Multiscale time integration of thermo-mechanically coupled problems with energy-momentum consistent finite element methods in time

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Presently, researchers invest considerable time and effort in finding more robust and reliable time integration methods than standard time integration methods for thermo-mechanically coupled dynamics. One possible way to achieve this aim is the design of so-called structure-preserving time integrators, which ensure that physical properties of the underlying problem are inherited by the discrete solution. However, the ability of these methods to take into account different time scales in the different sub-solutions has hitherto based on a fractional step finite difference method. In this paper, we present an alternative way of using a Galerkin-based structure-preserving method.

1 Introduction

Recently, structure-preserving methods have proven their excellent robustness and reliability of finding meaningful solutions for different thermo-mechanical problems (see e.g. Romero [1], Krüger et al. [2] and Gross & Betsch [3]). However, the ability of these methods to take into account the usely present different time scales in the different sub-solutions has hitherto based on a fractional step method (see Kassiotis et al. [4] and Romero [5]). Using a fractional step method, we obtain a so-called h-adaption of the time axis because the problem is solved sequentially on so-called micro time steps (compare also Leyendecker & Ober-Blöbaum [6] for multiscale in time problems in constraint systems).

An alternative way is to use a Galerkin-based structure-preserving method in order to increase the degree of the trial functions in time. This corresponds to a p-adaptive method solving the problem simultaneously on all micro time steps (see Fig. 1). In this paper, we present such a Galerkin-based method designed for problems on different time scales. For instance, we consider finite thermo-viscoelastodynamics, which sub-solutions of the corresponding system of differential equations are represented by the deformation mapping, the velocity field, the temperature field and a locally defined tensor-valued internal variable. The time scales of these variables are generally different. The shortest time scale is usually present in the internal variables time evolution and the largest time scale is associated with the motion. The time scale of the temperature lies in between, through the thermo-mechanical coupling. By means of numerical examples, we show the advantage of this strategy and also compare the numerical effort.

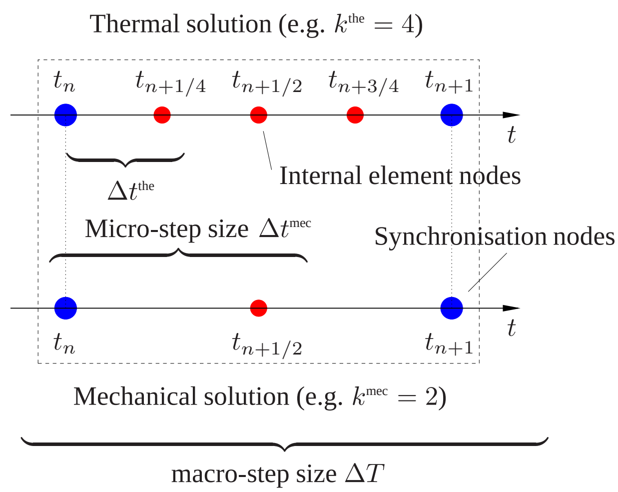

Fig. 1: Multiscale time approximation in a Galerkin-based time integration (cp. Kassiotis et al. [4] for finite differences in time).

2 Problem definition

We consider material points $\boldsymbol{X} \in \mathcal{B}_0$ of a homogenous body $\mathcal{B}_0$ with mass density ρ_0, mapped by the deformation mapping $\boldsymbol{\varphi}_t : \mathcal{B}_0 \to \mathcal{B}_t$ in the current configuration $\mathcal{B}_t$ at time t. The continuum motion $t \mapsto \boldsymbol{\varphi}_t$ is then determined by the Lagrangian local momentum balance $\rho_0 \partial_t \boldsymbol{v}_t = \mathrm{DIV}[\boldsymbol{F}_t \boldsymbol{S}_t] + \boldsymbol{b}_t$ with the velocity field $\boldsymbol{v}_t := \partial_t \boldsymbol{\varphi}_t$ and the deformation gradient field $\boldsymbol{F}_t := \partial_{\boldsymbol{X}} \boldsymbol{\varphi}_t$. The considered isotropic stress tensor field $\boldsymbol{S}_t := \boldsymbol{S}_t^{\text{ela}} + \boldsymbol{S}_t^{\text{vis}} + \boldsymbol{S}_t^{\text{the}}$ is a sum of an elastic part $\boldsymbol{S}_t^{\text{ela}} := 2\,\partial_{\boldsymbol{C}} \hat{\Psi}_{\text{ela}}(\boldsymbol{C}_t)$, a viscoelastic part $\boldsymbol{S}_t^{\text{vis}} := 2\,\partial_{\boldsymbol{C}} \hat{\Psi}_{\text{vis}}\left(\boldsymbol{C}_t \boldsymbol{\Gamma}_t^{-1}\right)$, dependent on a symmetric internal variable tensor $\boldsymbol{\Gamma}_t$, and a thermoelastic part $\boldsymbol{S}_t^{\text{the}} := 2\,\partial_{\boldsymbol{C}} \hat{\Psi}_{\text{the}}(\boldsymbol{C}_t, \theta_t)$, dependent on the temperature field $\theta_t : \mathcal{B}_0 \to \mathbb{R}$ of the body. Hence, the motion is coupled (i) to the local time evolution equation $\boldsymbol{M}_t^{\text{vis}} = n_{\text{dim}} V_{\text{sph}}\, \mathrm{SPH}([\boldsymbol{L}_t^{\text{vis}}]^T) + 2V_{\text{dev}}\, \mathrm{DEV}([\boldsymbol{L}_t^{\text{vis}}]^T)$ with respect to the viscous Mandel stress tensor $\boldsymbol{M}_t^{\text{vis}} := \boldsymbol{C}_t \boldsymbol{S}_t^{\text{vis}}$ and the viscous strain rate tensor $\boldsymbol{L}_t^{\text{vis}} := [\boldsymbol{\Gamma}_t]^{-1} \partial_t \boldsymbol{\Gamma}_t / 2$, and (ii) to the

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Lagrangian local entropy balance $\theta_t \partial_t \eta_t = -\mathrm{DIV}[\boldsymbol{Q}_t] + D_t^{\mathrm{vis}}$ subject to the prescribed entropy $\eta_t := -\partial_\theta \hat{\Psi}_{\mathrm{the}}$, Fourierian heat flux $\boldsymbol{Q}_t := -k_0\, \boldsymbol{C}_t^{-1} \partial_{\boldsymbol{X}} \theta_t$ and viscous internal dissipation $D_t^{\mathrm{vis}} = \boldsymbol{M}_t^{\mathrm{vis}} : \boldsymbol{L}_t^{\mathrm{vis}}$. This system of differential equations has to be completed by the usual initial boundary conditions and the initial condition $\boldsymbol{\Gamma}_{t_0} = \boldsymbol{I}$ of the internal variable.

3 Time integration

We use continuous finite elements in $\mathscr{B}_0$, continuous time finite elements for $\boldsymbol{\varphi}_t$ and $\boldsymbol{v}_t$, and modified discontinuous time finite elements for θ_t (ehG method). Then, we introduce nodal approximations $\boldsymbol{C}_t^*$ and η_t^*, respectively, whose nodal values are determined to allow energy consistency and a multiscale time integration. The aimed energy consistency demands algorithmic enhancements of $\boldsymbol{S}_t$ and D_t^{vis}. In the nodal approximation η_t^*, the nodal entropies cannot be determined in the usual way due to the distinct polynomial degrees. But, we found two possibilities to avoid this problem. First, we can introduce a new thermal right Cauchy-Green tensor $\boldsymbol{C}_t^{**}$ by a L_2-projection of $\boldsymbol{C}_t^*$ in the mechanical problem. Second, we can apply the energy consistency condition on η_t for a L_2-projection of η_t^* at the Gauss points in order to determine new nodal entropies.

4 Numerical results

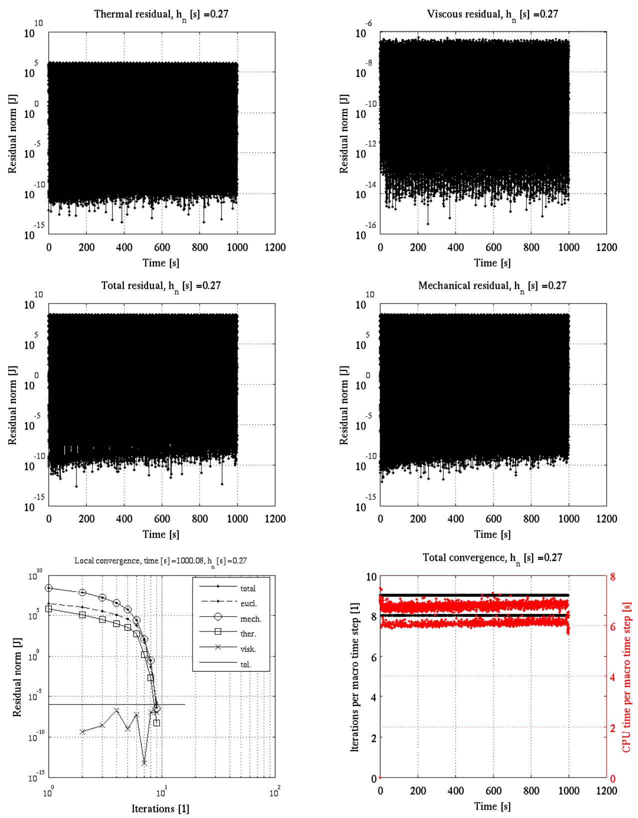

Fig. 2: Convergence behaviour ($k^{\mathrm{mec}} = 1; k^{\mathrm{the}} = 4; k^{\mathrm{vis}} = 1$; ehG method with entropy projection) with the largest time step size.

In Fig. 2, we show the very stable convergence behaviour of the global and local Newton-Raphson method, respectively. Fig. 3 depicts a convergence and effort comparison of both projections for linear time finite elements in the motion as well as the velocity, and higher finite elements in the temperature and the internal variable. We recognise no remarkable differences between both projections, but we clearly observe the positive influence of higher order finite elements in time.

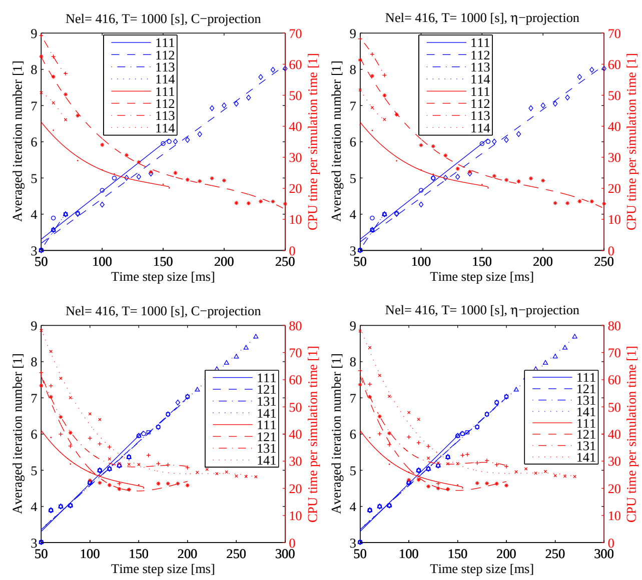

Fig. 3: Convergence and effort comparison ($k^{\mathrm{mec}} = 1; k^{\mathrm{the}} = 1, \ldots, 4; k^{\mathrm{vis}} = 1, \ldots, 4$; ehG method) of both projections.3

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