# Variational-based higher-order energy-momentum schemes with incompatible modes for fiber-reinforced materials

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# Motivation and goals

## Motivation

1. Growing need for physically accurate dynamic simulations of fiber-reinforced materials in light-weight structures
2. Higher-order accurate energy-momentum schemes,
   (symmetric Gauss-Runge-Kutta-based schemes with additional independent time approximations)
   (see Hairer, Lubich, Warner [2006] for second-order accurate versions)
   but only combined with locking-free finite elements

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## Goals

1. In order to include the variational method of incompatible modes, which is
   (see Wilson et al. [1973], Ibrahimbegovic & Wilson [1991], Simo, Armero & Taylor [1993])
2. appropriate for linear and quadratic hexaeder and tetraeder finite elements,
   (see Taylor[2005], Ibrahimbegovic[2009], Caylak & Mahnken [2011])
3. we aim at a variational design of energy-momentum schemes, based on
   - fiber-reinforced (i.e. anisotropic) continuum formulations,
   - higher-order energy-momentum consistent time approximations,
   - different higher-order accurate space approximations.
4. Derivation of energy-momentum-based design criteria for spatial elements.

# Current state and strategy of research

## Current state of research

1. Higher-order accurate energy-momentum schemes
   - variationally-consistent (Bartelt & G. [2016]; G., Ramesh & Dietzsch [2016])
   - but only with locking finite elements (Erler & G. [2015])
2. Second-order accurate energy-momentum schemes with locking-free
   - shell-type eight-noded brick elements (Miehe & Schröder [2001])
   - quadrilaterals with incompatible modes (Müller & Betsch [2007])
   - mixed assumed Jacobian elements (Armero [2008])
   - mixed assumed shell elements (Betsch & Janz [2016])

## Strategy of research

1. Discrete variational principle, which include existing Galerkin methods, (cp. Schröder & Kuhl [2015])
2. but allow for introducing the variational method of incompatible modes. (see Simo & Armero [1992])
3. Investigation of the energy-momentum behaviour of locking finite elements,
4. which may render new design criteria for locking free finite elements.

# Continuum model for fiber-reinforced materials



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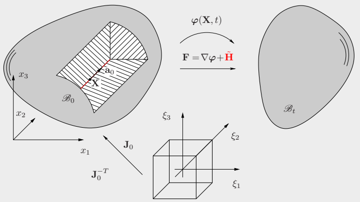
## Transversely isotropic material

Schröder, Neff & Balzani [2005]




## Incompatible displacement gradient tensor

Simo, Armero & Taylor [1993]
Müller & Betsch [2007]

$$[\tilde{\boldsymbol{H}}_{\square}(\boldsymbol{\xi})]^{\alpha}{}_{\beta} := \sum_{I=1}^{n_{\mathrm{dim}}} [\boldsymbol{\Gamma}^I]^{\alpha} \, [\nabla_{\boldsymbol{\xi}} \tilde{N}_I(\boldsymbol{\xi})]_{\beta}$$

$$[\tilde{\boldsymbol{H}}_0(\boldsymbol{\xi})]^{B}{}_{A} := \frac{j_0}{j(\boldsymbol{\xi})} \, [\boldsymbol{J}_0]^{B}{}_{\alpha} [\tilde{\boldsymbol{H}}_{\square}(\boldsymbol{\xi})]^{\alpha}{}_{\beta} [\boldsymbol{J}_0^{-1}]^{\beta}{}_{A}$$

$$[\tilde{\boldsymbol{H}}_t(\boldsymbol{\xi})]^{a}{}_{A} := [\nabla_{\boldsymbol{X}} \boldsymbol{\varphi}_t^0]^{a}{}_{B} [\tilde{\boldsymbol{H}}_0(\boldsymbol{\xi})]^{B}{}_{A}$$

$$\boxed{[\tilde{\boldsymbol{H}}_t(\boldsymbol{\xi})]^{a}{}_{A} = \sum_{I=1}^{n_{\mathrm{dim}}} [\boldsymbol{\alpha}_t^I]^{a} \, [\tilde{\nabla}_{\boldsymbol{X}} \tilde{N}_I(\boldsymbol{\xi})]_{A}}$$

## Matrix and fiber deformation

Klinkel, Sansour & Wagner [2005]

1. Deformation gradients

$$\boldsymbol{F}_F := \boldsymbol{a} \otimes \boldsymbol{a}_0 = \boldsymbol{F} \boldsymbol{A}_0 \qquad \boldsymbol{a} = \boldsymbol{F} \boldsymbol{a}_0 \qquad \boldsymbol{A}_0 := \boldsymbol{a}_0 \otimes \boldsymbol{a}_0$$

2. Right Cauchy-Green tensors

$$\boldsymbol{C}_F := \boldsymbol{F}_F^T \boldsymbol{F}_F := C_F \boldsymbol{A}_0 \qquad C_F := \boldsymbol{C} : \boldsymbol{A}_0 \equiv I_4^C \qquad \boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$$

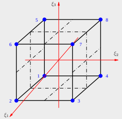
3. Total second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} := 2 \sum_{i=1}^{3} \frac{\partial \hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial I_i^C} \frac{\partial I_i^C}{\partial \boldsymbol{C}} + 2 \frac{\partial \hat{W}(I_1^C, I_2^C, I_3^C, C_F)}{\partial C_F} \boldsymbol{A}_0 + 2 \frac{\partial W_F(C_F)}{\partial C_F} \boldsymbol{A}_0$$

# Incompatible modes of brick elements

## 8-node brick

Hughes [2000]

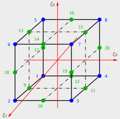



| #nodes | $\xi_1$ | $\xi_2$ | $\xi_3$ |
|---|---|---|---|
| 1 | −1 | −1 | −1 |
| 2 | 1 | −1 | −1 |
| 3 | 1 | 1 | −1 |
| 4 | −1 | 1 | −1 |
| 5 | −1 | −1 | 1 |
| 6 | 1 | −1 | 1 |
| 7 | 1 | 1 | 1 |
| 8 | −1 | 1 | 1 |

linear Lagrange element

## 20-node brick

Hughes [2000]

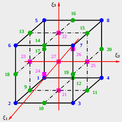



| #nodes | $\xi_1$ | $\xi_2$ | $\xi_3$ |
|---|---|---|---|
| 1 | −1 | −1 | −1 |
| 2 | 1 | −1 | −1 |
| 3 | 1 | 1 | −1 |
| 4 | −1 | 1 | −1 |
| 5 | −1 | −1 | 1 |
| 6 | 1 | −1 | 1 |
| 7 | 1 | 1 | 1 |
| 8 | −1 | 1 | 1 |
| 9 | 0 | −1 | −1 |
| 10 | 1 | 0 | −1 |
| 11 | 0 | 1 | −1 |
| 12 | −1 | 0 | −1 |
| 13 | 0 | −1 | 1 |
| 14 | 1 | 0 | 1 |
| 15 | 0 | 1 | 1 |
| 16 | −1 | 0 | 1 |
| 17 | −1 | −1 | 0 |
| 18 | 1 | −1 | 0 |
| 19 | 1 | 1 | 0 |
| 20 | −1 | 1 | 0 |

quadratic serentipity element

## 27-node brick

Hughes [2000]




| #nodes | $\xi_1$ | $\xi_2$ | $\xi_3$ |
|---|---|---|---|
| 1 | −1 | −1 | −1 |
| 2 | 1 | −1 | −1 |
| 3 | 1 | 1 | −1 |
| 4 | −1 | 1 | −1 |
| 5 | −1 | −1 | 1 |
| 6 | 1 | −1 | 1 |
| 7 | 1 | 1 | 1 |
| 8 | −1 | 1 | 1 |
| 9 | 0 | −1 | −1 |
| 10 | 1 | 0 | −1 |
| 11 | 0 | 1 | −1 |
| 12 | −1 | 0 | −1 |
| 13 | 0 | −1 | 1 |
| 14 | 1 | 0 | 1 |
| 15 | 0 | 1 | 1 |
| 16 | −1 | 0 | 1 |
| 17 | −1 | −1 | 0 |
| 18 | 1 | −1 | 0 |
| 19 | 1 | 1 | 0 |
| 20 | −1 | 1 | 0 |
| 21 | 0 | 0 | −1 |
| 22 | 0 | 0 | 1 |
| 23 | 0 | −1 | 0 |
| 24 | 1 | 0 | 0 |
| 25 | 0 | 1 | 0 |
| 26 | −1 | 0 | 0 |
| 27 | 0 | 0 | 0 |

## Shape functions of incompatible modes

1. Linear 8-node brick element

   Simo, Armero & Taylor [1993]

$$\tilde{N}_1 := \frac{1}{2}\left(\xi_1^2 - 1\right) \quad \tilde{N}_2 := \frac{1}{2}\left(\xi_2^2 - 1\right)$$

$$\tilde{N}_3 := \frac{1}{2}\left(\xi_3^2 - 1\right)$$

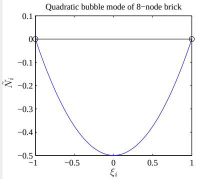
2. Quadratic 20-/27-node brick element

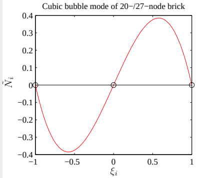
   Wilson et al. [1973], Ibrahimbegovic [2009]

$$\tilde{N}_1 := \xi_1\left(1 - \xi_1^2\right) \quad \tilde{N}_2 := \xi_2\left(1 - \xi_2^2\right)$$

$$\tilde{N}_3 := \xi_3\left(1 - \xi_3^2\right)$$

## Edge bubble modes



# Functional form of the total energy balance

## Total energy balance $\dot{\mathcal{H}} = 0$

$$\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F) + \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}}) = 0$$

## Kinetic power functional

$$\dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \dot{\boldsymbol{v}} \,\mathrm{d}V - \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \,\mathrm{d}V + \int_{\mathscr{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{u}} \,\mathrm{d}V$$

## Stress power functional $\dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F)$

$$\begin{aligned}\dot{\Pi}^{\text{int}} := \quad & \frac{1}{2}\int_{\mathscr{B}_0} \Big\{ \big[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} - \boldsymbol{S}\big] : \dot{\tilde{\boldsymbol{C}}} + \big[2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F - \boldsymbol{S}_F : \boldsymbol{A}_0\big] : \dot{\tilde{C}}_F \Big\} \,\mathrm{d}V \\ & - \frac{1}{2}\int_{\mathscr{B}_0} \dot{\boldsymbol{S}} : \big[\tilde{\boldsymbol{C}} - \boldsymbol{C}(\boldsymbol{u}, \tilde{\boldsymbol{H}})\big] \,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0} \boldsymbol{S} : \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \,\mathrm{d}V \\ & - \frac{1}{2}\int_{\mathscr{B}_0} \dot{\boldsymbol{S}}_F : \big[\tilde{C}_F \,\boldsymbol{A}_0 - \boldsymbol{C}_F(\boldsymbol{u}, \tilde{\boldsymbol{H}})\big] \,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0} \boldsymbol{S}_F : \dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \,\mathrm{d}V\end{aligned}$$

## External power functional $\dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}})$

$$\dot{\Pi}^{\text{ext}} := -\int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \big\{ \boldsymbol{h} \cdot (\dot{\boldsymbol{u}} - \dot{\bar{\boldsymbol{u}}}) + \dot{\boldsymbol{h}} \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}}) \big\} \,\mathrm{d}A$$

# Mixed Jourdain principle

(cp. Schröder & Kuhl [2015], Klinkel & Wagner [1997], Hackl [1997])

## Mixed principle of virtual power $\delta_* \dot{\mathcal{H}} = 0$

$$\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \delta_* \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F) + \delta_* \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}}) = 0$$

## Virtual kinetic power $\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}})$

$$\delta_* \dot{\mathcal{T}} := \int_{\mathscr{B}_0} [\rho_0 \boldsymbol{v} - \boldsymbol{p}] \cdot \delta_* \dot{\boldsymbol{v}} \,\mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{u}}] \,\mathrm{d}V + \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}V$$

## Virtual stress power $\delta_* \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{C}}_F, \boldsymbol{S}, \boldsymbol{S}_F; \tilde{\boldsymbol{S}}, \tilde{S}_F)$

$$\begin{aligned} \delta_* \dot{\Pi}^{\text{int}} := {} & \frac{1}{2} \int_{\mathscr{B}_0} \Big\{ \big[ 2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} - \boldsymbol{S} \big] : \delta_* \dot{\tilde{\boldsymbol{C}}} + \big[ 2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F - \boldsymbol{S}_F : \boldsymbol{A}_0 \big] : \delta_* \dot{\tilde{C}}_F \Big\} \,\mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S} : \Big[ \dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \Big] \,\mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{F}\,\boldsymbol{S} : \Big[ \nabla_{\boldsymbol{X}}(\delta_* \dot{\boldsymbol{u}}) + \delta_* \dot{\tilde{\boldsymbol{H}}} \Big] \,\mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \delta_* \boldsymbol{S}_F : \Big[ \dot{\tilde{C}}_F \boldsymbol{A}_0 - \dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) \Big] \,\mathrm{d}V + \frac{1}{2} \int_{\mathscr{B}_0} \boldsymbol{F}_F\,\boldsymbol{S}_F : \Big[ \nabla_{\boldsymbol{X}}(\delta_* \dot{\boldsymbol{u}}) + \delta_* \dot{\tilde{\boldsymbol{H}}} \Big] \boldsymbol{A}_0 \,\mathrm{d}V \end{aligned}$$

## Virtual external power $\delta_* \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{h}})$

$$\delta_* \dot{\Pi}^{\text{ext}} := - \int_{\mathscr{B}_0} \rho_0 \boldsymbol{b} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}V - \int_{\partial_t \mathscr{B}_0} \boldsymbol{t} \cdot \delta_* \dot{\boldsymbol{u}} \,\mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \{ \boldsymbol{h} \cdot \delta_* \dot{\boldsymbol{u}} + \delta_* \dot{\boldsymbol{h}} \cdot [\boldsymbol{u} - \bar{\boldsymbol{u}}] \} \,\mathrm{d}A$$

# Euler-Lagrange equations

## Constitutive equations

$$\rho_0 \boldsymbol{v} = \boldsymbol{p} \quad \forall t \geqslant t_0 \qquad \delta_* \dot{\tilde{\boldsymbol{H}}} = \boldsymbol{O} \text{ and } \tilde{\boldsymbol{H}}(t_0) = \boldsymbol{O}$$

$$2\,\mathrm{D}W(\tilde{\boldsymbol{C}}) + \tilde{\boldsymbol{S}} = \boldsymbol{S} \quad \forall t \geqslant t_0 \qquad \boldsymbol{S}_F = \left[2\,\mathrm{D}W_F(\tilde{C}_F) + \tilde{S}_F\right] \boldsymbol{A}_0$$

$$\dot{\boldsymbol{C}}(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) = \dot{\tilde{\boldsymbol{C}}} \quad \text{with} \quad \tilde{\boldsymbol{C}}(t_0) = \boldsymbol{C}(\boldsymbol{u}_0, \tilde{\boldsymbol{H}}(t_0))$$

$$\dot{\boldsymbol{C}}_F(\dot{\boldsymbol{u}}, \dot{\tilde{\boldsymbol{H}}}) : \boldsymbol{A}_0 = \dot{\tilde{C}}_F \quad \text{with} \quad \tilde{C}_F(t_0) = \boldsymbol{C}_F(\boldsymbol{u}_0, \tilde{\boldsymbol{H}}(t_0)) : \boldsymbol{A}_0$$

## Boundary conditions

$$[\boldsymbol{F}\boldsymbol{S} + \boldsymbol{F}_F\,(\boldsymbol{S}_F : \boldsymbol{A}_0)]\,\boldsymbol{N} = \boldsymbol{t} \qquad \forall t \geqslant t_0 \quad \text{on} \quad \partial_t \mathscr{B}_0$$

$$\delta_* \dot{\boldsymbol{u}} = \boldsymbol{0} \text{ and } \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ with } \boldsymbol{u}(t_0) = \bar{\boldsymbol{u}}(t_0) \quad \text{on} \quad \partial_u \mathscr{B}_0$$

## Equations of motion

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{with} \quad \boldsymbol{u}(t_0) = \boldsymbol{u}_0$$

$$\mathrm{DIV}[\boldsymbol{F}\boldsymbol{S} + \boldsymbol{F}_F\,(\boldsymbol{S}_F : \boldsymbol{A}_0)] + \rho_0 \boldsymbol{b} = \dot{\boldsymbol{p}} \quad \text{with} \quad \boldsymbol{p}(t_0) = \boldsymbol{p}_0 \equiv \rho_0 \boldsymbol{v}_0$$

## Time evolution characteristics

1. continuous time evolutions of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{p}$ as well as $\tilde{\boldsymbol{C}}$, $\tilde{C}_F$, $\tilde{\boldsymbol{H}} = \boldsymbol{0}$
2. discontinuous time evolution of the stresses $\boldsymbol{S}$, $\boldsymbol{S}_F$ and $\tilde{\boldsymbol{S}} = \boldsymbol{O}$, $\tilde{S}_F = O$

# Discrete variational principle (2nd-order accurate)

## Discrete principle of virtual power at $\xi_1 = 0.5$

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_1), \dot{\boldsymbol{v}}_h^n(\xi_1), \dot{\boldsymbol{p}}_h^n(\xi_1), \dot{\tilde{\boldsymbol{H}}}(\xi_1), \dot{\tilde{\boldsymbol{C}}}_h^n(\xi_1), \dot{\tilde{C}}_{F_h}^n(\xi_1), \boldsymbol{S}_h^n(\xi_1), \boldsymbol{S}_{F_h}^n(\xi_1); \tilde{\boldsymbol{S}}_h^n(\xi_1), \tilde{S}_{F_h}^n(\xi_1))\, h_n =$$

$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}_d(\boldsymbol{u}_{n+1}, \boldsymbol{v}_{n+1}, \boldsymbol{p}_{n+1}, \boldsymbol{H}_{n+1}, \boldsymbol{C}_{n+1}, C_{F_{n+1}}, \boldsymbol{S}_{n+\frac{1}{2}}, \boldsymbol{S}_{F_{n+\frac{1}{2}}}; \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}, \tilde{S}_{F_{n+\frac{1}{2}}})\, h_n = 0$$

## Galerkin approximations (with shape functions $M_1 = 1-\alpha,\ M_2 = \alpha$)

$$\boldsymbol{u}_h^n(\alpha) := \boldsymbol{u}_n + \alpha\,(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) \qquad \boldsymbol{v}_h^n(\alpha) := \boldsymbol{v}_n + \alpha\,(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) \qquad \boldsymbol{p}_h^n(\alpha) := \boldsymbol{p}_n + \alpha\,(\boldsymbol{p}_{n+1} - \boldsymbol{p}_n)$$

$$\tilde{\boldsymbol{H}}_h^n(\alpha) := \boldsymbol{H}_n + \alpha\,(\boldsymbol{H}_{n+1} - \boldsymbol{H}_n) \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) := \boldsymbol{C}_n + \alpha\,(\boldsymbol{C}_{n+1} - \boldsymbol{C}_n) \qquad \tilde{C}_{F_h}^n(\alpha) := C_{F_n} + \alpha\,(C_{F_{n+1}} - C_{F_n})$$

## Semidiscrete variational forms

$$\int_{\mathscr{B}_0} \Big[\boldsymbol{S}_{n+\frac{1}{2}} - 2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) - \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\Big] : \delta\boldsymbol{C}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \Big[\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0 - 2\,\mathrm{D}W_F(\tilde{C}_{F n+\frac{1}{2}}) - \tilde{S}_{F_{n+\frac{1}{2}}}\Big] : \delta C_{F_{n+1}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \Big[\rho_0 \boldsymbol{v}_{n+\frac{1}{2}} - \boldsymbol{p}_{n+\frac{1}{2}}\Big] \cdot \delta\boldsymbol{v}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \Big[(C_{F_{n+1}} - C_{F_n})\,\boldsymbol{A}_0 - \Big(\boldsymbol{F}_{F_{n+1}}^T + \boldsymbol{F}_{F_n}^T\Big)\,(\boldsymbol{F}_{F_{n+1}} - \boldsymbol{F}_{F_n})\Big] : \delta\boldsymbol{S}_{F_{n+\frac{1}{2}}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \Big[\boldsymbol{v}_{n+\frac{1}{2}} - \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{h_n}\Big] \cdot \delta\boldsymbol{p}_{n+1}\,\mathrm{d}V = 0 = \int_{\mathscr{B}_0} \Big[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n - \Big(\boldsymbol{F}_{n+1}^T + \boldsymbol{F}_n^T\Big)\,(\boldsymbol{F}_{n+1} - \boldsymbol{F}_n)\Big] : \delta\boldsymbol{S}_{n+\frac{1}{2}}\,\mathrm{d}V$$

$$\int_{\mathscr{B}_0} \delta\boldsymbol{h}_{n+1} \cdot [\boldsymbol{u}_{n+1} - \bar{\boldsymbol{u}}_{n+1}]\,\mathrm{d}A = 0 = \int_{\mathscr{B}_0} \boldsymbol{F}_{n+\frac{1}{2}} \Big[\boldsymbol{S}_{n+\frac{1}{2}} + \Big(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\Big)\,\boldsymbol{A}_0\Big] : \delta\boldsymbol{H}_{n+1}\,\mathrm{d}V$$

$$\int_{\partial_t\mathscr{B}_0} \boldsymbol{t}_{n+\frac{1}{2}} \cdot \delta\boldsymbol{u}_{n+1}\,\mathrm{d}A + \int_{\partial_u\mathscr{B}_0} \boldsymbol{h}_{n+\frac{1}{2}} \cdot \delta\boldsymbol{u}_{n+1}\,\mathrm{d}A$$

$$= \int_{\mathscr{B}_0} \Big\{\frac{\boldsymbol{p}_{n+1} - \boldsymbol{p}_n}{h_n} + \boldsymbol{B}_{n+\frac{1}{2}}^T \Big[\boldsymbol{S}_{n+\frac{1}{2}} + \Big(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\Big)\,\boldsymbol{A}_0\Big] - \rho_0 \boldsymbol{b}_{n+\frac{1}{2}}\Big\} \cdot \delta\boldsymbol{u}_{n+1}\,\mathrm{d}V$$

# Discrete Euler-Lagrange equations (2nd-order accurate)

## Initial conditions

$$\boldsymbol{u}(t_0) = \boldsymbol{u}_0 \qquad \boldsymbol{p}(t_0) = \rho_0 \boldsymbol{v}_0 \qquad \boldsymbol{v}(t_0) = \boldsymbol{v}_0 \qquad \tilde{\boldsymbol{C}}(t_0) = (\nabla \boldsymbol{u}_0 + \boldsymbol{I})^T (\nabla \boldsymbol{u}_0 + \boldsymbol{I}) \qquad \tilde{\boldsymbol{H}}(t_0) = \boldsymbol{O}$$

## Discrete strong forms

(compare Betsch & Janz [2016])

$$\rho_0 \left[\boldsymbol{v}_n + \boldsymbol{v}_{n+1}\right] = \boldsymbol{p}_n + \boldsymbol{p}_{n+1} \qquad h_n \left[\boldsymbol{v}_n + \boldsymbol{v}_{n+1}\right] = 2\left(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n\right)$$

$$2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) + \tilde{\boldsymbol{S}}_{n+\frac{1}{2}} = \boldsymbol{S}_{n+\frac{1}{2}} \qquad \boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0 = 2\,\mathrm{D}W_F(\tilde{C}_{F_{n+\frac{1}{2}}}) + \tilde{S}_{F_{n+\frac{1}{2}}}$$

$$2\,\mathbb{I}^{\mathrm{sym}} : \boldsymbol{F}^T_{n+\frac{1}{2}} \left(\boldsymbol{F}_{n+1} - \boldsymbol{F}_n\right) = \boldsymbol{C}_{n+1} - \boldsymbol{C}_n \qquad \left(C_{F_{n+1}} - C_{F_n}\right) \boldsymbol{A}_0 = 2\,\mathbb{I}^{\mathrm{sym}} : \boldsymbol{F}^T_{F_{n+\frac{1}{2}}} \left(\boldsymbol{F}_{F_{n+1}} - \boldsymbol{F}_{F_n}\right)$$

## Discrete weak forms

(compare Simo, Armero & Taylor [1993]
Müller & Betsch [2007])

$$\frac{2}{h_n} \mathbf{M} \left[\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h_n} - \mathbf{v}_n\right] + \int_{\mathscr{B}_0} \mathbf{B}^T_{n+\frac{1}{2}} \left[\mathbf{S}_{n+\frac{1}{2}} + \left(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\right) \mathbf{A}_0\right] \mathrm{d}V = \mathbf{H}_t\, \mathbf{t}_{n+\frac{1}{2}} + \mathbf{H}_u\, \mathbf{h}_{n+\frac{1}{2}} + \mathbf{M}\, \mathbf{b}_{n+\frac{1}{2}}$$

$$\mathbf{M} := \int_{\mathscr{B}_0} \rho_0\, \mathbf{N}^T \mathbf{N}\, \mathrm{d}V \qquad \mathbf{H}_t := \int_{\partial_t \mathscr{B}_0} \bar{\mathbf{N}}^T \bar{\mathbf{N}}\, \mathrm{d}V \qquad \mathbf{H}_u := \int_{\partial_u \mathscr{B}_0} \bar{\mathbf{N}}^T \bar{\mathbf{N}}\, \mathrm{d}V$$

$$\boldsymbol{B}_{n+\frac{1}{2}} \left[\boldsymbol{u}_{n+1} - \boldsymbol{u}_n\right] := \mathbb{I}^{\mathrm{sym}} : \boldsymbol{F}^T_{n+\frac{1}{2}} \left[\nabla \boldsymbol{u}_{n+1} - \nabla \boldsymbol{u}_n\right] \qquad \boldsymbol{G}_{n+\frac{1}{2}} \left[\boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n\right] := \mathbb{I}^{\mathrm{sym}} : \boldsymbol{F}^T_{n+\frac{1}{2}} \left[\boldsymbol{H}_{n+1} - \boldsymbol{H}_n\right]$$

$$\int_{\mathscr{B}_0} \mathbf{G}^T_{n+\frac{1}{2}} \left[\mathbf{S}_{n+\frac{1}{2}} + \left(\boldsymbol{S}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\right) \mathbf{A}_0\right] \mathrm{d}V = \mathbf{0} \qquad \boldsymbol{H}_{n+1} = \sum_{A=1}^{n_{\mathrm{dim}}} \boldsymbol{\alpha}^A_{n+1} \otimes \tilde{\nabla} \tilde{N}_A$$

## Discrete total energy balance

$$\frac{\mathcal{T}_{n+1} - \mathcal{T}_n}{h_n} + \frac{\Pi^{\mathrm{ext}}_{n+1} + \Pi^{\mathrm{ext}}_n}{h_n} = -\frac{1}{2} \int_{\mathscr{B}_0} \left[\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} + \left(\tilde{\boldsymbol{S}}_{F_{n+\frac{1}{2}}} : \boldsymbol{A}_0\right) \boldsymbol{A}_0\right] : \underbrace{2\left[\boldsymbol{B}_{n+\frac{1}{2}} \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{h_n} + \boldsymbol{G}_{n+\frac{1}{2}} \frac{\boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n}{h_n}\right]}_{(\boldsymbol{C}_{n+1} - \boldsymbol{C}_n)/h_n} \mathrm{d}V$$

Variational-based higher-order energy-momentum schemes with incompatible modes for fiber-reinforced materials

Michael Groß & Julian Dietzsch

Introduction
Variational setting
Continuum model
Incompatible mode method
Total energy balance
Variational principle
Euler-Lagrange equations
Discrete setting
Discrete variation
Discrete E-L-equations
Superimposed stress
Higher-order approx.
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Conclusions

# Discrete superimposed fiber stress (2nd-order accurate)

## Algorithmic claim for the fiber

(one equation for 1 unknown)

$$\frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\mathrm{D}W_F(\tilde{C}_{F_{n+\frac{1}{2}}}) + \tilde{S}_{F_{n+\frac{1}{2}}}\Big]\underbrace{\boldsymbol{A}_0 : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]}_{C_{F_{n+1}} - C_{F_n}}\,\mathrm{d}V = \int_{\mathscr{B}_0}\big[W_F(C_{F_{n+1}}) - W_F(C_{F_n})\big]\,\mathrm{d}V$$

## Local constraint

(compare G., Betsch & Steinmann [2005])

$$\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) := W_{F_{n+1}} - W_{F_n} - \frac{1}{2}\Big[2\,\mathrm{D}W_F(\tilde{C}_{n+\frac{1}{2}}) + \tilde{S}_{F_{n+\frac{1}{2}}}\Big]\big[C_{F_{n+1}} - C_{F_n}\big] = 0$$

## Constrained variational problem

(compare Gauss' principle in Ramm [2011])

$$\mathcal{L}(\mu, \tilde{S}_{F_{n+\frac{1}{2}}}) := \frac{1}{2}\Big(\tilde{S}_{F_{n+\frac{1}{2}}}\Big)^2 + \mu\,\mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) \qquad \delta_*\mathcal{L}(\mu, \tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

## Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{S}_{F_{n+\frac{1}{2}}}} \equiv \tilde{S}_{F_{n+\frac{1}{2}}} - \frac{\mu}{2}\big[C_{F_{n+1}} - C_{F_n}\big] = 0 \qquad \frac{\partial\mathcal{L}}{\partial\mu} \equiv \mathcal{G}(\tilde{S}_{F_{n+\frac{1}{2}}}) = 0$$

## Discrete superimposed fiber stress

Gonzalez [2000]

$$\tilde{S}_{F_{n+\frac{1}{2}}} = 2\,\frac{\mathcal{G}(O)}{\big[C_{F_{n+1}} - C_{F_n}\big]\,\cancel{\big[C_{F_{n+1}} - C_{F_n}\big]}}\,\cancel{\big[C_{F_{n+1}} - C_{F_n}\big]}$$

# Discrete superimposed matrix stress (2nd-order accurate)

## Algorithmic claim for the matrix (one equation for 6 unknowns)

$$\frac{1}{2}\int_{\mathscr{B}_0}\left[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) + \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\mathrm{d}V = \int_{\mathscr{B}_0}[W(\boldsymbol{C}_{n+1}) - W(\boldsymbol{C}_n)]\,\mathrm{d}V$$

## Local constraint

$$\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := W_{n+1} - W_n - \frac{1}{2}\left[2\,\mathrm{D}W(\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}) + \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\right] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] = 0$$

## Constrained variational problem

$$\mathcal{L}(\mu, \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) := \frac{1}{2}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} : \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} + \mu\,\mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) \qquad \delta_*\mathcal{L}(\mu, \tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

## Discrete Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}} \equiv \tilde{\boldsymbol{C}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}\,\tilde{\boldsymbol{C}}_{n+\frac{1}{2}} - \frac{\mu}{2}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] = \boldsymbol{O} \qquad \frac{\partial\mathcal{L}}{\partial\mu} \equiv \mathcal{G}(\tilde{\boldsymbol{S}}_{n+\frac{1}{2}}) = 0$$

## Discrete superimposed stress tensor Armero & Zambrana-Rojas [2007]

$$\tilde{\boldsymbol{S}}_{n+\frac{1}{2}} = 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n] : [\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}[\boldsymbol{C}_{n+1} - \boldsymbol{C}_n]\,\tilde{\boldsymbol{C}}^{-1}_{n+\frac{1}{2}}$$

# Higher-order accurate time approximation

## Discrete variation principle

$$\sum_{n=0}^{N-1}\sum_{i=1}^{k}\delta_*\dot{\mathcal{H}}(\dot{\boldsymbol{u}}_h^n(\xi_i),\dot{\boldsymbol{v}}_h^n(\xi_i),\dot{\boldsymbol{p}}_h^n(\xi_i),\dot{\tilde{\boldsymbol{H}}}(\xi_i),\dot{\tilde{\boldsymbol{C}}}_h^n(\xi_i),\dot{\tilde{C}}_{F_h}^n(\xi_i),\boldsymbol{S}_h^n(\xi_i),\boldsymbol{S}_{F_h}^n(\xi_i);\tilde{\boldsymbol{S}}_h^n(\xi_i),\tilde{S}_{F_h}^n(\xi_i))\,w_i\,h_n=0$$

## Galerkin-based approximations (Lagrange polynomials $M_j$)

$$\boldsymbol{u}_h^n(\alpha):=\sum_{j=1}^{k+1}M_j(\alpha)\,\boldsymbol{u}_j^n \qquad \boldsymbol{v}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{v}_j^n \qquad \boldsymbol{p}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{p}_j^n$$

$$\tilde{\boldsymbol{H}}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{H}_j^n \qquad \tilde{\boldsymbol{C}}_h^n(\alpha):=\sum M_j(\alpha)\,\boldsymbol{C}_j^n \qquad \tilde{C}_{F_h}^n(\alpha):=\sum M_j(\alpha)\,C_{F_j}^n$$

## Discrete weak assumed strain equation

$$\sum_{i=1}^{k}\int_{\mathcal{B}_0}\delta_*\boldsymbol{S}_h^n(\xi_i):\left[\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha}-\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))\right]w_i\,\mathrm{d}V=0 \qquad i=1,\dots,k$$

## Discrete local assumed strain equation (Euler-Lagrange equation)

$$\frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\xi_i)}{\mathrm{d}\alpha}-\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))=\boldsymbol{O} \qquad i=1,\dots,k \qquad \frac{\mathrm{d}\tilde{\boldsymbol{C}}_h^n(\alpha)}{\mathrm{d}\alpha}=\sum_{j=1}^{k+1}\mathring{M}_j(\alpha)\,\boldsymbol{C}_j^n\equiv\sum_{i=1}^{k}\tilde{M}_i(\alpha)\,\tilde{\boldsymbol{C}}_i^n$$

## Unknown nodal values $C_l^n$, $l=2,\dots,k$

$$\boldsymbol{C}_l^n:=\sum_{i=1}^{k}m_{li}\,\mathring{\boldsymbol{C}}(\mathring{\boldsymbol{u}}_h^n(\xi_i))+\boldsymbol{C}_1^n \qquad \text{with} \qquad \boldsymbol{m}=\begin{bmatrix}\mathring{M}_2(\xi_1) & \dots & \mathring{M}_{k+1}(\xi_1)\\ \vdots & \dots & \vdots\\ \mathring{M}_2(\xi_k) & \dots & \mathring{M}_{k+1}(\xi_k)\end{bmatrix}^{-1}$$

# Higher-order accurate stress approximation

## 2nd-order (Simo & Tarnow [1992])

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_2^n := (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

## 4th-order accurate (cp. G., Betsch & Steinmann [2005])

$$\boldsymbol{C}_n \equiv \boldsymbol{C}_1^n := (\boldsymbol{F}_1^n)^T \boldsymbol{F}_1^n$$

$$\boldsymbol{C}_2^n := \frac{1}{3}\left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right]^T \left[\frac{\boldsymbol{F}_1^n + \boldsymbol{F}_3^n}{2} - \boldsymbol{F}_2^n\right] + (\boldsymbol{F}_2^n)^T \boldsymbol{F}_2^n$$

$$\boldsymbol{C}_{n+1} \equiv \boldsymbol{C}_3^n := (\boldsymbol{F}_3^n)^T \boldsymbol{F}_3^n$$

## Old superimposed stress with mixed 'strain' approximation in time

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l}\ \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)$$

(energy consistent, **but not** variationally consistent approximation)

G., Betsch & Steinmann [2005]

with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\boldsymbol{C}}(\overset{\circ}{\boldsymbol{u}}{}_h^n)(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, [\boldsymbol{F}_j^n]^T \boldsymbol{F}_j^n$$

## New superimposed stress with uniform 'strain' approximation in $t$

$$\tilde{\boldsymbol{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\boldsymbol{O})}{\sum_{l=1}^{k} [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l) : \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l)\, [\tilde{\boldsymbol{C}}_h^n(\xi_l)]^{-1}\, w_l}\ [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}\, \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_i)\, [\tilde{\boldsymbol{C}}_h^n(\xi_i)]^{-1}$$

with

$$\mathcal{G}(\boldsymbol{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \frac{\partial W(\tilde{\boldsymbol{C}}_h^n(\xi_l); \boldsymbol{A}_0, \boldsymbol{\kappa}_0)}{\partial \tilde{\boldsymbol{C}}_h^n(\xi_l)} : \overset{\circ}{\tilde{\boldsymbol{C}}}{}_h^n(\xi_l)\, w_l = 0 \qquad \tilde{\boldsymbol{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\, \boldsymbol{C}_j^n$$

# Free rotating pipe with inner taper (H8E9-element) (energy-momentum consistency; 2nd-order accurate in time)





## Pipe material

Isotropic strain energy functions

$$W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\mathrm{vol}}(\boldsymbol{C}) := \frac{\kappa^{\mathrm{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\mathrm{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\mathrm{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\mathrm{iso}}))^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\mathrm{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

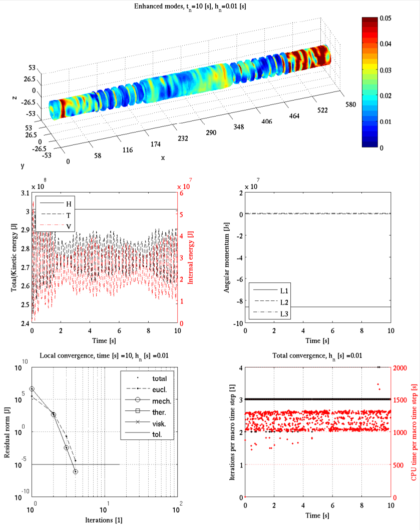
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -7\,\boldsymbol{e}_1$$

no external forces

## Conservation properties at $T = 10$

Colours indicate norm of incompatible modes ($h_n = 0.01$)



# Rotating ring with pressure load (H8E9-element)
## (influence of reinforcing fibers; 2nd-order accurate in time)



### Ring material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

### Initial conditions

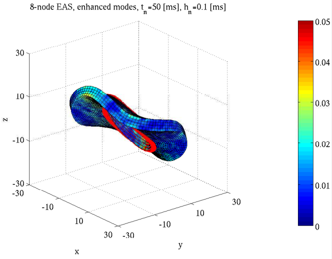
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

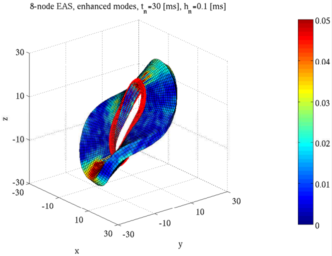
$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1$$

internal pressure p=120

### Current configurations at $T = 0 \ldots 0.05$

Colours indicate norm of incompatible modes ($h_n = 0.0001$)



# Rotating ring with pressure (I) (H20/H20E9-element)
# (influence of incompatible modes; 2nd-order accurate in time)

## Ring material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\mathrm{iso}}(\boldsymbol{C}^{\mathrm{iso}}) := \; & \mu_{10}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right] \\
& + \mu_{20}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
& + \mu_{30}\left[I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
& + \mu_{01}\left[I_2(\boldsymbol{C}^{\mathrm{iso}}) - I_2(\boldsymbol{I})\right] \\
& + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\mathrm{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\mathrm{vol}}(\boldsymbol{C}) := \frac{\kappa^{\mathrm{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\mathrm{ani}}(\boldsymbol{C}^{\mathrm{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\mathrm{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\mathrm{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\mathrm{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

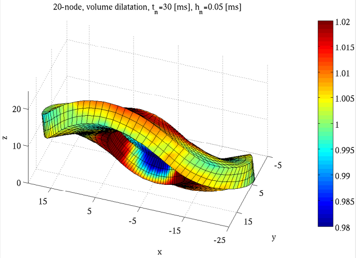
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

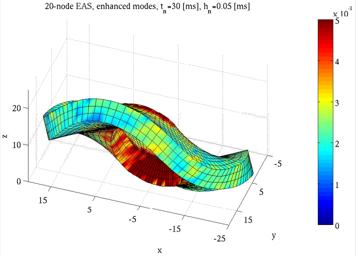
$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1
$$

outer pressure p=1000

## Current configurations at $T = 0.03$

Colours indicate volume dilatation or incompatible modes



# Rotating ring with pressure (II) (H20E9-element)

## (energy consistency; 2nd-order accurate in time)



## Ring material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := {} & \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
& + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
& + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
& + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
& + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

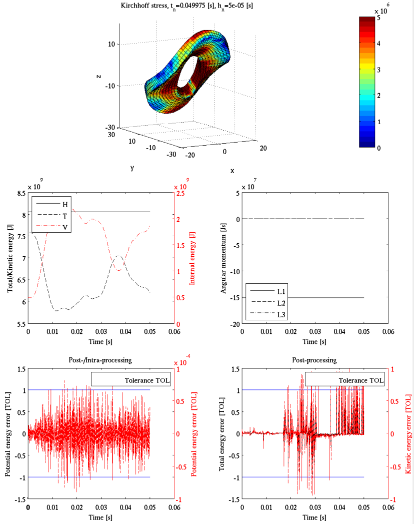
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1$$

outer pressure p=1000

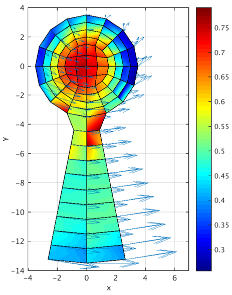
## Conservation properties at $T = 0.05$

Colours indicate von-Mises stress ($h_n = 5 \cdot 10^{-5}$)



# Free flying blade (recall of properties) (H8-element)
# (temporal convergence; 2nd-order/4th-order accurate in time)

## Initial configuration




## Blade material (Erler & G. [2015])

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}) := \frac{c_1}{2}\left[I_3(\boldsymbol{C})^{-1/3} I_1(\boldsymbol{C}) - I_1(\boldsymbol{I})\right]$$

$$W^{\text{vol}}(\boldsymbol{C}) := c_2\left[I_3(\boldsymbol{C}) - 1\right]^2$$

Hartmann & Neff [2003]

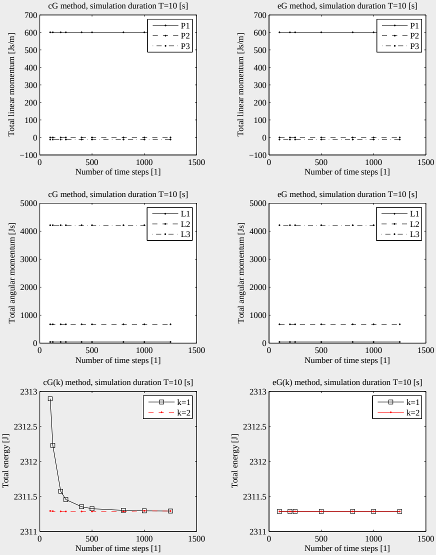
Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{c_3}{2\,c_4}\exp\left[c_4\left(I_3(\boldsymbol{C})^{-1/3} I_4(\boldsymbol{C}) - 1\right)^2 - 1\right]$$

$$\boldsymbol{a}_0 := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

Schröder, Neff & Balzani [2005]

## Conservation properties vs. time steps



# Cook-like cube under pressure (I) (H27E9-element)
# (loaded tumbling motion; 2th-order accurate in time)

## Cube material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) \;:=\;& \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
&+ \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
&+ \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
&+ \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
&+ \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) \;:=\; \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) \;:=\; \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\,\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
$$

Al-Kinani, Hartmann & Netz [2014]

$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) \;:=\; \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 \;:=\; \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

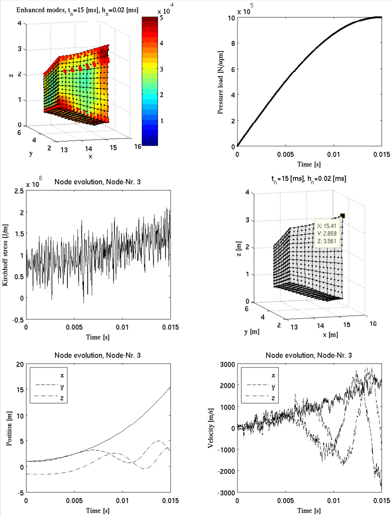
$$
\boldsymbol{u}_0^A = \boldsymbol{0} \qquad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1
$$

outer pressure $p = 10^6$

## Time evolutions to $T = 0.15$

Colours indicate incompatible modes $(h_n = 2 \cdot 10^{-5})$



# Cook-like cube under pressure (II) (H27E9-element) (loaded tumbling motion; 2th-order accurate in time)

## Cube material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c + 1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix}\sin\phi \\ \cos\phi \\ 0\end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

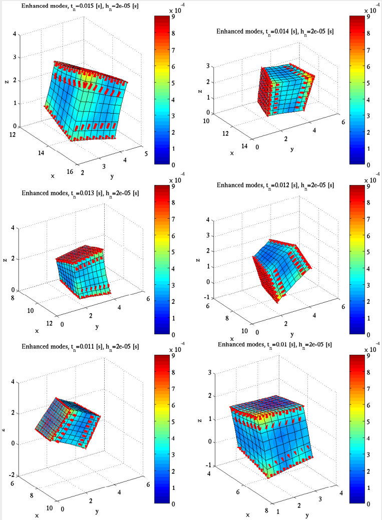
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1$$

outer pressure $p = 10^6$

## Current configurations $(T = 0.1 \ldots 0.15)$

Colours indicate norm of the incompatible modes $(h_n = 2 \cdot 10^{-5})$



# Cook-like cube (III) (H8/H8E9/H20/H20E9/H27/H27E9-element) (spatial convergence; 2th-order accurate in time)





## Cube material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\,[I_4(\boldsymbol{C})]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

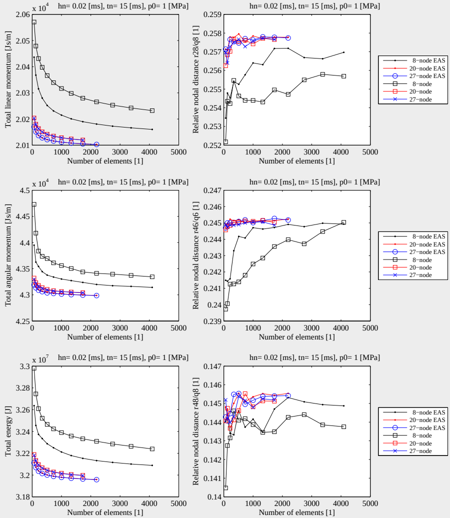
$$\boldsymbol{u}_0^A = \boldsymbol{0} \qquad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -100\,\boldsymbol{e}_1$$

outer pressure $p = 10^6$

## Convergence of kinetics and kinematics

Simulation to $T = 0.15$ with sinusodial pressure $(h_n = 2 \cdot 10^{-5})$



# Simple cube under pressure (I) (H27E9-element)
## (rotation with compression; 2th-order accurate in time)



## Cube material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \;& \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
&+ \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
&+ \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
&+ \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
&+ \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
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Al-Kinani, Hartmann & Netz [2014]

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## Initial conditions

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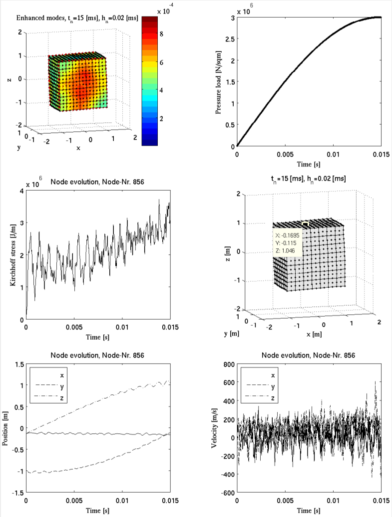
$$

$$
\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -100\,\boldsymbol{e}_1
$$

outer pressure $p = 3 \cdot 10^6$

## Time evolutions to $T = 0.15$

Colours indicate incompatible modes $(h_n = 2 \cdot 10^{-5})$



# Simple cube under pressure (II) (H27E9-element)
# (rotation with compression; 2th-order accurate in time)

## Cube material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := {} & \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
& + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
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& + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
& + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

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W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
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Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

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$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]
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$$
\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

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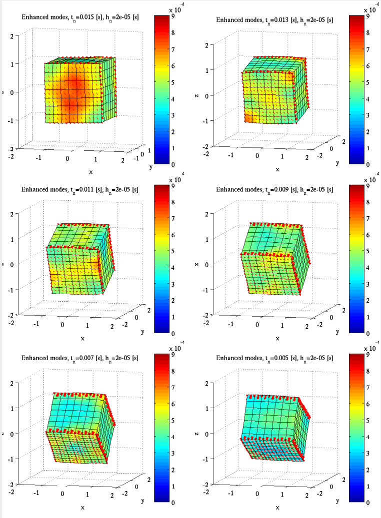
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

$$
\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -100\, \boldsymbol{e}_1
$$

outer pressure $p = 3 \cdot 10^6$

## Current configurations $(T = 0.005 \ldots 0.15)$

Colours indicate norm of the incompatible modes $(h_n = 2 \cdot 10^{-5})$



# Simple cube (III) (H8/H8E9/H20/H20E9/H27/H27E9-element)

# (spatial convergence; 2th-order accurate in time)

## Cube material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}})-I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}})-I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}})-I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}})-I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1-\exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}})-I_1(\boldsymbol{I})\right)\right]\right\}$$

$$W^{\text{vol}}(\boldsymbol{C}) := \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2}+(I_3(\boldsymbol{C}))^{-5/2}-2\right]$$

Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

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$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}},\boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2}+3\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2}-4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C},\boldsymbol{A}_0) := \frac{g_0}{g_c+1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix}\sin\phi\\ \cos\phi\\ 0\end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

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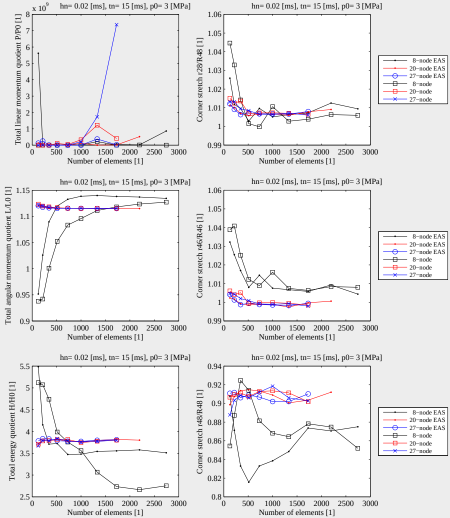
$$\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -100\,\boldsymbol{e}_1$$

outer pressure $p = 3\cdot 10^6$

## Convergence of kinetics and kinematics

Simulation to $T = 0.15$ with sinusodial pressure $(h_n = 2\cdot 10^{-5})$



# Rotating pipe with inner taper (I) (H8-element)

## (higher-order accuracy in time; 2nd-/4th-/6th-/8th-order)

## Pipe material

Isotropic strain energy functions

$$
\begin{aligned}
W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) \; := \; & \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] \\
& + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 \\
& + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 \\
& + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] \\
& + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}
\end{aligned}
$$

$$
W^{\text{vol}}(\boldsymbol{C}) \; := \; \frac{\kappa^{\text{vol}}}{50}\left[(I_3(\boldsymbol{C}))^{5/2} + (I_3(\boldsymbol{C}))^{-5/2} - 2\right]
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Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$
W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) \; := \; \frac{\kappa^{\text{ani}}}{3}\left[\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{3/2} + 3\,\left(I_4(\boldsymbol{C}^{\text{iso}})\right)^{-1/2} - 4\right]
$$

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$$
W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) \; := \; \frac{g_0}{g_c + 1}\,\left[I_4(\boldsymbol{C})\right]^{g_c+1}
$$

Schröder, Wriggers & Balzani [2011]

$$
\boldsymbol{a}_0 \; := \; \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ
$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

$$
\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A
$$

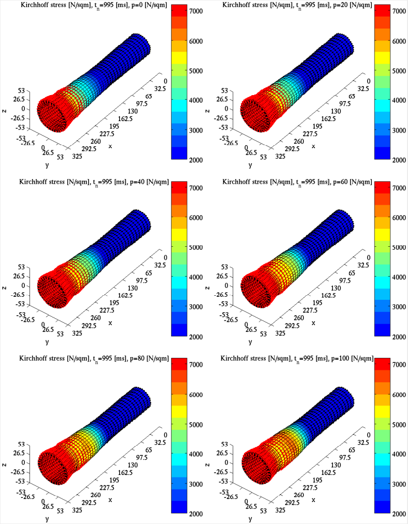
$$
\boldsymbol{\omega} = \omega_0\,\boldsymbol{e}_1 = -7\,\boldsymbol{e}_1
$$

Fluid pressure in the pipe

$$
\hat{p}(r) = p + \frac{1}{2}\,\rho_F\,\omega_0^2\,r^2
$$

## Pipe configurations at $T = 1.0$ $(h_n = 0.01)$

Colours indicate von-Mises stress and red arrows the pressure load



# Rotating pipe with inner taper (II) (H8-element)

## (higher-order accuracy in time; 2nd-/4th-/6th-/8th-order)

## Pipe material

Isotropic strain energy functions

$$W^{\text{iso}}(\boldsymbol{C}^{\text{iso}}) := \mu_{10}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right] + \mu_{20}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^2 + \mu_{30}\left[I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right]^3 + \mu_{01}\left[I_2(\boldsymbol{C}^{\text{iso}}) - I_2(\boldsymbol{I})\right] + \frac{Y_1}{Y_2}\left\{1 - \exp\left[-Y_2\left(I_1(\boldsymbol{C}^{\text{iso}}) - I_1(\boldsymbol{I})\right)\right]\right\}$$

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Yeoh [1993], Hartmann & Neff [2003], Heimes [2005]

Anisotropic strain energy functions

$$W^{\text{ani}}(\boldsymbol{C}^{\text{iso}}, \boldsymbol{A}_0) := \frac{\kappa^{\text{ani}}}{3}\left[(I_4(\boldsymbol{C}^{\text{iso}}))^{3/2} + 3\,(I_4(\boldsymbol{C}^{\text{iso}}))^{-1/2} - 4\right]$$

Al-Kinani, Hartmann & Netz [2014]

$$W^{\text{fib}}(\boldsymbol{C}, \boldsymbol{A}_0) := \frac{g_0}{g_c + 1}\left[I_4(\boldsymbol{C})\right]^{g_c+1}$$

Schröder, Wriggers & Balzani [2011]

$$\boldsymbol{a}_0 := \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{fiber angle} \quad \phi = 50^\circ, 60^\circ$$

Holzapfel, Gasser & Ogden [2000]

## Initial conditions

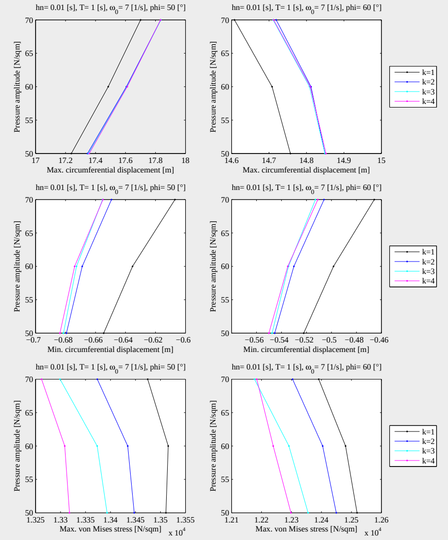
$$\boldsymbol{u}_0^A = \boldsymbol{0} \quad \boldsymbol{v}_0^A = \boldsymbol{\omega} \times \boldsymbol{X}^A$$

$$\boldsymbol{\omega} = \omega_0\, \boldsymbol{e}_1 = -7\, \boldsymbol{e}_1$$

Fluid pressure in the pipe

$$\hat{p}(r) = p + \frac{1}{2}\,\rho_F\,\omega_0^2\, r^2$$

## Displacement/stress (higher-order acc.)



# Summary and outlook

1. Motivation:
   - Dynamic simulation methods for fiber-reinforced materials
   - with more accurate time and space approximations
2. Goals:
   - Higher-order accurate energy-momentum schemes
   - with energy-momentum consistent space approximations

     (or at least with a fast spatial convergence to the energy-momentum limit)
3. Strategy:
   - Formulation of a mixed variational principle
   - Introduction of time approximations which preserve balance laws
   - and the transfer of this strategy to the space approximation

     (by reffering to the analogy between temporal and spatial convergence)
4. Intermediate results:
   - Higher-order accurate energy-momentum schemes
   - with incompatible modes up to quadratic finite elements