Dynamic thermo-mechanical coupling in fiber-reinforced bodies simulated by higher-order variational energy-momentum schemes

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Structure-preserving numerical methods for coupled problems

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Motivation, goals and strategy

Motivation

1. Dynamic simulations of fiber-reinforced materials in light-weight structures
2. Variational design of energy-momentum schemes on arbitrary coupled fields

Goal 1: energy-momentum schemes for coupled fields

Mechanical fields

$\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{F}, \boldsymbol{C}, \boldsymbol{S}$ $\quad C_v$

$\boldsymbol{T}, \boldsymbol{B}, \boldsymbol{R}$ $\quad \boldsymbol{A}_0, \boldsymbol{\kappa}_M$

Coupling

Thermal fields

$\theta, \eta, \nabla\theta$

$R, Q, h, \lambda \quad \kappa_T$

Coupling

Mixed fields

$\tilde{\boldsymbol{C}}, \tilde{C}_F$ local in space

$\tilde{\vartheta}, \tilde{\boldsymbol{S}}$ space-time

Coupling

Goal 2: energy-momentum schemes based on variation

1. Statics: Multifield principles of virtual work (mixed finite elem.)
2. Dynamics: Multifield principles of least action (Hamilton's principle) $\rightsquigarrow$ momentum-consistent (VI)
3. Dynamics: Multifield principles of virtual power (energy-momentum schemes for coupled fields)

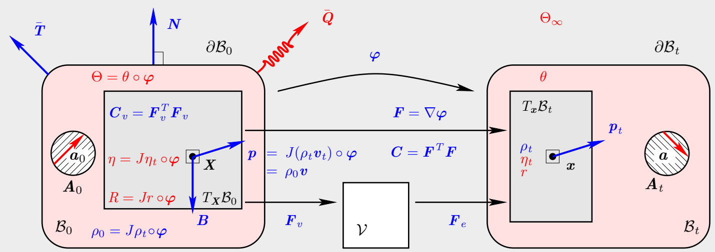
Strategy

(cf. Miehe & Schröder [2001], Armero [2008], Betsch & Janz [2016], Schlögl & Leyendecker [2016])

1. Energy-momentum schemes arise as discrete Euler-Lagrange equations

Continuum model of a fiber-reinforced body I

Transversely isotropic material in motion (cf. Schröder, Neff & Balzani [2005])



Deformation of fibers and matrix (cf. Klinkel, Sansour & Wagner [2005])

1. Deformation gradients

$$\boldsymbol{F}_F := \boldsymbol{a} \otimes \boldsymbol{a}_0 = \boldsymbol{F}\boldsymbol{A}_0 \qquad \boldsymbol{a} = \boldsymbol{F}\boldsymbol{a}_0 \qquad \boldsymbol{A}_0 := \boldsymbol{a}_0 \otimes \boldsymbol{a}_0$$

2. Right Cauchy-Green tensors

$$\boldsymbol{C}_F := \boldsymbol{F}_F^T \boldsymbol{F}_F := C_F \boldsymbol{A}_0 \qquad C_F := \boldsymbol{C} : \boldsymbol{A}_0 \equiv I_4^C \qquad \boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$$

3. Isotropic viscoelastic matrix

$$\boldsymbol{F} = \boldsymbol{F}_v \boldsymbol{F}_e \qquad \boldsymbol{C}_v = \boldsymbol{F}_v^T \boldsymbol{F}_v \qquad \rightsquigarrow \text{symmetric internal variable}$$

Continuum model of a fiber-reinforced body II

Total free energy

1. Elastic free energy

$$\Psi^{\text{ela}}(\boldsymbol{C}, C_F; \boldsymbol{A}_0) = \hat{\Psi}^{\text{ela}}_M(I_1^C, I_2^C, I_3^C, I_4^C) + \hat{\Psi}^{\text{ela}}_F(C_F)$$

2. Thermoelastic free energy

$$\Psi^{\text{the}}(\Theta, \boldsymbol{C}, C_F) = \hat{\Psi}^{\text{cap}}(\Theta) - n_{\text{dim}}\, \beta_M\, (\Theta - \Theta_\infty)\, \hat{\Psi}^{\text{vol}}_M(I_3^C) - 2\sqrt{C_F}\beta_F\, (\Theta - \Theta_\infty)\, \hat{\Psi}^{\text{ela}}_F(C_F)$$

3. Viscoelastic free energy

$$\Psi^{\text{vis}}_M(\boldsymbol{\Lambda}) = \hat{\Psi}^{\text{ela}}_M(I_1^\Lambda, I_2^\Lambda, I_3^\Lambda) \qquad \boldsymbol{\Lambda} = \boldsymbol{C}\boldsymbol{C}_v^{-1} \qquad \boldsymbol{C}_e = \boldsymbol{F}_e^T \boldsymbol{F}_e$$

Total equilibrium and non-equilibrium stress

1. Total second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} := 2\,\frac{\partial \hat{\Psi}}{\partial \boldsymbol{C}} = \boldsymbol{S}^{\text{ela}}_M + \boldsymbol{S}^{\text{the}}_M + \boldsymbol{S}^{\text{vis}}_M + S_F(\Theta, C_F)\, \boldsymbol{A}_0$$

2. Viscous evolution equation and non-equilibrium stress tensor

$$\boxed{\boldsymbol{Y} = \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v} \qquad \boldsymbol{Y} := -\frac{\partial \Psi^{\text{vis}}_M}{\partial \boldsymbol{C}_v} \qquad D^{\text{int}} := \boldsymbol{Y} : \dot{\boldsymbol{C}}_v \geq 0$$

$$\mathbb{V}(\boldsymbol{C}_v) = \frac{1}{4}\left(V_{\text{vol}} - \frac{V_{\text{dev}}}{n_{\text{dim}}}\right) \boldsymbol{C}_v^{-1} \otimes \boldsymbol{C}_v^{-1} + \frac{V_{\text{dev}}}{4}\, \mathbb{I}^{\text{sym}} : \boldsymbol{C}_v^{-1} \qquad \text{(pos.-def.)}$$

Basis of the principle of virtual power I

Total energy balance in functional form

$$\dot{\mathcal{T}}(\dot{\boldsymbol{\varphi}},\dot{\boldsymbol{v}},\dot{\boldsymbol{p}})+\dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{\varphi}},\dot{\boldsymbol{C}}_v,\dot{\Theta},\tilde{\Theta},\boldsymbol{R},h,\lambda)+\dot{\Pi}^{\text{int}}(\dot{\boldsymbol{\varphi}},\dot{\boldsymbol{C}}_v,\dot{\tilde{\boldsymbol{C}}},\dot{\tilde{C}}_F,\boldsymbol{S},\boldsymbol{S}_F,\dot{\Theta},\dot{\eta},\tilde{\Theta};\tilde{\boldsymbol{S}},\tilde{S}_F)=0$$

Kinetic energy

$$T=\frac{\rho_0}{2}\,\boldsymbol{v}\cdot\boldsymbol{v}\qquad\qquad \mathcal{T}:=\int_{\mathscr{B}_0}T\,\mathrm{d}V-\int_{\mathscr{B}_0}[\boldsymbol{v}-\dot{\boldsymbol{\varphi}}]\cdot\boldsymbol{p}\,\mathrm{d}V$$

Kinetic power functional

$$\dot{\mathcal{T}}(\dot{\boldsymbol{\varphi}},\dot{\boldsymbol{v}},\dot{\boldsymbol{p}}):=\int_{\mathscr{B}_0}\left[\frac{\partial T}{\partial\boldsymbol{v}}-\boldsymbol{p}\right]\cdot\dot{\boldsymbol{v}}\,\mathrm{d}V-\int_{\mathscr{B}_0}\dot{\boldsymbol{p}}\cdot[\boldsymbol{v}-\dot{\boldsymbol{\varphi}}]\,\mathrm{d}V+\int_{\mathscr{B}_0}\boldsymbol{p}\cdot\ddot{\boldsymbol{\varphi}}\,\mathrm{d}V$$

External power functional

(cf. Romero [2010])

$$\begin{aligned}\dot{\Pi}^{\text{ext}}:=&-\int_{\mathscr{B}_0}\rho_0\boldsymbol{B}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}V-\int_{\partial_T\mathscr{B}_0}\bar{\boldsymbol{T}}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}A-\int_{\partial_\varphi\mathscr{B}_0}\boldsymbol{R}\cdot(\dot{\boldsymbol{\varphi}}-\dot{\bar{\boldsymbol{\varphi}}})\,\mathrm{d}A\\&+\int_{\mathscr{B}_0}\frac{1}{\Theta}\nabla\tilde{\Theta}\cdot\boldsymbol{Q}\,\mathrm{d}V+\int_{\mathscr{B}_0}\frac{\tilde{\Theta}}{\Theta}\left[R+D^{\text{tot}}\right]\mathrm{d}V+\int_{\partial_Q\mathscr{B}_0}\frac{\tilde{\Theta}}{\Theta}\bar{Q}\,\mathrm{d}A\\&+\int_{\partial_\Theta\mathscr{B}_0}\lambda\,(\tilde{\Theta}-\bar{\Theta})\,\mathrm{d}A-\int_{\partial_{\dot{\Theta}}\mathscr{B}_0}h\left(\dot{\Theta}-\dot{\bar{\Theta}}\right)\mathrm{d}A+\frac{1}{2}\int_{\mathscr{B}_0}\dot{\boldsymbol{C}}_v:\mathbb{V}(\boldsymbol{C}_v):\dot{\boldsymbol{C}}_v\,\mathrm{d}V\end{aligned}$$

with $\quad D^{\text{tot}}:=D^{\text{int}}+D^{\text{cdu}}=D^{\text{int}}-\dfrac{1}{\Theta}\nabla\Theta\cdot\boldsymbol{Q}$

Basis of the principle of virtual power II

Fourier's law of transversely isotropic heat conduction

Al-Kinani [2014]

$$\boldsymbol{q} := -\boldsymbol{\kappa}\boldsymbol{F}^{-T}\nabla\Theta \qquad \boldsymbol{\kappa} := k\,\boldsymbol{I} + \frac{k_F - k}{\|\boldsymbol{a}\|^2}\boldsymbol{a}\otimes\boldsymbol{a} \qquad \boldsymbol{Q} := -\left[\frac{k_F - k}{C_F}\,\boldsymbol{A}_0 + k_0\,\boldsymbol{C}^{-1}\right]\nabla\Theta$$

Internal energy

$$\begin{aligned}\Pi^{\text{int}} := &\int_{\mathscr{B}_0}[\Psi_M(\Theta,\tilde{\boldsymbol{C}},\boldsymbol{C}_v;\boldsymbol{A}_0) + \Psi_F(\Theta,\tilde{C}_F;\boldsymbol{A}_0)]\,\mathrm{d}V - \int_{\mathscr{B}_0}\eta\,[\tilde{\Theta}-\Theta]\,\mathrm{d}V\\ &-\frac{1}{2}\int_{\mathscr{B}_0}\boldsymbol{S}_M : [\tilde{\boldsymbol{C}} - \boldsymbol{C}(\boldsymbol{\varphi})]\,\mathrm{d}V - \int_{\mathscr{B}_0} S_F : [\tilde{C}_F - \boldsymbol{C}(\boldsymbol{\varphi}) : \boldsymbol{A}_0]\,\mathrm{d}V\\ &+\frac{1}{2}\int_{\mathscr{B}_0}\tilde{\boldsymbol{S}} : \tilde{\boldsymbol{C}}\,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\tilde{S}_F\,\tilde{C}_F\,\mathrm{d}V\end{aligned}$$

Internal power functional

$$\begin{aligned}\dot{\Pi}^{\text{int}} := &\frac{1}{2}\int_{\mathscr{B}_0}\left\{\left[2\,\frac{\partial\Psi}{\partial\tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} - \boldsymbol{S}\right] : \dot{\tilde{\boldsymbol{C}}} + \left[2\,\frac{\partial\Psi}{\partial\tilde{C}_F} + \tilde{S}_F - \boldsymbol{S}_F : \boldsymbol{A}_0\right] : \dot{\tilde{C}}_F\right\}\mathrm{d}V\\ &-\frac{1}{2}\int_{\mathscr{B}_0}\dot{\boldsymbol{S}} : [\tilde{\boldsymbol{C}} - \boldsymbol{C}(\boldsymbol{\varphi})]\,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\boldsymbol{S} : \dot{\boldsymbol{C}}(\dot{\boldsymbol{\varphi}})\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{\partial\Psi}{\partial\boldsymbol{C}_v} : \dot{\boldsymbol{C}}_v\,\mathrm{d}V\\ &-\frac{1}{2}\int_{\mathscr{B}_0}\dot{\boldsymbol{S}}_F : [\tilde{C}_F\,\boldsymbol{A}_0 - \boldsymbol{C}_F(\boldsymbol{\varphi})]\,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\boldsymbol{S}_F : \dot{\boldsymbol{C}}_F(\dot{\boldsymbol{\varphi}})\,\mathrm{d}V\\ &+\int_{\mathscr{B}_0}\dot{\eta}\,(\tilde{\Theta}-\Theta)\,\mathrm{d}V + \int_{\mathscr{B}_0}\dot{\Theta}\left[\eta + \frac{\partial\Psi}{\partial\Theta}\right]\mathrm{d}V + \int_{\mathscr{B}_0}\eta\,\dot{\tilde{\Theta}}\,\mathrm{d}V\end{aligned}$$

Principle of virtual power (cf. Schröder & Kuhl [2015], Klinkel & Wagner [1997], Hackl [1997])

Mixed principle of virtual power $\delta_* \dot{\mathcal{H}} = 0$

$$\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \delta_* \dot{\Pi}^{\mathrm{ext}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{C}}_v, \dot{\Theta}, \tilde{\Theta}, \boldsymbol{R}, h, \lambda) + \delta_* \dot{\Pi}^{\mathrm{int}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \dot{\Theta}, \dot{\eta}, \tilde{\Theta}, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F) = 0$$

Virtual kinetic power

$$\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) := \int_{\mathscr{B}_0} \left[\frac{\partial T}{\partial \boldsymbol{v}} - \boldsymbol{p} \right] \cdot \delta_* \dot{\boldsymbol{v}} \,\mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{\varphi}}] \,\mathrm{d}V + \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot \delta_* \dot{\boldsymbol{\varphi}} \,\mathrm{d}V$$

Virtual external power

$$\begin{aligned} \delta_* \dot{\Pi}^{\mathrm{ext}} := & -\int_{\mathscr{B}_0} \rho_0 \boldsymbol{B} \cdot \delta_* \dot{\boldsymbol{\varphi}} \,\mathrm{d}V - \int_{\partial_T \mathscr{B}_0} \boldsymbol{T} \cdot \delta_* \dot{\boldsymbol{\varphi}} \,\mathrm{d}A - \int_{\partial_\varphi \mathscr{B}_0} \delta_* \boldsymbol{R} \cdot [\dot{\boldsymbol{\varphi}} - \dot{\bar{\boldsymbol{\varphi}}}] \,\mathrm{d}A - \int_{\partial_\varphi \mathscr{B}_0} \boldsymbol{R} \cdot \delta_* \dot{\boldsymbol{\varphi}} \\ & + \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \,\mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \tilde{\Theta} \, \frac{R + D^{\mathrm{tot}}}{\Theta} \,\mathrm{d}V + \int_{\mathscr{B}_0} \frac{1}{\Theta} \nabla(\delta_* \tilde{\Theta}) \cdot \boldsymbol{Q} \,\mathrm{d}V + \int_{\partial_\Theta \mathscr{B}_0} \lambda \, \delta_* \tilde{\Theta} \,\mathrm{d}A \\ & + \int_{\partial_Q \mathscr{B}_0} \frac{\delta_* \tilde{\Theta}}{\Theta} \bar{Q} \,\mathrm{d}A + \int_{\partial_\Theta \mathscr{B}_0} \delta_* \lambda [\tilde{\Theta} - \bar{\Theta}] \,\mathrm{d}A - \int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \delta_* h \left[\dot{\Theta} - \dot{\bar{\Theta}} \right] \mathrm{d}A - \int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \delta_* \dot{\Theta} \, h \,\mathrm{d}A \end{aligned}$$

Virtual internal power

$$\begin{aligned} \delta_* \dot{\Pi}^{\mathrm{int}} := & \frac{1}{2} \int_{\mathscr{B}_0} \left\{ \left[2 \frac{\partial \Psi_M}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} - \boldsymbol{S} \right] : \delta_* \dot{\tilde{\boldsymbol{C}}} + \left[2 \frac{\partial \Psi_F}{\partial \tilde{C}_F} + \tilde{S}_F - S_F \right] : \delta_* \dot{\tilde{C}}_F - \dot{\eta} \, \delta_* \tilde{\Theta} \right\} \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S} : \left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}(\boldsymbol{\varphi}) \right] \mathrm{d}V + \int_{\mathscr{B}_0} \left\{ \nabla(\delta_* \dot{\boldsymbol{\varphi}}) : \boldsymbol{F} [\boldsymbol{S} + S_F \boldsymbol{A}_0] + \delta_* \dot{\boldsymbol{C}}_v : \frac{\partial \Psi}{\partial \boldsymbol{C}_v} \right\} \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S}_F : \left[\dot{\tilde{C}}_F - \dot{\boldsymbol{C}}(\boldsymbol{\varphi}) : \boldsymbol{A}_0 \right] \mathrm{d}V + \int_{\mathscr{B}_0} \left\{ \delta_* \dot{\Theta} \left[\frac{\partial \Psi}{\partial \Theta} + \eta \right] - \delta_* \dot{\eta} \, [\tilde{\Theta} - \Theta] \right\} \mathrm{d}V \end{aligned}$$

Euler-Lagrange equations

Constitutive equations

$$\frac{\partial T}{\partial \boldsymbol{v}} = \boldsymbol{p} \qquad \forall t \geqslant t_0 \qquad \eta = -\frac{\partial \Psi}{\partial \Theta}$$

$$2\,\frac{\partial \Psi_M}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} = \boldsymbol{S} \qquad \forall t \geqslant t_0 \qquad \boldsymbol{S}_F = \left[2\,\frac{\partial \Psi_F}{\partial \tilde{C}_F} + \tilde{S}_F\right] \boldsymbol{A}_0$$

$$\dot{\boldsymbol{C}}(\dot{\boldsymbol{\varphi}}) = \dot{\tilde{\boldsymbol{C}}} \quad \text{with} \quad \tilde{\boldsymbol{C}}(t_0) = \boldsymbol{C}(\boldsymbol{\varphi}_0) \qquad -\frac{\partial \Psi_M}{\partial \boldsymbol{C}_v} = \mathbb{V}(\boldsymbol{C}_v)\,\dot{\boldsymbol{C}}_v$$

$$\dot{\boldsymbol{C}}_F(\dot{\boldsymbol{\varphi}}) : \boldsymbol{A}_0 = \dot{\tilde{C}}_F \quad \text{with} \quad \tilde{C}_F(t_0) = \boldsymbol{C}_F(\boldsymbol{\varphi}_0) : \boldsymbol{A}_0 \qquad \tilde{\Theta} = \Theta$$

Neumann and Dirichlet boundary conditions

$$\boldsymbol{F}\,[\boldsymbol{S} + S_F \boldsymbol{A}_0]\,\boldsymbol{N} = \bar{\boldsymbol{T}} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_T \mathscr{B}_0$$

$$\boldsymbol{F}\,[\boldsymbol{S} + S_F \boldsymbol{A}_0]\,\boldsymbol{N} = \boldsymbol{R} \qquad \dot{\boldsymbol{\varphi}} = \dot{\bar{\boldsymbol{\varphi}}} \quad \text{with} \quad \boldsymbol{\varphi}(t_0) = \bar{\boldsymbol{\varphi}}(t_0) \quad \text{on} \quad \partial_{\varphi} \mathscr{B}_0$$

$$-\boldsymbol{Q} \cdot \boldsymbol{N} = \bar{Q} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_Q \mathscr{B}_0$$

$$-\frac{\boldsymbol{Q} \cdot \boldsymbol{N}}{\Theta} = \lambda \qquad \tilde{\Theta} = \bar{\Theta} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_{\Theta} \mathscr{B}_0$$

$$0 = h \qquad \dot{\Theta} = \dot{\bar{\Theta}} \quad \text{with} \quad \Theta(t_0) = \bar{\Theta}(t_0) \quad \text{on} \quad \partial_{\dot{\Theta}} \mathscr{B}_0$$

Time evolution equations in first order form (cf. Romero [2010])

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{with} \quad \boldsymbol{\varphi}(t_0) = \boldsymbol{\varphi}_0$$

$$\mathrm{Div}[\boldsymbol{F}\boldsymbol{S} + \boldsymbol{F}_F S_F] + \rho_0 \boldsymbol{B} = \dot{\boldsymbol{p}} \quad \text{with} \quad \boldsymbol{p}(t_0) = \boldsymbol{p}_0 \equiv \frac{\partial T(\boldsymbol{v}_0)}{\partial \boldsymbol{v}}$$

$$-\frac{\mathrm{Div}[\boldsymbol{Q}]}{\Theta} + \frac{R + D^{\mathrm{int}}}{\Theta} = \dot{\eta} \quad \text{with} \quad \eta(t_0) = \eta_0 \equiv -\frac{\partial \Psi(\boldsymbol{\varphi}_0, \Theta_0)}{\partial \Theta}$$

Discrete variational principle

Discrete principle of virtual power

$$\sum_{n=0}^{N-1} \sum_{l,j,i=1}^{k_{\text{mec}},k_{\text{the}},k_{\text{vis}}} \delta_* \dot{\mathcal{H}}(\mathbf{z}_h^n(\xi_{l,j,i}), \mathbf{w}_h^n(\xi_{l,j}); \tilde{\boldsymbol{S}}_h^n(\xi_l), \tilde{S}_{F_h}^n(\xi_l))\, w_{l,j,i}\, h_n = 0 \quad \text{with} \quad \begin{aligned} \mathbf{z} &= (\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \dot{\Theta}, \dot{\eta}) \\ \mathbf{w} &= (\tilde{\Theta}, \boldsymbol{S}, S_F, \boldsymbol{R}, h, \lambda) \end{aligned}$$

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}_d(\mathbf{q}, \mathbf{v}, \mathbf{p}, \mathbf{C}_v, \mathbf{C}, \mathsf{C}_F, \mathsf{o}, \mathsf{s}, \tilde{\mathsf{o}}, \mathbf{R}, \mathbf{h}, \boldsymbol{\lambda}, \tilde{\mathbf{S}}, \tilde{\mathsf{S}}_F; \tilde{\boldsymbol{S}}_h^n(\xi_i), \tilde{S}_{F_h}^n(\xi_i))\, h_n = 0 \quad \text{with} \quad \begin{aligned} \mathbf{q} &= (\boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_{k_{\text{mec}}+1}) \\ \tilde{\mathsf{o}} &= (\tilde{\Theta}_1, \ldots, \tilde{\Theta}_{k_{\text{the}}}) \end{aligned}$$

Galerkin time approximation with Lagrange polynomials

$$\boldsymbol{\varphi}_h^n(\alpha) := \sum_{j=1}^{k_{\text{mec}}+1} M_j(\alpha)\, \boldsymbol{\varphi}_j^n \qquad \delta_* \dot{\boldsymbol{\varphi}}_h^n(\alpha) := \frac{1}{h_n} \sum_{i=1}^{k_{\text{mec}}} \tilde{M}_j(\alpha)\, \tilde{\boldsymbol{\varphi}}_i^n \qquad \delta_* \tilde{\Theta}_h^n(\alpha) := \sum_{i=1}^{k_{\text{the}}} \tilde{M}_i(\alpha)\, \tilde{\Theta}_i^n \qquad \tilde{\Theta}_h^n(\alpha) := \sum_{i=1}^{k_{\text{the}}} \tilde{M}_i(\alpha)\, \tilde{\Theta}_i^n$$

Discrete Euler-Lagrange equations with space-time finite elements

Momentum equation: $\left[\tilde{\mathbf{B}} \otimes \mathbf{H}_q \otimes \boldsymbol{I}\right] [\rho_0 \mathbf{v} - \mathbf{p}] + \tilde{\mathbf{b}} \otimes \left[\mathbf{H}_q \otimes \boldsymbol{I}\right] [\rho_0 \mathbf{v}_1 - \mathbf{p}_1] = \mathbf{0}$

Velocity equation: $\left[\tilde{\mathbf{B}} \otimes \mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{v} + \tilde{\mathbf{b}} \otimes \left[\mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{v}_1 = \left[\frac{1}{h_n} \tilde{\mathbf{A}} \otimes \mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{q} + \frac{1}{h_n} \tilde{\mathbf{a}} \otimes \left[\mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{q}_1$

Positions on $\partial_\varphi \mathscr{B}_0$: $\left[\tilde{\mathbf{A}} \otimes \bar{\mathbf{H}}_q \otimes \boldsymbol{I}\right] \bar{\mathbf{q}} + \tilde{\mathbf{a}} \otimes \left[\bar{\mathbf{H}}_q \otimes \boldsymbol{I}\right] \bar{\mathbf{q}}_1 = \dot{\bar{\mathbf{q}}}$

momentum balance: $\left[\frac{1}{h_n} \tilde{\mathbf{A}} \otimes \mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{p} + \frac{1}{h_n} \tilde{\mathbf{a}} \otimes \left[\mathbf{H}_q \otimes \boldsymbol{I}\right] \mathbf{p}_1 - \mathbf{F}^{\text{int}} - \mathbf{F}^{\text{ext}} - \bar{\mathbf{F}}^{\text{ext}} = \left[\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_q \otimes \boldsymbol{I}\right] \mathbf{R}$

Temperature | Entropy equation: $\left[\tilde{\mathbf{C}} \otimes \mathbf{H}_o\right] \tilde{\mathbf{o}} = \left[\tilde{\mathbf{B}} \otimes \mathbf{H}_o\right] \mathbf{o} + \tilde{\mathbf{b}} \otimes \mathbf{H}_o \mathbf{o}_1$ $\qquad \boldsymbol{\Phi} + \left[\tilde{\mathbf{B}} \otimes \mathbf{H}_o\right] \mathbf{s} + \tilde{\mathbf{b}} \otimes \mathbf{H}_o \mathbf{s}_1 = \left[\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o\right] \bar{\mathbf{h}}$

Temperatures on $\partial_{\dot{\Theta}} \mathscr{B}_0 | \partial_\Theta \mathscr{B}_0$: $\left[\tilde{\mathbf{A}} \otimes \bar{\mathbf{H}}_o\right] \bar{\mathbf{o}} + \tilde{\mathbf{a}} \otimes \left[\bar{\mathbf{H}}_o\right] \bar{\mathbf{o}}_1 = \dot{\bar{\mathbf{o}}}$ $\qquad \bar{\mathbf{o}} = \mathbf{e}_k \otimes \bar{\mathbf{o}}$

Entropy balance: $\left[\frac{1}{h_n} \tilde{\mathbf{A}} \otimes \mathbf{H}_o\right] \mathbf{s} + \frac{1}{h_n} \tilde{\mathbf{a}} \otimes \mathbf{H}_o \mathbf{s}_1 - \mathbf{S}^{\text{con}} - \mathbf{S}^{\text{cdu}} - \mathbf{S}^{\text{int}} - \bar{\mathbf{S}}^{\text{ext}} = \left[\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o\right] \boldsymbol{\lambda}$

Local discrete Euler-Lagrange equations

Stress equations at the Gauss points in space and time

$$\boldsymbol{S}(\xi_i,\chi_I) = 2\,\frac{\partial \Psi_M(\xi_i,\chi_I)}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}}_h^n(\xi_i,\chi_I) \qquad S_F(\xi_i,\chi_I) = 2\,\frac{\partial \Psi_F(\xi_i,\chi_I)}{\partial \tilde{C}_F} + \tilde{S}_{F_h}^n(\xi_i,\chi_I) \qquad (\xi_i = 1,\ldots,k_{\mathrm{mec}})$$

Viscous time evolution equations at the Gauss points in space

$$\mathbf{Y}(\mathbf{C}_v(\chi_I)) = \frac{1}{h_n}\mathbf{V}(\mathbf{C}_v(\chi_I)) \rightarrow$$ local level Newton-Raphson method $$\rightsquigarrow \mathbf{C}_v(\chi_I) \in \mathbb{R}^{6\,k_{\mathrm{vis}}}$$

Discrete weak assumed strain equations on the element

$$\sum_{i=1}^{k_{\mathrm{mec}}} \delta_* \boldsymbol{S}_h^{n,e}(\xi_i,\chi_I) : \left[\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^{n,e}(\xi_i,\chi_I)}{\mathrm{d}\alpha} - \mathring{\boldsymbol{C}}(\mathring{\boldsymbol{\varphi}}_h^{n,e}(\xi_i,\chi_I))\right] w_i = 0 \qquad (i = 1,\ldots,k_{\mathrm{mec}})$$

Discrete assumed strain equations at the Gauss points in space

$$\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_l,\chi_I)}{\mathrm{d}\alpha} = \mathring{\boldsymbol{C}}(\mathring{\boldsymbol{\varphi}}_h^n(\xi_l,\chi_I)) \qquad l = 1,\ldots,k_{\mathrm{mec}} \qquad \frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_l,\chi_I)}{\mathrm{d}\alpha} = \sum_{j=1}^{k_{\mathrm{mec}}+1} \mathring{M}_j(\xi_l)\,\boldsymbol{C}_j^n(\chi_I) \equiv \sum_{i=1}^{k_{\mathrm{mec}}} \tilde{M}_i(\xi_l)\,\tilde{\boldsymbol{C}}_i^n(\chi_I)$$

Assumed strains at the time nodes $\boldsymbol{C}_l^n(\chi_I)$, $l = 2,\ldots,k_{\mathrm{mec}}$

$$\boldsymbol{C}_l^n(\chi_I) := \sum_{i=1}^{k_{\mathrm{mec}}} m_{li}\,\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{\varphi}}_h^n(\xi_i,\chi_I)) + \boldsymbol{C}_1^n(\chi_I) \qquad \text{with} \qquad \boldsymbol{m} = \begin{bmatrix} \mathring{M}_2(\xi_1) & \ldots & \mathring{M}_{k+1}(\xi_1) \\ \vdots & \ldots & \vdots \\ \mathring{M}_2(\xi_k) & \ldots & \mathring{M}_{k+1}(\xi_k) \end{bmatrix}^{-1}$$

Discrete mechanical energy balance

Discrete kinetic energy balance (cf. G. & Betsch [2011])

$$\frac{\mathcal{T}_{n+1}-\mathcal{T}_n}{h_n} = \frac{1}{2\,h_n}\left\{\mathbf{v}_{k_{\mathrm{mec}}+1}^T\left[\mathbf{M}\otimes\boldsymbol{I}\right]\mathbf{v}_{k_{\mathrm{mec}}+1} - \mathbf{v}_1^T\left[\mathbf{M}\otimes\boldsymbol{I}\right]\mathbf{v}_1\right\}$$

$$= \frac{1}{2\,h_n}\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}\end{bmatrix}^T\left[2\mathbf{J}^{\mathrm{sym}}\otimes\mathbf{M}\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}\end{bmatrix} \quad \text{with } 2\mathbf{J}^{\mathrm{sym}} := \mathrm{diag}\,\overbrace{(-1,0,\dots,0,1)}^{k_{\mathrm{mec}}}$$

$$= \begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix}^T\left[\frac{1}{h_n}\mathbf{J}^T\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}\end{bmatrix} \quad \text{with momentum equation and } \boldsymbol{a}\boldsymbol{A}\boldsymbol{a} = \boldsymbol{a}\boldsymbol{A}^T\boldsymbol{a}$$

$$= \begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix}^T\left[\frac{1}{(h_n)^2}\left[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}\right]^T\tilde{\mathbf{C}}^{-T}\left[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}\right]\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix} \quad \text{with velocity equation}$$

$$= \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\frac{1}{(h_n)^2}\tilde{\mathbf{T}}_1^T\left[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}\right]\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix} \quad \text{with } \tilde{\mathbf{T}}_1 := \tilde{\mathbf{C}}^{-1}\left[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}\right] \text{ and } \boldsymbol{a}\boldsymbol{A}\boldsymbol{b} = \boldsymbol{b}\boldsymbol{A}^T\boldsymbol{a}$$

$$\frac{\mathcal{T}_{n+1}-\mathcal{T}_n}{h_n} = \frac{1}{h_n}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_1^T\otimes\mathbf{I}_q\otimes\boldsymbol{I}\right]\left\{\mathbf{F}^{\mathrm{int}}+\mathbf{F}^{\mathrm{ext}}+\bar{\mathbf{F}}^{\mathrm{ext}}\right\} \quad \text{with momentum balance}$$

$$+ \frac{1}{h_n}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_1^T\otimes\mathbf{I}_q\otimes\boldsymbol{I}\right]\cancelto{0\ (\perp)}{\left[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes\boldsymbol{I}\right]}\mathbf{R} \qquad \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_1^T\otimes\mathbf{I}_q\otimes\boldsymbol{I}\right] \equiv \mathring{\mathbf{q}}^T$$

Discrete potential energy balance with dead loads $\mathbf{F}^{\mathrm{ext}}$

$$\sum_{I=1}^{N_q}\left[\Psi_{n+1}(\chi_I)-\Psi_n(\chi_I)\right]W_I = -\,\mathring{\mathbf{q}}^T\left[\mathbf{F}^{\mathrm{int}}+\mathbf{F}^{\mathrm{ext}}\right] - \left[\Pi_{n+1}^{\mathrm{ext}}-\Pi_n^{\mathrm{ext}}\right]$$

$$+ \sum_{i=1}^{k_{\mathrm{vis}}}\sum_{I=1}^{N_q}\left[\frac{\partial\Psi_M}{\partial\boldsymbol{C}_v}:\mathring{\boldsymbol{C}}_v\right](\xi_i,\chi_I)\,W_I\,w_i + \sum_{j=1}^{k_{\mathrm{the}}}\sum_{I=1}^{N_q}\left[\frac{\partial\Psi}{\partial\Theta}\,\mathring{\Theta}\right](\xi_j,\chi_I)\,W_I\,w_j$$

$$\Pi_{n+1}^{\mathrm{pot}}-\Pi_n^{\mathrm{pot}} \quad - \sum_{i,j,l}\sum_{I}\left[\frac{\partial\Psi_M}{\partial\tilde{\boldsymbol{C}}}:\mathring{\tilde{\boldsymbol{C}}} + \frac{\partial\Psi_F}{\partial\tilde{C}_F}\,\mathring{\tilde{C}}_F + \frac{\partial\Psi_M}{\partial\boldsymbol{C}_v}:\mathring{\boldsymbol{C}}_v + \frac{\partial\Psi}{\partial\Theta}\,\mathring{\Theta}\right](\xi_{i,j,l},\chi_I)\,W_I\,w_{i,j,l}$$

$$= \sum_{l=1}^{k_{\mathrm{mec}}}\sum_{I=1}^{N_q}\frac{1}{2}\left[\tilde{\boldsymbol{S}}:\mathring{\tilde{\boldsymbol{C}}} + \tilde{S}_F\,\mathring{\tilde{C}}_F\right](\xi_l,\chi_I)\,W_I\,w_l \qquad \text{with stress equations}$$

Discrete superimposed fiber stress $(k_{\text{mec}} = k_{\text{the}} = k_{\text{vis}} = 1)$

Discrete claim for the fiber stress at the Gauss points in space

$$\left[\frac{\partial \Psi_F(\tilde{C}_F,\Theta)_{n+\frac{1}{2}}}{\partial \tilde{C}_F} + \frac{1}{2}\,\tilde{S}_{F_{n+\frac{1}{2}}}\right] \underbrace{\boldsymbol{A}_0 : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]}_{C_{F_{n+1}} - C_{F_n}} + \frac{\partial \Psi_F(\tilde{C}_F,\Theta)_{n+\frac{1}{2}}}{\partial \Theta}\,[\Theta_{n+1} - \Theta_n] = \Psi_F(\tilde{C}_F,\Theta)_{n+1} - \Psi_F(\tilde{C}_F,\Theta)_n$$

Variational problem

(cf. Gauss's principle in Ramm [2011])

$$\mathcal{L}(\mu,\tilde{S}_{F_{n+\frac{1}{2}}}) := \frac{1}{2}\left(\tilde{S}_{F_{n+\frac{1}{2}}}\right)^2 + \mu\,\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) \qquad \delta_*\mathcal{L}(\mu,\tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

Discrete claim as constraint

(cf. Hesch & Betsch [2011], G. & Betsch [2011])

$$\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) := \Psi_F(\tilde{C}_F,\Theta)_{n+1} - \Psi_F(\tilde{C}_F,\Theta)_n - \left[\frac{\partial \Psi_F(\tilde{C}_F,\Theta)_{n+\frac{1}{2}}}{\partial \tilde{C}_F} + \frac{1}{2}\,\tilde{S}_{F_{n+\frac{1}{2}}}\right][C_{F_{n+1}} - C_{F_n}] - \frac{\partial \Psi_F(\tilde{C}_F,\Theta)_{n+\frac{1}{2}}}{\partial \Theta}\,[\Theta_{n+1} - \Theta_n] = 0$$

Discrete Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \tilde{S}_{F_{n+\frac{1}{2}}}} \equiv \tilde{S}_{F_{n+\frac{1}{2}}} - \frac{\mu}{2}\,[C_{F_{n+1}} - C_{F_n}] = 0 \qquad \frac{\partial \mathcal{L}}{\partial \mu} \equiv \mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

Discrete superimposed fiber stress

(cf. Gonzalez [2000], Hesch & Betsch [2011])

$$\tilde{S}_{F_{n+\frac{1}{2}}} = 2\,\frac{\mathcal{G}(O)}{[C_{F_{n+1}} - C_{F_n}]\,\cancel{[C_{F_{n+1}} - C_{F_n}]}}\,\cancel{[C_{F_{n+1}} - C_{F_n}]}$$

Dynamic thermo-mechanical coupling in fiber-reinforced bodies simulated by higher-order variational energy-momentum schemes

Michael Groß and Julian Dietzsch

Introduction

Variational setting

Continuum model

Variational principle

Euler-Lagrange equations

Discrete setting

Global E-L equations

Local E-L equations

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Total energy balance

Total momenta balances

Total entropy balance

Numerical studies

Summary

Discrete superimposed stress tensor $(k_{\text{mec}} = k_{\text{the}} = k_{\text{vis}} = 1)$

Discrete claim for the matrix stress tensor (one equation for 6 unknowns)

$$\left[\frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\tilde{\boldsymbol{C}}} + \frac{1}{2}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] + \frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\boldsymbol{C}_v} + \frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\Theta} = \Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+1} - \Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_n$$

Constrained variational problem with the Mandel stress tensor

$$\mathcal{L}(\mu,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := \frac{1}{2}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} : \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} + \mu\,\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) \qquad \delta_*\mathcal{L}(\mu,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

Discrete claim as constraint (cf. Hesch & Betsch [2011], G. & Betsch [2011])

$$\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := \Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+1} - \Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_n - \left[\frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\tilde{\boldsymbol{C}}} + \frac{1}{2}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] - \frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\boldsymbol{C}_v} - \frac{\partial\Psi_M(\tilde{\boldsymbol{C}},\boldsymbol{C}_v,\Theta)_{n+\frac{1}{2}}}{\partial\Theta}$$

Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}} \equiv \tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} - \frac{\mu}{2}\,[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] = \boldsymbol{O} \qquad \frac{\partial\mathcal{L}}{\partial\mu} \equiv \mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

Discrete superimposed stress tensor (cf. Armero & Zambrana-Rojas [2007])

$$\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} = 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}\,[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}\,[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}$$

Higher-order variational vs. Galerkin schemes

Galerkin scheme ($k_{\text{mec}} = 2$)

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_2^n := (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_3^n := (\boldsymbol{F}_3^n)^T \boldsymbol{F}_3^n$$

Variational scheme for $k_{\text{mec}} = 2$

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_2^n := \frac{1}{3}\left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right]^T \left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right] + (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_3^n := (\boldsymbol{F}_3^n)^T \boldsymbol{F}_3^n$$

Superimposed stress tensor of Galerkin schemes (cf. Mohr, Menzel, Steinmann [2008])

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\displaystyle\sum_{l=1}^{k_{\text{mec}}} \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{\varphi}}{}_h^n)(\xi_l)\, w_l}\; \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)$$

(energy consistent, **but not** variationally consistent approximation)

with

$$\tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k_{\text{mec}}+1} M_{j+1}(\alpha)\, [\boldsymbol{F}_j^n]^T \boldsymbol{F}_j^n \qquad (i = 1, \ldots, k_{\text{mec}})$$

Superimposed stress tensor of variational schemes (cf. G. & Dietzsch [2017])

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\displaystyle\sum_{l=1}^{k_{\text{mec}}} [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l)\, [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, w_l}\; [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)\, [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}$$

with

$$\tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k_{\text{mec}}+1} M_{j+1}(\alpha)\, \boldsymbol{C}_j^n \qquad (i = 1, \ldots, k_{\text{mec}})$$

Discrete balances of thermal and total energy

Discrete thermal energy balance

$$
\begin{aligned}
\frac{\Pi^{\text{the}}_{n+1} - \Pi^{\text{the}}_{n}}{h_n} &= \frac{1}{h_n}\left\{\mathbf{o}^T_{k_{\text{the}}+1}\,\mathbf{H}_o\,\mathbf{s}_{k_{\text{the}}+1} - \mathbf{o}^T_1\,\mathbf{H}_o\,\mathbf{s}_1\right\} && \text{with}\quad \Pi^{\text{the}}_{\alpha} := \sum_{I=1}^{N_q}\Theta(\chi_I,\alpha)\,\eta(\chi_I,\alpha)\,W_I \\
&= \begin{bmatrix}\mathbf{o}_1\\ \mathbf{o}\end{bmatrix}^T \left[\frac{2}{h_n}\,\mathbf{J}^{\text{sym}} \otimes \mathbf{H}_o\right] \begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}\end{bmatrix} && \text{with}\quad 2\mathbf{J}^{\text{sym}} := \operatorname{diag}\overbrace{(-1,0,\dots,0,1)}^{k_{\text{the}}} \\
&= \begin{bmatrix}\mathbf{o}_1\\ \mathbf{o}\end{bmatrix}^T \left[\frac{1}{h_n}\left(\mathbf{J}+\mathbf{J}^T\right) \otimes \mathbf{H}_o\right] \begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}\end{bmatrix} && \text{with}\quad \mathbf{J}^T \equiv [\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]^T\,\overbrace{\tilde{\mathbf{C}}^{-T}\,[\tilde{\mathbf{b}}|\tilde{\mathbf{B}}]}^{\tilde{\mathbf{T}}_2} \equiv \tilde{\mathbf{T}}^T_1\,[\tilde{\mathbf{b}}|\tilde{\mathbf{B}}] \\
&= \underbrace{\begin{bmatrix}\mathbf{o}_1\\ \mathbf{o}\end{bmatrix}^T \left[\tilde{\mathbf{T}}^T_2 \otimes \mathbf{I}_o\right]}_{\tilde{\boldsymbol{\Theta}}^T} \underbrace{\frac{1}{h_n}\left[[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}] \otimes \mathbf{H}_o\right] \begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}\end{bmatrix}}_{\text{entropy balance}} + \frac{1}{h_n} \underbrace{\begin{bmatrix}\mathbf{o}_1\\ \mathbf{o}\end{bmatrix}^T \left[\tilde{\mathbf{T}}^T_1 \otimes \mathbf{I}_o\right]}_{\mathring{\boldsymbol{\Theta}}^T} \underbrace{\left[[\tilde{\mathbf{b}}|\tilde{\mathbf{B}}] \otimes \mathbf{H}_o\right] \begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}\end{bmatrix}}_{\text{entropy equation}} \\
\frac{\Pi^{\text{the}}_{n+1} - \Pi^{\text{the}}_{n}}{h_n} &= \tilde{\boldsymbol{\Theta}}^T\left[\cancel{\mathbf{S}^{\text{con}} + \mathbf{S}^{\text{cdu}}}\right] + \tilde{\boldsymbol{\Theta}}^T\left\{\mathbf{S}^{\text{int}} + \bar{\mathbf{S}}^{\text{ext}} + [\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}\right\} - \frac{1}{h_n}\,\mathring{\boldsymbol{\Theta}}^T\left\{\boldsymbol{\Phi} - [\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o]\,\bar{\mathbf{h}}\right\}
\end{aligned}
$$

Discrete total energy balance

$$
\begin{aligned}
\frac{\mathcal{H}_{n+1} - \mathcal{H}_{n}}{h_n} &= \frac{\mathcal{T}_{n+1} - \mathcal{T}_{n}}{h_n} + \frac{\Pi^{\text{pot}}_{n+1} - \Pi^{\text{pot}}_{n}}{h_n} + \frac{\Pi^{\text{ext}}_{n+1} - \Pi^{\text{ext}}_{n}}{h_n} + \frac{\Pi^{\text{the}}_{n+1} - \Pi^{\text{the}}_{n}}{h_n} \\
&= \frac{1}{h_n}\,\mathring{\mathbf{q}}^T\left\{\cancel{\mathbf{F}^{\text{int}} + \mathbf{F}^{\text{ext}}} + \bar{\mathbf{F}}^{\text{ext}}\right\} \\
&- \frac{1}{h_n}\,\mathring{\mathbf{q}}^T\left\{\cancel{\mathbf{F}^{\text{int}} + \mathbf{F}^{\text{ext}}}\right\} + \sum_{I=1}^{N_q}\left\{\cancel{\sum_{i=1}^{k_{\text{vis}}}\left[\frac{\partial\Psi_M}{\partial\boldsymbol{C}_v} : \mathring{\boldsymbol{C}}_v\right](\xi_i,\chi_I)\,\frac{w_i}{h_n}} + \xcancel{\sum_{j=1}^{k_{\text{the}}}\left[\frac{\partial\Psi}{\partial\Theta}\,\mathring{\Theta}\right](\xi_j,\chi_I)\,\frac{w_j}{h_n}}\right\} W_I \\
&- \frac{1}{h_n}\,\mathring{\boldsymbol{\Theta}}^T\left\{\xcancel{\boldsymbol{\Phi}} - [\tilde{\mathbf{C}} \otimes \mathbf{H}_o]\,\bar{\mathbf{h}}\right\} + \tilde{\boldsymbol{\Theta}}^T\left\{\cancel{\mathbf{S}^{\text{int}}} + \bar{\mathbf{S}}^{\text{ext}} + [\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}\right\} \\
\frac{\mathcal{H}_{n+1} - \mathcal{H}_{n}}{h_n} &= \frac{1}{h_n}\,\mathring{\mathbf{q}}^T\,\bar{\mathbf{F}}^{\text{ext}} + \tilde{\boldsymbol{\Theta}}^T\left\{\bar{\mathbf{S}}^{\text{ext}} + [\tilde{\mathbf{C}} \otimes \bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}\right\} + \frac{1}{h_n}\,\mathring{\boldsymbol{\Theta}}^T\,[\tilde{\mathbf{C}} \otimes \mathbf{H}_o]\,\bar{\mathbf{h}}
\end{aligned}
$$

Discrete balances of total momenta

Discrete balance of total angular momentum (cf. G. & Betsch [2011])

$$\frac{\mathcal{J}_{n+1}-\mathcal{J}_n}{h_n} = \frac{1}{h_n}\left\{\mathbf{q}_{k_{\mathrm{mec}}+1}^T\left[\mathbf{M}\otimes\mathbf{W}_{\xi_0}^T\right]\mathbf{v}_{k_{\mathrm{mec}}+1}-\mathbf{q}_1^T\left[\mathbf{M}\otimes\mathbf{W}_{\xi_0}^T\right]\mathbf{v}_1\right\} \quad \text{with} \quad \mathbf{W}_{\xi_0}:=\operatorname{spin}[\boldsymbol{\xi}_0]$$

$$= \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\frac{2}{h_n}\mathbf{J}^{\mathrm{sym}}\otimes\mathbf{M}\otimes\mathbf{W}_{\xi_0}^T\right]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}\end{bmatrix} \quad \text{with} \quad 2\mathbf{J}^{\mathrm{sym}}:=\operatorname{diag}\overbrace{(-1,0,\dots,0,1)}^{k_{\mathrm{the}}}$$

$$= \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\frac{1}{h_n}\left(\mathbf{J}+\mathbf{J}^T\right)\otimes\mathbf{H}_q\otimes\mathbf{W}_{\xi_0}^T\right]\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix} \quad \text{with momentum equation}$$

$$= \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_2^T\otimes\mathbf{I}_q\otimes\mathbf{W}_{\xi_0}^T\right]\left\{\frac{1}{h_n}\left[[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix}\right\} \text{momentum balance}$$

$$+ \frac{1}{h_n}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_1^T\otimes\rho_0\,\mathbf{I}_q\otimes\mathbf{W}_{\xi_0}^T\right]\left\{\left[[\tilde{\mathbf{b}}|\tilde{\mathbf{B}}]\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}\end{bmatrix}\right\} \text{velocity equation}$$

$$\frac{\mathcal{J}_{n+1}-\mathcal{J}_n}{h_n} = \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[\tilde{\mathbf{T}}_2^T\otimes\mathbf{I}_q\otimes\mathbf{W}_{\xi_0}^T\right]\left\{\cancel{\mathbf{F}^{\mathrm{int}}}+\mathbf{F}^{\mathrm{ext}}+\bar{\mathbf{F}}^{\mathrm{ext}}+\left[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes\boldsymbol{I}\right]\mathbf{R}\right\} \quad \text{with} \quad \boldsymbol{S}=\boldsymbol{S}^T$$

$$+ \cancel{\frac{1}{(h_n)^2}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}^T\left[[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]^T\tilde{\mathbf{C}}^{-T}[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]\otimes\mathbf{M}\otimes\mathbf{W}_{\xi_0}^T\right]\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}\end{bmatrix}} \quad \text{with} \quad \mathbf{M}=\mathbf{M}^T$$

Discrete balance of total linear momentum (cf. G. & Betsch [2011])

$$\frac{\mathcal{P}_{n+1}-\mathcal{P}_n}{h_n} = \frac{1}{h_n}\mathbf{c}^T\left\{\left[\mathbf{H}_q\otimes\boldsymbol{I}\right]\mathbf{p}_{k_{\mathrm{mec}}+1}-\left[\mathbf{H}_q\otimes\boldsymbol{I}\right]\mathbf{p}_1\right\} \quad \text{with} \quad \mathbf{c}^T:=\left[\boldsymbol{\xi}_0^T,\dots,\boldsymbol{\xi}_0^T\right]\in\mathbb{R}^{1\times n_{\mathrm{dof}}}$$

$$= \left[\mathbf{e}_{k_{\mathrm{mec}}}\otimes\mathbf{c}\right]^T\left\{\frac{1}{h_n}\left[[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]\otimes\mathbf{H}_q\otimes\boldsymbol{I}\right]\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}\end{bmatrix}\right\} \text{momentum balance}$$

$$= \left[\mathbf{e}\otimes\mathbf{c}\right]^T\left\{\cancel{\mathbf{F}^{\mathrm{int}}}+\mathbf{F}^{\mathrm{ext}}+\bar{\mathbf{F}}^{\mathrm{ext}}+\left[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes\boldsymbol{I}\right]\mathbf{R}\right\} \quad \text{with} \quad \sum_{I=1}^{N_q}\nabla N_I=0$$

$$\frac{\mathcal{P}_{n+1}-\mathcal{P}_n}{h_n} = \left[\mathbf{e}\otimes\mathbf{c}\right]^T\left\{\mathbf{F}^{\mathrm{ext}}+\bar{\mathbf{F}}^{\mathrm{ext}}+\left[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes\boldsymbol{I}\right]\mathbf{R}\right\} \qquad \mathbf{e}_{k_{\mathrm{mec}}}^T:=[1,\dots,1]\in\mathbb{R}^{1\times k_{\mathrm{mec}}}$$

Discrete balance of total entropy

Discrete balance of total entropy or thermal momentum

$$\frac{\mathcal{S}_{n+1}-\mathcal{S}_n}{h_n} = \frac{1}{h_n}\mathbf{e}_\Theta^T\{[\mathbf{H}_q]\,\mathbf{s}_{k_{\mathrm{the}}+1}-[\mathbf{H}_q]\,\mathbf{s}_1\} \qquad \text{with} \quad \mathbf{e}_\Theta^T := [1,\dots,1]\in\mathbb{R}^{1\times n_{\mathrm{node}}}$$

$$= [\mathbf{e}_{k_{\mathrm{the}}}\otimes\mathbf{e}_\Theta]^T \underbrace{\frac{1}{h_n}\left[[\tilde{\mathbf{a}}|\tilde{\mathbf{A}}]\otimes\mathbf{H}_q\right]\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}\end{bmatrix}}_{\text{momentum balance}} \qquad \text{with} \quad \mathbf{e}_{k_{\mathrm{the}}}^T := [1,\dots,1]\in\mathbb{R}^{1\times k_{\mathrm{the}}}$$

$$\frac{\mathcal{S}_{n+1}-\mathcal{S}_n}{h_n} = [\mathbf{e}_{k_{\mathrm{the}}}\otimes\mathbf{e}_\Theta]^T\left\{\cancel{\mathbf{S}^{\mathrm{con}}}+\mathbf{S}^{\mathrm{cdu}}+\mathbf{S}^{\mathrm{int}}+\bar{\mathbf{S}}^{\mathrm{ext}}+[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}\right\} \qquad \text{with} \quad \sum_{I=1}^{N_q}\nabla N_I = 0$$

In the case without transient Dirichlet and Neumann boundary conditions, we obtain at $\partial_\Theta\mathscr{B}_0$

$$[\mathbf{e}_{k_{\mathrm{the}}}\otimes\bar{\mathbf{e}}_\Theta]^T\left\{\cancel{\left[\frac{1}{h_n}\tilde{\mathbf{A}}\otimes\bar{\mathbf{H}}_o\right]\mathbf{s}+\frac{1}{h_n}}\,\tilde{\mathbf{a}}\otimes\bar{\mathbf{H}}_o\,\mathbf{s}_1-\cancel{\bar{\mathbf{S}}^{\mathrm{con}}}-\bar{\mathbf{S}}^{\mathrm{cdu}}-\bar{\mathbf{S}}^{\mathrm{int}}-\cancel{\bar{\mathbf{S}}^{\mathrm{ext}}}\right\} = [\mathbf{e}_{k_{\mathrm{the}}}\otimes\bar{\mathbf{e}}_\Theta]^T[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}\leq 0$$

Discrete balance of Lyapunov function

$$\frac{\mathcal{F}_{n+1}-\mathcal{F}_n}{h_n} = \frac{\mathcal{H}_{n+1}-\mathcal{H}_n}{h_n}-\Theta_\infty\frac{\mathcal{S}_{n+1}-\mathcal{S}_n}{h_n} \qquad \text{with} \quad \boldsymbol{\Theta}_\infty^T := \Theta_\infty\,[\mathbf{e}_{k_{\mathrm{the}}}\otimes\mathbf{e}_\Theta]^T$$

$$= \frac{1}{h_n}\,\mathring{\mathbf{q}}^T\left\{\bar{\mathbf{F}}^{\mathrm{ext}}+[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes I]\,\mathbf{R}\right\}+[\tilde{\boldsymbol{\Theta}}-\boldsymbol{\Theta}_\infty]^T\left\{\bar{\mathbf{S}}^{\mathrm{ext}}+\cancel{[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_o]\,\boldsymbol{\lambda}}\right\}$$

$$-\ \boldsymbol{\Theta}_\infty^T\left\{\mathbf{S}^{\mathrm{cdu}}+\mathbf{S}^{\mathrm{int}}\right\}+\frac{1}{h_n}\,\mathring{\boldsymbol{\Theta}}^T[\tilde{\mathbf{C}}\otimes\mathbf{H}_o]\,\bar{\mathbf{h}}$$

In the case without transient Dirichlet and Neumann boundary conditions, we arrive at

$$\frac{\mathcal{F}_{n+1}-\mathcal{F}_n}{h_n} = \frac{1}{h_n}\,\mathring{\mathbf{q}}^T\cancel{\left\{\bar{\mathbf{F}}^{\mathrm{ext}}+[\tilde{\mathbf{C}}\otimes\bar{\mathbf{H}}_q\otimes I]\,\mathbf{R}\right\}}+[\tilde{\boldsymbol{\Theta}}-\boldsymbol{\Theta}_\infty]^T\cancel{\bar{\mathbf{S}}^{\mathrm{ext}}}$$

$$-\ \boldsymbol{\Theta}_\infty^T\left\{\mathbf{S}^{\mathrm{cdu}}+\mathbf{S}^{\mathrm{int}}\right\}+\frac{1}{h_n}\,\mathring{\boldsymbol{\Theta}}^T\cancel{[\tilde{\mathbf{C}}\otimes\mathbf{H}_o]\,\bar{\mathbf{h}}}$$

$$\frac{\mathcal{F}_{n+1}-\mathcal{F}_n}{h_n} = -\,\boldsymbol{\Theta}_\infty^T\left\{\mathbf{S}^{\mathrm{cdu}}+\mathbf{S}^{\mathrm{int}}\right\}\leq 0$$

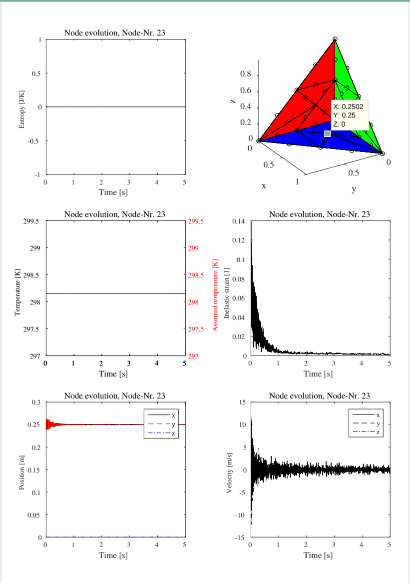
Dynamic Taylor impact test with a tetrahedron

('Decreasing'-Test of balance laws; $k_{\text{mec}} = 2$, $k_{\text{the}} = 2$, $k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \mathbf{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 23 (bottom)



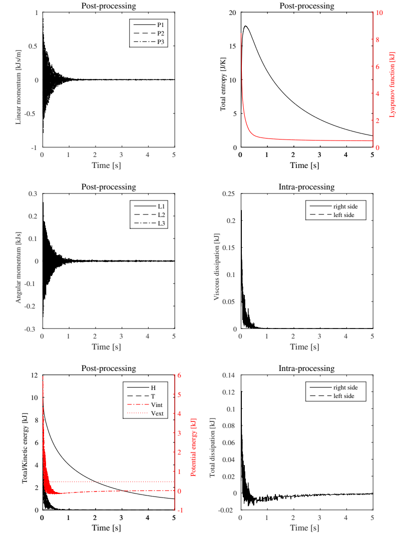
Dynamic Taylor impact test with a tetrahedron

('Decreasing'-Test of balance laws; $k_{\mathrm{mec}} = 2, k_{\mathrm{the}} = 2, k_{\mathrm{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\mathrm{el}} = 20$
Node number:	$n_{\mathrm{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Balance properties $\mathrm{tol} = 10^{-2}\,\mathrm{J/tolevo} = 10^{-10}\,\mathrm{J}$

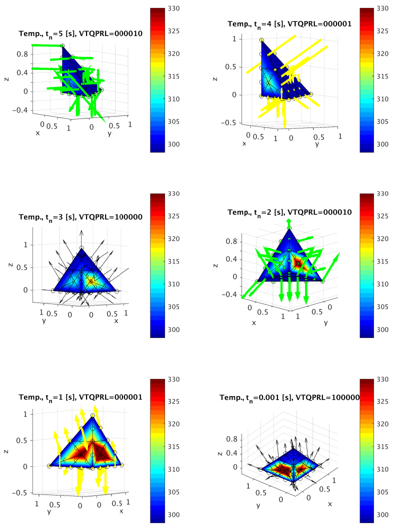


Dynamic Taylor impact test with a tetrahedron
('Decreasing'-Test of balance laws; $k_{\text{mec}} = 2$, $k_{\text{the}} = 2$, $k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Snapshots of the motion with reactions



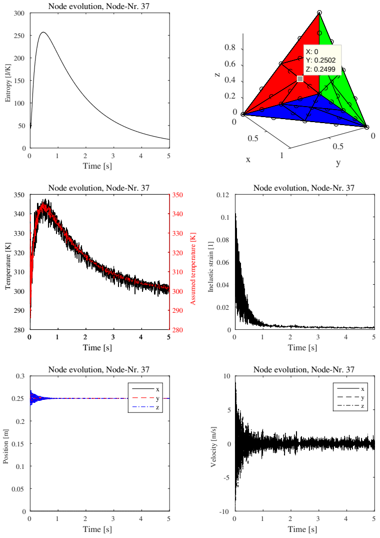
Dynamic Taylor impact test with a tetrahedron

('Decreasing'-Test of balance laws; $k_{\text{mec}} = 2,\, k_{\text{the}} = 2,\, k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 37 (right)



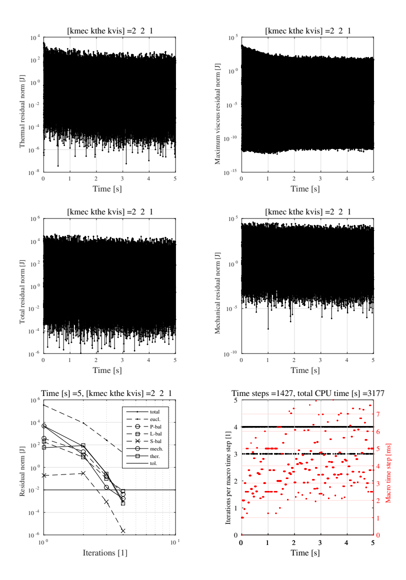
Dynamic Taylor impact test with a tetrahedron

('Decreasing'-Test of balance laws; $k_{\text{mec}} = 2$, $k_{\text{the}} = 2$, $k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time steps and CPU time $h_n^{\min} = 10^{-6}\,\mathrm{s}$

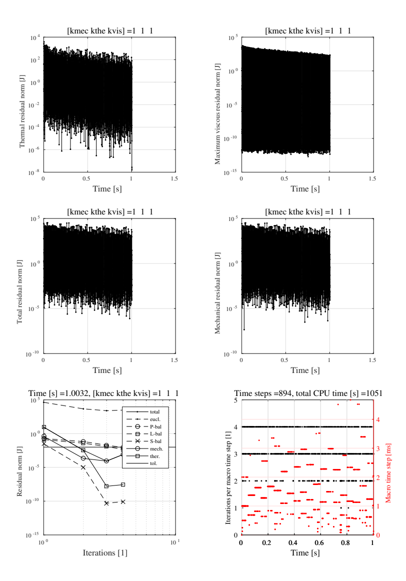


Dynamic Taylor impact test with a tetrahedron ('Decreasing'-Test of balance laws; $k_{\text{mec}} = 1, k_{\text{the}} = 1, k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time steps and CPU time $h_n^{\min} = 10^{-6}\,\text{s}$

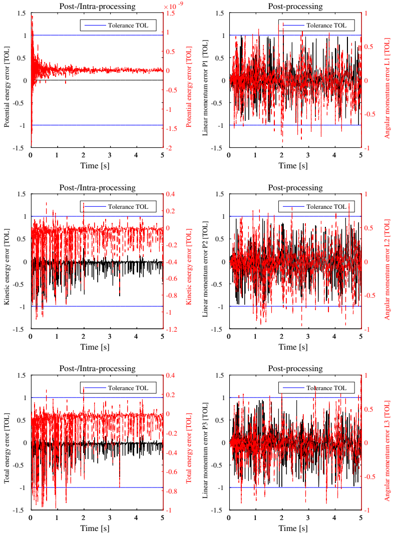


Dynamic Taylor impact test with a tetrahedron ('Decreasing'-Test of balance laws; $k_{\text{mec}} = 2, k_{\text{the}} = 2, k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 20$
Node number:	$n_{\text{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Mechanical balance errors $\text{tol} = 10^{-2}\,\text{J}$

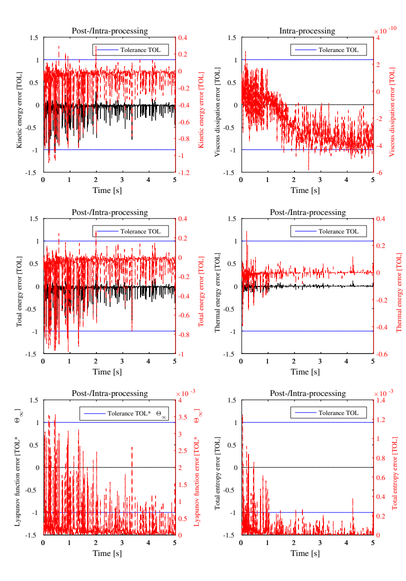


Dynamic Taylor impact test with a tetrahedron
('Decreasing'-Test of balance laws; $k_{\mathrm{mec}} = 2$, $k_{\mathrm{the}} = 2$, $k_{\mathrm{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\mathrm{el}} = 20$
Node number:	$n_{\mathrm{no}} = 59$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$\Theta^A = \Theta_\infty$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = -10\,\boldsymbol{e}_z$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Thermal balance errors $\mathrm{tol} = 10^{-2}\,\mathrm{J}$



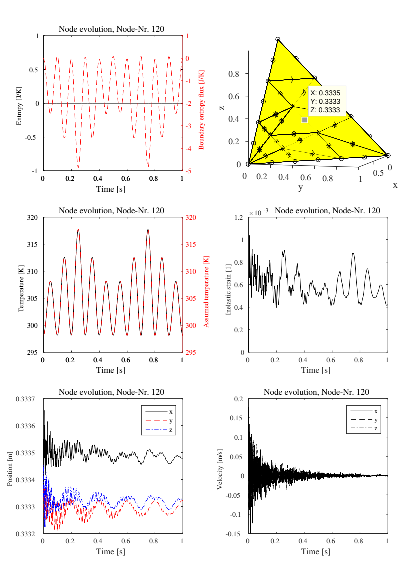
Pressure and temperature load on a tetrahedron

(Mechanical Neumann/Thermal Dirichlet load; $k_{\mathrm{mec}} = 2$, $k_{\mathrm{the}} = 2$, $k_{\mathrm{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\mathrm{el}} = 65$
Node number:	$n_{\mathrm{no}} = 158$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$T_{\mathrm{load}} = 0.5$
	$\omega_{\mathrm{load}} = 10\pi$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = \mathbf{0}$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \mathbf{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	$\hat{\Theta} = 10$
Neumann loads	
Traction load:	no
Pressure load:	$\hat{p} = 4 \cdot 10^4$
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 120 (front)



Pressure and temperature load on a tetrahedron

(Mechanical Neumann/Thermal Dirichlet load; $k_{\text{mec}} = 2, k_{\text{the}} = 2, k_{\text{vis}} = 1$)

Dynamic thermo-mechanical coupling in fiber-reinforced bodies simulated by higher-order variational energy-momentum schemes

Michael Groß and Julian Dietzsch

Introduction

Variational setting
Continuum model
Variational principle
Euler-Lagrange equations

Discrete setting
Global E-L equations
Local E-L equations
Mechanical energy balance
Total energy balance
Total momenta balances
Total entropy balance

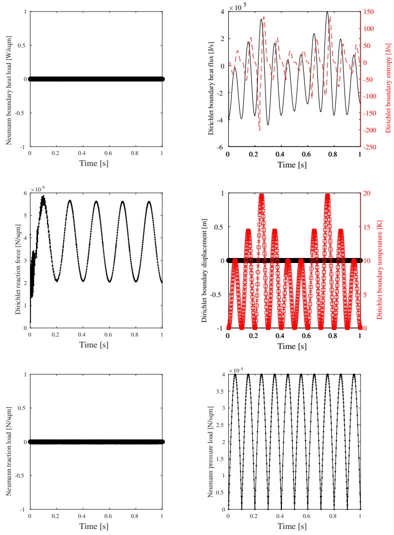
Numerical studies

Summary

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 65$
Node number:	$n_{\text{no}} = 158$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$T_{\text{load}} = 0.5$
	$\omega_{\text{load}} = 10\pi$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = \boldsymbol{0}$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	$\hat{\Theta} = 10$
Neumann loads	
Traction load:	no
Pressure load:	$\hat{p} = 4 \cdot 10^4$
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of loads and reactions



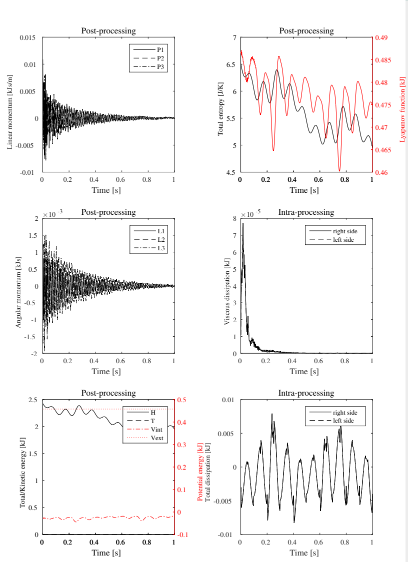
Pressure and temperature load on a tetrahedron

(Mechanical Neumann/Thermal Dirichlet load; $k_{\text{mec}} = 2$, $k_{\text{the}} = 2$, $k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\text{edge}} = 1$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 65$
Node number:	$n_{\text{no}} = 158$
Boundary conditions	
Red face:	x-dof fixed
Green face:	y-dof fixed
Blue face:	z-dof fixed
	$\Theta^A = \Theta_\infty$
Yellow face:	$T_{\text{load}} = 0.5$
	$\omega_{\text{load}} = 10\pi$
Initial conditions	
Initial velocity:	$\boldsymbol{v}_0^A = \boldsymbol{0}$
Initial angular velocity:	$\boldsymbol{\omega}_0^A = \boldsymbol{0}$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	$\hat{\Theta} = 10$
Neumann loads	
Traction load:	no
Pressure load:	$\hat{p} = 4 \cdot 10^4$
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Balance properties



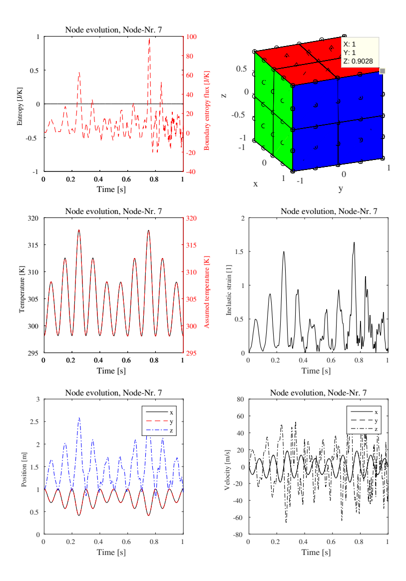
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $k_{\mathrm{mec}} = 1, k_{\mathrm{the}} = 2, k_{\mathrm{vis}} = 2$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 2$
Mesh	
Element:	27-node
Element number:	$n_{\mathrm{el}} = 8$
Node number:	$n_{\mathrm{no}} = 125$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \mathbf{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements with $f(t)$:	$T_{\mathrm{load}} = 0.5$
Temperatures with $f(t)$:	$\omega_{\mathrm{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 7 (corner)



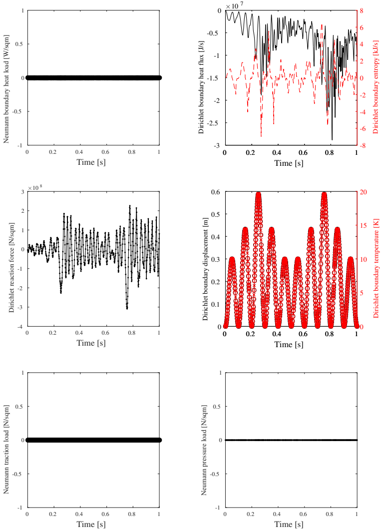
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $k_{\text{mec}} = 1, k_{\text{the}} = 2, k_{\text{vis}} = 2$)

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 2$
Mesh	
Element:	27-node
Element number:	$n_{\text{el}} = 8$
Node number:	$n_{\text{no}} = 125$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \boldsymbol{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements with $f(t)$:	$T_{\text{load}} = 0.5$
Temperatures with $f(t)$:	$\omega_{\text{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of loads and reactions



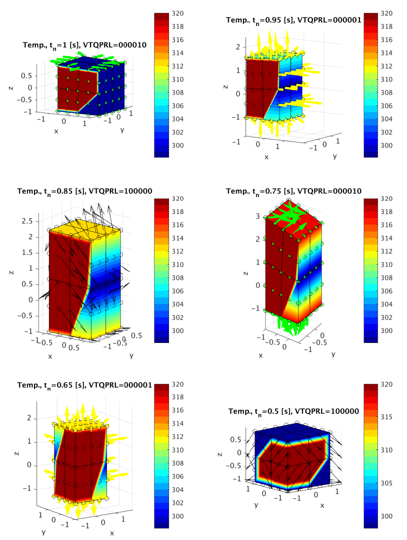
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $k_{\text{mec}} = 1, k_{\text{the}} = 2, k_{\text{vis}} = 2$)

Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge length	$l_{\text{edge}} = 2$
Mesh	
Element:	27-node
Element number:	$n_{\text{el}} = 8$
Node number:	$n_{\text{no}} = 125$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \boldsymbol{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements with $f(t)$:	$T_{\text{load}} = 0.5$
Temperatures with $f(t)$:	$\omega_{\text{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Snapshots of the motion with reactions



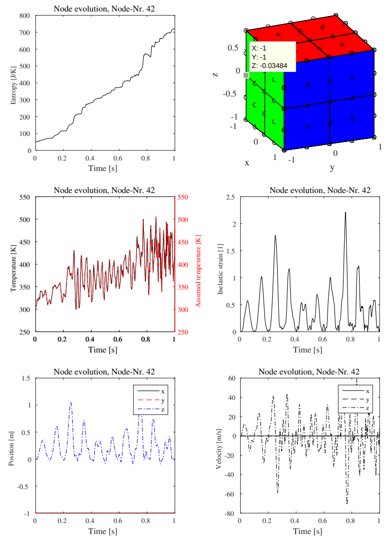
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $k_{\text{mec}} = 1, k_{\text{the}} = 2, k_{\text{vis}} = 2$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\text{edge}} = 2$
Mesh	
Element:	27-node
Element number:	$n_{\text{el}} = 8$
Node number:	$n_{\text{no}} = 125$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \boldsymbol{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements with $f(t)$:	$T_{\text{load}} = 0.5$
Temperatures with $f(t)$:	$\omega_{\text{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 42 (corner)



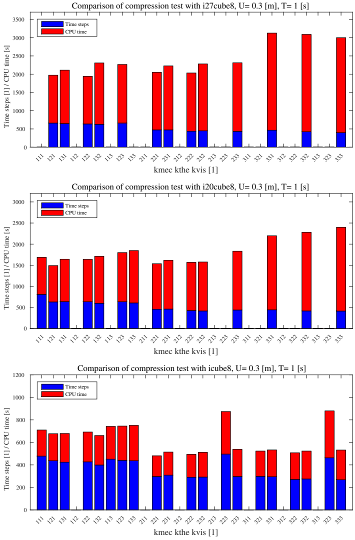
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $n_{\mathrm{el}} = 8$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 2$
Mesh	
Element:	8/20/27-node
Element number:	$n_{\mathrm{el}} = 8$
Node number:	$n_{\mathrm{no}} = 27/81/125$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \mathbf{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements $[f(t)]$:	$T_{\mathrm{load}} = 0.5$
Temperatures $[f(t)]$:	$\omega_{\mathrm{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time steps and CPU time $h_n^{\min} = 10^{-6}\,\mathrm{s}$



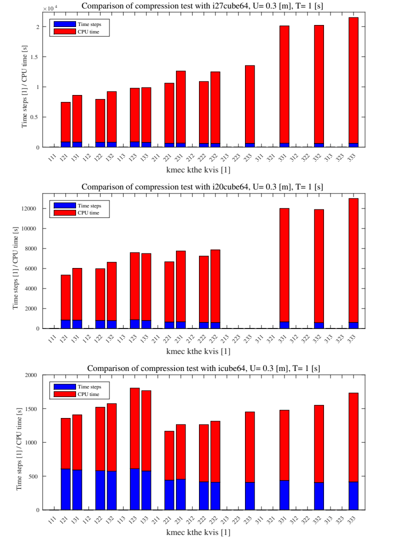
Compression test with heating on a cube

(Mechanical Dirichlet/Thermal Dirichlet load; $n_{\mathrm{el}} = 64$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge length	$l_{\mathrm{edge}} = 2$
Mesh	
Element:	8/20/27-node
Element number:	$n_{\mathrm{el}} = 64$
Node number:	$n_{\mathrm{no}} = 125/425/729$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
	$\Theta^A = 10\,f(t)$
Red xy-face:	$\Theta^A = 10\,f(t)$
Blue yz-face	$\Theta^A = \Theta_\infty$
	$u_x^A = -0.3\,f(t)$
Blue xz-face	$\Theta^A = \Theta_\infty$
	$u_y^A = -0.3\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \mathbf{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacements $[f(t)]$:	$T_{\mathrm{load}} = 0.5$
Temperatures $[f(t)]$:	$\omega_{\mathrm{load}} = 10\pi$
Neumann loads	
Traction load:	no
Pressure load:	no
Heat flux load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time steps and CPU time $h_n^{\min} = 10^{-6}\,\mathrm{s}$



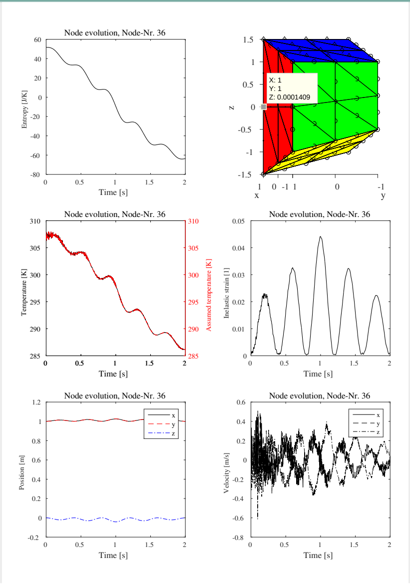
Compression with cooling on a 'Cook-like' cube

(Mechanical Neumann/Thermal Neumann load; $k_{\mathrm{mec}} = 1$, $k_{\mathrm{the}} = 2$, $k_{\mathrm{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge lengths	$l_{\mathrm{edge}_1} = 1.5$
	$l_{\mathrm{edge}_2} = 2$
	$l_{\mathrm{edge}_3} = 2.5$
Mesh	
Element:	10-node
Element number:	$n_{\mathrm{el}} = 46$
Node number:	$n_{\mathrm{no}} = 121$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
Red faces:	$Q^A = -2000\,f(t)$
Blue face	$\Theta^A = \Theta_\infty$
	$T_z^A = -400000\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \mathbf{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Heat flux load [$f(t)$]:	$T_{\mathrm{load}} = 2$
Traction load [$f(t)$]:	$\omega_{\mathrm{load}} = 2.5\pi$
Pressure load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of node 36 (long edge)



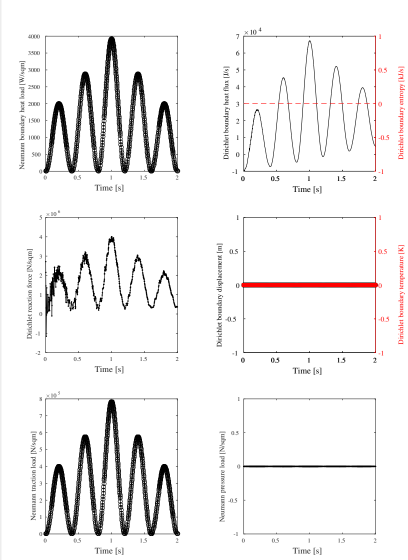
Compression with cooling on a 'Cook-like' cube

(Mechanical Neumann/Thermal Neumann load; $k_{\mathrm{mec}} = 1, k_{\mathrm{the}} = 2, k_{\mathrm{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge lengths	$l_{\mathrm{edge}_1} = 1.5$ $l_{\mathrm{edge}_2} = 2$ $l_{\mathrm{edge}_3} = 2.5$
Mesh	
Element:	10-node
Element number:	$n_{\mathrm{el}} = 46$
Node number:	$n_{\mathrm{no}} = 121$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
Red faces:	$Q^A = -2000\,f(t)$
Blue face	$\Theta^A = \Theta_\infty$ $T_z^A = -400000\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \mathbf{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Heat flux load [$f(t)$]:	$T_{\mathrm{load}} = 2$
Traction load [$f(t)$]:	$\omega_{\mathrm{load}} = 2.5\pi$
Pressure load:	no
Volume loads	
Standard gravity	$g = 9.81$

Time evolutions of loads and reactions



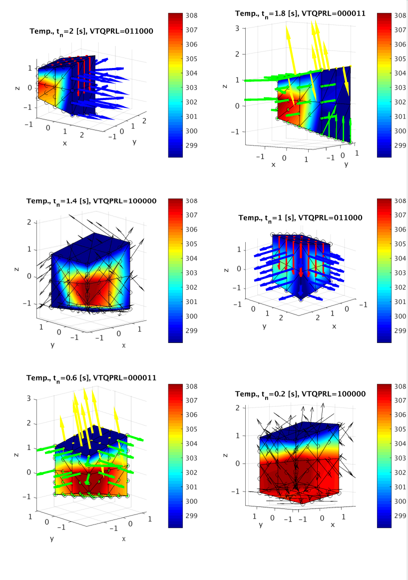
Compression with cooling on a 'Cook-like' cube

(Mechanical Neumann/Thermal Neumann load; $k_{\text{mec}} = 1, k_{\text{the}} = 2, k_{\text{vis}} = 1$)

Characteristics $\Theta_\infty = 298.15\,\mathrm{K}$

Geometry	
Edge lengths	$l_{\text{edge}_1} = 1.5$ $l_{\text{edge}_2} = 2$ $l_{\text{edge}_3} = 2.5$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 46$
Node number:	$n_{\text{no}} = 121$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
Red faces:	$Q^A = -2000\,f(t)$
Blue face	$\Theta^A = \Theta_\infty$ $T_z^A = -400000\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \boldsymbol{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Heat flux load $[f(t)]$:	$T_{\text{load}} = 2$
Traction load $[f(t)]$:	$\omega_{\text{load}} = 2.5\pi$
Pressure load:	no
Volume loads	
Standard gravity	$g = 9.81$

Snapshots of the motion with reactions



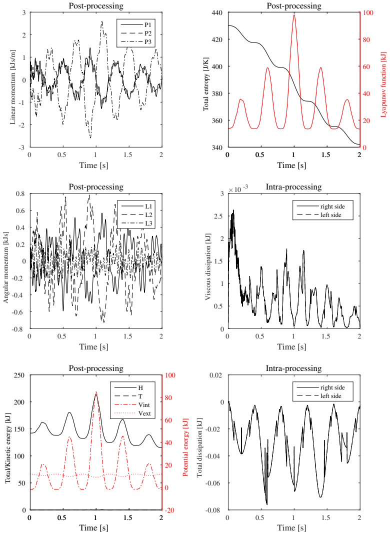
Compression with cooling on a 'Cook-like' cube

(Mechanical Neumann/Thermal Neumann load; $k_{\text{mec}} = 1, k_{\text{the}} = 2, k_{\text{vis}} = 1$)

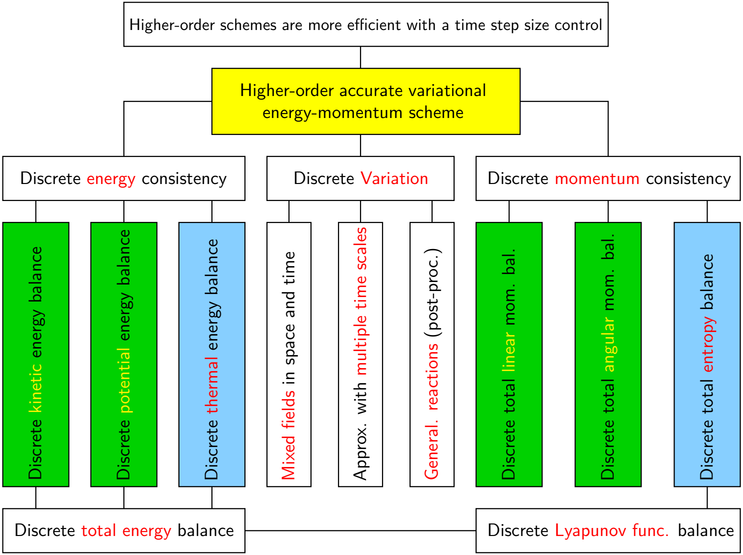
Characteristics $\Theta_\infty = 298.15\,\text{K}$

Geometry	
Edge lengths	$l_{\text{edge}_1} = 1.5$
	$l_{\text{edge}_2} = 2$
	$l_{\text{edge}_3} = 2.5$
Mesh	
Element:	10-node
Element number:	$n_{\text{el}} = 46$
Node number:	$n_{\text{no}} = 121$
Boundary conditions	
Green yz-face:	x-dof fixed
Green xz-face:	y-dof fixed
Yellow xy-face:	z-dof fixed
Red faces:	$Q^A = -2000\,f(t)$
Blue face	$\Theta^A = \Theta_\infty$
	$T_z^A = -400000\,f(t)$
Initial conditions	
Initial velocities:	$\boldsymbol{v}_0^A = \boldsymbol{0} = \boldsymbol{\omega}_0^A$
Initial temperature:	$\Theta_0^A = 308.15$
Dirichlet loads	
Displacement profile:	no
Temperature profile:	no
Neumann loads	
Heat flux load [$f(t)$]:	$T_{\text{load}} = 2$
Traction load [$f(t)$]:	$\omega_{\text{load}} = 2.5\pi$
Pressure load:	no
Volume loads	
Standard gravity	$g = 9.81$

Balance properties



Summary and Outlook



Outlook: Mixed finite element stress fields