A new mixed finite element formulation for reorientation in liquid crystalline elastomers

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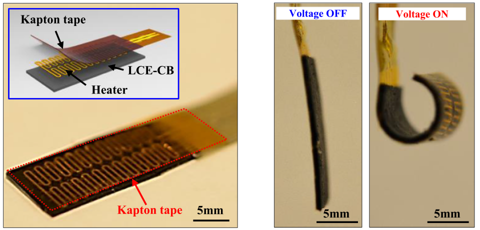
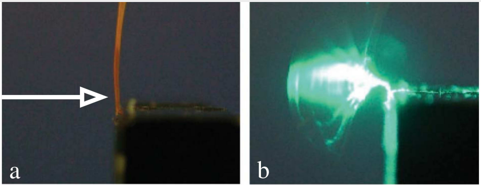
Motivation and goals

Goal 1: FE simulation of thermal actuation of motion of a LCE material



Introducing Joule heat power by heating pads, see Cui Y. et al. [2018]

Goal 2: FE simulation of UV light acuation of motion of a LCE material



Using UV light for inducing vibrations, see Corbett & Warner [2009]

Step 1: FE formulation for actuation of continuum motions by boundary or volume loads

We design a dynamic mixed FE method for continuum motions with internal reorientation

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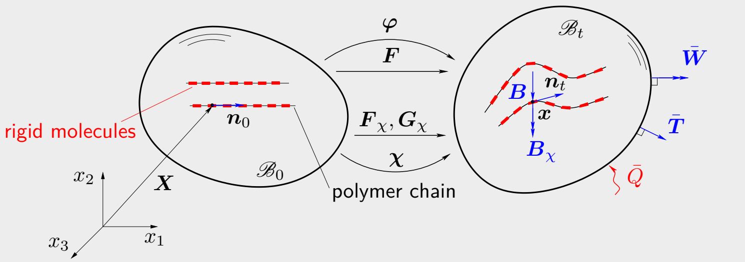
Continuum formulation with reorientation effects

(cp. Frank [1958], Leslie [1968], Anderson et al. [1999], Himpel et al. [2008], De Luca et al. [2013])

Continuum configurations of a LCE with boundary/volume loads



1. Orientation mapping

$$\boldsymbol{\chi} : \mathscr{B}_0 \times \mathscr{T} \to \mathbb{R}^{n_{\dim}} \text{ with}$$
$$\boldsymbol{\chi}(\boldsymbol{X}, 0) = \boldsymbol{n}_0(\boldsymbol{X}) \text{ and } \boldsymbol{n}_0 \cdot \boldsymbol{n}_0 = 1$$

2. Orientation tensor

$$\boldsymbol{F}_\chi := \boldsymbol{\chi} \otimes \boldsymbol{n}_0 \qquad \boldsymbol{n}_t = \boldsymbol{F}_\chi \, \boldsymbol{n}_0$$

3. Orient. deformation tensor

$$\boldsymbol{C}_\chi := \boldsymbol{F}^t \, \boldsymbol{g} \, \boldsymbol{F}_\chi = \boldsymbol{F}^t \, \boldsymbol{g}_\chi \, \boldsymbol{F}$$

4. Distorsion tensor

$$\boldsymbol{K}_\chi := \boldsymbol{F}^t \, \boldsymbol{g} \, \boldsymbol{G}_\chi = \boldsymbol{F}^t \, \boldsymbol{g}_K \, \boldsymbol{F}$$

5. Orient. velocity vector

$$\boldsymbol{v}_\chi(\boldsymbol{X}, t) := \dot{\boldsymbol{\chi}}(\boldsymbol{X}, t) = \dot{\boldsymbol{n}}_t$$

6. Orient. momentum vector

$$\boldsymbol{p}_\chi := \rho_0 \left[(l_\chi^2 - l_0^2) \boldsymbol{A}_0 + l_0^2 \, \boldsymbol{I} \right] \boldsymbol{v}_\chi$$
$$\boldsymbol{A}_0 := \boldsymbol{n}_0 \otimes \boldsymbol{n}_0$$

Free energy functions as stress potentials

(cp. Frank [1958], Leslie [1968], Warner et al. [1993], Anderson et al. [1999], Himpel et al. [2008], De Luca et al. [2013])

Free energy of thermo-orientational deformations

1. Interactive free energy

$$\Psi_i(\boldsymbol{F}^t\,\boldsymbol{g}\,\boldsymbol{\chi}, \Theta) \equiv \Psi^{\mathrm{ori}}(\boldsymbol{C}_\chi, \Theta) := \hat{\Psi}^{\mathrm{ori}}(I_1^{\mathrm{ori}}, J_2^{\mathrm{ori}}, \Theta)$$

2. Orientational invariants

$$I_1^{\mathrm{ori}} := \boldsymbol{C}_\chi \boldsymbol{A}_0 : \boldsymbol{G}^{-1} \qquad J_2^{\mathrm{ori}} := \boldsymbol{C}_\chi \boldsymbol{A}_0 : \boldsymbol{C}_\chi \boldsymbol{A}_0$$

Free energy associated with distorsions of the orientation field

1. Frank free energy

$$\Psi^{\mathrm{dis}}(\boldsymbol{K}_\chi) := \hat{\Psi}^{\mathrm{dis}}(I_1^{\mathrm{dis}}, J_2^{\mathrm{dis}})$$

2. Distorsional invariants

$$I_1^{\mathrm{dis}} := (\boldsymbol{K}_\chi - \mathrm{Grad}[\boldsymbol{n}_0]) : \boldsymbol{G}^{-1} \qquad J_2^{\mathrm{dis}} := (\boldsymbol{K}_\chi - \mathrm{Grad}[\boldsymbol{n}_0]) : (\boldsymbol{K}_\chi - \mathrm{Grad}[\boldsymbol{n}_0])$$

Free energy of thermo-elastic deformations

1. Compressible free energy

$$\Psi^{\mathrm{ela}}(\boldsymbol{C}, \Theta) := \hat{\Psi}^{\mathrm{ela}}(I_1^{\mathrm{ela}}, J_2^{\mathrm{ela}}, I_3^{\mathrm{ela}}, \Theta)$$

2. Deformation invariants

$$I_1^{\mathrm{ela}} := \boldsymbol{C} : \boldsymbol{G}^{-1} \qquad J_2^{\mathrm{ela}} := \boldsymbol{C} : \boldsymbol{C} \qquad I_3^{\mathrm{ela}} := \det[\boldsymbol{C}]$$

Heat conduction and thermomechanical coupling

(cp. Anderson et al. [1999], Holzapfel [2000], Al-Kinani et al. [2015])

Duhamel's law for transverse isotropy with reorientation

1. Cauchy heat flux vector

$$\boldsymbol{q} := -\boldsymbol{k}^{\mathrm{cdu}}\,\mathrm{grad}[\Theta] \qquad \boldsymbol{k}^{\mathrm{cdu}} := k_n\,\boldsymbol{\chi}\otimes\boldsymbol{\chi} + k_0\left(\boldsymbol{g}^{-1} - \boldsymbol{\chi}\otimes\boldsymbol{\chi}\right)$$

2. Piola transformation and orientation tensor

$$\det[\boldsymbol{F}]^{-1}\,\boldsymbol{F}\,\boldsymbol{Q} = -\boldsymbol{k}^{\mathrm{cdu}}\,\boldsymbol{F}^{-t}\,\mathrm{Grad}[\Theta] \qquad \boldsymbol{\chi} := \boldsymbol{F}_\chi\,\boldsymbol{n}_0$$

3. Piola heat flux vector

$$\boldsymbol{Q} := -\boldsymbol{K}^{\mathrm{cdu}}\,\mathrm{Grad}[\Theta] \qquad \boldsymbol{K}^{\mathrm{cdu}} := \det[\boldsymbol{F}]\left[(k_n - k_0)\,\boldsymbol{C}^{-1}\boldsymbol{C}_\chi\,\boldsymbol{A}_0\,\boldsymbol{C}_\chi^t\,\boldsymbol{C}^{-t} + k_0\,\boldsymbol{C}^{-1}\right]$$

Thermomech. coupling for transverse isotropy with reorientation

1. Coupling parameters

$$\alpha_0\,(\Theta - \Theta_\infty) := \sqrt{I_3^{\mathrm{ela}}} - 1 \qquad \beta_n\,(\Theta - \Theta_\infty) := \sqrt{J_2^{\mathrm{ori}}} - 1$$

2. Coupling free energy functions

$$\Psi_{\mathrm{the}}^{\mathrm{ela}}(\boldsymbol{C},\Theta) := -2\sqrt{I_3^{\mathrm{ela}}}\,\alpha_0\,(\Theta - \Theta_\infty)\,\frac{\partial\Psi_{\mathrm{vol}}^{\mathrm{ela}}(I_3^{\mathrm{ela}})}{\partial I_3^{\mathrm{ela}}}$$

$$\Psi_{\mathrm{the}}^{\mathrm{ori}}(\boldsymbol{C}_\chi,\Theta) := -2\sqrt{J_2^{\mathrm{ori}}}\,\beta_n\,(\Theta - \Theta_\infty)\,\frac{\partial\Psi_{\mathrm{II}}^{\mathrm{ori}}(J_2^{\mathrm{ori}})}{\partial J_2^{\mathrm{ori}}}$$

Reorientation with drilling degrees of freedom

Reorientation modelled as dissipative process (cp. Garikipati et al. [2006])

1. (Non-isothermal) Clausius-Planck inequality

$$D_\chi^{\text{int}} := \boldsymbol{N}_\chi : \boldsymbol{g}\,\dot{\boldsymbol{F}} - \dot{\Psi}^{\text{ori}}(\boldsymbol{C}_\chi, \Theta) - \eta_\chi\,\dot{\Theta} \geq 0$$

2. Normalized orientation vectors guaranteed by drilling degrees of freedom

$$\mathbb{I}^{\text{skw}} : \boldsymbol{g}\,\dot{\boldsymbol{F}}_\chi\,\boldsymbol{F}_\chi^{-1} = \boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\alpha}} \qquad \dot{\boldsymbol{\alpha}} := \dot{\alpha}^k\,\boldsymbol{g}_k \circ \boldsymbol{\varphi}(\boldsymbol{X}, t)$$

3. Reorientation dissipation

$$D_\chi^{\text{int}} := \left[\boldsymbol{N}_\chi - \boldsymbol{F}_\chi\,\boldsymbol{S}_\chi^t\right] : \boldsymbol{g}\,\dot{\boldsymbol{F}} - \left[\frac{\partial \Psi^{\text{ori}}(\boldsymbol{C}_\chi, \Theta)}{\partial \Theta} + \eta_\chi\right]\dot{\Theta} - \boldsymbol{\tau}_\chi : \boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\alpha}} \geq 0$$

4. Coleman-Noll procedure

$$\boldsymbol{N}_\chi := \boldsymbol{F}_\chi\,\boldsymbol{S}_\chi^t \qquad \boldsymbol{\tau}_\chi := \boldsymbol{F}\,\boldsymbol{S}_\chi\,\boldsymbol{F}_\chi^t \qquad \eta_\chi := -\frac{\partial \Psi^{\text{ori}}(\boldsymbol{C}_\chi, \Theta)}{\partial \Theta}$$

Reorientation equations

1. Orientational nonequilibrium stress equation (solved weakly on the element)

$$\boxed{-\frac{1}{2}\boldsymbol{\epsilon} : \boldsymbol{\tau}_\chi = \boldsymbol{\Sigma}_\chi} \qquad \boldsymbol{\Sigma}_\chi = V_\chi\,\dot{\boldsymbol{\alpha}} \qquad D_\chi^{\text{int}} := 2\,\boldsymbol{\Sigma}_\chi\cdot\dot{\boldsymbol{\alpha}} \geq 0$$

2. Global orientation equation with Dirichlet boundary conditions in general

$$\boxed{\dot{\boldsymbol{\chi}} = -\boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\alpha}}\cdot\boldsymbol{\chi}}$$

Variational-based weak formulation (I)

Principle of virtual power extended to mixed fields

1. Incremental principle of virtual power

$$\int_{t_n}^{t_{n+1}} \delta_* \dot{\mathcal{H}}(\dot{\tilde{\boldsymbol{U}}}_1, \ldots \dot{\tilde{\boldsymbol{U}}}_s, \tilde{\boldsymbol{V}}_1, \ldots \tilde{\boldsymbol{V}}_p)\,\mathrm{d}t = 0$$

2. Total virtual power of deformation φ, temperature Θ and orientation $\boldsymbol{\chi}$

$$\delta_* \dot{\mathcal{H}} := \delta_* \mathcal{P}_\varphi + \delta_* \mathcal{P}_\Theta + \delta_* \mathcal{P}_\chi \qquad \mathcal{H} := \mathcal{T} + \Pi^{\mathrm{int}} + \Pi^{\mathrm{ext}}$$

Virtual power associated with the motion (I)

1. Virtual power of motion

$$\delta_* \mathcal{P}_\varphi := \delta_* \dot{\mathcal{T}}_\varphi(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \delta_* \dot{\Pi}^{\mathrm{ext}}_\varphi(\dot{\boldsymbol{\varphi}}, \tilde{\boldsymbol{R}}) + \delta_* \dot{\Pi}^{\mathrm{int}}_\varphi(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{C}}}, \tilde{\boldsymbol{P}}, \tilde{\boldsymbol{S}})$$

2. Path-independent virtual kinetic power

$$\delta_* \dot{\mathcal{T}}_\varphi(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) := \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{v}} \cdot [\rho_0\, \boldsymbol{v} - \boldsymbol{p}]\,\mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}} \cdot [\dot{\boldsymbol{\varphi}} - \boldsymbol{v}]\,\mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \dot{\boldsymbol{p}}\,\mathrm{d}V$$

3. Path-(in)dependent virtual external power

$$\begin{aligned}\delta_* \dot{\Pi}^{\mathrm{ext}}_\varphi(\dot{\boldsymbol{\varphi}}, \tilde{\boldsymbol{R}}) := &-\int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \boldsymbol{B}\,\mathrm{d}V - \int_{\partial_T \mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \bar{\boldsymbol{T}}\,\mathrm{d}A \\ &- \int_{\partial_\varphi \mathscr{B}_0} \delta_* \tilde{\boldsymbol{R}} \cdot [\dot{\boldsymbol{\varphi}} - \dot{\bar{\varphi}}]\,\mathrm{d}A - \int_{\partial_\varphi \mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \tilde{\boldsymbol{R}}\,\mathrm{d}A\end{aligned}$$

Variational-based weak formulation (II)

Virtual power associated with the motion (II)

Path-independent virtual internal power $\delta_* \dot{\mathit{\Pi}}^{\mathrm{int}}_{\varphi}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{C}}}, \tilde{\boldsymbol{P}}, \tilde{\boldsymbol{S}}) := \delta_* \mathcal{P}^{\mathrm{int}}_{\varphi}$

$$\begin{aligned}\delta_* \mathcal{P}^{\mathrm{int}}_{\varphi} := &\int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{P}} : \left[\mathrm{Grad}[\dot{\boldsymbol{\varphi}}] - \dot{\tilde{\boldsymbol{F}}}\right] \mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{S}} : \left[\frac{\partial}{\partial t}\left(\tilde{\boldsymbol{F}}^t \tilde{\boldsymbol{F}}\right) - \dot{\tilde{\boldsymbol{C}}}\right] \mathrm{d}V \\ &+ \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{C}}} : \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}} - \frac{1}{2}\tilde{\boldsymbol{S}}\right] \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{F}}} : \left[\tilde{\boldsymbol{F}}\,\tilde{\boldsymbol{S}} - \tilde{\boldsymbol{P}}\right] \mathrm{d}V + \int_{\mathscr{B}_0} \tilde{\boldsymbol{P}} : \mathrm{Grad}[\delta_* \dot{\boldsymbol{\varphi}}]\, \mathrm{d}V\end{aligned}$$

Virtual power associated with the thermal evolution (I)

1. Virtual power of thermal evolution

$$\delta_* \mathcal{P}_{\Theta} := \delta_* \dot{\mathit{\Pi}}^{\mathrm{ext}}_{\Theta}(\dot{\Theta}, \tilde{\Theta}, \tilde{\lambda}, \tilde{h}) + \delta_* \dot{\mathit{\Pi}}^{\mathrm{int}}_{\Theta}(\dot{\Theta}, \dot{\eta}, \tilde{\Theta})$$

2. Path-dependent virtual external power

$$\begin{aligned}\delta_* \dot{\mathit{\Pi}}^{\mathrm{ext}}_{\Theta}(\dot{\Theta}, \tilde{\Theta}, \tilde{\lambda}, \tilde{h}) := &\int_{\mathscr{B}_0} \delta_* \tilde{\Theta}\, \frac{D^{\mathrm{tot}}}{\Theta}\, \mathrm{d}V + \int_{\mathscr{B}_0} \frac{1}{\Theta}\, \mathrm{Grad}[\delta_* \tilde{\Theta}] \cdot \boldsymbol{Q}\, \mathrm{d}V \\ &+ \int_{\partial_Q \mathscr{B}_0} \delta_* \tilde{\Theta}\, \frac{\bar{Q}}{\Theta}\, \mathrm{d}A + \int_{\partial_\Theta \mathscr{B}_0} \delta_* \tilde{\Theta}\, \tilde{\lambda}\, \mathrm{d}A - \int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \delta_* \dot{\Theta}\, \tilde{h}\, \mathrm{d}A \\ &+ \int_{\partial_\Theta \mathscr{B}_0} \delta_* \tilde{\lambda}\, [\tilde{\Theta} - \Theta_\infty]\, \mathrm{d}A - \int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \delta_* \tilde{h} \left[\dot{\Theta} - \dot{\bar{\Theta}}\right] \mathrm{d}A\end{aligned}$$

with

$$D^{\mathrm{tot}} := -\frac{1}{\Theta}\, \mathrm{Grad}[\tilde{\Theta}] \cdot \boldsymbol{Q} + 2\, \dot{\boldsymbol{\alpha}} \cdot \boldsymbol{\Sigma}_{\chi}$$

Variational-based weak formulation (III)

Virtual power associated with the thermal evolution (II)

Path-independent virtual internal power

$$\delta_* \dot{\Pi}_{\Theta}^{\mathrm{int}}(\dot{\Theta}, \dot{\eta}, \tilde{\Theta}) := \int_{\mathscr{B}_0} \delta_* \dot{\Theta} \left(\frac{\partial \Psi}{\partial \Theta} + \eta \right) \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\eta} \, (\Theta - \tilde{\Theta}) \, \mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \tilde{\Theta} \, \dot{\eta} \, \mathrm{d}V$$

Virtual power associated with the reorientation (I)

1. Virtual power of reorientation

$$\delta_* \mathcal{P}_{\chi} := \delta_* \dot{\mathcal{T}}_{\chi}(\dot{\boldsymbol{\chi}}, \dot{\boldsymbol{v}}_{\chi}, \dot{\boldsymbol{p}}_{\chi}) + \delta_* \mathcal{P}_{\chi}^{\mathrm{ext}} + \delta_* \mathcal{P}_{\chi}^{\mathrm{int}}$$

2. Path-independent virtual kinetic power

$$\delta_* \dot{\mathcal{T}}_{\chi}(\dot{\boldsymbol{\chi}}, \dot{\boldsymbol{v}}_{\chi}, \dot{\boldsymbol{p}}_{\chi}) := \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{v}}_{\chi} \cdot \left(\rho_0 \left[(l_{\chi}^2 - l_0^2) \boldsymbol{A}_0 + l_0^2 \, \boldsymbol{I} \right] \boldsymbol{v}_{\chi} - \boldsymbol{p}_{\chi} \right) \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{p}}_{\chi} \cdot [\dot{\boldsymbol{\chi}} - \boldsymbol{v}_{\chi}] \, \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{\chi}} \cdot \dot{\boldsymbol{p}}_{\chi} \, \mathrm{d}V$$

3. Path-dependent virtual external power

$$\delta_* \dot{\Pi}_{\chi}^{\mathrm{ext}}(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\chi}}, \tilde{\boldsymbol{Z}}, \tilde{\boldsymbol{\tau}}_n, \tilde{\boldsymbol{\nu}}) =: \delta_* \mathcal{P}_{\chi}^{\mathrm{ext}}$$

where

$$\delta_* \mathcal{P}_{\chi}^{\mathrm{ext}} := -\int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{\chi}} \cdot \boldsymbol{B}_{\chi} \, \mathrm{d}V - \int_{\partial_W \mathscr{B}_0} \delta_* \dot{\boldsymbol{\chi}} \cdot \bar{\boldsymbol{W}} \, \mathrm{d}A - \int_{\partial_{\chi} \mathscr{B}_0} \delta_* \tilde{\boldsymbol{Z}} \cdot [\dot{\boldsymbol{\chi}} - \dot{\bar{\boldsymbol{\chi}}}] \, \mathrm{d}A - \int_{\partial_{\chi} \mathscr{B}_0} \delta_* \dot{\boldsymbol{\chi}} \cdot \tilde{\boldsymbol{Z}} \, \mathrm{d}A - \int_{\partial_{\chi} \mathscr{B}_0} 2 \, \delta_* \tilde{\boldsymbol{\tau}}_n \cdot \tilde{\boldsymbol{\nu}} \, \mathrm{d}A - \int_{\partial_{\chi} \mathscr{B}_0} 2 \, \delta_* \tilde{\boldsymbol{\nu}} \cdot \tilde{\boldsymbol{\tau}}_n \, \mathrm{d}A + \int_{\mathscr{B}_0} 2 \, \delta_* \dot{\boldsymbol{\alpha}} \cdot \boldsymbol{\Sigma}_{\chi} \, \mathrm{d}V$$

Variational-based weak formulation (IV)

Virtual power associated with the reorientation (II)

Path-independent virtual internal power

$$\delta_* \dot{\Pi}^{\text{int}}_{\chi}(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\chi}}, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{F}}}_{\chi}, \dot{\tilde{\boldsymbol{G}}}_{\chi}, \dot{\tilde{\boldsymbol{C}}}_{\chi}, \dot{\tilde{\boldsymbol{K}}}_{\chi}, \tilde{\boldsymbol{\tau}}_n, \tilde{\boldsymbol{P}}_{\chi}, \tilde{\boldsymbol{P}}_K, \tilde{\boldsymbol{S}}_{\chi}, \tilde{\boldsymbol{S}}_K) := \delta_* \mathcal{P}^{\text{int}}_{\chi}$$

where

$$\begin{aligned}
\delta_* \mathcal{P}^{\text{int}}_{\chi} := & \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{F}}} : \left[\tilde{\boldsymbol{F}}_{\chi}\, \tilde{\boldsymbol{S}}^t_{\chi} + \tilde{\boldsymbol{G}}_{\chi}\, \tilde{\boldsymbol{S}}^t_K\right] \mathrm{d}V + \int_{\mathscr{B}_0} 2\, \delta_* \tilde{\boldsymbol{\tau}}_n \cdot [\dot{\boldsymbol{\chi}} + \boldsymbol{\epsilon} \cdot \dot{\boldsymbol{\alpha}} \cdot \boldsymbol{\chi}]\, \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{P}}_{\chi} : \left[\dot{\boldsymbol{\chi}} \otimes \boldsymbol{n}_0 - \dot{\tilde{\boldsymbol{F}}}_{\chi}\right] \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{P}}_K : \left[\mathrm{Grad}[\dot{\boldsymbol{\chi}}] - \dot{\tilde{\boldsymbol{G}}}_{\chi}\right] \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{S}}_{\chi} : \left[\frac{\partial}{\partial t}\left(\tilde{\boldsymbol{F}}^t \tilde{\boldsymbol{F}}_{\chi}\right) - \dot{\tilde{\boldsymbol{C}}}_{\chi}\right] \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{S}}_K : \left[\frac{\partial}{\partial t}\left(\tilde{\boldsymbol{F}}^t \tilde{\boldsymbol{G}}_{\chi}\right) - \dot{\tilde{\boldsymbol{K}}}_{\chi}\right] \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{C}}}_{\chi} : \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}_{\chi}} - \tilde{\boldsymbol{S}}_{\chi}\right] \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{K}}}_{\chi} : \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{K}}_{\chi}} - \tilde{\boldsymbol{S}}_K\right] \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{F}}}_{\chi} : [\tilde{\boldsymbol{F}}\, \tilde{\boldsymbol{S}}_{\chi} - \tilde{\boldsymbol{P}}_{\chi}]\, \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{G}}}_{\chi} : [\tilde{\boldsymbol{F}}\, \tilde{\boldsymbol{S}}_K - \tilde{\boldsymbol{P}}_K]\, \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \tilde{\boldsymbol{P}}_{\chi} : [\delta_* \dot{\boldsymbol{\chi}} \otimes \boldsymbol{n}_0]\, \mathrm{d}V + \int_{\mathscr{B}_0} \tilde{\boldsymbol{P}}_K : \mathrm{Grad}[\delta_* \dot{\boldsymbol{\chi}}]\, \mathrm{d}V + \int_{\mathscr{B}_0} \delta_* \tilde{\boldsymbol{S}}_{\chi} : \tilde{\boldsymbol{F}}^t\, (\boldsymbol{\epsilon} \cdot \boldsymbol{\alpha})\, \tilde{\boldsymbol{F}}_{\chi}\, \mathrm{d}V \\
& + \int_{\mathscr{B}_0} \left[\frac{1}{2} \boldsymbol{\epsilon} : \boldsymbol{\tau}_{\chi} - \tilde{\boldsymbol{\tau}}_n \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\chi}\right] \cdot 2\, \delta_* \dot{\boldsymbol{\alpha}}\, \mathrm{d}V + \int_{\mathscr{B}_0} 2\, \tilde{\boldsymbol{\tau}}_n \cdot \delta_* \dot{\boldsymbol{\chi}}\, \mathrm{d}V
\end{aligned}$$

Total virtual power in the incremental principle

$$\int_{\mathscr{T}_n} \left[\delta_* \mathcal{P}_{\varphi} + \delta_* \mathcal{P}_{\Theta} + \delta_* \dot{\mathcal{T}}_{\chi}(\dot{\boldsymbol{\chi}}, \dot{\boldsymbol{v}}_{\chi}, \dot{\boldsymbol{p}}_{\chi}) + \delta_* \mathcal{P}^{\text{ext}}_{\chi} + \delta_* \mathcal{P}^{\text{int}}_{\chi}\right] \mathrm{d}t = 0$$

Global weak forms of motion with reorientation

Weak balance of linear momentum

$$\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \delta_*\dot{\boldsymbol{\varphi}} \cdot [\dot{\boldsymbol{p}} - \boldsymbol{B}]\,\mathrm{d}V\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\partial_T\mathscr{B}_0} \delta_*\dot{\boldsymbol{\varphi}} \cdot \bar{\boldsymbol{T}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \mathrm{Grad}[\delta_*\dot{\boldsymbol{\varphi}}] : \tilde{\boldsymbol{P}}\,\mathrm{d}V\,\mathrm{d}t = \int_{\mathscr{T}_n}\int_{\partial_\varphi\mathscr{B}_0} \delta_*\dot{\boldsymbol{\varphi}} \cdot \tilde{\boldsymbol{R}}\,\mathrm{d}A\,\mathrm{d}t$$

Weak balance of thermal momentum

(cf. Romero [2010], Schiebl & Betsch [2021])

$$\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \delta_*\tilde{\Theta}\left[\dot{\eta} - \frac{D^{\mathrm{tot}}}{\Theta}\right]\mathrm{d}V\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\partial_Q\mathscr{B}_0} \delta_*\tilde{\Theta}\,\frac{\bar{Q}}{\Theta}\,\mathrm{d}A\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \mathrm{Grad}[\delta_*\tilde{\Theta}] \cdot \frac{1}{\Theta}\,\boldsymbol{Q}\,\mathrm{d}V\,\mathrm{d}t = \int_{\mathscr{T}_n}\int_{\partial_\Theta\mathscr{B}_0} \delta_*\tilde{\Theta}\,\tilde{\lambda}\,\mathrm{d}A\,\mathrm{d}t$$

Weak balance of orientational momentum

$$\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \delta_*\dot{\boldsymbol{\chi}} \cdot [\dot{\boldsymbol{p}}_\chi + 2\,\tilde{\boldsymbol{\tau}}_n - \boldsymbol{B}_\chi]\,\mathrm{d}V\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\partial_W\mathscr{B}_0} \bar{\boldsymbol{W}} \cdot \delta_*\dot{\boldsymbol{\chi}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \tilde{\boldsymbol{P}}_K : \mathrm{Grad}[\delta_*\dot{\boldsymbol{\chi}}]\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \tilde{\boldsymbol{P}}_\chi : [\delta_*\dot{\boldsymbol{\chi}} \otimes \boldsymbol{n}_0]\,\mathrm{d}V\,\mathrm{d}t = \int_{\mathscr{T}_n}\int_{\partial_\chi\mathscr{B}_0} \tilde{\boldsymbol{Z}} \cdot \delta_*\dot{\boldsymbol{\chi}}\,\mathrm{d}A\,\mathrm{d}t$$

Weak balance of orientation rate

$$\int_{\mathscr{T}_n}\int_{\mathscr{B}_0} 2\,\delta_*\tilde{\boldsymbol{\tau}}_n \cdot [\dot{\boldsymbol{\chi}} + \boldsymbol{\epsilon} \cdot \dot{\boldsymbol{\alpha}} \cdot \boldsymbol{\chi}]\,\mathrm{d}V\,\mathrm{d}t = \int_{\mathscr{T}_n}\int_{\partial_\chi\mathscr{B}_0} 2\,\delta_*\tilde{\boldsymbol{\tau}}_n \cdot \tilde{\boldsymbol{\nu}}\,\mathrm{d}A\,\mathrm{d}t$$

Balance laws of the weak formulation (I)

Energy and momentum functions of the LCE extended continuum

Kinetic energy	Kinetic energy of orientation	Potential energy
$\mathcal{T}(t) := \int_{\mathscr{B}_0} \frac{1}{2}\, \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}V$	$\mathcal{T}_\chi(t) := \int_{\mathscr{B}_0} \frac{1}{2}\, \boldsymbol{v}_\chi\cdot\boldsymbol{p}_\chi\,\mathrm{d}V$	$\Pi^{\mathrm{int}}(t) := \int_{\mathscr{B}_0} \Psi\,\mathrm{d}V$
Linear momentum	Angular momentum	Momentum of orientation
$\boldsymbol{L}(t) := \int_{\mathscr{B}_0} \boldsymbol{p}\,\mathrm{d}V$	$\boldsymbol{J}(t) := \int_{\mathscr{B}_0} \boldsymbol{\varphi}\times\boldsymbol{p}\,\mathrm{d}V$	$\boldsymbol{L}_\chi(t) := \int_{\mathscr{B}_0} \boldsymbol{p}_\chi\,\mathrm{d}V$
Moment of momentum	Reorientation function	Thermal energy
$\boldsymbol{J}_\chi(t) := \int_{\mathscr{B}_0} \boldsymbol{\chi}\times\boldsymbol{p}_\chi\,\mathrm{d}V$	$\mathcal{C}^{\mathrm{ori}}(t) := \int_{\mathscr{B}_0} \left[\lVert\boldsymbol{\chi}\rVert^2 - 1\right]\mathrm{d}V$	$\Pi^{\mathrm{the}}(t) := \int_{\mathscr{B}_0} \Theta\,\eta\,\mathrm{d}V$
Entropy	Total energy	Lyapunov function
$\mathcal{S}(t) := \int_{\mathscr{B}_0} \eta\,\mathrm{d}V$	$\mathcal{H} := \mathcal{T} + \mathcal{T}_\chi + \Pi^{\mathrm{int}} + \Pi^{\mathrm{the}} + \Pi^{\mathrm{ext}}$	$\mathcal{F} := \mathcal{H} - \Theta_\infty\,\mathcal{S}$

Linear momentum (symmetry of virtual translations)

$$\boldsymbol{L}(t_{n+1}) - \boldsymbol{L}(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_T\mathscr{B}_0} \bar{\boldsymbol{T}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\varphi\mathscr{B}_0} \tilde{\boldsymbol{R}}\,\mathrm{d}A\,\mathrm{d}t$$

Orientational momentum (symmetry of virtual orientations)

$$\boldsymbol{L}_\chi(t_{n+1}) - \boldsymbol{L}_\chi(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} [\boldsymbol{B}_\chi - 2\,\tilde{\boldsymbol{\tau}}_n - \tilde{\boldsymbol{P}}_\chi\,\boldsymbol{n}_0]\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_W\mathscr{B}_0} \bar{\boldsymbol{W}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\chi\mathscr{B}_0} \tilde{\boldsymbol{Z}}\,\mathrm{d}A\,\mathrm{d}t$$

Balance laws of the weak formulation (II)

Thermal momentum

(cf. Romero [2010], Schiebl & Betsch [2021])

$$\mathcal{S}(t_{n+1}) - \mathcal{S}(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \frac{D^{\text{tot}}}{\Theta}\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_Q\mathscr{B}_0} \frac{\bar{Q}}{\Theta}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\Theta\mathscr{B}_0} \tilde{\lambda}\,\mathrm{d}A\,\mathrm{d}t$$

Angular momentum

(symmetry of virtual rotations)

$$\boldsymbol{J}(t_{n+1}) - \boldsymbol{J}(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \boldsymbol{\varphi}\times\boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_T\mathscr{B}_0} \boldsymbol{\varphi}\times\bar{\boldsymbol{T}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\varphi\mathscr{B}_0} \boldsymbol{\varphi}\times\tilde{\boldsymbol{R}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} [\tilde{\boldsymbol{F}}_\chi\tilde{\boldsymbol{S}}_\chi^t + \tilde{\boldsymbol{G}}_\chi\tilde{\boldsymbol{S}}_K^t]\times\tilde{\boldsymbol{F}}\,\mathrm{d}V\,\mathrm{d}t$$

Moment of orientational momentum

(symmetry of virtual reorientations)

$$\boldsymbol{J}_\chi(t_{n+1}) - \boldsymbol{J}_\chi(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \boldsymbol{\chi}\times\boldsymbol{B}_\chi\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_W\mathscr{B}_0} \boldsymbol{\chi}\times\bar{\boldsymbol{W}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\chi\mathscr{B}_0} \boldsymbol{\chi}\times\tilde{\boldsymbol{Z}}\,\mathrm{d}A\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} [\tilde{\boldsymbol{F}}_\chi\tilde{\boldsymbol{S}}_\chi^t + \tilde{\boldsymbol{G}}_\chi\tilde{\boldsymbol{S}}_K^t]\times\tilde{\boldsymbol{F}}\,\mathrm{d}V\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \boldsymbol{\chi}\times 2\,\tilde{\boldsymbol{\tau}}_n\,\mathrm{d}V\,\mathrm{d}t$$

Kinetic energy of motion

(symmetry of virtual time shifts)

$$\mathcal{T}(t_{n+1}) - \mathcal{T}(t_n) = \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \boldsymbol{B}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_T\mathscr{B}_0} \bar{\boldsymbol{T}}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\varphi\mathscr{B}_0} \tilde{\boldsymbol{R}}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}A\,\mathrm{d}t - \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} [\tilde{\boldsymbol{S}} : \dot{\tilde{\boldsymbol{F}}}^t\tilde{\boldsymbol{F}} + \tilde{\boldsymbol{S}}_\chi : \dot{\tilde{\boldsymbol{F}}}^t\tilde{\boldsymbol{F}}_\chi + \tilde{\boldsymbol{S}}_K : \dot{\tilde{\boldsymbol{F}}}^t\tilde{\boldsymbol{G}}_\chi]\,\mathrm{d}V\,\mathrm{d}t$$

Balance laws of the weak formulation (III)

Kinetic energy of orientation (symmetry of virtual time shifts)

$$\begin{aligned}\mathcal{T}_\chi(t_{n+1}) - \mathcal{T}_\chi(t_n) = & \int_{\mathcal{T}_n}\int_{\mathcal{B}_0} \boldsymbol{B}_\chi \cdot \dot{\boldsymbol{\chi}}\,\mathrm{d}V\,\mathrm{d}t + \int_{\mathcal{T}_n}\int_{\partial_\varphi\mathcal{B}_0} \bar{\boldsymbol{W}} \cdot \dot{\boldsymbol{\chi}}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathcal{T}_n}\int_{\partial_\varphi\mathcal{B}_0} \tilde{\boldsymbol{Z}} \cdot \dot{\boldsymbol{\chi}}\,\mathrm{d}A\,\mathrm{d}t \\ & - \int_{\mathcal{T}_n}\int_{\mathcal{B}_0} [\tilde{\boldsymbol{S}}_\chi : \tilde{\boldsymbol{F}}^t \left(\dot{\tilde{\boldsymbol{F}}}_\chi + \boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\alpha}}\cdot\tilde{\boldsymbol{F}}_\chi\right) + \tilde{\boldsymbol{S}}_K : \tilde{\boldsymbol{F}}^t \dot{\tilde{\boldsymbol{G}}}_\chi + D_\chi^{\mathrm{int}}]\,\mathrm{d}V\,\mathrm{d}t\end{aligned}$$

Thermal energy (symmetry of virtual time shifts)

$$\begin{aligned}\Pi^{\mathrm{the}}(t_{n+1}) - \Pi^{\mathrm{the}}(t_n) = & \int_{\mathcal{T}_n}\int_{\mathcal{B}_0} \left[-\frac{\partial\Psi}{\partial\Theta}\,\dot{\Theta} + D_\chi^{\mathrm{int}}\right]\mathrm{d}V\,\mathrm{d}t + \int_{\mathcal{T}_n}\int_{\partial_{\dot{\Theta}}\mathcal{B}_0} \dot{\Theta}\,\tilde{h}\,\mathrm{d}A\,\mathrm{d}t \\ & + \int_{\mathcal{T}_n}\int_{\partial_\Theta\mathcal{B}_0} \Theta\,\tilde{\lambda}\,\mathrm{d}A\,\mathrm{d}t + \int_{\mathcal{T}_n}\int_{\partial_Q\mathcal{B}_0} \bar{Q}\,\mathrm{d}A\,\mathrm{d}t\end{aligned}$$

Potential energy (symmetry of virtual time shifts)

$$\begin{aligned}\Pi(t_{n+1}) - \Pi(t_n) = & \int_{\mathcal{T}_n}\int_{\mathcal{B}_0} [\tilde{\boldsymbol{S}}_\chi : \frac{\partial}{\partial t}\left(\tilde{\boldsymbol{F}}^t\tilde{\boldsymbol{F}}_\chi\right) + \tilde{\boldsymbol{S}}_K : \frac{\partial}{\partial t}\left(\tilde{\boldsymbol{F}}^t\tilde{\boldsymbol{G}}_\chi\right) + \tilde{\boldsymbol{S}}_\chi : \tilde{\boldsymbol{F}}^t(\boldsymbol{\epsilon}\cdot\dot{\boldsymbol{\alpha}})\tilde{\boldsymbol{F}}_\chi]\,\mathrm{d}V\,\mathrm{d}t \\ & + \int_{\mathcal{T}_n}\int_{\mathcal{B}_0} \left[\tilde{\boldsymbol{S}} : \dot{\tilde{\boldsymbol{F}}}^t\tilde{\boldsymbol{F}} - \boldsymbol{B}\cdot\dot{\boldsymbol{\varphi}} - \boldsymbol{B}_\chi\cdot\dot{\boldsymbol{\chi}}\right]\mathrm{d}V\,\mathrm{d}t\end{aligned}$$

Path-independent volume dead loads

$$\Pi^{\mathrm{ext}}(t) := -\int_{\mathcal{B}_0} \boldsymbol{B}\cdot\boldsymbol{\varphi}\,\mathrm{d}V\,\mathrm{d}t - \int_{\mathcal{B}_0} \boldsymbol{B}_\chi\cdot\boldsymbol{\chi}\,\mathrm{d}V$$

Balance laws of the weak formulation (IV)

Total energy

(cf. Holzapfel [2000], Romero [2010], Schiebl & Betsch [2021])

$$\begin{aligned}\mathcal{H}(t_{n+1}) - \mathcal{H}(t_n) = & \int_{\mathscr{T}_n}\int_{\partial_\varphi \mathscr{B}_0} \tilde{\boldsymbol{R}} \cdot \dot{\boldsymbol{\varphi}} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\chi \mathscr{B}_0} \tilde{\boldsymbol{Z}} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}A\,\mathrm{d}t \\ & - \int_{\mathscr{T}_n}\int_{\partial_T \mathscr{B}_0} \bar{\boldsymbol{T}} \cdot \dot{\boldsymbol{\varphi}} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_W \mathscr{B}_0} \bar{\boldsymbol{W}} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}A\,\mathrm{d}t \\ & - \int_{\mathscr{T}_n}\int_{\partial_\Theta \mathscr{B}_0} \tilde{\lambda}\,\Theta \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \tilde{h}\,\dot{\Theta} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_Q \mathscr{B}_0} \bar{Q} \,\mathrm{d}A\,\mathrm{d}t\end{aligned}$$

Lyapunov function

(cf. Romero [2010], Schiebl & Betsch [2021])

$$\begin{aligned}\mathcal{F}(t_{n+1}) - \mathcal{F}(t_n) = & \int_{\mathscr{T}_n}\int_{\partial_T \mathscr{B}_0} \bar{\boldsymbol{T}} \cdot \dot{\boldsymbol{\varphi}} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\varphi \mathscr{B}_0} \tilde{\boldsymbol{R}} \cdot \dot{\boldsymbol{\varphi}} \,\mathrm{d}A\,\mathrm{d}t \\ & + \int_{\mathscr{T}_n}\int_{\partial_W \mathscr{B}_0} \bar{\boldsymbol{W}} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_\chi \mathscr{B}_0} \tilde{\boldsymbol{Z}} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}A\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_{\dot{\Theta}} \mathscr{B}_0} \tilde{h}\,\dot{\Theta} \,\mathrm{d}A\,\mathrm{d}t \\ & - \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} \frac{\Theta_\infty}{\Theta} D^{\mathrm{tot}} \,\mathrm{d}V\,\mathrm{d}t + \int_{\mathscr{T}_n}\int_{\partial_Q \mathscr{B}_0} \frac{\Theta - \Theta_\infty}{\Theta} \bar{Q} \,\mathrm{d}A\,\mathrm{d}t\end{aligned}$$

Reorientation function

(cf. Betsch & Steinmann [2002])

$$\mathcal{C}^{\mathrm{ori}}(t_{n+1}) - \mathcal{C}^{\mathrm{ori}}(t_n) \equiv \int_{\mathscr{T}_n}\int_{\mathscr{B}_0} 2\,\boldsymbol{\chi} \cdot \dot{\boldsymbol{\chi}} \,\mathrm{d}V\,\mathrm{d}t = \int_{\mathscr{T}_n}\int_{\partial_\chi \mathscr{B}_0} 2\,\boldsymbol{\chi} \cdot \tilde{\boldsymbol{\nu}} \,\mathrm{d}A\,\mathrm{d}t$$

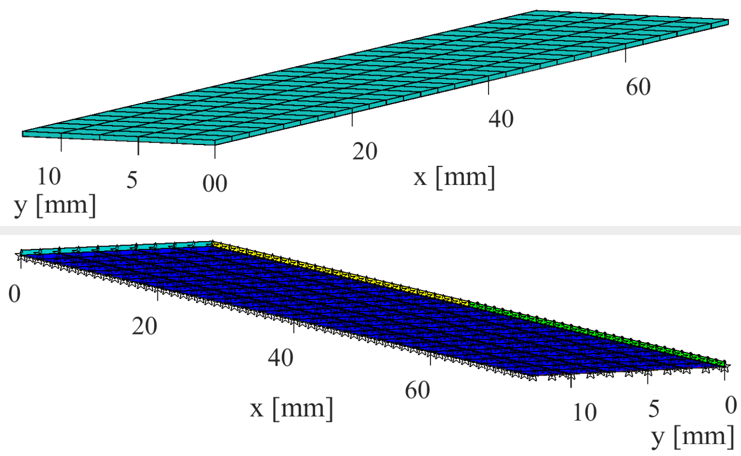
Thin LCE strip subject to initial rotation

Boundary and initial conditions

121-em with H20-mixed-Bbar

Mesh and boundary conditions

$\boldsymbol{n}_0^A = \boldsymbol{e}_y$, $\boldsymbol{\omega}_0^A = 32\,\boldsymbol{e}_z [1/\mathrm{s}]$



Activated Dirichlet and Neumann boundaries

Blue bottom as $\partial_\chi \mathcal{B}_0 \equiv \partial_\Theta \mathcal{B}_0$: Fixed orientation $n_z^A = 0$ and temperature $\Theta^A = \Theta_\infty$

Thin LCE strip subject to initial rotation

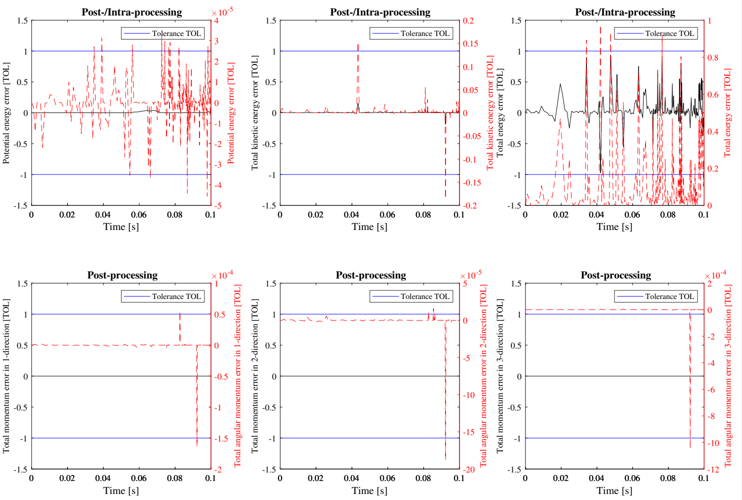
Unsteady right-left-rotation due to reorientation

Movie of a soft strip $(\mathrm{E} \approx 0.914\,[\mathrm{MPa}],\ \nu \approx 0.493)$

Thin LCE film subject to initial rotation

Unsteady right-left-rotation due to reorientation

Balance laws versus time $(E \approx 0.914\,[\mathrm{MPa}],\ \nu \approx 0.493)$



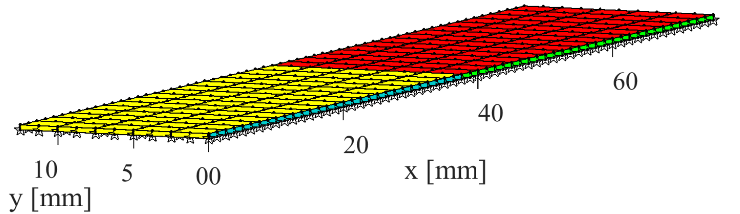
Thin LCE strip subject to temperature control

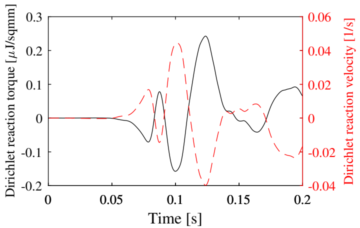
Boundary and initial conditions

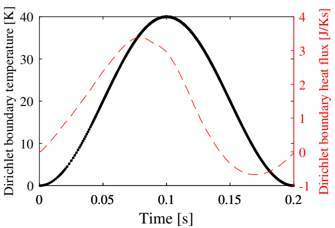
121-em with H20-mixed-Bbar

Boundary conditions and loads

$\Theta_0 = \Theta_\infty$, $\hat{\Theta} = 40\,[\mathrm{K}]$







Activated Dirichlet and Neumann boundaries

Blue bottom as $\partial_\chi \mathcal{B}_0 \equiv \partial_\Theta \mathcal{B}_0$: Fixed orientation $n_z^A = 0$ and temperature $\Theta^A = \Theta_\infty$

Yellow top as $\partial_{\dot{\Theta}} \mathcal{B}_0$: $\Theta^A = \hat{\Theta}\, f(t)$ Red top as boundary $\partial_Q \mathcal{B}_0$: thermally isolated

Thin LCE strip subject to temperature control

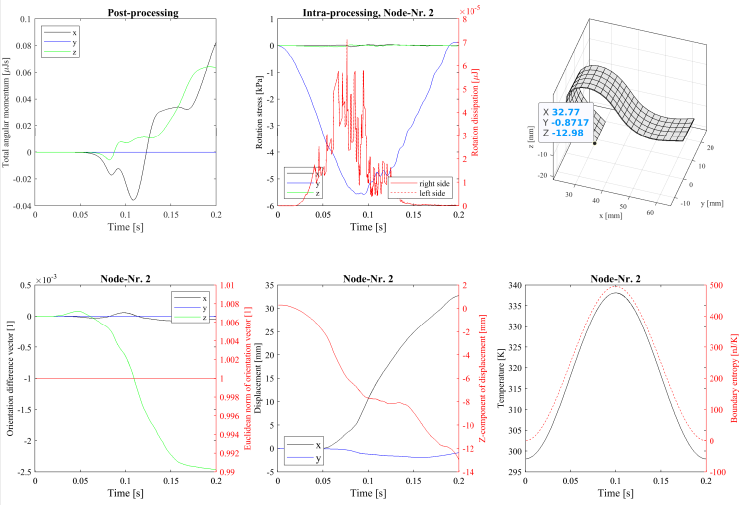
Bending motion in direction to the boundary normal of $\partial_\chi \mathcal{B}_0$

Movie of a soft strip ($\boldsymbol{n}_0 = \boldsymbol{e}_y$, $\mathrm{E} \approx 0.914\,[\mathrm{MPa}]$, $\nu \approx 0.493$)

Thin LCE strip subject to temperature control

Bending motion in direction to the boundary normal of $\partial_\chi \mathcal{B}_0$

Nodal time evolutions ($\boldsymbol{n}_0 = \boldsymbol{e}_y$, E $\approx 0.914\,[\mathrm{MPa}]$, $\nu \approx 0.493$)



Thin LCE strip subject to temperature control

Bending motion in direction to the boundary normal of $\partial_\chi \mathcal{B}_0$

Movie of a soft strip ($\boldsymbol{n}_0 = \boldsymbol{e}_x$, $\mathrm{E} \approx 0.914\,[\mathrm{MPa}]$, $\nu \approx 0.493$)

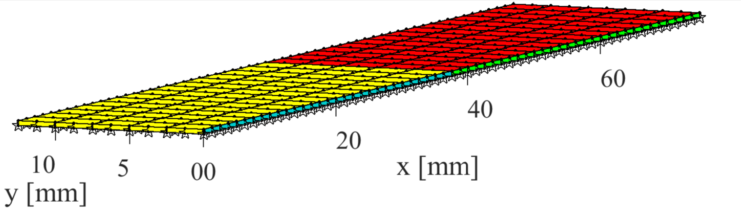
Thin LCE strip subject to transient heat flux load

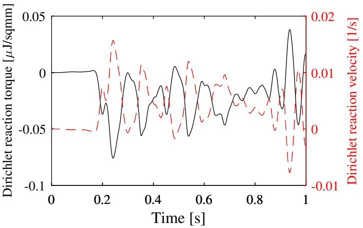
Boundary and initial conditions

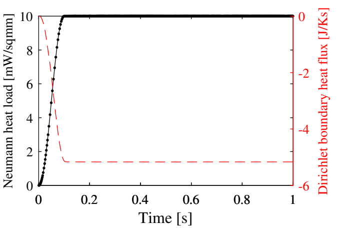
121-em with H20-mixed-Bbar

Boundary conditions and loads

$\boldsymbol{n}_0 = \boldsymbol{e}_y$, $\Theta_0 = \Theta_\infty$







Activated Dirichlet and Neumann boundaries

Blue bottom as $\partial_\chi \mathcal{B}_0 \equiv \partial_\Theta \mathcal{B}_0$: Fixed orientation $n_z^A = 0$ and temperature $\Theta^A = \Theta_\infty$

Yellow top as $\partial_{\dot{\Theta}} \mathcal{B}_0$: $\Theta^A = \Theta_0$

Red top as boundary $\partial_Q \mathcal{B}_0$: $\bar{Q} = \hat{Q}\, f_Q(t)$

Thin LCE strip subject to transient heat flux load

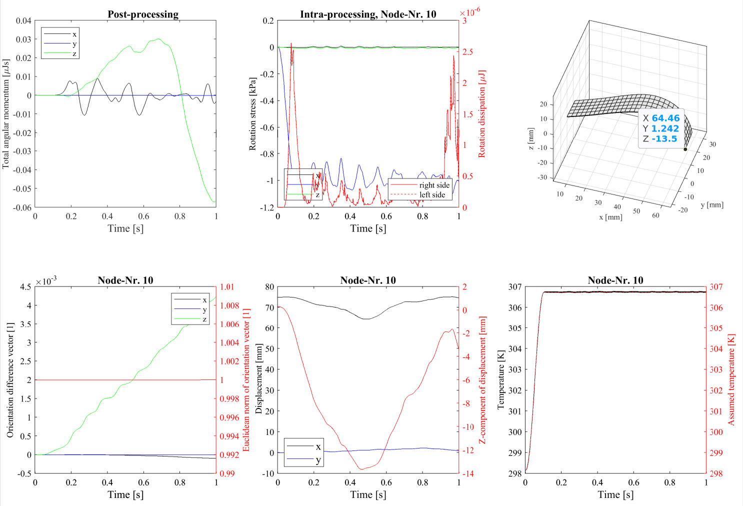
Bending motion in direction to the boundary normal of $\partial_\chi \mathcal{B}_0$

Movie of a soft strip $(\mathrm{E} \approx 0.914\,[\mathrm{MPa}],\ \nu \approx 0.493)$

Thin LCE strip subject to volume load

Bending motion in direction to the boundary normal of $\partial_\chi \mathcal{B}_0$

Nodal time evolutions and balance laws $(E \approx 0.914\,[\mathrm{MPa}],\ \nu \approx 0.493)$



Summary

1. Aims: Dynamic simulations of motion actuations
   - of liquid crystalline elastomers by means of
   - thermal Dirichlet and Neumann boundaries.
2. Numerical goals: Dynamic finite element simulations
   - with the approach of a mixed finite element method and
   - reorientations by means of drilling degrees of freedom.
3. Numerical strategy:
   - introducing an independent global orientation field,
   - formulating local rotations by drilling degrees of freedom,
   - using local evolution equations for stress-induced motions.
4. Numerical results: Motion actuations with
   - thermal Dirichlet and Neumann boundaries
   - activating bending motions as in experiments.
5. Next steps:
   - Simulation of UV light actuations by chemical processes