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DYNAMIC THERMO-MECHANICAL COUPLING IN FIBER-REINFORCED BODIES SIMULATED BY HIGHER-ORDER VARIATIONAL ENERGY-MOMENTUM SCHEMES

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Abstract

The most widely-used materials in mechanical engineering are metals, which are able to dissipate relatively well generated heat as well as to conduct arising strong forces. But, solids made of metals are usally not used in leight-weight structures due to their considerable weight. In this case, metal foams or fiber-reinforced polymers are the materials of choice. In this paper, we consider polymers reinforced by one familie of fibers conducting heat as well as strong forces independently in two directions. The heat conduction model is based on Duhamel's law of transversely isotropic heat conduction, which prescribe heat conduction in fiber direction with an own conductivity coefficient as well as heat conduction normal to the fibers in the matrix material. The heat capacity coefficient of the fiber-reinforced material is assumed linear in temperature. The thermal expansion of the matrix material is defined as volume change depending linearly on the temperature, but the independent thermal expansion of the fibers are assumed as length change depending linearly on the temperature. We simulate nonlinear rotordynamics examples as fast rotating bodies with large deformations. Therefore, we model a flexible body as transversely isotropic thermoelastic continuum, and discredize it by linear or quadratic finite elements. The transient simulation is performed by a higher-order accurate time integration method, called ehG method, which determines independently of the time step size a non-positive thermal dissipation arising from Duhamel's law, and therefore fulfills the corresponding stability estimate based on the Lyapunov function of thermoelasticity numerically exactly.

1. Problem definition

We consider non-isothermal deformations of a transversely isotropic continuum moving in the Euclidean space $\mathbb{R}^{n_{\text{dim}}}$ with the ambient temperature Θ_∞. The initial configuration $\mathscr{B}_0 = \mathscr{B}_0^M \cup \mathscr{B}_0^F$ of the fiber-reinforced continuum body $\mathscr{B}$ (see Fig. 1 on the right) is defined as homogenization of the set $\mathscr{B}_0^M$ of the matrix material with the set $\mathscr{B}_0^F$ of the fibers. Therefore, we obtain an imaginary fiber at any point $\mathbf{X} \in \mathscr{B}_0$, which is directed along the normalized fiber vector $\mathbf{a}_0$ with $\|\mathbf{a}_0\| = 1$. The corresponding stretched fiber vector $\mathbf{a}$ in the current configuration $\mathscr{B}_t$ is given by

$$\mathbf{a} = \mathbf{F}\,\mathbf{a}_0 \tag{1}$$

where

$$\mathbf{F} := \nabla\mathbf{u} + \mathbf{I} \tag{2}$$

denotes the deformation gradient and $\mathbf{u}$ the displacement vector field. The tensor $\mathbf{I}$ designates the second-order unit tensor, and the symbol ∇ indicates the partial derivative with respect to the material point $\mathbf{X} \in \mathscr{B}_0$. The deformation gradient $\mathbf{F}_F$ in fiber direction takes the form

$$\mathbf{F}_F := \mathbf{a} \otimes \mathbf{a}_0 = \mathbf{F}\,\mathbf{a}_0 \otimes \mathbf{a}_0 = \mathbf{F}\,\mathbf{A}_0 \tag{3}$$

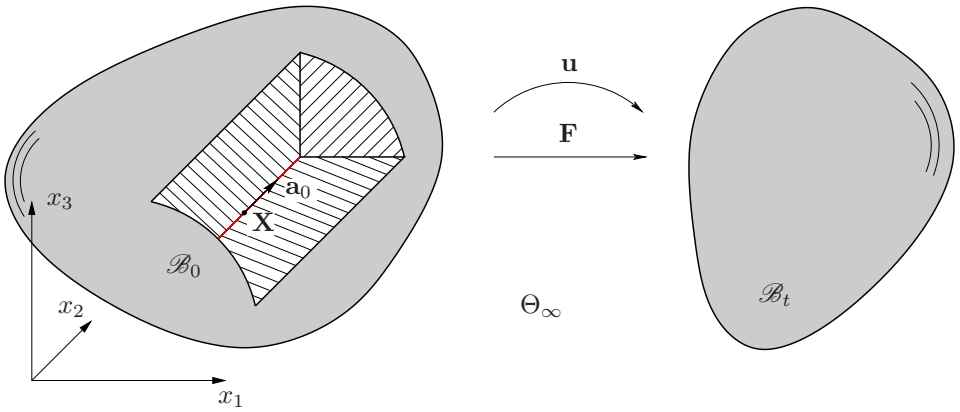

Figure 1: Initial and current configuration of a transversely isotropic continuum.

with

$$\mathbf{A}_0 := \mathbf{a}_0 \otimes \mathbf{a}_0 = \mathbf{F}^{-1}\,[\mathbf{a}\otimes\mathbf{a}]\,\mathbf{F}^{-T} = \mathbf{F}^{-1}\,\mathbf{A}\,\mathbf{F}^{-T} \tag{4}$$

The tensors $\mathbf{A}_0$ and $\mathbf{A} := \mathbf{a}\otimes\mathbf{a}$ are called the material and spatial structural tensors, respectively, related by pull-back or push-forward operations. As deformation tensor of the fibers, we define

$$\mathbf{C}_F := \mathbf{F}_F^T\,\mathbf{F}_F = [\mathbf{F}\,\mathbf{a}_0\otimes\mathbf{a}_0]^T\,[\mathbf{F}\,\mathbf{a}_0\otimes\mathbf{a}_0] = [\mathbf{a}_0\otimes\mathbf{a}_0]\,\mathbf{C}\,[\mathbf{a}_0\otimes\mathbf{a}_0] = \mathbf{A}_0\,\mathbf{C}\,\mathbf{A}_0 = [\mathbf{C}:\mathbf{A}_0]\,\mathbf{A}_0 \tag{5}$$

which is related to the right Cauchy-Green tensor $\mathbf{C} := \mathbf{F}^T\,\mathbf{F}$ by a projection on $\mathbf{A}_0$. We consider the free energy density $\Psi := \Psi_{\text{ani}} + \Psi_{\text{fib}}$ as a superposition of the isothermal function $\Psi_{\text{ani}}(\mathbf{C};\mathbf{A}_0)$, where the semicolon in the argument separates the parameter tensor $\mathbf{A}_0$ acting at any $\mathbf{X}\in\mathscr{B}_0$ from the variables $\mathbf{C}$ and Θ, and the non-isothermal function

$$\Psi_{\text{fib}}(\mathbf{C},\Theta;\mathbf{A}_0) = \Psi_M(\mathbf{C},\Theta) + \Psi_F(\mathbf{C}_F,\Theta) \tag{6}$$

The field Θ denotes the absolute temperature of the body parametrized by the material point $\mathbf{X}\in\mathscr{B}_0$. Vectors including material parameters with respect to $\mathscr{B}_0$ are neglected in the arguments for the sake of clarity. The total second Piola-Kirchhoff stress tensor is given by

$$\mathbf{S} := 2\,\frac{\partial\Psi(\mathbf{C},\Theta;\mathbf{A}_0)}{\partial\mathbf{C}} = 2\,\frac{\partial\Psi_{\text{ani}}(\mathbf{C};\mathbf{A}_0)}{\partial\mathbf{C}} + 2\,\frac{\partial\Psi_M(\mathbf{C},\Theta)}{\partial\mathbf{C}} + \mathbf{S}_F \tag{7}$$

with

$$\mathbf{S}_F := 2\,\frac{\partial\Psi_F(\mathbf{C}_F,\Theta)}{\partial\mathbf{C}_F} : \frac{\partial\mathbf{C}_F}{\partial\mathbf{C}} = 2\,\mathbf{A}_0\,\frac{\partial\Psi_F(\mathbf{C}_F,\Theta)}{\partial\mathbf{C}_F}\,\mathbf{A}_0 = \left[2\,\frac{\partial\Psi_F(\mathbf{C}_F,\Theta)}{\partial\mathbf{C}_F} : \mathbf{A}_0\right]\mathbf{A}_0 \tag{8}$$

The free energy functions Ψ_{ani}, Ψ_M and Ψ_F directly depends on the invariants of the corresponding deformation tensors. The free energy density Ψ_{ani} can therefore be written as $\Psi_{\text{ani}}(\mathbf{C};\mathbf{A}_0) = \hat{\Psi}_{\text{ani}}(I_1,I_2,I_3,I_4)$, whereby, on the other hand, the non-isothermal free energy densities can be formulated as $\Psi_M = \hat{\Psi}_M(I_1,I_2,I_3,\Theta)$ and $\Psi_F = \hat{\Psi}_F(I_4,\Theta)$. The tensor invariants

$$I_1 := \mathbf{C}:\mathbf{I} \qquad\qquad I_2 := \frac{1}{2}\left[(I_1)^2 - \mathbf{C}^2:\mathbf{I}\right] \qquad\qquad I_3 := \det\mathbf{C} \tag{9}$$

denotes the tensor invariants of the right Cauchy-Green tensor $\mathbf{C}$, and the invariant

$$I_4 \equiv \mathbf{C}_F:\mathbf{A}_0 = \mathbf{A}_0\,\mathbf{C}\,\mathbf{A}_0:\mathbf{A}_0 = [\mathbf{a}_0\cdot\mathbf{C}\,\mathbf{a}_0]\,\mathbf{a}_0\cdot\mathbf{a}_0 = \mathbf{a}_0\cdot\mathbf{C}\,\mathbf{a}_0 = \mathbf{C}:\mathbf{A}_0 = \mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2 =: C_F \tag{10}$$

designates the squared fiber stretch $C_F = \lambda_F^2$. Consequently, the deformation tensor $\mathbf{C}_F$ and the stress tensor $\mathbf{S}_F$, respectively, can be simply written as

$$\mathbf{C}_F = C_F\,\mathbf{A}_0 \qquad\qquad \mathbf{S}_F = 2\left[\frac{\partial\hat{\Psi}_F(C_F,\Theta)}{\partial C_F}\,\frac{\partial C_F}{\partial\mathbf{C}_F} : \mathbf{A}_0\right]\mathbf{A}_0 = 2\,\frac{\partial\hat{\Psi}_F(C_F,\Theta)}{\partial C_F}\,\mathbf{A}_0 \tag{11}$$

We introduce the total entropy of the fiber-reinforced material as energetic conjugate of the temperature difference $\vartheta := \Theta - \Theta_\infty$ by means of the relative internal energy density $e(\mathbf{C},\eta) := \Psi(\mathbf{C},\Theta) + \eta\,(\Theta-\Theta_\infty)$ such that from the total differential of $e(\mathbf{C},\eta)$ follows the Gibbs relation

$$\eta := -\frac{\partial\Psi(\mathbf{C},\Theta;\mathbf{A}_0)}{\partial\Theta} = -\frac{\partial\hat{\Psi}_M(I_1,I_2,I_3,\Theta)}{\partial\Theta} - \frac{\partial\hat{\Psi}_F(C_F,\Theta)}{\partial\Theta} \tag{12}$$

We consider the thermal expansion of the matrix material as volume change of $\mathscr{B}_0$, and the thermal expansion of the fibers as length change in direction of the fiber vector $\mathbf{a}_0$. Both effects are assumed to depend linearly on the body temperature, so that

$$n_{\mathrm{dim}}\,\beta_M\,(\Theta-\Theta_0)=\frac{V_t-V_0}{V_0}=J-1 \qquad (\beta_F-\beta_M)\,(\Theta-\Theta_0)=\frac{\|\mathbf{a}\|-\|\mathbf{a}_0\|}{\|\mathbf{a}_0\|}=\lambda_F-1 \quad (13)$$

where $J:=\sqrt{I_3}$ denotes the determinant of the deformation gradient $\mathbf{F}$, and Θ_0 the initial temperature in $\mathscr{B}_0$. The constant β_F denotes the linear heat expansion coefficient in direction of the fiber vector $\mathbf{a}_0$, and the constant $n_{\mathrm{dim}}\beta_M$ denotes the volume expansion coefficient of the matrix material. Hence, these assumptions lead to constant partial derivatives

$$n_{\mathrm{dim}}\,\beta_M=\frac{\partial J}{\partial\Theta} \qquad \beta_F-\beta_M=\frac{\partial\lambda_F}{\partial\Theta} \quad (14)$$

and to the fact that η depends directly on J, λ_F and the temperature Θ. In this way, we arrive at

$$\eta=-\frac{\partial\hat{\Psi}_M^{\mathrm{the}}(J,\Theta)}{\partial\Theta}-\frac{\partial\hat{\Psi}_F^{\mathrm{the}}(\lambda_F,\Theta)}{\partial\Theta} \quad (15)$$

For deriving the explicit form of the independent free energy functions $\hat{\Psi}_M^{\mathrm{the}}(J,\Theta)$ and $\hat{\Psi}_F^{\mathrm{the}}(\lambda_F,\Theta)$, we introduce by means of the relative Gibbs free energy densities

$$g_M^{\mathrm{the}}(p,\Theta):=\hat{\Psi}_M^{\mathrm{the}}(J,\Theta)-p\,(J-1) \qquad g_F^{\mathrm{the}}(P_F,\Theta):=\hat{\Psi}_F^{\mathrm{the}}(\lambda_F,\Theta)-P_F\,(\lambda_F-1) \quad (16)$$

the hydrostatic pressure p and the fiber stress P_F along the fiber vector $\mathbf{a}_0$ as independent variables. In this way, p and P_F can be stated below by independent assumptions. The total differentials of these relative Gibbs free energy densities then leads to the Gibbs relations

$$1-J=\frac{\partial g_M^{\mathrm{the}}(p,\Theta)}{\partial p} \qquad 1-\lambda_F=\frac{\partial g_F^{\mathrm{the}}(P_F,\Theta)}{\partial P_F} \quad (17)$$

The equality of the mixed partial derivatives of the Gibbs free energies renders the Maxwell relations

$$\left.\frac{\partial J}{\partial\Theta}\right|_p=\left.\frac{\partial\eta}{\partial p}\right|_\Theta \qquad \left.\frac{\partial\lambda_F}{\partial\Theta}\right|_{P_F}=\left.\frac{\partial\eta}{\partial P_F}\right|_\Theta \quad (18)$$

(compare Reference [1]). Taking into account the constant partial derivatives in Eq. (14), these Maxwell relations leads to the entropy density function

$$\eta(p,P_F,\theta)=\alpha_M\,p+(\beta_F-\beta_M)\,P_F+T(\Theta)+K_\eta \quad (19)$$

with the integration constant K_η. On the other hand, assuming a linear heat capacity, we obtain from

$$c_0\,[1+c_1\,(\Theta-\Theta_\infty)]=\Theta\left.\frac{\partial\eta(p,P_F,\theta)}{\partial\Theta}\right|_{p,P_F} \quad (20)$$

by integration the entropy density function

$$\eta(p,P_F,\theta)=M(p,P_F)+c_0\,c_1\Theta+c_0\,(1-\Theta_\infty\,c_1)\ln\Theta+K_\eta \quad (21)$$

By comparing Eq. (19) and Eq. (21) and determining the integration constant K_η by means of the condition that at ambient temperature Θ_∞ only mechanical entropy $M(p,P_F)$ remains, we arrive at

$$\eta(p,P_F,\theta)=\alpha_M\,p+(\beta_F-\alpha_M)\,P_F+c_0\,c_1\,(\Theta-\Theta_\infty)+c_0\,(1-\Theta_\infty\,c_1)\ln\frac{\Theta}{\Theta_\infty} \quad (22)$$

For the sake of simplicity, we assume for p and P_F the isothermal definitions

$$p:=\frac{\partial\hat{W}^{\mathrm{vol}}(J)}{\partial J} \qquad P_F:=\frac{\partial\hat{W}^{\mathrm{fib}}(C_F)}{\partial C_F}\frac{\partial C_F(\lambda_F)}{\partial\lambda_F}=2\,\lambda_F\,\frac{\partial\hat{W}^{\mathrm{fib}}(C_F)}{\partial C_F} \quad (23)$$

based on a volumetric strain energy function $\hat{W}^{\mathrm{vol}}(J)$ and a fiber strain energy function $\hat{W}^{\mathrm{fib}}(C_F)$. Then, integrating Eq. (22) over the temperature Θ, and determining the corresponding integration

constant such that the free energy function vanish at ambient temperature, independent of the fiber stretch λ_F and the volume change J, we obtain the free energy functions

$$\hat{\Psi}_M^{\text{the}}(J,\Theta) := c_0 \left[(1-\Theta_\infty) \left(\Theta - \Theta_\infty - \Theta \ln \frac{\Theta}{\Theta_\infty} \right) - \frac{1}{2} c_1 (\Theta - \Theta_\infty)^2 \right] - (\Theta - \Theta_\infty) n_{\text{dim}} \beta_M \frac{\partial \hat{W}^{\text{vol}}(J)}{\partial J}$$

$$\hat{\Psi}_F^{\text{the}}(\lambda_F,\Theta) := -2\lambda_F (\Theta - \Theta_\infty)(\beta_F - \beta_M) \frac{\partial \hat{W}^{\text{fib}}(C_F(\lambda_F))}{\partial C_F} \tag{24}$$

The heat conduction in the body is modelled by Duhamel's law of transversely isotropic heat conduction according to References [1, 2], which introduce in Duhamel's law $\mathbf{q} = -\boldsymbol{\kappa}\mathbf{F}^{-T}\nabla\Theta$ the spatial thermal conductivity tensor

$$\boldsymbol{\kappa} = (k_F - k) \frac{1}{\|\mathbf{a}\|^2} \mathbf{a} \otimes \mathbf{a} + k\mathbf{I} \tag{25}$$

with the thermal conductivity coefficient k_F of the fibers and k of the matrix material. The Piola transformation of the Cauchy (spatial) heat flux vector $\mathbf{q}$ by taking into account Eq. (4) leads to the Piola (material) heat flux vector

$$\mathbf{Q} := -\left[\frac{1}{C_F} (k_{F_0} - k_0) \mathbf{A}_0 + k_0 \mathbf{C}^{-1} \right] \nabla\Theta \tag{26}$$

where $k_{F_0} := J k_F$ and $k_0 := J k$ denotes the material conductivity coefficients of the material.

2. Time discretization

The time discretization is performed by the extension of the ehG method in Reference [3] to cover the material formulation summarized above and the modifications of the time approximations of the algorithmic strain and stress tensor in Reference [4]. Within the framework of thermoelasticity, we modified the time approximation of the entropy density of the ehG method according to the equation

$$\int_0^1 \frac{\partial \Theta_h^n(\alpha)}{\partial \alpha} \left[\frac{\partial \Psi(\tilde{C}_h^n(\alpha), \Theta_h^n(\alpha))}{\partial \Theta} + \tilde{\eta}_h^n(\alpha) \right] \mathrm{d}\alpha = 0 \tag{27}$$

with

$$\tilde{\eta}_h^n(\alpha) := \sum_{i=1}^{k+1} M_i(\alpha)\, \eta_i^n \qquad \tilde{\mathbf{C}}_h^n(\alpha) := \sum_{i=1}^{k+1} M_i(\alpha)\, \mathbf{C}_i^n \qquad \Theta_h^n(\alpha) := \sum_{i=1}^{k+1} M_i(\alpha)\, \Theta_i^n \tag{28}$$

The polynomials $M_i(\alpha)$ are the Lagrange polynomials of order k tabulated in Reference [3]. By using Eq. (27), we determine the value η_i^n of the assumed entropy approximation $\tilde{\eta}_h^n(\alpha)$ at the i-th time node of the n-th time step, and at each Gauss point of a spatial finite element. A similar time approximation condition is stated for the assumed right Cauchy-Green tensor $\tilde{\mathbf{C}}_h^n(\alpha)$ (see Reference [4]). The solution of Eq. (27) is straightforward introducing $k+1$ Gauss points in time leading to

$$\sum_{l=1}^{k+1} \frac{\partial \Theta_h^n(\xi_l)}{\partial \alpha} \left[\frac{\partial \Psi(\tilde{C}_h^n(\xi_l), \Theta_h^n(\xi_l))}{\partial \Theta} + \tilde{\eta}_h^n(\xi_l) \right] w_l = 0 \tag{29}$$

Thus, the nodal value η_1^n, which does not arise in the first time step from the initial temperature Θ_0 at the initial time point $t_0 = 0$ due to the discontinuous temporal approximation of the temperature Θ, is also determined. In this way, the ehG method is able to approximate the temperature field and the motion with finite elements in time of different degrees. Eq. (29) leads to a weak fulfillment of Eq. (15) in time, and also emanates from a variational principle as $\tilde{\mathbf{C}}_h^n$ in Reference [4].

References

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