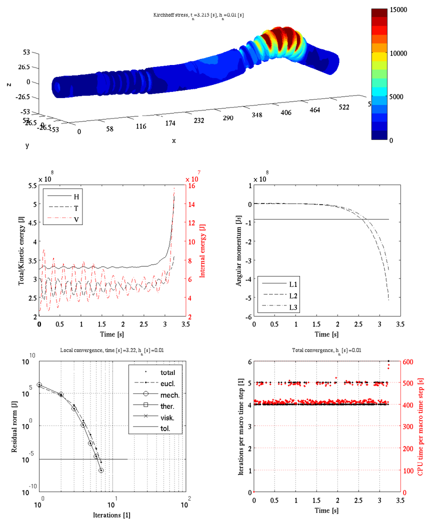
$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -7\,\boldsymbol{e}_1$$

Fluid pressure in the pipe

$$p(r) = p_0 + \tfrac{1}{2}\,\rho_F\,\omega_0^2\,r^2$$

Pipe configurations at $T = 3.22$ ($h_n = 0.01$)

Colours indicate von Mises stress



Rotating disc (I) (Q1E9-element)

(shear locking in dynamics; linear finite elements in time)

Michael Groß, Rajesh Ramesh and Julian Dietzsch

Pipe material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 0^\circ$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

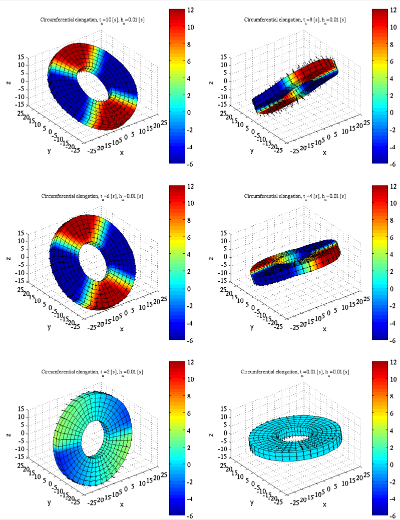
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1$$

no external forces

Current configurations at $t = 0, \ldots, 10$

Colour indicates circumferential elongation ($h_n = 0.01$)



Rotating disc (II) (Q1-element)

(shear locking in dynamics; linear finite elements in time)

Energy and Momentum conserving variational based time integration of anisotropic hyperelastic continua

Michael Groß, Rajesh Ramesh and Julian Dietzsch

Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) &:= \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
&\quad + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
&\quad + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
&\quad + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
&\quad + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c + 1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 0^\circ
$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

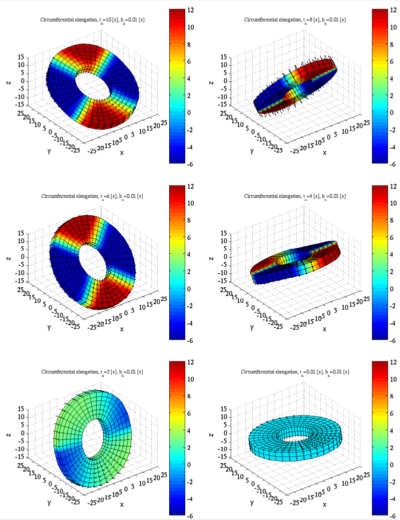
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1
$$

no external forces

Current configurations at $t = 0, \ldots, 10$

Colour indicates circumferential elongation ($h_n = 0.01$)



Rotating disc (III) (Q1E9-element)

(shear locking in dynamics; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$\begin{aligned} W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) := \; & \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] \\ & + \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 \\ & + \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 \\ & + \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] \\ & + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\} \end{aligned}$$

$$W^{\mathrm{vol}}(\boldsymbol{C}) := \frac{\kappa^{\mathrm{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\mathrm{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\mathrm{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c + 1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle } \phi = 0^\circ$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

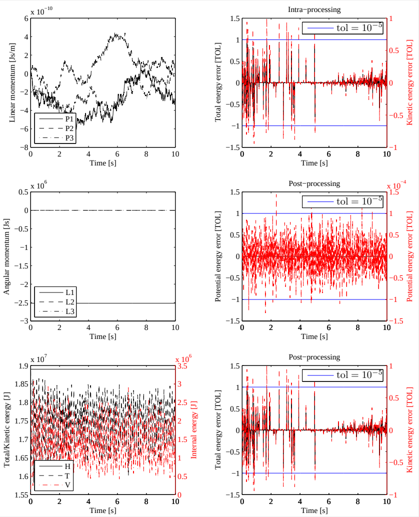
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1$$

no external forces

Conservation properties

time step size $h_n = 0.01$ and Newton tolerance 10^{-5}



Rotating disc (IV) (Q1-element)

(shear locking in dynamics; linear finite elements in time)

Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) \;:=\;& \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] \\
&+ \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
&+ \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
&+ \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] \\
&+ \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\mathrm{vol}}(\boldsymbol{C}) \;:=\; \frac{\kappa^{\mathrm{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) \;:=\; \frac{\kappa^{\mathrm{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\mathrm{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) \;:=\; \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c + 1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 \;:=\; \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \qquad \text{fiber angle} \quad \phi = 0^\circ
$$

Holzapfel, Gasser & Ogden [2000]

Initial conditions

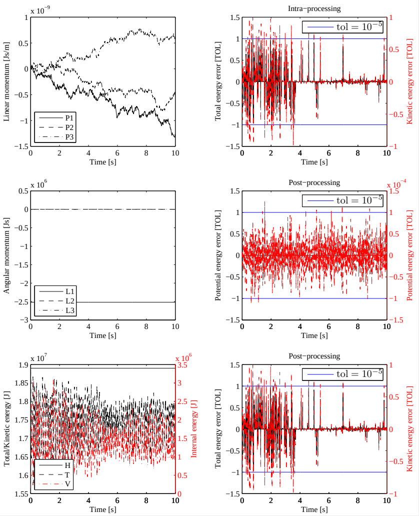
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \qquad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -15\, \boldsymbol{e}_1
$$

no external forces

Conservation properties

time step size $h_n = 0.01$ and Newton tolerance 10^{-5}



Summary

1. Motivation:
   - Dynamic simulation methods for fiber-reinforced structures
   - related with improved time and space approximations
2. Goals:
   - Variationally consistent energy-momentum schemes
   - with higher-order Galerkin-based approximations
   - and enhanced displacement gradients
3. Strategy:
   - Formulation of a mixed variational principle
   - Introduction of a Galerkin-based time/space discretization
   - Discrete variation at the time/space mesh points
4. Important results:
   - Variationally consistent assumed strain approximations in time
   - a new energy-consistent higher-order stress approximation
   - an energy-consistent higher-order accurate Q1E9-element